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The compact support property for solutions to the heat equation with noise

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Summary. We consider all solutions of a martingale problem associated with the stochastic pde $u_t = \frac{1}{2}u_{xx} + u^\gamma \dot{W}$ and show that $u(t, \cdot)$ has compact support for all $t \geqq 0$ if $u(0, \cdot)$ does and if $\gamma < 1$. This extends a result of T. Shiga who derived this compact support property for $\gamma \leqq 1/2$ and complements a result of C. Mueller who proved this property fails if $\gamma \geqq 1$.

1 Introduction

Consider solutions $u(t, x)$ with $t \geqq 0$, $x \in \mathbb{R}$, to the stochastic partial differential equation

$$(1.1)\qquad \begin{aligned} u_t &= \tfrac{1}{2}u_{xx} + u^\gamma \dot{W} \\ u(0, x) &= u_o(x) \end{aligned}$$

where $\dot{W} = \dot{W}(t, x)$ is 2-parameter white noise and u_t denotes the partial derivative of u with respect to t. Assume that $u(0, x)$ is continuous with compact support. D. Dawson asked us the question: For which γ does $u(t, x)$ have compact support as a function of x? In this paper, we prove compact support for $\gamma < 1$, with a certain caveat to be mentioned later. Compact support had been shown by Iscoe (1988, Theorem 1) for $\gamma = \frac{1}{2}$, and by Shiga for $\gamma < \frac{1}{2}$. Mueller (1990) showed non-compact support for $\gamma \geqq 1$. Iscoe's proof used properties of super-Brownian motion, and Shiga expanded on Iscoe's proof. The case $\frac{1}{2} < \gamma < 1$ is more delicate, as we explain below. We mention in passing that our theorem deals with a more general equation than (1.1), namely

$$(1.2)\qquad \begin{aligned} u_t &= \tfrac{1}{2}u_{xx} + a(u)\dot{W} \\ u(0, x) &= u_o(x) \end{aligned}$$

where $a(u) \geqq a_K u^\gamma$ for $u \leqq K$ and some $a_K > 0$, and $a(u) \leqq c(u^\theta + u)$ for some $0 < \theta < 1$ and $c > 0$. See Corollary 3.9 for the precise statement.

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The solution $u(t, x)$ can be viewed as the density of a super-Brownian motion, with birth and death rate depending on $u(t, x)$. Loosely speaking, super-Brownian motion $X_t(dx)$ is a measure-valued process arising as a limit of branching Brownian motions. To be precise, let $B_1^N(t), \ldots, B_{n(t)}^N(t)$ be independent Brownian motions which split in two or die, with equal probability, at times $\left\{\frac{k}{N}\right\}_{k=1}^{\infty}$. Let

$$X_T^N(dx) = \frac{1}{N} \sum_{i=1}^{n(t)} \delta_{B_i^N(t)}(dx) .$$

Choose $X_0^N(dx)$ to converge weakly, as $N \to \infty$, to a finite, nonnegative initial measure X_0. Then X_t^N converges weakly to the super-Brownian motion X_t. For more details, see Watanabe (1968) and Perkins (1988, Theorem 2.8). Although X_t can be defined for $x \in \mathbb{R}^d$, it is only when d = 1 that the measure X_t has a density $\frac{dX_t(x)}{dx} = u(t, x)$. This density $u(t, x)$ satisfies (1.1) with $\gamma = \frac{1}{2}$ (see Konno and Shiga 1988 or Reimers 1989). Iscoe showed that X_t has compact support for all t a.s., if X_0 has compact support. This was the motivation for our question.

By itself, (1.1) has no meaning since solutions $u(t, x)$ are not differentiable. We reformulate it as an integral equation:

$$\int u(t, x)\phi(x)dx = \int_0^t \int u(s, x)\phi''(x)/2dx\, ds + \int_0^t \int u(s, x)^\gamma \phi(x)\, dW(s, x) \quad \forall \phi \in D(\Delta/2),$$

where $D(\Delta/2) = \{\phi \in \bar{C} : \phi'' \in \bar{C}\}$ and $\bar{C}$ is the space of continuous functions on the one point compactification of $\mathbb{R}$. (Equivalently $D(\Delta/2)$ is the generator of one-dimensional Brownian motion on the Banach space $\bar{C}$.) The second integral is given meaning via the theory of martingale measures of Walsh (1986). Walsh also discusses existence and uniqueness results for such equations with Lipschitz coefficients. In our case, if $\gamma = 1$, then existence and uniqueness hold for all time. If $\gamma > 1$, existence and uniqueness hold up to an explosion time (which may be ∞). Finally, for $\gamma < 1$, solutions exist for all time, but uniqueness is not known. For $\gamma = \frac{1}{2}$, however, uniqueness in law follows from the martingale problem, which is tractable for this value of γ (see Roelly-Coppoletta 1986).

Here is the intuitive reason that the compact support problem is more delicate for $\gamma > \frac{1}{2}$. We note that $u_t = \frac{1}{2}u_{xx} + cu^{\frac{1}{2}}\dot{W}$ gives a super-Brownian motion with rate of birth and death equal to c. Our equation may be written as

$$u_t = \tfrac{1}{2}u_{xx} + u^{\gamma - \frac{1}{2}} u^{\frac{1}{2}} \dot{W} .$$

The behavior of $u(t, x)$ near the boundary of its support, where $u(t, x)$ is small, is critical for determining the nature of the support. Thus, if u is small and $\gamma < \frac{1}{2}$, then $u(t, x)$ is a super-Brownian motion with rate of birth and death speeded up. This means that particles are likely to die before they get very far, and the support will be compact. Our case was harder, since if $\gamma > \frac{1}{2}$ and $u(t, x)$ is small, then $u(t, x)$ is a super-Brownian motion with birth and death slowed down. Of course, if the birth and death rates slowed down to 0, then $u(t, x)$ would satisfy the heat equation without noise, and hence $u(t, x)$ would have noncompact support.

Our proof makes use of a "historical process" H_t, similar to the process introduced in Dawson and Perkins (1991) for studying the super-Brownian motion (the case $\gamma = \frac{1}{2}$). For fixed t, $H_t(dy)$ is a measure on continuous paths $y(s)$ which are held constant after t. This measure gives the ancestry of particles which are at position $y(t)$ at time t. We view these particles as infinitesimal components of the measure $u(t, x)dx$. To be more precise,

$$u(t, x)\, dx = H_t\{y(\cdot) : y(t) \in dx\} .$$

This particle picture is consistent with our description of $u(t, x)$ as a super-Brownian motion with birth and death rate depending on u. In order to use stochastic calculus, we construct H_t as a solution of a martingale problem. We do not know if H_t is the unique solution, but show that all densities $u(t, x)$ which arise from this martingale problem do satisfy (1.1) and do have compact support for all $t \geqq 0$. (This is the caveat mentioned above.) We conjecture that uniqueness in law holds for solutions of (1.1), and therefore that all solutions of (1.1), have compact support.

Working with these richer processes will allow us to prove a stronger result. We give a uniform modulus of continuity for all paths $y(\cdot)$ in the closed support of $H_t(dy)$ for all t in a compact interval. More precisely, for all $c > c_\gamma = (2/(1 - \gamma \vee \frac{1}{2}))^{1/2}$ and $T > 0$ there is a $\delta(c, T, \omega) > 0$ a.s. such that

$$|y(r) - y(s)| \leqq c(|r - s|\log 1/|r - s|)^{1/2} \equiv ch(r - s)\ \forall y \in \text{closed support of } H_t,\ t \leqq T$$
$$\text{and } r, s \leqq T \text{ satisfying } |r - s| < \delta .$$

See Theorem 3.5 for the precise result. For $\gamma = 1/2$ this agrees with the modulus of Dawson and Perkins (1991) for which $c_{1/2} = 2$ is known to be the best constant. It is not hard to derive a modulus of continuity for the support of $u(t, \cdot)$, $S(u(t))$. More precisely if c, T, δ are as above and $S^\varepsilon = \{x \in \mathbb{R}: |x - x'| \leqq \varepsilon \text{ for some } x' \in S\}$ then

$$S(u(t)) \subset S(u(s))^{ch(t-s)}\ \forall s, t \in [0, T] \text{ satisfying } t - s < \delta \text{ and } s \text{ rational} .$$

See Corollary 3.8 (and the ensuing discussion on the unpleasant but harmless assumption that s, but not t, be rational). For $\gamma = \frac{1}{2}$, this result was proved in Dawson et al. (1989).

Now we give a summary of the proof. The reader will note that the important ideas are simple, as are the key calculations given in Sect. 3. Setting up the historical process H_t, however, involves many technicalities.

Recall that we need to show a modulus of continuity for paths $y(t)$ in the support of $H_t(dy)$. The proof proceeds as in Lévy's modulus of continuity for Brownian motion. That is, we get estimates for the probability that any of the differences $|y(r) - y(s)|$ are large, where r, s are dyadic rationals of the form $k/2^n$. Assume $r < s < t$, and consider the set $A = \{y(\cdot): |y(r) - y(s)| > K\}$. Here K is related to the modulus. We show that $H_\tau(A)$, $\tau > s$ can be compared to X_τ, where

$$dX_\tau = X_\tau^\gamma dB_\tau .$$

This involves using Jensen's inequality to "split off" the evolution of the particles in the set A from the others. We show that $H_\tau(A)$ is small for $\tau < t$, (at least in expectation). Now if X_0 is small, then it is not hard to show that X_τ hits 0 very quickly. Of course, once X_τ hits 0, it stays there. Thus, with high probability, $H_t(A) = 0$. This means that, with high probability, there are no particles alive at time t whose ancestors satisfied $|y(r) - y(s)| > K$. Some further analysis, following arguments in Dawson et al. (1989), gives us the modulus.

As far as we know, (1.1) might have solutions $u(t, x)$ not arising from the historical process H_t. Our original proof of compact support did not use the historical process. We outline it here, in hopes that an industrious reader can use it to show compact support for all solutions. Consider Eq. (1.1) on the interval $x \in [-J, J]$ with Dirichlet boundary conditions. If $u(t, \cdot)$ has compact support in $(-J, J)$, it will have compact support on all of $\mathbb{R}$. There are 3 steps.

1. Using integration by parts, show that $M(t) = \int_{-J}^{J} u(t, x)h(t, x)dx$ is a non-negative supermartingale, where $h(t, x) = \frac{1}{T+t} \exp\left\{\frac{x^2}{2(T+t)}\right\}$. (Note that $h_t = -\frac{1}{2}h_{xx}$.) Hence $M(t)$ is bounded uniformly in t by a finite random variable. Therefore the integral of $u(t, x)$ over $|x|$ large must be small.

2. Split $u(t, x)$ into two parts as follows. Let $\dot{W}_1$ and $\dot{W}_2$ be independent white noises and show that if

$$\bar{u}_t = \tfrac{1}{2}\bar{u}_{xx} + \bar{u}^\gamma \dot{W}_1, \ \bar{u}(0, x) = u(0, x)1(|x| \leqq L)$$

$$v_t = \tfrac{1}{2}v_{xx} + \sqrt{(\bar{u} + v)^{2\gamma} - \bar{u}^{2\gamma}}\, \dot{W}_2\,, \quad v(0, x) = u(0, x)1(|x| > L)$$

then $u = \bar{u} + v$ satisfies (1.1).

3. To show, for example, that $u(1, \cdot)$ has compact support argue as follows. Use Jensen's inequality to show that $\int_{-J}^{J} v(t, x)\mathrm{d}x$ dies out faster than the solution $X(t)$ of $\mathrm{d}X = X^\gamma \mathrm{d}B$ (we leave the latter for the reader, along with many other things). This means that as v will have died out with high probability by time 1 we need only consider $\bar{u}$. Now split $\bar{u}$ as in step 2 but with $t = 1/2$ in place of 0 and L_1 in place of L. Using step 1 and the above one argues that the contribution from the mass outside $[-L_1, L_1]$ is extinct by $t = 1$ with high probability. Continuing in this way one obtains a sequence $\bar{u}^{(n)}(t, x)$ such that $\bar{u}^{(n)}(1 - 2^{-n}, \cdot)$ is supported on $[-L_n, L_n]$ and $u(1, x) = \bar{u}^{(n)}(1, x)$ with high probability: Let $n \to \infty$ to show that with high probability $u(1, \cdot)$ is supported on $[-L, L]$ where $\lim L_n = L < J$. To do the argument for all times one would imitate the proof of the modulus of continuity for super-Brownian motion (eg. in Dawson et al. 1989).

The catch in the above is that in step 2 it is not clear how to decompose a given solution of (1.1) in this manner. The historical approach allows us to decompose solutions in a natural manner and also allows us to formulate a more precise result.

Section 2 of this work sets up a martingale problem for the historical process associated with an ordinary super-process ($\gamma = \frac{1}{2}$). We allow our spatial motions to be the paths of a càdlàg Borel right process Y taking values in a Polish space and consider branching mechanisms in which the branching rate at x is $\sigma(x)$ where $\sigma : \mathbb{R} \to [0, \infty)$ is a bounded Borel measurable function. Hence, in the terminology of Dawson and Perkins (1991), we introduce a martingale problem for the (Y, Φ)-historical process where $\Phi(x, \lambda) = -\sigma^2(x)\lambda^2/2$. Using the general results of Fitzsimmons (1988, 1990) it is easy to obtain a martingale problem which characterizes the (Y, Φ)-historical process. Section 2 enlarges the class of test functions to a simpler and more useful class. Although this extension is needed for the proofs of our main results in Sect. 3, it has also found application in the construction and characterization of "interacting super-processes" and we therefore have presented these results in a more general setting. The martingale problem for the historical process associated with (1.2) is introduced in Sect. 3, where the main results described above are proved. Section 4 contains the proof of existence of solutions for this martingale problem. Although this is just a tightness argument, things are fairly complicated because we are simultaneously constructing a random density $u(t, x)$ and a process H_t taking values in the space of finite measures on continuous paths. The final argument (in Proposition 4.11) uses nonstandard analysis which is an efficient took for handling complicated tightness arguments of this sort. The reader who is unfamiliar with nonstandard analysis and in particular the Loeb space construction may wish to consult Cutland (1983) or Albeverio et al. (1986). We will use standard terminology and notation regarding Loeb measures from these references in the proof of Proposition 4.11 without further comment. However, we suspect that most readers will find this result "intuitively obvious" given the technical lemmas which precede it.

2 A martingale problem for the historical process

Our goal in this section is to derive a martingale problem for the historical process of Dawson and Perkins (1991). It will be used in the next section to set up the corresponding martingale problem for the historical process associated with (1.2). We begin by giving some definitions and results from the general theory of processes.

Assume $Y = (D, \mathscr{D}, \mathscr{D}_{t+}, \theta_t, Y_t, P_y)$ is the canonical realization of a Borel right process with càdlàg E-valued paths on $[0, \infty)$. $\mathscr{D}$ is the Borel σ-field of $D = D([0, \infty), E)$, $Y_t(y) = y(t)$ $(y \in D)$ and $\mathscr{D}_t = \sigma(Y_s : s \leqq t)$. If $w, y \in D$ and $t \geqq 0$, let $y^t(s) = y(s \wedge t)$ and

$$(y/t/w)(u) = \begin{cases} y(u) & \text{if } u < t \\ w(u-t) & \text{if } u \geqq t \end{cases}.$$

We set $\hat{E} = \{(s, y) \in [0, \infty) \times D : y = y^s\}$ and let $\hat{\mathscr{E}}$ denotes its Borel σ-field. $\mathscr{E}$ denotes the Borel σ-field on E and $p\mathscr{E}$ (respectively, $b\mathscr{E}$) is the cone of non-negative (respectively, bounded) $\mathscr{E}$-measurable $\mathbb{R}$-valued functions. Throughout this section we work with a fixed $s \geqq 0$ and

$$m \in M_F(D)^s = \{m \in M_F(D) : y = y^s \; m\text{-a.s.}\} .$$

Define a finite measure $P_{s,m}$ on D by

$$P_{s,m}(A) = \int P_{y(s)}(y/s/Y \in A)\, dm(y) .$$

We will usually work under a fixed $P_{s,m}$ and let $\bar{\mathscr{D}}_t^m = \mathscr{D}_{t+} \vee \{P_{s,m}\text{-null sets in } D\}$.

If $\{\mathscr{F}_t : t \geqq s\}$ is a filtration on some measurable space $(\Omega, \mathscr{F}\}$ the σ-fields of $(\mathscr{F}_t)_{t \geqq s}$-optional sets $(\mathscr{O}(\mathscr{F}_t))$ and $(\mathscr{F}_t)_{t \geqq s}$-predictable sets $(\mathscr{P}(\mathscr{F}_t))$ in $[s, \infty) \times \Omega$ are defined as in (Dellacherie and Meyer 1978, Chap. IV) with s in place of 0 and $\mathscr{F}_s$ playing the role of $\mathscr{F}_{0-}$.

Part (a) of the following theorem may be "well-known" but we have been unable to find it in the literature.

Proposition 2.1. (a) *Every $(\bar{\mathscr{D}}_t^m)_{t \geqq s}$-optional process is $P_{s,m}$-indistinguishable from a $(\mathscr{D}_t)_{t \geqq s}$-optional process.*
(b) *A Borel map $X : [s, \infty) \times D \to \mathbb{R}$ is $(\mathscr{D}_t)_{t \geqq s}$-optional if and only if*

$$X(t, y) = X(t, y^t)\; \forall (t, y) \in [s, \infty) \times D .$$

Proof. Let $\phi: D \to \mathbb{R}$ be bounded and Borel measurable and let $T \geqq s$ be a $(\bar{\mathscr{D}}_t^m)_{t \geqq s}$-stopping time. Then the strong Markov property of Y implies

(2.1) $$P_{s,m}(\phi \circ \theta_T \,|\, \bar{\mathscr{D}}_T^m) = P_{Y(T)}(\phi(Y))\ P_{s,m}\text{-a.s.}$$

(eg. see Dawson and Perkins 1991, Theorem 2.21 and its proof). It follows easily that

(2.2) $$P_{s,m}(\phi \,|\, \bar{\mathscr{D}}_T^m)(y) = P_{y(T)}(\phi(y/T(y)/Y))P_{s,m}\text{-a.e. } y\ .$$

For example, argue as in Example 6.12 of Sharpe (1988) but take his G to be $\mathcal{O}((\bar{\mathscr{D}}_t^m)_{t \geqq s}) \times \mathscr{D}$-measurable. Let $(N_t^\phi)_{t \geqq s}$ be a càdlàg version of $P_{s,m}(\phi \,|\, \bar{\mathscr{D}}_t^m)$ and let $R^\phi(t, y) = P_{y(t)}(\phi(y/t/Y))$ $(t \geqq s, y \in D)$. R^ϕ is Borel measurable because Y is Borel right, and an application of (b) shows R^ϕ is $(\mathscr{D}_t)_{t \geqq s}$-optional. (2.2) and the section theorem imply that N^ϕ and R^ϕ are $P_{s,m}$-indistinguishable.

Let ${}^\circ X$, respectively ${}^+X$, denote the $P_{s,m}$-optional projections of a bounded measurable $X: [s, \infty) \times D \to \mathbb{R}$ with respect to $(\mathscr{D}_t)_{t \geqq s}$, respectively $(\mathscr{D}_{t+})_{t \geqq s}$ (see Dellacherie and Meyer (1980, p. 412)). If $X(t, y) = \phi_1(t)\phi_2(y)$ (ϕ_i bounded measurable), then up to $P_{s,m}$-evanescent sets in $[s, \infty) \times D$ we have

$$ {}^+X(t, y) = \phi_1(t)N^{\phi_2}(t, y) = \phi_1(t)R^{\phi_2}(t, y) \text{ (by the above).}$$

Hence ${}^+X$ is $P_{s,m}$-indistinguishable from the $(\mathscr{D}_t)_{t \geqq s}$-optional process $\phi_1(t)R^{\phi_2}(t, y)$, which is therefore ${}^\circ X$ (up to $P_{s,m}$ indistinguishability).

Let

$$\mathscr{C} = \{X: [s, \infty) \times D \to \mathbb{R} \text{ bounded, measurable: } {}^\circ X \text{ and } {}^+X \text{ are } P_{s,m}\text{-indistinguishable}\}\ .$$

Then $\mathscr{K} \equiv \{X(t, y) = \phi_1(t)\phi_2(y): \phi_i \text{ bounded measurable}\} \subset \mathscr{C}$ by the above. It is clear that $\mathscr{C}$ is a real vector space and is closed under bounded monotone limits. Therefore a monotone class theorem (Sharpe 1988, pp. 369) shows that $\mathscr{C}$ contains all bounded Borel measurable functions on $[s, \infty) \times D$. Therefore any $(\mathscr{D}_{t+})_{t \geqq s}$-optional process is $P_{s,m}$-indistinguishable from a $(\mathscr{D}_t)_{t \geqq s}$-optional process (it is trivial to remove the boundedness assumption). As every $(\bar{\mathscr{D}}_t^m)_{t \geqq s}$-optional process is $P_{s,m}$-indistinguishable from a $(\mathscr{D}_{t+})_{t \geqq s}$-optional process (Dellacherie and Meyer 1980, p. 413), the proof of (a) is complete.

(b) This is immediate from Dellacherie and Meyer (1978, IV. 97). □

We are now ready to describe the historical process H of Dawson and Perkins (1991). The reader should think of H as giving the past histories of particles alive at t. The particles move according to independent copies of Y and branch into two or die with equal probability at rate $\sigma(x)$, depending on the location, x, of the particle. This means that although some of our results extend to more general branching mechanisms, we consider only

$$\Phi(x, \lambda) = -\sigma^2(x)\lambda^2/2 \quad x \in E,\ \lambda \in \mathbb{R}\ ,$$

where $\sigma: E \to [0, \infty)$ is bounded and measurable. $H = (G, \mathscr{G}, \mathscr{G}[s, t+], H_t, \mathbb{Q}_{s,m})$ denotes the (Y, Φ)-historical process introduced in Dawson and Perkins (1991) (see Theorem 2.2.3 of that work for a precise characterization). Here $G = D(M_F(D))$ (càdlàg $M_F(D)$-valued paths), $\mathscr{G}$ is its Borel σ-field, H_t is the coordinate mapping,

$$\mathscr{G}[s, t+] = \bigcap_{u > t} \sigma(H_r: s \leqq r \leqq u)$$

and although $\mathbb{Q}_{s,m}$ is defined on $\mathscr{G}[s, \infty)$ for any $s \geqq 0$ and $m \in M_F(D)^s$ we are focussing on our particular (s, m). H is an inhomogeneous Borel strong Markov process with càdlàg paths in $M_F(D)$ and such that $H_t \in M_F(D)^t \; \forall t \geqq s \; \mathbb{Q}_{s,m}$-a.s. (see Dawson and Perkins 1991, Definition 2.1.0). Let $\mathscr{G}^m[s, t] = \mathscr{G}[s, t+] \vee \{\mathbb{Q}_{s,m}$-null sets in $G\}$.

To help understand H we introduce the Borel right process $\hat{Y}$ with càdlàg paths in $\hat{E}(\hat{Y} = (\hat{D}, \hat{\mathscr{D}}, \hat{\mathscr{D}}_{t+}, \hat{\theta}_t, \hat{P}_{s,y}))$ and its semigroup

$$\hat{P}_t g(u, y) = P_{y(u)}(g(u + t, y/u/Y^t))$$

(see Dawson and Perkins 1991, Proposition 2.1.2, Theorem 2.2.1). If $\hat{X}_t$ is the $(\Phi, \hat{Y})$-superprocess starting at $\delta_s \times m$, then $(\hat{X}_t)_{t \geqq 0}$ is equal in law to $(\delta_{s+t} \times \mathrm{H}_{s+t})_{t \geqq 0}$, where H has law $\mathbb{Q}_{s,m}$. This is the definition of $\mathbb{Q}_{s,m}$ given in Sect. 2.1 of Dawson and Perkins (1991) where the reader may find a more detailed description of H.

If $\hat{E}_s = ([s, \infty) \times D) \cap \hat{E}$ and ϕ: $\hat{E}_s \to [0, \infty)$ is Borel we let

$$H_t(\phi_t) = \int_D \phi(t, y^t) H_t(dy) \quad t \geqq s \,. \tag{2.3}$$

Proposition 2.2 *Let* ϕ: $\hat{E}_s \to [0, \infty)$ *be Borel measurable.*

(a) $\{H_t(\phi_t) : t \geqq s\}$ *is* $\{\mathscr{G}[s, t] : t \geqq s\}$*-optional.*
(b) *If* Y *is a Hunt process,* $\{H_t(\phi_t) : t \geqq s\}$ *is* $\{\mathscr{G}[s, t] : t \geqq s\}$*-predictable.*

Proof. (a) is proved via a standard monotone class argument, starting with $\phi(t, y) = \phi_1(t)\phi_2(y^t)$ where ϕ_1 and ϕ_2 are Borel on $[s, \infty)$ and D, respectively.
(b) If $\phi(t, y) = \phi_1(t)\phi_2(y^t)$ where ϕ_1: $[s, \infty) \to [0, \infty)$ and ϕ_2: $D \to [0, \infty)$ are bounded and continuous then $H_t(\phi_t) = \phi_1(t) H_t(\phi_2)$ is predictable since $H_t(\phi_2)$ is $\mathbb{Q}_{s,m}$-a.s. continuous for $t \geqq s$ by the weak continuity of H. The latter holds because Y is Hunt (Dawson and Perkins 1991, Theorem 2.2.3(c)). A monotone class argument gives the result for $\phi(t, y) = \psi(t, y^t)$ when ψ: $[s, \infty) \times D \to [0, \infty)$ is Borel and the result follows. □

Fitzsimmons (1988) established the path regularity of a broad class of superprocesses by a clever use of Rost's theorem on Skorokhod embedding. Since H_t is a trivial projection of a super-process $(\hat{X}_t)$ which falls under the purview of Fitzsimmons' work it is easy to apply his results to this setting. Our point of view is a little different as we prefer to work with a fixed initial measure. We sketch a few of the arguments in the following proof only because of some minor technical modifications we make to the arguments in Sect. 3 of Fitzsimmons (1988).

Notation. $F = F_{s,m} = \{\phi : \hat{E}_s \to \mathbb{R}$: ϕ Borel measurable $\phi(s, Y^s)$ is right-continuous $P_{s,m}$-a.s. and $\sup_{t \geqq s} |\phi(t, Y^t)| \leqq K \; P_{s,m}$-a.s. for some $K\}$.

Theorem 2.3. (a) *If* ϕ, ψ: $\hat{E}_s \to \mathbb{R}$ *are Borel and satisfy* $\phi(t, Y^t) = \psi(t, Y^t) \; \forall t \geqq s$ $P_{s,m}$-a.s., *then* $\phi_t(y) = \psi_t(y)$ H_t-a.a.y, *and hence* $H_t(\phi_t) = H_t(\psi_t)$, (*one exists* iff *the other does, in which case they are equal*) $\forall t \geqq s$ $\mathbb{Q}_{s,m}$-a.s.
(b) *If* $\phi \in F_{s,m}$, *then* $H_t(\phi_t)$ *is right-continuous on* $[s, \infty)$ $\mathbb{Q}_{s,m}$-a.s. *If, in addition* $\phi(t, Y^t)$ *has left-limits on* $[s, \infty) P_{s,m}$-a.s. *then so does* $H_t(\phi_t) \mathbb{Q}_{s,m}$-a.s.
(c) *If* Y *is a Hunt process and* $\phi \in F_{s,m}$ *is such that* $\{\phi(t, Y^t) : t \geqq s\}$ *is quasi-left-continuous under* $P_{s,m}$, *then* $H_t(\phi_t)$ *is continuous on* $[s, \infty) \mathbb{Q}_{s,m}$-a.s.

Proof. Let $T \geqq s$ be a bounded $\{\bar{\mathscr{G}}^m[s,t]: t \geqq s\}$-stopping time, $\alpha > 0$, and define ν on $(\hat{E}, \hat{\mathscr{E}})$ by

$$\nu(g) = \mathbb{Q}_{s,m}(e^{-\alpha T} H_T(g_T)) = \mathbb{Q}_{\delta_s \times m}(e^{-\alpha T} \hat{X}_{T-s}(g)) .$$

If U is the potential operator of $\hat{Y}$ but killed at rate α, then an easy calculation shows $\nu U \leqq (\delta_s \times m) U$ and so, using Rost's theorem as in Fitzsimmons (1988), one sees there is a randomized stopping time $V(x, \hat{Y})$ for $\hat{Y}$ ($x \in [0, 1]$) such that $\forall g \in p\hat{\mathscr{E}}$

$$\begin{aligned} \nu(g) &= \hat{P}_{\delta_s \times m}(e^{-\alpha V} g(\hat{Y}_V)) = \int P_{y(s)}(e^{-\alpha V} g(s+V, y/s/Y^V))dm(y) \\ &= e^{\alpha s} P_{s,m}(e^{-\alpha W} g(W, Y^W)) , \end{aligned} \tag{2.4}$$

where $W(x, Y) = s + V(x, (s + \cdot, Y^{s+\cdot}))$ is a randomized stopping time for Y and we have suppressed the integral over $([0, 1], dx)$. Apply (2.4) to $g(t,y) = |\phi(t,y) - \psi(t,y)|$ (set $g(t,y) = 0$ for $t < s$) and use the hypotheses on ϕ, ψ to conclude that $H_T(|\phi_T - \psi_T|) = 0$ $\mathbb{Q}_{s,m}$-a.s. The result now follows by the section theorem and Proposition 2.2.

(b) We omit the proof as it will be clear from that of (c) below and the proof of (Fitzsimmons 1988, Theorem 3.5(a)).

(c) Let $s \leqq T_n \uparrow T_\infty$ be bounded $\{\hat{\mathscr{G}}^m[s,t]: s \geqq t\}$-stopping times, let $\alpha > 0$, and define ν_n, ν_∞ on $(\hat{E}, \hat{\mathscr{E}})$ by

$$\nu_n(g) = \mathbb{Q}_{s,m}(e^{-\alpha T_n} H_{T_n}(g_{T_n})), \quad n \in \mathbb{N} \cup \{\infty\} .$$

If U is as in (a) then one can easily check $(\delta_s \times m) U \geqq \nu_n U \downarrow \nu U$, and so, as in Fitzsimmons (1988, Proof of (3.4)) there are randomized stopping times $\{V_n, V_\infty\}$ for $\hat{Y}$ such that $V_n \uparrow V_\infty$ pointwise and $\forall g \in p\hat{\mathscr{E}}$, $n \in \mathbb{N} \cup \{\infty\}$,

$$\begin{aligned} \nu_n(g) &= \hat{P}_{\delta_s \times m}(e^{-\alpha V_n} g(\hat{Y}(V_n))) \\ &= e^{\alpha s} P_{s,m}(e^{-\alpha W_n} g(W_n, Y^{W_n})) \end{aligned} \tag{2.5}$$

where $W_n(x, Y) = s + V(x, (s + \cdot, Y^{s+\cdot}))$ is a randomized stopping time for Y, as in (a). (2.5) extends immediately to $g: \hat{E}_s \to \mathbb{R}$ which are Borel and such that $\sup_{t \geqq s} |g(t, Y^t)| \leqq K$ $P_{s,m}$-a.s. Setting $g = \phi$ in (2.5) and using the $P_{s,m}$ quasi-left-continuity of ϕ, we see from (2.5) that $\nu_n(\phi) \to \nu_\infty(\phi)$. Therefore $e^{-\alpha t} H_t(\phi_t)$ is $\mathbb{Q}_{s,m}$-a.s. left-continuous on (s, ∞) by (Dellacherie and Meyer 1978, IV. 44) and the predictability of $H_t(\phi_t)$ established in Proposition 2.2. The right-continuity of $H_t(\phi_t)$ (from (b)) completes the proof. □

In Fitzsimmons (1988, 1990) a broad class of superprocesses were characterized as solutions of a martingale problem, described in terms of a weak generator of the underlying process Y. A direct interpretation of these results in our setting gives a class of functions $\phi(t, Y^t)$ which is a little too restrictive for our purposes. For our fixed $P_{s,m}$ we introduce the domain

$$\begin{aligned} D(A_{s,m}) = \{\phi \in F_{s,m} : \exists A_{s,m}\phi \in F_{s,m} \text{ such that } \\ M^\phi(t, Y) = \phi(t, Y^t) - \phi(s, Y^s) - \int_s^t A_{s,m}\phi(r, Y^r) dr \text{ is a } (\bar{\mathscr{D}}_t^m)_{t \geqq s} \\ - \text{martingale under } P_{s,m}\} . \end{aligned}$$

Note that $A_{s,m}\phi$ (or $A\phi$ if there is no ambiguity) is uniquely defined up to $P_{s,m}$-evanescent sets. We call $A_{s,m}$ th $P_{s,m}$-weak generator of Y.

Remark 2.4 If $\hat{A}$ is the weak generator of the $\hat{E}$-valued Markov process $\hat{Y}$, defined in (Fitzsimmons 1988, Sect. 4) it is easy to check that $\phi \in D(\hat{A})$ implies $\phi|_{\hat{E}_s} \in D(A_{s,m})$ and $A_{s,m}\phi = \hat{A}\phi|_{\hat{E}_s}$. Since $D(\hat{A})$ is dense in $b\hat{\mathscr{E}}$ (for bounded pointwise convergence), we see that $D(A_{s,m})$ is dense in $b\hat{\mathscr{E}}_s$, again in the bounded pointwise sense. A slightly stronger result for one-dimensional Brownian motion (the proof in fact holds for any Feller process) will be derived in Sect. 4 (Lemma 4.10).

Corollary 2.5 *If Y is a Hunt process and $\phi \in D(A_{s,m})$, then $H_t(\phi_t)$ is continuous on* $[s, \infty)\mathbb{Q}_{s,m}$-a.s.

Proof. Since Y is Hunt it follows that $(\hat{\mathscr{D}}_t^m)_{t \geq s}$ is quasi-left-continuous (Sharpe 1988, (47.6)) and so M_t^ϕ (in the definition of $D(A_{s,m})$) is quasi-left-continuous under $P_{s,m}$. It follows from the definition of $D(A_{s,m})$ that $\phi(t, Y^t)$ is quasi-left-continuous under $P_{s,m}$ and so the result follows from Theorem 2.3. □

Remarks 2.6 (a) The restriction to Borel maps on $\hat{E}_s$, or equivalently (by 2.2(b)) $(\mathscr{D}_t)_{t \geq s}$-optional processes is necessary. It is *not* true that if ϕ and ψ are $P_{s,m}$-indistinguishable bounded $(\mathscr{D}_{t+})_{t \geq s}$-optional processes, then $H_t(\phi_t)$ and $H_t(\psi_t)$ are $\mathbb{Q}_{s,m}$-indistinguishable. To see this let $(s, m) = (0, \delta_0)$, $t_0 \geq 0$ and set

$$\psi(t, y) = 1(t \geq t_0,\ \overline{\lim}_{h \downarrow 0}|y(t_0 + h) - y(t_0)|(h \log\log 1/h)^{-1/2} = 1)$$

where Y is Brownian motion on $\mathbb{R}$. Then $\psi(t, y)$ and $\phi(t, y) = 1(t \geq t_0)$ are $P_{s,m}$-indistinguishable $(\mathscr{D}_{t+})$-optional processes but $H_{t_0}(\psi_{t_0}) = 0$ because $y^{t_0} = y$ H_{t_0}-a.a. y and $H_{t_0}(\phi_{t_0}) = H_{t_0}(1)$. This explains the importance of Proposition 2.1(a).
(b) The converse to Theorem 2.3(a) is false. To see this let (s, m, Y) be as in (a) above and let $\phi(t, y) = 1(\{y: y_t \in A\})$ where A is a non-empty Lebesgue null set of $\mathbb{R} - \{0\}$ and let $\psi(t, y) \equiv 0$. Then $H_t(\phi_t) = X_t(A)$ where X is super-Brownian motion. By Reimers (1989) $X_t(B) = 0\ \forall t > 0\ \mathbb{Q}_{0,m}$-a.s. if and only if B is Lebesgue null (see also Perkins 1991). Hence $H_t(\phi_t)$ *and* $H_t(\psi_t)$ are $Q_{s,m}$-indistinguishable but clearly $\phi(t, Y^t) = 1(Y(t) \in A)$ and 0 are not P_{0,δ_0}-indistinguishable. In view of Reimers' theorem, the natural conjecture is:

(2.6) If $A \in \hat{\mathscr{E}}_s$ then $\int 1_A(t, y) dH_t(y) = 0 \quad \forall t > s\ \mathbb{Q}_{s,m}$-a.s.

if and only if $P_{s,m}(1_A(t, Y^t)) = 0 \quad \forall t > s$

(at least for an appropriate class of Y's). The necessity of the latter condition is of course trivial.

Theorem 2.7 *Assume Y is a Hunt process. $\mathbb{Q}_{s,m}$ is the unique probability on $\mathscr{G}[s, \infty)$ such that $\forall \phi \in D(A_{s,m})$*

$$Z_t(\phi) = H_t(\phi_t) - m(\phi_s) - \int_s^t H_r(A\phi_r)\,dr, \quad t \geq s$$

is a continuous $\{\mathscr{G}[s, t+]: t \geq s\}$-martingale such that $Z_s(\phi) = 0$ *and*

$$\langle Z(\phi)\rangle_t = \int_s^t \int \sigma^2(y_r)\phi(r, y)^2 H_r(dy)\,dr\,.$$

Proof. Since $A_{s,m}$ is an extension of Fitzsimmons' weak infinitesimal generator of $\hat{Y}$, $\hat{A}$, (see Remark 2.4), the uniqueness of $\mathbb{Q}_{s,m}$ now follows easily from the uniqueness of the martingale problem for $\hat{X}$ (under $\hat{\mathbb{Q}}_{\delta_s \times m}$) established in Fitzsimmons (1990).

Let $t > r \geqq s$ and $\phi \in D(A_{s,m})$. Therefore

$$P_{s,m}(\phi(t, Y^t) - \phi(r, Y^r)|\mathscr{D}_r^m) = \int_r^t P_{s,m}(A\phi(u, Y^u)|\mathscr{D}_r^m)\,du$$

and so by the Markov property (see (2.2)) for $P_{s,m}$-a.a.y.

(2.7) $$P_{y(r)}(\phi(t, y/r/Y^{t-r})) = \phi(r, y^r) + \int_r^t P_{y(r)}(A\phi(u, y/r/Y^{u-r}))\,du\,.$$

By the superprocess property (Dawson and Perkins 1991, Theorem 2.1.5(d)) we have

(2.8) $\mathbb{Q}_{s,m}(H_r(g)) = P_{s,m}(g(Y^r))\ \forall r \geqq s,\ g\colon D \to [0, \infty)$ Borel measurable,

and so

(2.9) (2.7) holds for H_r – a.a.y $\mathbb{Q}_{s,m}$-a.s.

The Markov property of H (Dawson and Perkins 1991, Theorem 2.1.5(a)) implies that $\mathbb{Q}_{s,m}$-a.s.,

(2.10)
$$\begin{aligned}\mathbb{Q}_{s,m}(H_t(\phi_t)|\mathscr{G}^m[s,r]) &= \mathbb{Q}_{r,H_r}(H_t(\phi_t))\\ &= \int P_{y(r)}(\phi(t, y/r/Y^{t-r}))H_r(dy) \quad \text{(by (2.8))}\\ &= H_r(\phi_r) + \int_r^t \int P_{y(r)}(A\phi(u, y/r/Y^{u-r}))H_r(dy)\,du \quad \text{(by (2.9))}\\ &= H_r(\phi_r) + \int_r^t \mathbb{Q}_{s,m}(H_u(A\phi_u)|\mathscr{G}^m[s,r])\,du\end{aligned}$$

where we have used (2.10) with $A\phi$ in place of ϕ and u in place of t. We have shown that $Z_t(\phi) = H_t(\phi_t) - m(\phi_s) - \int_s^t H_u(A\phi_u)\,du$ is a $\{\mathscr{G}^m[s,t]: t \geqq s\}$- and hence a $\{\mathscr{G}[s, t+]: t \geqq s\}$-martingale. Since $t \to \int_s^t H_u(A\phi_u)du$ is $\mathbb{Q}_{s,m}$-a.s. continuous ($H_u(|A\phi_u|) \leqq KH_u(1)$ for all $u \geqq 0$ $\mathbb{Q}_{s,m}$-a.s. for some K by Theorem 2.3(a) and the latter is $Q_{s,m}$-a.s. continuous by Corollary 2.5), Corollary 2.5 implies $Z_t(\phi)$ is a continuous martingale under $\mathbb{Q}_{s,m}$.

We may, and shall, assume ϕ and $A\phi$ are uniformly bounded on $\hat{E}_s$ (not just $P_{s,m}$-a.s.) since Theorem 2.3(a) shows an appropriate truncation will not affect the processes of interest. This will avoid any integrability concerns in what follows.

A direct calculation using the definition of H in terms of the $\hat{Y}$-superprocess, $\hat{X}$, and the known formula for $\hat{\mathbb{Q}}_{\delta_r \times v}(\hat{X}_{t-r}(\phi)^2)$ (see Fitzsimmons 1988, (2.7)) gives for $s \leqq r \leqq t$

$$\begin{aligned}\mathbb{Q}_{r,H_r}(H_t(\phi_t)^2) &= P_{r,H_r}(\phi(t, Y^t))^2\\ &\quad + \int_0^{t-r} \int P_{y(r)}(\sigma^2(Y_u)P_{Y(u)}(\phi(t, (y/r/Y^u)/r + u/\cdot^{t-r-u}))^2)H_r(dy)\,du\\ &= P_{r,H_r}(\phi(t, Y^t))^2 + P_{r,H_r}\left(\int_s^t \sigma^2(Y_v)P_{Y(v)}(\phi(t, Y/v/\cdot^{t-v}))^2\,dv\right)\end{aligned}$$

$$(2.11)\qquad Q_{r,H_r}(H_t(\phi_t)^2) = P_{r,H_r}(\phi(t, Y^t))^2 + P_{r,H_r}\left(\int_r^t \sigma^2(Y_v)\left[\phi(v, Y^v) + \int_v^t P_{Y(v)}(A\phi(u, Y/v/\cdot^{u-v}))\,du\right]^2 dv\right)$$

(this follows from (2.7) as for (2.9)). Therefore

$$\left|\mathbb{Q}_{s,m}\left(\left(H_t(\phi_t) - H_r(\phi_r)\right)^2 - \int_r^t\int \sigma^2(y_v)\phi(v, y^v)^2\, H_v(dy)dv\,|\,\mathscr{G}^m[s,r]\right)\right|$$

$$= \left|\mathbb{Q}_{r,H_r}(H_t(\phi_t)^2) - 2H_r(\phi_r)P_{r,H_r}(\phi(t, Y^t)) + H_r(\phi_r)^2 - P_{r,H_r}\left(\int_s^t \sigma^2(Y_v)\phi^2(v, Y^v)dv\right)\right| \quad \text{(by the Markov property of } Y \text{ and (2.8))}$$

$$= \left|[P_{r,H_r}(\phi(t, Y^t) - \phi(r, Y^r))]^2 + P_{r,H_r}\left(\int_r^t \sigma^2(Y_v)2\phi(v, Y^v) \times \int_v^t P_{Y(v)}(A\ \phi(Y/v/\cdot^{u-v}))\,du\,dv\right) + P_{r,H_r}\left(\int_r^t \sigma^2(Y_v)\left[\int_v^t P_{Y(v)}(A\phi(Y/v/\cdot^{u-v}))\,du\right]^2 dv\right)\right| \quad \text{(by (2.11))}$$

$$\leqq \left(\int_r^t P_{r,H_r}(A\phi(u, Y^u))\,du\right)^2 + [2\|\phi\|_\infty\|\sigma^2\|_\infty\|A\phi\|_\infty(t-r)^2/2 + \|\sigma^2\|_\infty\|A\phi\|_\infty^2(t-r)^3]H_r(1)$$

(use (2.9) in the first term)

$$\leqq c(\sigma^2, \phi)H_r(1)[(t-r)^2 + (t-r)^3].$$

It follows that the quadratic variation of the continuous semimartingale $H_t(\phi_t)$ is $\langle H(\phi)\rangle_t = \int_s^t \int \sigma^2(y_v)\phi^2(v, y^v)H_v dy)\,dv$ (note this process is continuous in t) and we are done. □

3 The compact support property

Now we focus on Eq. (1.2) described in the Introduction and introduce a martingale problem which should be solved by a "historical process" H associated with (1.2). The existence of a solution to this martingale problem will be established in the next section. The reader should pay close attention to the estimates (3.15) and (3.16) since these are key steps in our argument.

$Y = (C, \mathscr{C}, \mathscr{C}_{t+}, \theta_t, Y_t, P_y)$ is the canonical representation of Brownian motion on the space C of continuous $\mathbb{R}$-valued functions on $[0, \infty)$ with its Borel σ-field $\mathscr{C}$. (C is given the compact-open topology). $\Delta/2$ denotes the generator of Y on $D(\Delta/2)$ in the Banach space $\bar{C}$ of continuous functions on $\bar{\mathbb{R}}$, the one point compactification of $\mathbb{R}$. Let $a: [0, \infty)) \to \mathbb{R}$ be a continuous function satisfying

(3.1) $$|a(u)| \leqq c_{3.1}(u + u^\theta) \quad \text{for some} \quad 0 < \theta < 1 .$$

In order to study solutions of

(3.2) $$\frac{\partial u}{\partial t}(t) = (\Delta/2)u_t(x) + a(u(t, x))\dot{W}$$

where $\dot{W}$ is white noise on $[0, \infty) \times \mathbb{R}$ we introduce a martingale problem for an $M_F(C)$-valued process. We further specialize the notation of the last section by letting Ω denote the space of weakly continuous $M_F(C)$-valued paths on $[0, \infty)$ with its Borel σ-field $\mathscr{F}$ and canonical right-continuous filtration $(\mathscr{F}_{t+})$. We slightly abuse the notation of the previous section and continue to let $H_t(\omega) = \omega(t)$ denote the coordinate mappings on Ω. Let π_t: $C \to \mathbb{R}$ be the projection map $\pi_t(y) = y(t)$ and define $X_t \in M_F(\mathbb{R})$ by $X_t(A) = H_t(\pi_t^{-1}(A))$. Fix $u_0(x) \geqq 0$, a bounded, continuous, Lebesgue-integrable function on the line and let $m(dx) = u_0(x)\,dx \in M_F(\mathbb{R})$. We use the notations $A = A_{0,m}$ and $P_m = P_{0,m}$ from the previous section with Y as above. Let

$$b(u) = a(u)^2 u^{-1} 1(u > 0) .$$

Assume $\mathbb{Q}_m$ is a probability on $(\Omega, \mathscr{F})$ which solves the following martingale problem:

(M_m) $$\forall \phi \in D(A) \ Z_t(\phi) = H_t(\phi_t) - m(\phi_0) - \int_0^t H_r(A\phi_r)\,dr$$

is a continuous, square-integrable $(\mathscr{F}_{t+})$-martingale such that $Z_0(\phi) = 0$ and

$$\langle Z(\phi)\rangle_t = \int_0^t \int b(u(r, y_r))\phi(r, y)^2 dH_r(y)\,dr.$$

Here $u(t, x)$ is a jointly continuous non-negative density such that

$$X_t(A) = \int_A u(t, x)\,dx \ \forall A \in \mathscr{B}(\mathbb{R}), \ t \geqq 0 \ \mathbb{Q}_m\text{-a.s.}$$

Also $y = y^t$ for H_t-a.a. y $\forall t \geqq 0$ $\mathbb{Q}_m$-a.s.

The existence of such a $\mathbb{Q}_m$ will be established in the next section under an addition condition on u_0. To be consistent with the previous section let

$$\hat{\mathbb{R}} = \{(t, y) \in [0, \infty) \times C : y = y^t\}, \ \hat{\mathscr{B}} = \text{the Borel } \sigma\text{-field of } \hat{\mathbb{R}}.$$

Lemma 3.1 $\mathbb{Q}_m(H_t(\phi_t)) = P_m(\phi(t, Y^t)) \ \forall \phi \in b\hat{\mathscr{B}}, t \geqq 0$.

Proof. Fix ϕ, t as above and let $\psi(s, y)$ be a right-continuous, bounded, $(\mathscr{C}_s)$-optional version of the martingale $P_m(\phi(t, Y^t)|\bar{\mathscr{C}}_s^m)(y)$. (Here we are trivially extending Proposition 2.1, as well as its notation, to the setting of continuous paths.) We may, and shall, also assume $\psi(s, y) = \phi(t, y^t)$ $\forall s \geqq t, y \in C$. Then $\psi \in D(A)$, $A\psi = 0$, and so (M_m) implies

$$\mathbb{Q}_m(H_t(\phi_t)) = \mathbb{Q}_m(H_t(\psi_t)) = m(\psi_o) = P_m(\phi(t, Y^t)) . \quad \square$$

If $\phi_n \in D(A)$, $\phi \in b\hat{\mathscr{B}}$ and $\phi_n \xrightarrow{bp} \phi$ (bounded pointwise convergence), then

$$\mathbb{Q}_m(\langle Z(\phi_n) - Z(\phi_k)\rangle_t) = \mathbb{Q}_m\left(\int_0^t \int b(u(r, y_r))(\phi_n(r, y) - \phi_k(r, y))^2 H_r(dy)\, dr\right)$$

$$\to 0 \quad \text{as} \quad n, k \to \infty\,.$$

the last by dominated convergence (recall $\mathbb{Q}_m(\langle Z(1)\rangle_t) < \infty$). Hence $Z(\phi_n)(t)$ converges uniformly in t on compacts in L^2 to a continuous square integrable martingale. Using the denseness of $D(A)$ in $b\hat{\mathscr{B}}$ (see Remark 2.4) we extend Z to an orthogonal martingale measure $\{Z(\phi)\colon \phi \in b\hat{\mathscr{B}}\}$ (see Walsh 1986) such that

$$\langle Z(\phi)\rangle_t = \int_0^t \int b(u(r, y_r))\phi(r, y)^2 H_r(dy)\, dr \quad \forall \phi \in b\hat{\mathscr{B}}\,. \tag{3.3}$$

As in Walsh (1986, Chap. 2) if $\phi\colon [0, \infty) \times \Omega \times C \to \mathbb{R}$ is $\mathscr{P}(\mathscr{F}_{t+}) \times \mathscr{C}$-measurable and satisfies

$$\mathbb{Q}_m\left(\int_0^t \int b(u(r, y_r))\phi(r, y)^2 H_r(dy)\, dr\right) < \infty \quad \forall t > 0\,, \tag{3.4}$$

then one may further extend Z to stochastic integrals of the form

$$Z_t(\phi) = \int_0^t \int \phi(r, w, y^r)\, dZ(r, y)\,. \tag{3.5}$$

Here $Z_t(\phi)$ is a continuous square integrable martingale which satisfies (3.3) (with ϕ now depending on ω as well).

By working on an appropriate product space we may introduce a white noise $\tilde{W}$ on $[0, \infty) \times \mathbb{R}$ (based on Lebesgue measure) which is independent of H. If $\phi\colon \mathbb{R} \to \mathbb{R}$ is square integrable, define

$$\begin{aligned} W_t(\phi) = {} & \int_0^t \int a(u(r, y_r))^{-1} \phi(y(r)) 1(u(r, y_r) \neq 0)\, dZ(r, y) \\ & + \int_0^t \int 1(u(r, x) = 0)\phi(x)\, d\tilde{W}(r, x)\,. \end{aligned} \tag{3.6}$$

It is easy to check the integrability condition (3.4) required for the existence of the first stochastic integral. Then $W_t(\phi)$ is a continuous orthogonal martingale measure with square function $\langle W(\phi)\rangle_t = t\int \phi^2(x)\, dx$. Hence $\dot{W}$ is a white noise on $[0, \infty) \times \mathbb{R}$ with respect to Lebesgue measure (Walsh 1986, 2.10).

Let $p(t, x)$ denote the transition density of standard one-dimensional Brownian motion and let $P_t f(x) = \int f(y) p(t, y - x)\, dy$ denote the Brownian semigroup.

We now show that the density $u(t, x)$ in (M_m) does satisfy the stochastic pde (3.2).

Proposition 3.2 The density $u(t, x)$ in (M_m) satisfies

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{2} u_{xx}(t, x) + a(u(t, x)\dot{W}, \quad u(0) = u_0 \tag{3.7}$$

$$u(t, x) = P_t u_0(x) + \int_0^t \int p(t - s, x' - x) a(u(s, x'))\, dW(s, x') \ \mathbb{Q}_m\text{-a.s. } \forall (t, x)\,. \tag{3.8}$$

Proof. If $\psi \in D(\Delta/2)$, Dynkin's formula implies that $\phi(t, y) = \psi(y(t)) \in D(A)$ and $A\phi(t, y) = (\Delta/2)\psi(y(t))$. Therefore (M_m) implies

$$X_t(\psi) = H_t(\phi_t) = m(\phi) + Z_t(\phi) + \int_0^t H_r(A\phi_r)\,dr$$

$$= m(\psi) + \int_0^t\int a(u(r, x))\psi(x)\,dW(r, x) + \int_0^t X_r((\Delta/2)\psi)\,dr\,.$$

The last line is an easy consequence of (3.6) and $a(0) = 0$. This proves (3.7).

The derivation of (3.8) is then standard. In fact it is probably simpler to derive (3.8) directly from (M_m) by taking $\phi(s, y) = p_{t-s\wedge t+\varepsilon}(y(s \wedge t) - x)$ in $(M_m)(A\phi = 0)$ and letting $\varepsilon \to 0$. The argument is then almost identical to the derivation of Proposition 4.2(b) in the next section. The required integrability conditions on u follow easily from the estimates in Sect. 6 of Shiga (1991) (as will be discussed in detail on Sect. 4 below). □

We will need to know that our solutions $u(t, x)$ are sufficiently rapidly decreasing in x. For $g \in C(\mathbb{R})$ (continuous functions from $\mathbb{R}$ to $\mathbb{R}$), let

$$|g|_p = \sup_x e^{p|x|}|g(x)|$$

and define

$$C^+_{\text{rap}} = \{g \in C(\mathbb{R}) : g \geqq 0 \quad \text{and} \quad |g|_p < \infty \quad \forall p > 0\}\,.$$

We topologize C^+_{rap} by the collection of norms $\{|\ |_p : p > 0\}$ in the usual way.

The following result follows from Theorem 2.4 of Shiga (1990) and its proof. It is the first time (3.1) is used.

Proposition 3.3 *If*

$$u_0 \in C^+_{\text{rap}}\,, \tag{3.9}$$

then

$$t \to u_t(\cdot) \text{ is a continuous } C^+_{\text{rap}}\text{-valued mapping.} \tag{3.10}$$

Note that we have not verified that $t \to u_t$ is a continuous C^+_{tem}-valued mapping- see Shiga (1990) for this terminology and see the proof of Theorem 2.4 of that article for the apparent need of this hypothesis. An examination of Shiga's proof shows that this condition is not needed to derive his moment condition (6.5) and it is easy to see that this, together with the joint continuity of u will give (3.10).

Now we begin the heart of the argument. We will use Lemma 3.4 to show that certain parts of H die out rapidly.

Notation. If M_t is a stochastic process let

$$U_M = \inf\{t : M_t = 0\} \quad (\inf\phi = +\infty)\,.$$

Lemma 3.4 *Let* $\beta \in (1,2)$ *and* $c_{3.2}(\beta) = 2((2-\beta)^{-1} + (\beta-1)^{-1})$. *If* $(M_t, \mathscr{H}_t)$ *is a non-negative continuous martingale and* T *is an* $(\mathscr{H}_t)$*-stopping time such that for some* $c > 0$,

$$\langle M\rangle_t - \langle M\rangle_s \geqq c\int_s^t 1(r < T)M_r^\beta\,dr \quad \forall s < t\,, \tag{3.11}$$

then

$$P(U_M \wedge T > t | \mathscr{H}_0) \leqq c_{3.2}(\beta) c^{-1} M_0^{2-\beta} t^{-1} \ \forall t > 0 .$$

Proof. Let $V = U_M \wedge T$, $W = \langle M \rangle(V)$ and let $\tau_t = \inf\{u : \langle M \rangle_u \geqq t\}$ ($\inf \emptyset = \infty$). Enlarge the space to introduce an independent Brownian motion $\hat{B}_t$ ($\hat{B}_0 = 0$) and define

$$B_t = \begin{cases} M(\tau_t) & \text{if } t \leqq W \\ M(\tau_W) + \hat{B}_{t-W} & \text{if } t > W . \end{cases}$$

By the Dubins-Schwarz theorem, conditional on $\mathscr{H}_0$, B is a Brownian motion starting at $M_0(\omega)$. It is also easy to check that

$$W \leqq U_B . \tag{3.12}$$

Integrate $1(r \leqq V) M_r^{-\beta}$ with respect to either side of (3.11) to conclude

$$\begin{aligned} cV &\leqq \int_0^V M_r^{-\beta} d\langle M \rangle_r \\ &= \int_0^W B_t^{-\beta} dt \\ &\leqq \int_0^{U_B} B_t^{-\beta} dt \qquad \text{(by (3.12))} . \end{aligned}$$

If L_t^r is Brownian local time, then

$$\begin{aligned} P(V | \mathscr{H}_0) &\leqq c^{-1} P\left(\int_0^{U_B} B_t^{-\beta} dt \,|\, \mathscr{H}_0 \right) \\ &= c^{-1} \int_0^\infty P_{M_0(\omega)}(L_{U_B}^x) x^{-\beta} dx \\ &= c^{-1} \int_0^\infty 2(M_0(\omega) \wedge x) x^{-\beta} dx \quad \text{(Ray-Knight Theorem)} \\ &= c_{3.2}(\beta) c^{-1} M_0^{2-\beta} . \end{aligned}$$

The result follows by Markov's inequality. □

Notation. If ν is a measure on a topological space, $S(\nu)$ denotes its closed support. Let $h(t) = (t \log^+(1/t))^{1/2}$ and for $\delta, c, T > 0$ define

$$K(\delta, c, T) = \{ y \in C : |y(t) - y(s)| \leqq ch(t-s) \ \forall s, t \in [0, T], 0 < t - s < \delta \} .$$

Here then is our main result.

Theorem 3.5 *In addition to* (3.9) *and* (3.1) *assume there is a* $\gamma \in (0, 1)$ *and a positive sequence* $\{a_K\}$ *such that*

$$a(u) \geqq a_K u^\gamma \quad \text{for} \quad 0 \leqq u \leqq K . \tag{3.13}$$

For $\mathbb{Q}_m$-a.a. ω $\forall c > \sqrt{2}(1 - (\gamma \vee 1/2))^{-1/2} \equiv c_\gamma$, $\forall L \in \mathbb{N}$ *there is a* $\delta = \delta(c, L, \omega) > 0$ *such that* $S(H_t) \subset K(\delta, c, L)$ $\forall t \leqq L$.

Proof. We may assume without loss of generality that $\gamma > 1/2$. Let $c > c_\gamma$ and choose $\varepsilon > 0$ sufficiently small so that

$$\beta = (2\gamma - \varepsilon)(1 - \varepsilon)^{-1} \in (1, 2 - 4c^{-2}) .$$

Next select $N \in \mathbb{N}$ larger than both $\|u_o\|_\infty$ and $\int u_o(x)^\varepsilon dx$ (see (3.9)). Let

$$T_N = \inf\left\{t: \|u_t\|_\infty \geqq N \quad \text{or} \quad \int u_t(x)^\varepsilon dx \geqq N\right\} .$$

It is clear from (3.10) ($\int u_t(x)^\varepsilon dx \leqq |u_t|_1^\varepsilon 2\varepsilon^{-1}$) that

(3.14) $$\lim_{N + \infty} T_N = +\infty \text{ a.s.}$$

For non-negative integers $0 \leqq j < k$, n, let

$$G_{n,j,k} = \{y \in C: |y(k2^{-n}) - y(j2^{-n})| > ch((k-j)2^{-n})\} ,$$

let $M_{n,j,k}(t) = H_{(k2^{-n}+t)\wedge T_N}(G_{n,j,k})$ and let $\phi_{n,j,k}(s, y)$ be a $(\mathscr{C}_s)$-optional, bounded, right-continuous version of the martingale $P_m(G_{n,j,k}|\bar{\mathscr{C}}_s^m)(y)$ such that $\phi_{n,j,k}(t, y) = 1_{G_{n,j,k}}(y)$ for $t \geqq k2^{-n}$, $y \in C$ (see Proposition 2.1). To relieve eye-strain we suppress the subscripts until further notice. Then $A\phi = 0$ and (M_m) implies M is an $(\mathscr{F}_{t+}^{n,k})$-martingale, where $\mathscr{F}_t^{n,k} = \mathscr{F}_{k2^{-n}+t}$, and

$$\langle M \rangle_t = \int_{k2^{-n}}^{k2^{-n}+t} \int 1(r < T_N) b(u(r, y_r)) 1_G(y) H_r(dy)\, dr .$$

The next two calculations are the key steps in our argument. We use Lemma 3.4 and Jensen's inequality to show that the "bad" parts of H (where the particles have large modulus) die not quickly.

Let $v(s, \omega, x)$ be a $\mathscr{P}((\mathscr{F}_{t+})) \times \mathscr{B}(\mathbb{R})$-measurable process such that

$$v(s, x)\, dx = H_s(y: y \in G, y(s) \in dx) , \quad v(s, x) \leqq u(s, x)\ \forall (s, x) ,$$

and let $I(s) = \int v(s, x)^\varepsilon dx$. Then for $s < t$,

(3.15)

$$\langle M \rangle_t - \langle M \rangle_s = \int\int 1(k2^{-n} + s \leqq r < k2^{-n} + t, r < T_N) b(u(r, x)) v(r, x)\, dx\, dr$$

$$\geqq a_N \int\int 1(s \leqq r - k2^{-n} < t, r < T_N)(u(r, x))^{2\gamma - 1} v(r, x)\, dx\, dr$$

(by (3.13))

$$\geqq a_N \int\int 1(s \leqq r - k2^{-n} < t, r < T_N)(v(r, x)^{1-\varepsilon})^\beta v(r, x)^\varepsilon\, dx\, dr$$

$$\geqq a_N \int \left(\int 1(s \leqq r - k2^{-n} < t, r < T_N) v(r, x)\, dx\right)^\beta I(r)^{1-\beta}\, dr$$

(Jensen's inequality)

$$\geqq a_N N^{1-\beta} \int_s^t 1(r < \tilde{T}_N) M_r^\beta\, dr ,$$

where $\tilde{T}_N = (T_N - k2^{-n})^+$ is an $(\mathscr{F}^{n,k}_{t+})$-stopping time. Now use Lemma 3.4 to conclude

$$
\begin{aligned}
(3.16)\quad & \mathbb{Q}_m(U_M > 2^{-n}, T_N > (k+1)2^{-n}) \\
& \leqq c_{3.2}(\beta)N^{\beta-1}a_N^{-1}2^n\mathbb{Q}_m(1(T_N > k2^{-n})H_{k2^{-n}}(G)^{2-\beta}) \\
& \leqq c_{3.2}N^{\beta-1}a_N^{-1}2^nP_m(|Y(k2^{-n}) - Y(j2^{-n})| > ch((k-j)2^{-n}))^{2-\beta} \\
& \qquad \text{(Jensen's inequality and Lemma 3.1)} \\
& \leqq c_{3.2}N^{\beta-1}a_N^{-1}m(\mathbb{R})2^{n-nc^2(2-\beta)/2}(k-j)^{c^2(2-\beta)/2},
\end{aligned}
$$

where the last line holds for all $k - j \leqq 2^{n/2}$ and $n \geqq n_o(c)$. The choice of β allows us to choose $\eta \in (0, 1/2)$ sufficiently small so that

$$2 + \eta + \eta c^2(2-\beta)/2 - c^2(2-\beta)/2 < 0\,.$$

Since $H_{(k+1)2^{-n}+t}(G)$ is a continuous non-negative martingale (by (M_m)) it sticks at zero as soon as it hits zero. Therefore we have proved that for $L, N \in \mathbb{N}$

$$
\begin{aligned}
& \mathbb{Q}_m\Big(T_N > L+1, \max_{0 \leqq j < k \leqq 2^nL, k-j \leqq 2^{n\eta}} \sup_{t \geqq 0} H_{(k+1)2^{-n}+t}(G_{n,j,k}) > 0\Big) \\
& \leqq c_{3.2}N^{\beta-1}a_N^{-1}m(\mathbb{R})L2^{n(2+\eta)-nc^2(2-\beta)/2+n\eta c^2(2-\beta)/2}
\end{aligned}
$$

which is summable over n by the choice of η. By Borel-Cantelli and (3.14) we may fix ω outside a $\mathbb{Q}_m$-null set such that $T_N(\omega) \uparrow +\infty$ and for any $L, N \in \mathbb{N}$ there is an $n_o(L, N, \omega)$ such that for all $n \geqq n_o(L, N)$ if $T_N(\omega) > L + 1$, then

$$H_t(G_{n,j,k}) = 0 \quad \forall 0 \leqq j < k \leqq 2^nL, k-j \leqq 2^{n\eta},\ t \geqq (k+1)2^{-n}\,.$$

Fix $L \in \mathbb{N}$, choose $N = N(\omega)$ such that $T_N(\omega) > L + 1$ and set $n_o(L, \omega) = n_o(L, N(\omega), \omega)$. If $n \geqq n_o(L)$ then

$$
\begin{aligned}
(3.17)\quad & |y(k2^{-n}) - y(j2^{-n})| \leqq ch(k-j)2^{-n})\ \forall j2^{-n} < k2^{-n} \leqq L, \\
& k - j \leqq 2^{n\eta},\ (k+1)2^{-n} \leqq t \text{ for } H_t\text{-a.a. } y\ \forall t \geqq 0\,.
\end{aligned}
$$

We now modify Lévy's derivation of the exact modulus of continuity for Brownian motion (see Itô and McKean 1974, Sect. 1.9) to show

$$
\begin{aligned}
(3.18)\quad & |y(v) - y(u)| \leqq (1+2\eta)ch(v-u) \text{ for all } 0 < v - u < 2^{-n_o(L,\omega)(1-\eta)} \\
& \qquad \equiv \delta(L, c, \eta, \omega) \\
& 0 \leqq u < v \leqq t \quad \text{for } H_t\text{-a.a. } y \quad \forall t \leqq L\,.
\end{aligned}
$$

We must slightly modify Lévy's argument to accommodate the restriction $(k+1)2^{-n} \leqq t$ in (3.17) and therefore briefly sketch a proof of (3.18). Choose $n \geqq n_o(L, \omega)$ such that

$$2^{-(n+1)(1-\eta)} \leqq v - u \leqq 2^{-n(1-\eta)}\,.$$

Let $u_1 = j2^{-n} \equiv ([u2^n] + 1)2^{-n}$ and $v_1 = k2^{-n} = ([v2^n] - 1)2^{-n}$. Since $v_1 + 2^{-n} \leqq v \leqq t$, (3.17) implies

$$(3.19)\quad |y(v_1) - y(u_1)| \leqq ch(v-u)\,.$$

Choose $v_1 \leqq v_m \uparrow v$ such that $v_m = k_m 2^{-n_m}$, $v_{m+1} = (k_m + 1)2^{-n_m}$ and $v_{m+1} + 2^{-n_m} \leqq v$ (at each stage we can always reduce the distance to v by a factor of 3/4). As in Lévy's argument we can check that

$$|y(v) - y(v_1)| \leqq \sum_{m \geqq 0} |y(v_{m+1}) - y(v_m)| \leqq \eta c h(v - u) . \tag{3.20}$$

Arguing exactly as in Lévy's proof, one obtains

$$|y(u) - y(u_1)| \leqq \eta c h(v - u) \tag{3.21}$$

((3.20) and (3.21) also require $n \geqq n_1(\eta)$, which we may assume without loss of generality). (3.19)–(3.21) give (3.18). Hence we have shown that $H_t(K(\delta, (1 + 2\eta)c, L)^c) = 0 \ \forall t \leqq L \ (\delta = \delta(L, c, \eta, \omega) > 0)$. Now note that since $K(\delta, c, L)$ is closed we have $S(H_t) \subset K(\delta, (1 + 2\eta)c, L)$ for all $t \leqq L$, and as $c > c_\gamma$ and $\eta > 0$ are arbitrary we are done. □

Remark 3.6 If $a(u) = \sigma u^{1/2} (\sigma > 0)$, then $\mathbb{Q}_m$ is the law of the $(Y, \sigma^2\lambda^2/2)$-historical process, constructed in Sect. 2 of Dawson and Perkins (1991). In this case the critical constant $c_{1/2} = 2$ in Theorem 3.5 was shown to be sharp in Theorem 8.7 of Dawson and Perkins (1991). In light of Corollary 3.8 below and the non-compactness result of Mueller (1990) for $a(u) = u^\gamma$ with $\gamma \geqq 1$, it is gratifying to note $c_\gamma \uparrow +\infty$ *as* $\gamma \uparrow 1$. We have not, however, checked whether or not c_γ is critical for $\gamma \neq 1/2$.

Corollary 3.7 *Under the hypotheses of Theorem* 3.5,

$$S(X_t) = \pi_t(S(H_t)) \quad \forall t \geqq 0 \ \mathbb{Q}_m\text{-a.s.}$$

Proof. Fix ω so that the conclusion of Theorem 3.5 holds. Let $\{x_n\}$ be a sequence in $\pi_t(S(H_t))$ converging to x. Choose $L > t$, $L > |x_n|$ for all n ($L \in \mathbb{N}$). Then $x_n = y_n(t)$ for some

$$y_n \in S(H_t) \cap \{y \in c : |y(t)| \leqq L\} \subset K(\delta(c, L, \omega), c, L) \cap \{y \in C : |y(t)| \leqq L, y = y^L\} .$$

(Here $c > c_\gamma$.) This last set is compact in C by Arzela-Ascoli, and so we may choose a subsequence $y_{n_k} \to y$ in C. Clearly $y \in S(H_t)$ and $x = y(t) \in \pi_t(S(H_t))$. Therefore $\pi_t(S(H_t))$ is closed. The inclusion $S(X_t) \subset \pi_t(S(H_t))$ is now immediate from

$$X_t(\pi_t(S(H_t))^c) = H_t(\pi_t^{-1}(\pi_t(S(H_t)))^c) \leqq H_t(S(H_t)^c) = 0 .$$

Conversely if $y^\circ \in S(H_t)$ then

$$X_t(B(y_t^\circ, \varepsilon)) = H_t(\{y : |y_t - y_t^\circ| < \varepsilon\}) > 0 \quad \forall \varepsilon > 0$$

and so $y_t^\circ \in S(X_t)$. □

Notation. If $A \subset \mathbb{R}$, $A^\delta = \{x \in \mathbb{R} : |x - y| \leqq \delta$ for some $y \in A\}$.

Corollary 3.8 *Under the hypotheses of Theorem* 3.5 *for* $\mathbb{Q}_m$-*a.a.* $\omega \ \forall c > c_\gamma$, $L \in \mathbb{N}$ $\exists \delta(c, L, \omega) > 0$ *such that*

$$S(X_t) \subset S(X_s)^{ch(t-s)} \forall s, t \in [0, L], s \textit{ rational and } 0 \leqq t - s < \delta(c, L, \omega) . \tag{3.22}$$

Proof. Let $\mathscr{U}$ be a countable base for C, $G \in \mathscr{U}$ and $s \in \mathbb{Q} \cap [0, \infty)$. As in the proof of Theorem 3.5 it follows from (M_m) that $\{H_t(\{y: y^s \in G\}): t \geqq s\}$ is a continuous $(\mathscr{F}_{t+})_{t \geqq s}$-martingale. Therefore $H_s(G) = 0$ implies $H_t(\{y\colon y^s \in G\}) = 0 \; \forall t \geqq s \; \mathbb{Q}_m$-a.s. Fix ω outside a $\mathbb{Q}_m$-null set so that this holds simultaneously for all $s \in \mathbb{Q}^{\geqq 0}$ and G in $\mathscr{U}$, and so that the conclusions of Theorem 3.5 and Corollary 3.7 hold. Let c, L, s and t be as in the statement of the corollary where $\delta(c, L, \omega)$ is as in Theorem 3.5. Let $x \in S(X_t)$, so that $x = y_0(t)$ for some $y_0 \in S(H_t)$ (by Corollary 3.7). If $y_0^s \in G \in \mathscr{U}$, then $\tilde{G} = \{y: y^s \in G\}$ is an open neighbourhood of y_0 and hence $0 < H_t(\{y: y^s \in G\})$. The choice of ω implies $H_s(G) > 0$. This proves $y_0^s \in S(H_s)$ and so $y_0(s) \in S(X_s)$ (Corollary 3.7 again). Since $y_0 \in K(\delta, c, L)$, it follows that

$$|x - y_0(s)| = |y_0(t) - y_0(s)| \leqq ch(t - s)$$

and therefore $x \in S(X_s)^{ch(t-s)}$, as required. □

Remark. The restriction that $s \in \mathbb{Q}$ in the above result is a nuisance but as there is no such restriction on t, (3.22) still provides on effective means of controlling the expansion rate of $S(X_t)$. The problem is that we have been unable to rule out sudden exceptional *drops* in the size of $S(X_s)$ for $s \leqq t$. Lemma 4.9 of Dawson et al. (1989) handles this problem for super-Brownian motion ($a(u) = u^{1/2}$). The proof, however, relies on the uniqueness in law of the process, a luxury we cannot afford in the current setting.

Corollary 3.9 *Assume* (3.1), (3.13) *and* $u_o(x)$ *is a continuous function with compact support. Then* $S(H_t)$ *and* $S(X_t)$ *are compact in* C and $\mathbb{R}$ *respectively, for all* $t \geqq 0$ $\mathbb{Q}_m$-a.s.

Proof. If $K_0 = S(m)$ (compact) then $H_t(\{y: y(0) \notin K_0\}) = 0 \; \forall t \geqq 0 \; \mathbb{Q}_m$-a.s. since it is a right-continuous $(\mathscr{F}_{t+})$-martingale starting at 0 (again by (M_m)). Fix ω so that this martingale is identically 0 and so that the conclusion of Theorem 3.5 holds. Then for $c > c_\gamma$, $L \in \mathbb{N}$ and δ as in Theorem 3.5 we have

$$S(H_t) \subset \{y \in K(\delta, c, L): y(0) \in K_0,\ y = y^L\} \quad \forall t \leqq L\,.$$

This latter set is compact by Arzela-Ascoli and hence $S(H_t)$ is compact for all $t \geqq 0$ a.s. The corresponding result for $S(X_t)$ now follows from Corollary 3.7 (or is immediate from Corollary 3.8). □

We let $U_X = \inf\{t\colon X_t = 0\}$ denote the lifetime of X. Since $X_t(\mathbb{R})$ is a martingale $X_t = 0 \; \forall t \geqq U_X$ on $\{U_X < \infty\}$ a.s.

Theorem 3.10 *Assume* (3.1), (3.9) *and*

$$a(u) \geqq a_0 u^\gamma \quad \forall u \geqq 0 \quad \text{and some} \quad \gamma \in [\tfrac{1}{2}, 1)\,. \tag{3.23}$$

Then $U_X < \infty$ and $X_{t+U_X} = 0 \; \forall t \geqq 0 \; \mathbb{Q}_m$-a.s.

Proof. Let $\psi(x) = \psi^{(\alpha)}(x) = \cosh(\alpha x)$ and $\psi_n(x) = \psi_n^{(\alpha)}(x) = \cosh(\alpha x) e^{-x^2/n}$. Then $\psi_n \in D(\Delta/2)$ and so Proposition 3.2 implies

$$X_t(\psi_n) = m(\psi_n) + \int_0^t \int a(u(r, x)) \psi_n(x)\, dW(r, x) + \int_0^t X_r(\psi_n''/2)\, dr \;\; \forall t \geqq 0 \;\; \mathbb{Q}_m\text{-a.s.} \tag{3.24}$$

(3.10) and the estimate $|\psi_n''| + |\psi_n| \leqq c_1\psi$ allows us to use dominated convergence to let $n \to \infty$ in each of the terms in (3.24) except the martingale term. To handle this term, note that

$$\int_0^t \int a(u(r,x))^2(\psi_n(x) - \psi(x))^2\,dx\,dr \leqq c_2 \int_0^t \int (u(r,x)^2 + u(r,x)^{2\theta})(\psi_n(x) - \psi(x))^2\,dx\,dr$$

$$\to 0 \quad \text{as} \quad n \to \infty$$

by (3.10) and dominated convergence. Now we may let $n \to \infty$ in (3.24) to conclude that

$$X_t(\psi) = m(\psi) + \int_0^t \int a(u(r,x))\psi(x)\,dW(r,x) + \int_0^t (\alpha^2/2)X_r(\psi)\,dr \quad \forall t \geqq 0 \ \mathbb{Q}_m\text{-a.s.}$$

$$\equiv m(\psi) + M_t + (\alpha^2/2)\int_0^t X_r(\psi)\,dr\,. \tag{3.25}$$

M_t is a continuous martingale such that

$$\frac{d}{dt}\langle M\rangle_t \geqq a_0^2 \int u(t,x)^{2\gamma}\psi(x)^2\,dx \quad (\text{by (3.23)})\,.$$

Let us first assume $\gamma > 1/2$ and let $c_3(\gamma) = \int_{-\infty}^{\infty} (\cosh x)^{(2\gamma-2)/(2\gamma-1)}\,dx$. Then

$$\begin{aligned} \frac{d}{dt}\langle M\rangle_t &\geqq a_0^2 \int (\psi(x)^{1/(2\gamma-1)}u(t,x))^{2\gamma}\psi(x)^{(2\gamma-2)/(2\gamma-1)}\,dx \\ &\geqq a_0^2\left(\int \psi(x)u(t,x)\,dx\right)^{2\gamma}\left(\int \psi(x)^{(2\gamma-2)/(2\gamma-1)}\,dx\right)^{1-2\gamma} \quad (\text{Jensen}) \\ &= a_0^2 X_t(\psi)^{2\gamma}(c_3(\gamma)/\alpha)^{1-2\gamma} \\ &\equiv a_1 X_t(\psi)^{2\gamma}\,. \end{aligned} \tag{3.26}$$

If $\gamma = 1/2$ then simply use $\psi \geqq 1$ to get the same conclusion with $a_1 = a_0^2$. Let

$$C_t = \int_0^t X_s(\psi)^{-2\gamma}d\langle M\rangle_s \quad \text{for} \quad t < U_X$$

$$\tau_t = \inf\{u : C_u > t\} \quad \text{for} \quad t < C_{U_X}\,.$$

Then

$$\frac{d\tau}{dt}(t) = \left(\frac{dC}{du}(\tau_t)\right)^{-1} = X_{\tau_t}(\psi)^{2\gamma}\left(\frac{d\langle M\rangle}{du}(\tau_t)\right)^{-1} \leqq a_1^{-1} \quad \text{for} \quad t < C_{U_X} \tag{3.27}$$

(by (3.26)). Therefore $\tau(C_{U_X}) \equiv \tau(C_{U_X}-) < \infty$ if $C_{U_X} < \infty$. Let

$$\tilde{X}^{(\alpha)}(t) = \tilde{X}(t) = X(\tau(t \wedge C_{U_X}))(\psi^{(\alpha)})\,,$$

so that $U_{\tilde{X}} = C_{U_X}$, and let $\tilde{M}_t = M(\tau(t \wedge U_{\tilde{X}}))$. $\tilde{M}$ is a continuous martingale ($\tau_t \leqq a_1^{-1}t$) and for $t < \tilde{U}_X$,

$$\frac{d}{dt}\langle \tilde{M}\rangle_t = \frac{d\langle M\rangle}{du}(\tau_t)\frac{d\tau}{dt}(t) = \tilde{X}_t^{2\gamma} \qquad (\text{by (3.27)}).$$

Hence by enlarging the space we may assume there is a standard Brownian motion $B_t \in \mathbb{R}$ such that $\tilde{M}_t = \int_0^t \tilde{X}_s^\gamma dB_s$. Therefore (3.25) implies

(3.28)
$$\tilde{X}_t = m(\psi) + \int_0^t \tilde{X}_s^\gamma dB_s + (\alpha^2/2) \int_0^{\tau(t \wedge U_{\tilde{X}})} X_r(\psi)\, dr$$
$$= m(\psi) + \int_0^t \tilde{X}_s^\gamma dB_s + (\alpha^2/2) \int_0^t \tilde{X}_s \tau'(s)\, ds\,.$$

If $\hat{X}_t^{(\alpha)}$ is the pathwise unique solution of

$$\hat{X}_t = m(\psi) + \int_0^t \hat{X}_s^\gamma\, dB_s + \alpha^2(2a_1)^{-1} \int_0^t \hat{X}_s\, ds\,,$$

then since $\tau' \leqq a_1^{-1}$, a standard comparison theorem (see Rogers and Williams 1987, V.43.1) shows that $\tilde{X}_t^{(\alpha)} \leqq \hat{X}_t^{(\alpha)}\ \forall t \geqq 0$ a.s.

A direct calculation (see Knight 1981, p. 92) shows 0 is an accessible boundary point for the diffusion $\hat{X}$, and that $\hat{X}$ has scale function

$$s_c(x) = \int_1^x \exp\{-c(1-\gamma)^{-1}(y^{2-2\gamma} - 1)\}\, dy\,,$$

where $c = \alpha^2(2a_1)^{-1}$. Note that

$$c = \alpha^2(2a_1)^{-1} = \begin{cases} a_0^{-2} c_3(\gamma)^{2\gamma-1} \alpha^{3-2\gamma}/2 & \gamma > \frac{1}{2} \\ \alpha^2 a_0^{-2}/2 & \gamma = \frac{1}{2} \end{cases}$$
$$\to 0 \quad \text{as} \quad \alpha \to 0+\,.$$

Therefore (see Knight 1981, p. 95),

(3.28)
$$\mathbb{Q}_m(U_X < \infty) \geqq \varliminf_{\alpha \to 0+} \mathbb{Q}_m(U_{\tilde{X}^{(\alpha)}} < \infty) \quad (\tau(t) \leqq a_1^{-1} t)$$
$$\geqq \varliminf_{\alpha \to 0+} \mathbb{Q}_m(U_{\hat{X}^{(\alpha)}} < \infty) \quad (\tilde{X}^{(\alpha)} \leqq \hat{X}^{(\alpha)})$$
$$\leqq \lim_{\alpha \to 0+} \frac{s_c(\infty) - s_c(m(\psi^{(\alpha)}))}{s_c(\infty) - s_c(0)}$$
$$= 1. \quad \square$$

4 Existence of solutions to the martingale problem

In order to prove the existence of a solution to (M_m) we assume (3.9) throughout this section and will initially strengthen (3.1) to

(4.1)
$$|a(u)| \leqq c_{4.1}(u + u^{1/2}) \quad \forall u \geqq 0\,,$$

where $a: [0, \infty) \to \mathbb{R}$ is continuous. Then

(4.2)
$$b(u) = a(u)^2 u^{-1} 1(u > 0) \leqq c_{4.2}(u + 1) \quad \forall u \geqq 0\,.$$

Let $(\Omega, \mathscr{F}, \mathscr{F}_{t+}, H_t)$ be as in Sect. 3, let $\mathscr{F}[r,s] = \sigma(H_u : r \leq u \leq s)$ and let $\mathbb{Q}_{s,m,\sigma^2}$ denote the law of the $(B, \sigma^2\lambda^2/2)$-historical process where B is one-dimensional Brownian motion and $\sigma\colon \mathbb{R} \to \mathbb{R}$ is bounded and measurable. Define $\{u(t,x)(\omega) : t \geq 0, x \in \mathbb{R}\}$ by

$$u(t,x) = \begin{cases} \lim_{n\to\infty} 2^n H_t(y : y_t \in I_n(x)) & \text{if the limit exists ,} \\ 0 & \text{otherwise} \end{cases}$$

where $x \in I_n(x) = [j2^{-n}, (j+1)2^{-n}), j \in \mathbb{Z}$. Recall that $A_{s,m}$ denotes the weak generator constructed in Sect. 2, where our underlying Markov process is now Brownian motion in $\mathbb{R}$. Also let A denote $A_{o,m}$ as in Sect. 3.

Let $\delta_n \downarrow 0$ and for fixed $n \in \mathbb{N}$ construct $\mathbb{Q}^n$ on $\mathscr{F}_{i\delta_n}$ by induction on i as follows. $\mathbb{Q}^n|_{\mathscr{F}_{\delta_n}} = \mathbb{Q}_{0,m,b(u_o)}|_{\mathscr{F}_{\delta_n}}$ (note that $b(u_o)$ is bounded and continuous by (4.2)). By Konno and Shiga (1988, Theorem 1.4, Remark 1.6), $\{u(t,x) : t \leq \delta_n, x \in \mathbb{R}\}$ is $\mathbb{Q}^n$-a.s. continuous and

$$H_t(y : y_t \in A) = \int_A u(t,x)\,dx \quad \forall A \in \mathscr{B}(\mathbb{R}), t \in [0, \delta_n] \ \ \mathbb{Q}^n\text{-a.s.} \tag{4.3}$$

By Theorem 2.4 of Shiga (1990) (see also the comments following Proposition 3.3 of the previous section) $t \mapsto u_t$ is a C^+_{rap}-valued continuous mapping on $[0, \delta_n]$ $\mathbb{Q}^n$-a.s. Let $\mathbb{Q}_{\delta_n, H_{\delta_n}, b(u_{\delta_n})}|_{\mathscr{F}[\delta_n, 2\delta_n]}$ be the regular conditional distribution of $\mathbb{Q}^n|_{\mathscr{F}[\delta_n, 2\delta_n]}$ given $\mathscr{F}_{\delta_n}$. Since $b(u_{\delta_n})$ is $\mathbb{Q}^n$-a.s. a bounded continuous function this is well-defined and since $u_{\delta_n} \in C^+_{\text{rap}}$ a.s., we can argue as above to see that $t \mapsto u_t$ is a continuous C^+_{rap}-valued mapping on $[0, 2\delta_n]$ satisfying (4.3) on $[0, 2\delta_n]$. Continuing in this way, we define $\mathbb{Q}^n$ on $\mathscr{F}$ so that $t \mapsto u_t$ is a continuous map from $[0, \infty)$ to C^+_{rap} $\mathbb{Q}^n$-a.s. and (4.3) holds on $[0, \infty)$ a.s.

Recall the notations $\hat{\mathbb{R}}$ and $\hat{\mathbb{R}}_s = \hat{\mathbb{R}} \cap ([s, \infty) \times C)$ from the previous sections. Let $\phi \in D(A)$. If $m_{\delta_n}(\cdot) = P_m(B^{\delta_n} \in \cdot)$, then $P_m = P_{\delta_n, m_{\delta_n}}$ by the Markov property (see Dawson and Perkins 1991, Theorem 2.2.1). Therefore $\phi|_{\hat{\mathbb{R}}_{\delta_n}} \in F_{\delta_n, m_{\delta_n}}$ and so $\phi|_{\hat{\mathbb{R}}_{\delta_n}} \in F_{\delta_n, H_{\delta_n}}$ $\mathbb{Q}^n$-a.s. because m_{δ_n} is the mean of H_{δ_n} under $\mathbb{Q}^n$. The Markov property implies

$$M_t^{\phi, \delta_n} = \phi(t, B^t) - \phi(\delta_n, B^{\delta_n}) - \int_{\delta_n}^t A\phi(r, B^r)\,dr, \quad t \geq \delta_n$$

is a $(\mathscr{C}_{t+} : t \geq \delta_n)$-martingale under $P_{\delta_n, y}$ for m_{δ_n}-a.a.y. and hence also with respect to $P_{\delta_n, H_{\delta_n}}$ $\mathbb{Q}^n$-a.s. This proves $\phi|_{\hat{\mathbb{R}}_{\delta_n}} \in D(A_{\delta_n, H_{\delta_n}})$ and $A_{\delta_n, H_{\delta_n}}\phi(t,y) = A\phi(t,y)$ for $t \geq \delta_n$ $\mathbb{Q}^n$-a.s. we may therefore apply Theorem 2.7 conditional on $\mathscr{F}_{\delta_n}$ to see that

$$Z_t^{\delta_n}(\phi) = H_t(\phi_t) - H_{\delta_n}(\phi_{\delta_n}) - \int_{\delta_n}^t H_r(A\phi_r)\,dr, t \in [\delta_n, 2\delta_n]$$

is a continuous $\mathscr{F}_{t+}$-martingale under $\mathbb{Q}^n$ such that

$$\langle Z^{\delta_n}(\phi)\rangle_t = \int_{\delta_n}^t b(u(\delta_n, y_r))\,\phi(r,y)^2\,H_r(dy)\,dr, t \in [\delta_n, 2\delta_n].$$

Let $u^n(t,x) = u([t/\delta_n]\,\delta_n, x)$. Using the above on $[\delta_n, 2\delta_n]$ and Theorem 2.7 on $[0, \delta_n]$, we see that for $t \leqq 2\delta_n$,

(M_m^n) $\quad \forall \phi \in D(A)\ Z_t(\phi) = H_t(\phi_t) - m(\phi_0) - \int_0^t H_r(A\phi_r)\,dr$ is a continuous $(\mathscr{F}_{t+})$-martingale under $\mathbb{Q}^n$ satisfying

$$\langle Z(\phi)\rangle_t = \int_0^t \int b(u^n(r, y_r))\,\phi(r,y)^2\, H_r(dy)\,dr\ \ \mathbb{Q}^n\text{-a.s.}$$

As in the proof of Lemma 3.1 this implies $\mathbb{Q}^n(H_t(G)) = P_m(B^t \in G)$ for $t \leqq 2\delta_n$ (recall this fact for $t = \delta_n$ was used in the above). We can proceed inductively to arrive at (M_m^n) for all $t \geqq 0$ and also

$$\mathbb{Q}^n(H_t(\phi)) = P_m(\phi(B^t)) \quad \forall t \geqq 0,\ \phi \text{ bounded measurable.} \tag{4.4}$$

As in Sect. 3 we may extend $Z_t(\phi)$ first to an orthogonal martingale on $\hat{\mathbb{R}}$ and then to $\phi\colon [0,\infty)\times\Omega\times C \to \mathbb{R}$ which are $\mathscr{P}(\bar{\mathscr{F}}_t^n)\times C$-measurable ($\bar{\mathscr{F}}_t^n = \mathscr{F}_t^{\mathbb{Q}^n}$) and such that

$$\mathbb{Q}^n\left(\int_0^t \int b(u^n(r,y_r))\,\phi(r,y)^2\,H_r(dy)\,dr\right) < \infty \quad \forall t > 0\,.$$

By localizing we may define $Z_t(\phi)$ for ϕ $\mathscr{P}(\bar{\mathscr{F}}_t^n)\times\mathscr{C}$-measurable satisfying

$$\int_0^t \int b(u^n(r,y_r))\,\phi(r,y)^2\,H_r(dy)\,dr < \infty \quad \forall t > 0\ \ \mathbb{Q}^n\text{-a.s.}$$

$Z_t(\phi)$ is then a continuous local martingale whose square function is given by the above integral. We write

$$Z_t(\phi) = \int_0^t \int \phi(s,\omega,y)\,dZ(s,y)\,.$$

Lemma 4.1 *Assume* $g\colon [0,T]\to[0,\infty)$ *is bounded,* $f\colon [0,T]\to[0,\infty)$ *is non-decreasing, and* $g(t) \leqq c\left(f(t) + \int_0^t (t-s)^{-1/2} g(s)\,ds\right)$ $\forall t \leqq T$. *Then* $g(t) \leqq f(t)\exp\{4ct^{1/2}\}$ $\forall t \leqq T$.

Proof. Iterate.

Notation. $v_n(q,\lambda,t) = \sup_{s\leqq t}\int e^{\lambda|x|}\,\mathbb{Q}^n(u(s,x)^q)\,dx,\ q,\lambda,t > 0\,.$

Proposition 4.2. (a) $\sup_n v_n(q,\lambda,t) = v_\infty(q,\lambda,t) < \infty \quad \forall q,\lambda,t > 0\,.$

(b) $u(t,x) = P_t u_o(x) + \int_0^t \int p_{t-s}(y(s)-x)\,dZ(s,y)$ $\ \mathbb{Q}^n$-a.s. $\quad \forall (t,x)$

$\mathbb{Q}^n(u(t,x)) = P_t u_o(x)\,.$

Proof. Fix $\lambda > 0$ and choose $K > |u_0|_\lambda = \sup_x e^{\lambda|x|} u_0(x)$. Let $T(k) = \inf\{t : |u_t|_\lambda > k\}$. If $\phi\colon \mathbb{R}\to\mathbb{R}$ is bounded and measurable and $t > 0$ is fixed, then

$$\phi(s,y) = P_{t-s\wedge t}\,\phi(y(s\wedge t)) \in D(A)$$

and $A\phi = 0$. (M_m^n) implies that $H_u(\phi_u) = P_m(\phi(B_t)) + Z_u(\phi)$ and setting $u = t \wedge T(k)$ and $\phi(x) = p_\varepsilon(x - x_0)$, we get

$$1(T(k) \geqq t) \int p_\varepsilon(x - x_0) u(t, x)\, dx + 1(T(k) < t) \int p_{t - T(k) + \varepsilon}(y(T(k)) - x_0)\, H(T(k))(dy)$$
$$= \int p_{t+\varepsilon}(y - x_0) u_o(y)\, dy + \int_0^{t \wedge T(k)} \int p_{t-s+\varepsilon}(y(s) - x_0)\, dZ(s, y)\,.$$

It is easy to see that each of the terms in the above, except perhaps the last one on the right, has a $\mathbb{Q}^n$-a.s. limit as $\varepsilon \downarrow 0$. Hence we have

(4.5) $$1(T(k) \geqq t) u(t, x_0) + 1(T(k) < t) \int p_{t - T(k)}(y - x_0) u(T(k), y)\, dy$$
$$= P_t u_o(x_0) + \lim_{\varepsilon \downarrow 0} \int_0^{t \wedge T(k)} \int p_{t-s+\varepsilon}(y(s) - x_0)\, dZ(s, y) \quad \mathbb{Q}^n\text{-a.s.}$$

Now

$$\mathbb{Q}^n\left(\int_0^{t \wedge T(k)} \int (p_{t-s+\varepsilon}(y(s) - x_0) - p_{t-s}(y(s) - x_0))^2\, b(u^n(s, y_s))\, H_s(dy)\, ds \right)$$
$$\leqq c_{4.2}\, \mathbb{Q}^n\left(\int_0^{t \wedge T(k)} \int (p_{t-s+\varepsilon}(y - x_0) - p_{t-s}(y - x_0))^2 (u^n(s, y) + 1) u(s, y)\, dy\, ds \right)$$
$$\leqq c_{4.2}(k + 1)k \int_0^t \int (p_{t-s+\varepsilon}(y - x_0) - p_{t-s}(y - x_0))^2\, dy\, ds$$
$$\to 0 \quad \text{as} \quad \varepsilon \downarrow 0 \text{ (by dominated convergence).}$$

This identifies the limit in (4.5) and implies that

(4.6) $$1(T(k) \geqq t) u(t, x_0) + 1(T(k) < t) P_{t - T(k)}(u_{T(k)})(x_0)$$
$$= P_t u_o(x_0) + \int_0^{t \wedge T(k)} \int p_{t-s}(y(s) - x_0)\, dZ(s, y)\ \mathbb{Q}^n\text{-a.s.}$$

As $t \mapsto u(t, \cdot)$ is a C_{rap}^+-valued continuous mapping it is easy to see that

$$\int_0^t \int b(u^n(x, y)) p_{t-s}(y \cdot x)^2 u(s, y)\, dy\, ds < \infty \quad \forall t > 0\ \mathbb{Q}^n\text{-a.s.,}$$

and so we may let $k \to \infty$ in (4.6) to get the first part of (b). This together with Fatou's lemma implies that

$$\mathbb{Q}^n(u(t, x)) \leqq P_t u_0(x)$$

and so

$$v_n(1, \lambda, t) \leqq \sup_{s \leqq t} \int e^{\lambda |x|} P_s u_o(x)\, dx = \sup_{s \leqq t} \int P_s(e^{\lambda|\cdot|})(x) u_o(x)\, dx$$
$$\leqq c(t, \lambda) \int e^{\lambda |x|} u_o(x)\, dx$$
$$= c(t, \lambda, u_o) < \infty\ ,$$

where in the next to last line we have used Lemma 6.2 of Shiga (1990). This gives (a) for $q = 1$. For $q \geqq 2$ use (4.6) to see that (c_q changes from line to line)

$$\begin{aligned}
&\mathbb{Q}^n(u(t,x)^q 1(t \leqq T(k))) \\
&\quad \leqq c_q\Big(P_t u_o(x)^q + \mathbb{Q}_n\Big(\int_0^t \int 1(s \leqq T(k)) p_{t-s}(y-x)^2\, b(u^n(s,y))u(s,y)\, dy\, ds^{q/2}\Big)\Big) \\
&\quad \leqq c_q\Big(P_t u_o(x)^q + \mathbb{Q}_n\Big(\int_0^t \int 1(s \leqq T(k)) p_{t-s}(y-x)^2\, (u^n(s,y)^2 \\
&\qquad\qquad + u(s,y)^2 + u(s,y))\, dy\, ds^{q/2}\Big)\Big) \qquad \text{(use (4.2))} \\
&\quad \leqq c_q\Big(P_t u_o(x)^q + \mathbb{Q}_n\Big(\int_0^t \int 1(s \leqq T(k)) p_{t-s}(y-x)^2\, (u^n(s,y)^q \\
&\qquad\qquad + u(s,y)^q + u(s,y)^{q/2})\, dy\, ds\Big)\Big(\int_0^t \int p_{t-s}(y-x)^2\, dy\, ds\Big)^{(q/2)-1} \\
&\qquad\qquad \text{(Jensen's inequality)}\,.
\end{aligned}$$

As $p_{t-s}(y-x)^2 \leqq (t-s)^{-1/2}\, p_{t-s}(y-x)$, this implies

$$\begin{aligned}
\mathbb{Q}^n(u(t,x)^q 1(t \leqq T(k))) &\leqq c_q\Big(P_t u_o(x)^q \\
&\quad + t^{(q-2)/4} \int_0^t \int (t-s)^{-1/2}\, p_{t-s}(y-x)\, \mathbb{Q}^n(1(s \leqq T(k))\, (u^n(s,y)^q \\
&\quad + u(s,y)^q + u(s,y)^{q/2}))\, dy\, ds\Big)\,.
\end{aligned}$$

If $v_n(q,\lambda,t,k) = \sup_{s \leqq t} \int e^{\lambda|y|}\, \mathbb{Q}^n(u(s,y)^q 1(s \leqq T(k)))\, dy$, then the above implies (use Lemma 6.2 of Shiga (1990) again) that for $u \leqq T$

$$\begin{aligned}
v_n(q,\lambda,u,k) &\leqq c_q \sup_{t \leqq u}\Big(\int P_t(e^{\lambda|\cdot|})(x) u_o(x)^q\, dx \\
&\quad + t^{(q-2)/4} \int_0^t \int (t-s)^{-1/2}\, P_{t-s}(e^{\lambda|\cdot|})(y)\, \mathbb{Q}^n(1(s \leqq T(k)) \\
&\quad \times (u^n(s,y)^q + u(s,y)^q + u(s,y)^{q/2}))\, dy\, ds\Big) \\
&\leqq c(q,\lambda,T)\Big(\int e^{\lambda|x|} u_o(x)^q\, dx + \sup_{t \leqq u} \int_0^t (t-s)^{-1/2}\, (v_n(q,\lambda,s,k) \\
&\quad + v_n(q/2,\lambda,s,k))\, ds\Big) \\
&\leqq c(q,\lambda,T)\Big(|u_o^q|_{2\lambda}\lambda^{-1} + \int_0^u (u-s)^{-1/2}\, (v_n(q,\lambda,s,k) \\
&\quad + v_n(q/2,\lambda,s,k))\, ds\Big)\,.
\end{aligned}$$

It is easy to see from the definition of $T(k)$ that $v_n(q, \lambda, t, k) < \infty$ for $q > 1$. Now use Lemma 4.1, the above bound on $v_n(1, \lambda, t)$ and an obvious induction on $q = 2^m$ to see that

$$\sup_n v_n(q, \lambda, u, k) \leqq c(q, \lambda, u) < \infty \quad \forall u \geqq 0 .$$

Let $k \to \infty$ to complete the proof of (a) for $q = 2^m$. An elementary argument shows the result follows for all $q > 0$.

Finally it is easy to use the estimate in (a) to show the stochastic integral in the first part of (b) is square integrable and hence has mean 0. The second part of (b) follows. □

Lemma 4.3 *If $T, \lambda > 0$ there is a $C(T, \lambda) < \infty$ such that*

$$\int_0^t \int (p_{t-s}(y-x) - p_{t'-s}(y-x'))^2 e^{-\lambda|y|} \, dy \, ds$$
$$\leqq C(T, \lambda)(|x - x'| + (t - t')^{1/2}) e^{-\lambda|x|}$$
$$\forall 0 < t' < t \leqq T, |x - x'| \leqq 1, \lambda > 0 .$$

where $p_u(z)$ is defined to be 0 if $u < 0$.

Proof. This is a routine, if lengthy, calculation. The estimate

$$P_t(e^{-\lambda|\cdot|})(x) \leqq C(T, \lambda) e^{-\lambda|x|} \quad \text{for} \quad t \leqq T$$

from Lemma 6.2 of Shiga (1990) is used frequently. The result is in fact implicit in the proof of Theorem 2.4 of Shiga (1990). □

The following result is a slight extension of Lemma 6.3(ii) of Shiga (1990). The proof is a minor modification of the usual proof of Kolmogorov's continuity criterion and is omitted. If X is a stochastic process, P_X denotes the law of X on the appropriate space of paths.

Lemma 4.4 *Let $\{X_n(t, \cdot): t \geqq 0, n \in \mathbb{N}\}$ be a sequence of continuous C^+_{rap}-valued processes. Suppose $\exists q > 0, \gamma > 2$ and $\forall T, \lambda > 0 \; \exists C = C(T, \lambda) > 0$ such that*

(4.7)
$$P(|X_n(t, x) - X_n(t', x')|^q) \leqq C(|x - x'|^\gamma + |t - t'|^\gamma) e^{-\lambda|x|} \; \forall t, t' \in [0, T],$$
$$|x - x'| \leqq 1, \; n \in \mathbb{N} .$$

If $\{P_{X_n(0)}: n \in \mathbb{N}\}$ is tight on C^+_{rap}, then $\{P_{X_n}: n \in \mathbb{N}\}$ is tight on $C([0, \infty), C^+_{\text{rap}})$.

Proposition 4.5 *$\{\mathbb{Q}^n(u \in \cdot): n \in \mathbb{N}\}$ is tight on $C([0, \infty), C^+_{\text{rap}})$.*

Proof. If $\hat{u}(t, x) = u(t, x) - P_t u_o(x)$, then since $t \mapsto P_t u_o \in C([0, \infty), C^+_{\text{rap}})$, it suffices to show $\{\mathbb{Q}^n(\hat{u} \in \cdot): n \in \mathbb{N}\}$ is tight. Let $q \geqq 1$, $\lambda > 0$, $0 \leqq t' < t \leqq T$ and $|x - x'| \leqq 1$. Allowing c_q to vary from line to line, we have from Proposition 4.2(b), Burkholder's inequality and (4.2):

$$\mathbb{Q}^n(|\hat{u}(t, x) - \hat{u}(t', x')|^{2q})$$
$$\leqq c_q \mathbb{Q}^n\Bigg(\int_0^t \int (p_{t-s}(y-x) - p_{t'-s}(y-x'))^2 e^{-\lambda|y|}$$
$$\times e^{\lambda|y|}(u^n(s, y)\, u(s, y) + u(s, y)) \, dy \, ds^q \Bigg)$$

$$\leqq c_q \mathbb{Q}^n\left(\int_0^t \int (u^n(s,y)^q u(s,y)^q + u(s,y)^q)\, e^{\lambda(q-1)|y|}\right.$$
$$\left.\times (p_{t-s}(y-x) - p_{t'-s}(y-x'))^2\, dy\, ds\right)$$
$$\times\left(\int_0^t \int (p_{t-s}(y-x) - p_{t'-s}(y-x'))^2\, e^{-\lambda|y|}\, dy\, ds\right)^{q-1}$$
$$\leqq c_q \mathbb{Q}^n\left(\int_0^t \int (u^n(s,y)^{8q} + u(s,y)^{8q} + u(s,y)^{4q})\, e^{4\lambda(q-1)|y|}\, dy\, ds\right)^{1/4}$$
$$\times \int_0^t \int |p_{t-s}(y-x) - p_{t'-s}(y-x')|^{8/3}\, dy\, ds^{3/4}$$
$$\times C'(T,\lambda,q)\,(|x-x'|^{q-1} + |t-t'|^{(q-1)/2})\, e^{-\lambda(q-1)|x|}$$

(Hölder's inequality and Lemma 4.3)

$$\leqq C(T,\lambda,q)\,(|x-x'|^{q-1} + |t-t'|^{(q-1)/2})\, e^{-\lambda|x|}$$

by Proposition 4.2 and an elementary calculation. Since $u(0,\cdot) = u_0(\cdot)$ $\mathbb{Q}^n$-a.s., the result is now immediate from Lemma 4.4. □

Turn now to the tightness of $\{\mathbb{Q}^n\}$ itself. We first show that, uniformly in (n, t), almost all of H_t can be supported on a compact set with high $\mathbb{Q}^n$ probability.

Lemma 4.6 *For any bounded $(\bar{\mathscr{F}}_t^n)$-stopping time S and any bounded measurable $\psi: C \to \mathbb{R}$*

(4.8) $\int \psi(y^S)\, H_{t\vee S}(dy) - H_S(\psi) = \iint 1(S < r \leqq t)\psi(y^S)\, Z(dr, dy) \quad \forall t \geqq 0 \quad \mathbb{Q}^n$-a.s.

Proof. By first considering $\phi(r,\omega,y) = 1(r \geqq S(\omega))\psi_1(S(\omega))\psi_2(y^r)$ and then bootstrapping, it is easy to check that the integrand on the right-hand side is $P(\bar{\mathscr{F}}_t^n) \times \mathscr{C}$ – measurable and hence the stochastic integral is well-defined.

If $S = s_0$ is constant, the result is a simple consequence of (M_m^n) with $\phi(t,y)$ equal to a bounded $\mathscr{C}_t$-optional version of $P_m(\psi(B^{s_0}) | \bar{C}_t^m)(y)$ $(A\phi = 0)$. For general S, approximate it from above by stopping times $\{S_m\}$ taking on finitely many values, for which the result follows easily from the above. If ψ is also continuous we obtain the result by taking a.s. limits (as $S_m \downarrow S$) on the left side of (4.8) and L^2-limits on the right (use Proposition 4.2(a)). Finally take bounded pointwise limits in ψ to obtain the result for all bounded measurable ψ. □

Notation. If $G \subset C$, let $G^t = \{y^t: y \in G\}$, $G_\infty = \cup_{0 \leqq t \leqq \infty} G^t$.

Lemma 4.7 *For any ε, $T > 0$ there is a $\delta > 0$ such that*

$$G \in \mathscr{C},\ P_m(B^T \notin G^T) < \delta \Rightarrow \sup_n \mathbb{Q}^n\left(\sup_{t \leqq T} H_t((G^t)^c) > \varepsilon\right) \leqq \varepsilon\,.$$

Proof. Let ε, T, $\delta > 0$ and assume $P_m(B^T \notin G^T) < \delta$. By the section theorem there is a bounded $(\mathscr{F}_t)$-stopping time $S = S_n$ such that

$$[S] \subset \{(t,\omega):\ H_t((G^t)^c) > \varepsilon, t \leqq T\} \cup [T+1] \tag{4.9}$$
$$\mathbb{Q}^n(S \leqq T) \geqq \mathbb{Q}^n\left(\sup_{t \leqq T} H_t((G^t)^c) > \varepsilon\right) - \varepsilon\,.$$

Here $[S]$ denotes the graph of S. Let

$$C_t = C_t^n = \int_0^t \int u(r,x)\, b(u^n(r,x))\, dx\, dr$$

and $T(L) = T^n(L) = \inf\{t: C_t^n \geqq L\}$. Use the elementary inequality

$$X \geqq 0, \mathscr{G} \text{ a } \sigma\text{-field} \Rightarrow P(X > \varepsilon/2 | \mathscr{G}) \geqq ((P(X|\mathscr{G}) - \varepsilon/2)^+)^2 P(X^2|\mathscr{G})^{-1}$$

(this is a simple application of Hölder's inequality), to see that on $\{S \leqq T \wedge T(L)\}$,

$$\begin{aligned}
&\mathbb{Q}^n(H(T \wedge T(L))(y: y^S \notin G^S) > \varepsilon/2 | \bar{\mathscr{F}}_S^n) \\
&\geqq ((\mathbb{Q}^n(H(T \wedge T(L))(y: y^S \notin G^S) | \bar{\mathscr{F}}_S^n) - \varepsilon/2)^+)^2 \\
&\qquad \times (\mathbb{Q}^n(H(T \wedge T(L))(y: y^S \notin G^S)^2 | \bar{\mathscr{F}}_S^n))^{-1} \\
&= ((H_S((G^S)^c) - \varepsilon/2)^+)^2 (H_S(G^S)^c)^2 + \mathbb{Q}^n\Bigg(\int_0^T \int 1(S \leqq r \leqq T(L))\, 1(y^S \notin G^S) \\
&\qquad \times b(u^n(r, y(r)))\, H_r(dy)\, dr | \bar{\mathscr{F}}_S^n) \Bigg)^{-1} \quad \text{(by Lemma 4.6)} \\
&\geqq H_S((G^S)^c)^2 [4(H_S((G^S)^c)^2 + \mathbb{Q}^n(C_{T(L)} | \bar{\mathscr{F}}_S^n))]^{-1} \\
&\geqq \varepsilon^2 [4(\varepsilon^2 + L)]^{-1} \equiv p(\varepsilon, L)\,.
\end{aligned}$$

Integrate both sides over $\{S \leqq T \wedge T(L)\}$ to arrive at

$$\mathbb{Q}^n(H(T \wedge T(L))(y: y^S \notin G^S) > \varepsilon/2, S \leqq T \wedge T(L)) \geqq p(\varepsilon, L)\, \mathbb{Q}^n(S \leqq T \wedge T(L))$$

and therefore

$$\begin{aligned}
(4.10) \quad \mathbb{Q}^n(S \leqq T) &\leqq \mathbb{Q}^n(S \leqq T \wedge T(L)) + \mathbb{Q}^n(T(L) < T) \\
&\leqq p(\varepsilon, L)^{-1} \mathbb{Q}^n(H(T \wedge T(L))(y: y^S \notin G^S) > \varepsilon/2) + \mathbb{Q}^n(C_T^n \geqq L) \\
&\leqq p(\varepsilon, L)^{-1} \mathbb{Q}^n(H_T(y: y^S \notin G^S) > \varepsilon/2) \\
&\quad + (1 + p(\varepsilon, L)^{-1}) \mathbb{Q}^n((C_T^n)^2) L^{-2} \\
&\leqq p(\varepsilon, L)^{-1} \mathbb{Q}^n(H_T((G^T)^c) > \varepsilon/2) + 8(1 + L\varepsilon^{-2}) L^{-2} c(T)\,,
\end{aligned}$$

by Proposition 4.2(a) and (4.2). Choose $L = L(\varepsilon, T)$ so that $8(1 + L\varepsilon^{-2}) L^{-2} c(T) < \varepsilon$. Use the above and (4.4) in (4.10) and conclude

$$\mathbb{Q}^n(S \leqq T) \leqq p(\varepsilon, L)^{-1} P_m(B^T \notin G^T) 2\varepsilon^{-1} + \varepsilon \leqslant p(\varepsilon, L)^{-1} 2\varepsilon^{-1} \delta + \varepsilon\,.$$

Choose δ so that $p(\varepsilon, L)^{-1} 2\varepsilon^{-1} \delta < \varepsilon$. Finally use (4.9) to see that

$$\mathbb{Q}^n\left(\sup_{t \leqq T} H_t((G^t)^c) > \varepsilon \right) \leqq \mathbb{Q}^n(S \leqq T) + \varepsilon < 3\varepsilon\,. \quad \square$$

Corollary 4.8 *For any $\varepsilon, T > 0$ there is a compact subset K of C such that*

$$\sup_n \mathbb{Q}^n\left(\sup_{t \leqq T} H_t(K^c) > \varepsilon \right) \leqq \varepsilon.$$

Proof. For $\varepsilon, T > 0$ choose $K' \subset C$ compact such that $P_m(B \notin K') < \delta$, where δ is as in the previous result. Then $K = K'_\infty$ is also compact (see Dawson and Perkins 1991, Lemma 7.6) and since $K^t \subset K$, Lemma 4.7 implies

$$\sup_n \mathbb{Q}^n\left(\sup_{t \leqq T} H_t(K^c) > \varepsilon\right) \leqq \sup_n \mathbb{Q}^n\left(\sup_{t \leqq T} H_t((K^t)^c) > \varepsilon\right) \leqq \varepsilon . \quad \square$$

Lemma 4.9 $\forall \phi \in D(A)$, $T > 0$, $q \geqq 2$ $\exists C(T, \phi, q)$ such that $\forall 0 \leqq t \leqq u \leqq T$

$$\mathbb{Q}^n(|Z_u(\phi) - Z_t(\phi)|^q) + \mathbb{Q}^n(|H_u(\phi_u) - H_t(\phi_t)|^q) \leqq C(T, \phi, q)|u - t|^{q/2} .$$

Proof. Choose K such that $\sup_t|\phi(t, y^t)| \leqq K$ for P_m-a.a.y. Applying Lemma 4.7 with $G \subset \{y : \sup|\phi(t, y^t)| \leqq K\}$, $G \in \mathscr{C}$, $P_m(B \in G^c) = 0$ we see that $|\phi(t, y^t)| \leqq K$ for H_t-a.a.y $\forall t \geqq 0$ $\mathbb{Q}^n$-a.s. Hence, by truncating we may assume wolog that $|\phi|$ and $|A\phi|$ are uniformly bounded. Burkholder's inequality implies (c_q changes from line to line)

$$\begin{aligned}
&\mathbb{Q}^n(|Z_u(\phi) - Z_t(\phi)|^q)\\
&\leqq c_q \mathbb{Q}^n\left(\int_t^u \int b(u^n(r, y_r))\, \phi(r, y)^2\, H_r(dy)\, dr^{q/2}\right)\\
&\leqq c_q \|\phi\|_\infty^q \mathbb{Q}^n\left(\int_t^u \int (u^n(r, x) + 1) u(r, x)\, e^{2|x|} e^{-2|x|}\, dx\, dr^{q/2}\right) \quad \text{(by (4.2))}\\
&\leqq c_q \|\phi\|_\infty^q \int_t^u \int \mathbb{Q}^n (u^n(r, x)^q + u(r, x)^q\\
&\qquad + u(r, x)^{q/2}) e^{(q-2)|x|}\, dx\, dr\, |u - t|^{(q/2)-1} \quad \text{(by Jensen)}\\
&\leqq c'(T, q) \|\phi\|_\infty^q |u - t|^{q/2},
\end{aligned}$$

where we have used Proposition 4.2(a) in the last line. We also have

$$\begin{aligned}
\mathbb{Q}^n\left(\left|\int_t^u H_r(A\phi_r)\, dr\right|^q\right) &\leqq \|A\phi\|_\infty^q \mathbb{Q}^n\left(\int_t^u H_r(1)^q\, dr\right)|u - t|^{q-1}\\
&\leqq c_q \|A\phi\|_\infty^q \int_t^u m(1)^q + \mathbb{Q}^n(Z_r(1)^q)\, dr\, |u - t)^{q-1} \quad \text{(by } (M_m^n))\\
&\leqq c''(T, q) \|A\phi\|_\infty^q |u - t|^q
\end{aligned}$$

by the above. The result is now immediate from the above estimates and (M_m^n). $\square$

Notation. If $\phi \colon \hat{\mathbb{R}} \to \mathbb{R}$ and $\hat{K} \subset \hat{\mathbb{R}}$ let $\|\phi\|_{\hat{K}} = \sup\{|\phi(t, y)| : (t, y) \in \hat{K}\}$.

Lemma 4.10 *If $\hat{K} \subset \hat{\mathbb{R}}$ is compact, there is a countable set $\{\phi_n\} \subset D(A) \cap C_b(\hat{\mathbb{R}})$ such that $\{A\phi_n\} \subset C_b(\hat{\mathbb{R}})$ and $\{\phi_n|_{\hat{K}} : n \in \mathbb{N}\}$ is a dense subset of $C(\hat{K})$. More precisely $\forall \psi \in C(\hat{K})$, $\varepsilon > 0$ $\exists \phi_n$ such that $\|\phi_n\|_{\hat{\mathbb{R}}} \leqq \|\psi\|_{\hat{K}}$ and $\|\phi_n - \psi\|_{\hat{K}} < \varepsilon$.*

Proof. Let $\psi \in C_b(\hat{\mathbb{R}})$ and define

$$\psi^{(n)}(s, y) = n \int_0^{1/n} P_{y(s)}(\psi(s + t, y/s/B^t))\, dt .$$

It is easy to see that $\psi^{(n)} \in C_b(\hat{\mathbb{R}})$. A routine calculation using the Markov property shows that

$$\psi^{(n)}(t, B^t) - \int_0^t P_{B(s)}(\phi(s+n^{-1}, B/s/\cdot^{1/n})) - \phi(s, B^s)\, ds$$

is a continuous P_m-martingale. Therefore $\psi^{(n)} \in D(A)$ and

$$A\psi^{(n)}(s, y) = P_{y(s)}(\phi(s+n^{-1}, y/s/B^{1/n})) - \phi(s, y^s)$$

is in $C_b(\hat{\mathbb{R}})$. We claim that

$$\psi^{(n)} \to \psi \text{ uniformly on compact subsets of } \hat{\mathbb{R}} \text{ as } n \to \infty\,.$$

Let $\hat{K}$ be compact in $\hat{\mathbb{R}}$. Let $\hat{d}((s, y), (s', y')) = |s - s'| + \| y - y' \|_\infty$ for (s, y), $(s', y') \in \hat{\mathbb{R}}$. Note that $\hat{d}$ is a metric for the topology on $\hat{\mathbb{R}}$ (which is the subspace topology it inherits from $[0, \infty) \times C$). The usual proof of the uniform continuity of $\psi|_{\hat{K}}$ in fact shows that for $\varepsilon > 0$ (fixed) there is a $\delta > 0$ such that $\forall (s', y') \in \hat{\mathbb{R}}$, $(s, y) \in \hat{K}$

$$\hat{d}((s, y), (s', y')) < \delta \Rightarrow |\psi(s, y) - \psi(s', y')| < \varepsilon\,.$$

Therefore if $(s, y) \in \hat{K}$

$$|P_{y(s)}(\psi(s+t, y/s/B^t)) - \psi(s, y)| < \varepsilon + 2\|\psi\|_{\hat{\mathbb{R}}} P_{y(s)}\left(t + \sup_{u \leq t}|B_u - B_0| > \delta\right) < 2\varepsilon$$

providing $t < t_0$ (t_0 is independent of the choice of (s, y)). It follows that $\|\psi^{(n)} - \psi\|_{\hat{K}} < 2\varepsilon$ if $n^{-1} < t_0$ and the claim is proved.

Fix $\hat{K}$ as above and let $\{\psi_j : j \in \mathbb{N}\}$ be a countable dense set in $C(\hat{K})$ with $\psi_1 \equiv 0$. By the Tietze extension theorem we may extend each ψ_j to a function in $C_b(\hat{\mathbb{R}})$, which we also denote by ψ_j, such that $\|\psi_j\|_{\hat{K}} = \|\psi_j\|_{\hat{\mathbb{R}}}$. Claim $S = \{\psi_j^{(n)} : n, j \in \mathbb{N}\}$ is the required countable set. By the above results $S \subset D(A) \cap C_b(\hat{\mathbb{R}})$ and $A\psi \in C_b(\hat{\mathbb{R}})\ \forall \psi \in S$. Let $\psi \in C(\hat{K})$ and $\varepsilon > 0$. Choose ψ_j such that $\|\psi_j - \psi\|_{\hat{K}} < \varepsilon/2$ and $\|\psi_j\|_{\hat{K}} \leqq \|\psi\|_{\hat{K}}$ (find a good approximation to $(1-\varepsilon')\psi$ for appropriate ε' and note if $\psi \equiv 0$ we take $j = 1$). Choose n sufficiently large so that $\|\psi_j - \psi_j^{(n)}\|_{\hat{K}} < \varepsilon/2$. Then $\|\psi_j^{(n)} - \psi\|_{\hat{K}} < \varepsilon$ and $\|\psi_j^{(n)}\|_{\hat{\mathbb{R}}} \leqq \|\psi_j\|_{\hat{\mathbb{R}}} = \|\psi_j\|_{\hat{K}} \leqq \|\psi\|_{\hat{K}}$. □

Proposition 4.11 *Assume* (3.9) *and* (4.1). *Then* $\{\mathbb{Q}^n : \in \mathbb{N}\}$ *is tight on* Ω *and any weak limit point,* $\mathbb{Q}$, *is a solution of* (M_m) *and is such that* $t \mapsto u_t$ *is in* $C([0, \infty), C^+_{\text{rap}})$ $\mathbb{Q}$-a.s.

Proof. The basic plan is clear. The compact containment given by Corollary 4.8 reduces tightness of $\{\mathbb{Q}^n\}$ to tightness of $\{\mathbb{Q}^n(H.(\phi.) \in \cdot) : n \in \mathbb{N}\}$ for a suitable countable collection of ϕ's (e.g. see Roelly–Coppoletta 1986, Theorem 2.1). Such a collection is provided by Lemma 4.10, in conjunction with Corollary 4.8, and the required tightness is given by Lemma 4.9. The existence of a continuous C^+_{rap}-valued density u, follows from Proposition 4.5 and it remains only to let $n \to \infty$ in (M^n_m) to derive (M_m). It is this last step which is a little delicate as $D(A)$ contains functions which are not continuous and hence may not behave well with respect to weak convergence. This problem was also encountered in the proof of Theorem 7.13 in Dawson and Perkins (1991). As in that paper, Lemma 4.7 provides the key, and (also as in Dawson and Perkins 1991) we give a proof using nonstandard analysis as it seems simpler than the standard proof (which we have not been able to find).

We work in an ω_1-saturated enlargement of a superstructure containing $\mathbb{R}$. Fix $\eta \in {}^*\mathbb{N} - \mathbb{N}$ and work with respect to the Loeb measure $L({}^*\mathbb{Q}^\eta)$ on $({}^*\Omega, L({}^*\mathscr{F}))$. We let ${}^*H_t$ denote the canonical paths in ${}^*\Omega$. Let ${}^*\hat{X}_t = \delta_t \times {}^*H_t$. By Corollary 4.8 $\exists\{K_j\}$ an increasing subsequence of compact sets in C such that ${}^*\mathbb{Q}^\eta\left(\sup_{t \le j} {}^*H_t({}^*K_j^c) > 2^{-j}\right) \le 2^{-j}$ and hence $L({}^*H_t)(ns({}^*C)^c) = 0$ $\forall t \in ns({}^*[0, \infty))$ $L({}^*\mathbb{Q}^\eta)$-a.s. Therefore $H_t = st_M({}^*H_t)$ $(= L({}^*H_t)(st_C^{-1}(\;)))$ exists for all $t \in ns{}^*[0, \infty)$ $L({}^*\mathbb{Q}^\eta)$-a.s. and the same is true of $\hat{X}_t = st_M({}^*\hat{X}_t) = \delta_{{}^\circ t} \times H_{{}^\circ t}$. (Here st_C is the standard part map on C and st_M denotes the standard part map on the appropriate space of measures.) Let $\hat{K}_j = \{(t, y^t) : t \in [0, j], y \in K_j\}$. Note that $\hat{K}_j$ is compact in $\hat{\mathbb{R}}$. By Lemma 4.10 there is a countable set $\{\phi_n\}$ in $D(A) \cap C_b(\hat{\mathbb{R}})$ such that $\{\phi_n|_{\hat{K}_j} : n \in \mathbb{N}\}$ is dense in $C(\hat{K}_j)$ $\forall j$ and $\phi_1 = 1$. By Lemma 4.9 we may fix ω outside a $L({}^*\mathbb{Q}^\eta)$-null set such that ${}^*H_t(\phi_n)$ is S-continuous on ${}^*[0, \infty)$ for all n, $\sup_{t \le j} {}^*\hat{X}_t({}^*K_j^c) \le 2^{-j}$ for $j \in \mathbb{N}$ sufficiently large, and $L({}^*H_t)(ns({}^*C)^c) = 0$ $\forall t \in ns({}^*[0, \infty))$. If $s \approx t \in ns({}^*[0, \infty))$, and $s \le t$, then

$$\int \phi_n \, d\hat{X}_t = {}^\circ\!\int {}^*\phi_n(x)\, d{}^*\hat{X}_t = {}^{\circ *}H_t(\phi_n(t)) = {}^{\circ *}H_s(\phi_n(s)) = \int \phi_n \, d\hat{X}_s \,.$$

If $\phi \in C_b(\hat{\mathbb{R}})$ choose ϕ_{n_j} such that $\|\phi_{n_j} - \phi\|_{\hat{K}_j} < 2^{-j}$ and $\|\phi_{n_j}\|_{\hat{\mathbb{R}}} \le \|\phi\|_{\hat{\mathbb{R}}}$ (use Lemma 4.10). Then

$$\begin{aligned} \left|\int \phi \, d\hat{X}_t - \int \phi \, d\hat{X}_s\right| &\le \left|\int \phi - \phi_{n_j} \, d\hat{X}_t\right| + \left|\int \phi - \phi_{n_j} \, d\hat{X}_s\right| \\ &\le 2^{-j+1} \sup_{u \le t} \hat{X}_u(1) + 4\|\phi\|_{\hat{\mathbb{R}}} \sup_{u \le t} \hat{X}_u(K_j^c) \\ &\le 2^{-j+1} \sup_{u \le t} H_u(1) + 4\|\phi\|_{\hat{\mathbb{R}}}\, 2^{-j} \end{aligned}$$

for j sufficiently large. This proves that $\hat{X}_t = \hat{X}_s$ and hence ${}^*H.$ is S-continuous $L({}^*\mathbb{Q}^\eta)$-a.s. (This is equivalent to tightness of $\{\mathbb{Q}^n : n \in \mathbb{N}\}$.)

Proposition 4.5 implies that $t \mapsto {}^*u(t, \cdot)$ is $L({}^*\mathbb{Q}^\eta)$-a.s. an S-continuous ${}^*C^+_{\text{rap}}$-valued mapping. This implies ${}^*u$ is a.s. S-continuous on ${}^*([0, \infty) \times \mathbb{R})$ and

$$|{}^*u(t, x)| \le C(T) e^{-|x|} \; \forall (t, x) \in {}^*([0, T] \times \mathbb{R}) \quad \text{for some } C(T) \in [0, \infty) \; L({}^*\mathbb{Q}^\eta)\text{-a.s.}$$

Fix ${}^*H$ outside a null set so that these conclusions hold and ${}^*u(t, x)$ is an internal density for ${}^*H_t(y : y_t \in dx)$ for all $t \in {}^*[0, \infty)$. Let $u = st({}^*u)$ where st denotes the standard part map on $ns({}^*C([0, \infty), C^+_{\text{rap}}))$. If $\phi \in C_b(\mathbb{R})$, then for $t \in [0, \infty)$

$$\begin{aligned} \int_C \phi(y_t)\, H_t(dy) &= \int_{{}^*C} \phi({}^\circ y_t)\, L({}^*H_t)(dy) \\ &= {}^\circ\!\int_{{}^*C} {}^*\phi(y_t){}^*H_t(dy) \\ &= {}^\circ\!\int_{{}^*\mathbb{R}} {}^*\phi(x){}^*u(t, x){}^*\,dx \\ &= \int \phi({}^\circ x) u(t, {}^\circ x)\, L({}^*dx) \\ &= \int \phi(x) u(t, x)\, dx \,. \end{aligned}$$

Hence $u(t, x)$ is a jointly continuous density for $H_t(y: y_t \in dx)$ for all $t \geqq 0$ and $t \mapsto u_t$ is in $C([0, \infty), C^+_{\text{rap}})$ $L(^*\mathbb{Q}_\eta)$-a.s. Note that this shows that (up to null sets) our notation is consistent with our earlier definition of u as a measurable function of H.

As an immediate consequence of Lemma 4.7 we have

$$\forall G \in \sigma(^*\mathscr{C})\, L(^*P_m)\,(B \in G^c) = 0 \tag{4.11}$$
$$\Rightarrow L(^*H_t)\,(G^{tc}) = 0\, \forall t \in ns(^*[0, \infty))\, L(^*\mathbb{Q}^\eta)\text{-a.s.}$$

Let $\phi \in D(A)$. (M^η_m) implies that

$$^*Z_t(\phi) = {}^*H_t(\phi_t) - {}^*m(\phi_0) - \int_0^t {}^*H_r(A\phi_r)^*\, dr \tag{4.12}$$

is an internal $*$-continuous martingale with internal quadratic variation

$$\langle {}^*Z(\phi)\rangle_t = \int_0^t \int {}^*b(u^\eta(r, y_r))\, \phi(r, y)^2\, {}^*H_r(dy)\, {}^*dr\,.$$

Lemma 4.9 implies $^*Z.(\phi)$ is a.s. S-continuous and $|^*Z_t(\phi)|^p$ is S-integrable $\forall p > 0$ and nearstandard t. This implies that $Z_t(\phi) = st_C(^*Z(\phi))\,(t)$ is a continuous square integrable martingale. If $\hat{\phi}(y)\,(t) = \phi(t, y^t)\,(y \in C)$, then $(\hat{\phi}, \widehat{A\phi})\,(y) \in C \times D$ for P_m-a.a.y because all P_m-martingales are a.s. continuous. By the nonstandard form of Lusin's theorem (see Anderson 1982, Theorem 3.7)

$$st_{C \times D}(^*\hat{\phi}, {}^*\widehat{A\phi})\,(y) = (\hat{\phi}(st_C y), \widehat{A\phi}(st_C y)) \quad \text{for} \quad L(^*P_m)\text{-a.a.}y. \tag{4.13}$$

Choose K such that

$$|^*A\phi(t, y^t)| + |^*\phi(t, y^t)| \leqq K \quad \forall t \in {}^*[0, \infty)\, L(^*P_m)\text{-a.a.}y. \tag{4.14}$$

By (4.13) and (4.14) we may choose $G \in \sigma(^*\mathscr{C})$ such that $L(^*P_m)\,(B \notin G) = 0$ and

$$G \subset \Big\{ y: {}^*\hat{\phi} \in ns(^*C), {}^*\widehat{A\phi} \in ns(^*D), \sup_t |^*\phi(t, y^t)| + |^*A\phi(t, y^t)| \leqq K,$$
$$^{\circ *}\phi(t, y^t) = \phi(^\circ t, st_C(y^t))\, \forall t \in ns(^*[0, \infty)), st_D(^*\widehat{A\phi})\,(y) = \widehat{A\phi}(st_C(y)) \Big\}\,.$$

(4.11) implies that $L(^*H_t)\,(G^{tc}) = 0\ \forall^\circ t < \infty\, L(^*\mathbb{Q}^\eta)$-a.s. Fix such a $^*H$. If $^\circ t < \infty$ and $y \in G^t$ then $y^t = \tilde{y}^t$ for some $\tilde{y} \in G$ and so

$$^{\circ *}\phi(t, y^t) = {}^{\circ *}\phi(t, \tilde{y}^t) = \phi(^\circ t, st_C(\tilde{y}^t)) = \phi(^\circ t, st_C(y^t)) \tag{4.15}$$

$$|^*\phi(t, y^t)| + |^*A\phi(t, y^r)| = |^*\phi(t, \tilde{y}^t)| + |^*A\phi(t, \tilde{y}^t)| \leqq K \tag{4.16}$$

$$\begin{aligned} {}^{\circ *}A\phi(t, y^t) = {}^{\circ *}A\phi(t, \tilde{y}^t) &\in \{\widehat{A\phi}(st_C(\tilde{y}))\,(^\circ t), \widehat{A\phi}(st_C(\tilde{y}))(^\circ t-)\} \\ &= \{A\phi(^\circ t, st_C(\tilde{y})^{^\circ t}), \lim_{s \uparrow t} A\phi(s, st_C(\tilde{y})^s)\} \\ &= \{A\phi(^\circ t, st_C(y)^{^\circ t}), \lim_{s \uparrow t} A\phi(s, st_C(y)^s)\}\,. \end{aligned} \tag{4.17}$$

(4.15) and (4.16) imply

$$^{\circ *}H_t(\phi_t) = \int_{G^t} \phi(^\circ t, st_C(y^t))\, L(^*H_t)\,(dy) = H_{^\circ t}(\phi_{^\circ t})\, \forall^\circ t < \infty\, L(^*\mathbb{Q}^\eta)\text{-a.s.} \tag{4.18}$$

[This is really the key step. Note the above would be trivial if ϕ were continuous.] (4.17) and (4.16) imply that $L({}^*\mathbb{Q}^\eta)$-a.s. $\forall {}^\circ t < \infty$

$$(4.19)\qquad {}^\circ\int_0^t {}^*H_r({}^*A\phi)^*\,dr = \int_0^t \int_{G^r} {}^{\circ *}A\phi(r, y^r)\, L({}^*H_r)(dy)\, L({}^*dr)$$

$$= \int_0^t \int_{G^r} A\phi({}^\circ r, st_C(y^{{}^\circ r}))\, L({}^*H_r)(dy)\, L({}^*dr)$$

$$= \int_0^{{}^\circ t} \int A\phi(r, y^r)\, H_r(dy)\, dr\,.$$

The nonstandard form of Lusin's theorem implies

$$(4.20)\qquad {}^{\circ *}m({}^*\phi_0) = m(\phi_0)\,.$$

(4.18), (4.19) and (4.20) allow us to take standard parts in (4.12) and conclude

$$Z_t(\phi) = H_t(\phi_t) - m(\phi_0) - \int_0^t H_r(A\phi_r)\,dr$$

is a continuous square integrable martingale. Now ${}^*Z_t(\phi)^2 - \langle {}^*Z(\phi)\rangle_t$ is an S-integrable, S-continuous internal martingale (use (4.2), Proposition 4.5 and Lemma 4.9 to get S-integrability). Therefore for $L({}^*\mathbb{Q}^\eta)$-a.a. ${}^*H$ and all ${}^\circ t < \infty$

$$\langle Z(\phi)\rangle(t) = st\left(\int_0^{\cdot} \int {}^*b(u^\eta(r, y_r))^*\, \phi(r, y^r)^2\, {}^*H_r(dy)\, dr\right)(t)$$

$$= \int_0^t \int b(u(r, y_r))\, \phi(r, y^r)^2\, H_r(dy)\, dr$$

where we have used the fact that ${}^{\circ *}b(u^\eta(r, y_r)) = b(u({}^\circ r, st(y)({}^\circ r)))$ for any S-continuous y in ${}^*C$ $L({}^*\mathbb{Q}^\eta)$-a.s. (by the a.s. S-continuity of ${}^*u$). We have also used (4.15), and the moment estimates from Proposition 4.2(a) to take the standard part through the integral sign. We have shown that $\mathbb{Q} = L({}^*\mathbb{Q}^\eta)(st_M({}^*H) \in \cdot)$ is a solution of (M_m). As η ranges over ${}^*\mathbb{N} - \mathbb{N}$ these laws are precisely the limit points of $\{\mathbb{Q}^n : n \in \mathbb{N}\}$ and the proof is complete. □

We now assume the weaker condition (3.1) in place of (4.1). Note that

$$a_n(u) = (u/(u + n^{-1}))^{1/2}\, a(u)$$

satisfies (4.1) and $b_n(u) = a_n(u)^2 u^{-1}$ satisfies

$$(4.21)\qquad u b_n(u) \leqq c_{4.3}(u^2 + u^{2\theta}) \quad \forall u \geqq 0,\ n \in \mathbb{N}\,.$$

Let $\mathbb{Q}^{(n)}$ be a solution of (M_m) but with b_n in place of b. $\mathbb{Q}^{(n)}$ exists by Proposition 4.11. Here then is our main existence result.

Theorem 4.12 *Assume* (3.9) *and* (3.1) $\{\mathbb{Q}^{(n)} : n \in \mathbb{N}\}$ *is tight on* Ω. *Any weak limit point,* $\mathbb{Q}$, *is a solution of* (M_m) *and is such that* $t \mapsto u_t$ *is a continuous* C^+_{rap}*-valued map* $\mathbb{Q}$-a.s.

Proof. The proof proceeds exactly as for Proposition 4.11. The only reason we could not allow $\theta < 1/2$ in the argument was the lack of a uniform bound on

$b(u^n(r, y))u(r, y)$. (4.21) gives us precisely such a bound in the present context and the proof now proceeds as before using (4.21) in place of (4.2). □

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