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Identities without the star for $*$-bands

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Abstract. Adair determined the lattice of varieties of $*$-bands and provided a single identity for each of them. We construct a new set of identities for the same purpose. Both of these systems involve at least one variable which is starred. Using the second system of identities, we show that the system of identities constructed by Gerhard and the author for varieties of bands also represents a set of identities determining the varieties of $*$-bands. The same type of result is proved for varieties of band monoids and varieties of $*$-band monoids using the results of two papers by Wismath.

1. Introduction and summary

After the determination of the lattice of varieties of bands by Birjukov [2], Fennemore [3] and Gerhard [4], it was natural to consider the same problem for the $*$-bands. In fact, a $*$-band is a band provided with an involution $*$ which satisfies the identity $x = xx^*x$. The lattice of varieties of $*$-bands was then determined by Adair [1] with a nontrivial adaptation of Fennemore's method. As Fennemore did for bands, Adair provides a system of identities for the varieties of $*$-bands. Except for the trivial variety and the rectangular bands, Adair's system of identities, one for each variety, includes the words containing one variable with the star applied to it. She also proved that the first two of the remaining system of identities are equivalent to an identity for semilattices and normal bands, respectively. This was established again by the author [6] and in addition that the next identity in the Adair system is equivalent to an identity for regular bands.

This situation raises the natural question whether all varieties of $*$-bands can be defined by identities not involving the $*$-operation. Judging from the results of the author's paper [6], where each variety of completely regular $*$-semigroups considered turns out to be also definable by an identity not involving the $*$-operation, the answer for $*$-bands ought to be positive. Indeed, our main result here affirms this expectation.

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The lattice of varieties of band monoids was determined by Wismath [7] providing each variety with a single identity from the Fennemore [3] list of identities for varieties of bands. Finally, the lattice of $*$-band monoids was constructed by Wismath [8] thereby completing this cycle of works on diverse varieties of bands.

In Section 2 we state the Adair system of identities for varieties of $*$-bands, the Gerhard-Petrich system for a set of varieties of bands and a new system for varieties of $*$-bands; we refer to these as the A-, GP- and N-systems, respectively. A few basic lemmas conclude this section. The equivalence of the A- and N-systems is established in Section 3. It is proved in Section 4 that certain varieties of $*$-bands can be given by the identities for semilattices, normal bands and regular bands, respectively. Section 5 consists of three lemmas which are needed in the proof of the theorem in Section 6; this is the main result of the paper and asserts the equivalence of the GP- and the N-systems. The effect of putting a variable in the words considered equal to 1, the empty word, is considered briefly in Section 7. This concerns the varieties of band monoids and of $*$-band monoids. A related discussion can be found in Section 8 and concluding remarks in Section 9. Three diagrams illustrate the mappings from the lattice of varieties of bands to the lattices of varieties of $*$-bands, band monoids and $*$-band monoids, respectively.

2. Preliminaries

For the set of variables in the identities for bands, we let $X = \{x_1, x_2, \dots\}$. Hence we consider the free semigroup X^+ on X consisting of all (nonempty) words on X with juxtaposition as product. The least band congruence on X^+ is generated by the set $\{(x, x^2) \mid x \in X^+\}$. For $u, v \in X^+$, we write $u \sim v$ if u and v are related by this congruence. For identities $u_1 = v_1, u_2 = v_2, \dots$, we denote by $[u_1 = v_1, u_2 = v_2, \dots]$ the variety of bands determined by these identities.

A *$*$-band* is a band B provided with a unary operation $*$ satisfying the identities

$$(xy)^* = y^*x^*, \qquad (x^*)^* = x, \qquad x = xx^*x.$$

The identities for $*$-bands draw their variables from the set $X \cup X^*$, where X is as above and $X^* = \{x^* \mid x \in X\}$ with $x \leftrightarrow x^*$ a bijection of the disjoint sets X and X^*. The words over $X \cup X^*$ are considered as elements of the free involutorial semigroup on X which is the semigroup $(X \cup X^*)^+$ with the unary operation: $x \to x^*$ if $x \in X$ and $x^* \to x$ if $x^* \in X^*$, and for the remaining words

$$y_1 y_2 \cdots y_n \to y_n^* \cdots y_2^* y_1^* \qquad (y_1, y_2, \dots, y_n \in X \cup X^*).$$

We denote by $u = v \Rightarrow w = z$ the implication of (∗-) band identities and consistently omit the identities above and $x^2 = x$.

If w is any of the band or ∗-band words, as defined above, let $\bar{w}$ be its *mirror image*, that is, if $w = y_1 y_2 \cdots y_n$, then $\bar{w} = y_n \cdots y_2 y_1$. In view of the involution ∗ in a ∗-band, we have the following simple and useful observation.

LEMMA 2.1. *Let $u = v$ be an identity of ∗-bands. Then $u = v \Rightarrow \bar{u} = \bar{v}$.*

Proof. If $u = v$ is of the form $u_1 u_2 \cdots u_m = v_1 v_2 \cdots v_n$ where $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \in X \cup X^*$, then applying the operation ∗ and substituting $u_i \to u_i^*$, $v_i \to v_i^*$, we obtain $u_m \cdots u_2 u_1 = v_n \cdots v_2 v_1$, that is $\bar{u} = \bar{v}$.

For band monoids we consider the free monoid $X^+ \cup \{1\}$, where 1 denotes the empty word, on X and for ∗-band monoids the free monoid $(X \cup X^*)^+ \cup \{1\}$.

We now introduce the systems of words which play a basic role in our deliberations.

A-*system* (Adair [1]): let

$$R_1 = x_1, \qquad R_2 = x_1 x_2,$$

and for $n > 2$, by induction define

$$R_n = \begin{cases} \overline{R_{n-2}} x_{(n+1)/2} & \text{if } n \text{ is odd} \\ \overline{R_{n-2}} x_{(n+2)/2} & \text{if } n \text{ is even} \end{cases};$$

$$S_1 = x_1^* x_1, \qquad S_2 = x_1 x_2^* x_1 x_2,$$

and for $n > 2$, set

$$S_n = \begin{cases} \overline{S_{n-2}} x_{(n+1)/2} \overline{R_{n-2}} x_{(n+1)/2} & \text{if } n \text{ is odd} \\ \overline{S_{n-2}} x_{(n+2)/2} \overline{R_{n-2}} x_{(n+2)/2} & \text{if } n \text{ is even} \end{cases}.$$

The A-system consists of the identities $R_n = S_n$ for $n \geq 1$.

GP-*system* (Gerhard-Petrich [5]): let

$$G_2 = x_2 x_1, \qquad H_2 = x_2, \qquad I_2 = x_2 x_1 x_2$$

and for $n > 2$, by induction define

$$G_n = x_n \overline{G_{n-1}}, \qquad T_n = G_n x_n \overline{T_{n-1}} \qquad \text{for } T \in \{H, I\}.$$

The GP-system consists of identities $G_n = H_n$, $G_n = I_n$ for $n > 1$.

N-*system* (new): let G_n be as above and

$$V_2 = x_2 x_1 x_2^* x_1, \qquad W_1 = x_1 x_1^*$$

and by induction define

$$T_n = G_n x_n \overline{T_{n-1}} \qquad \text{for } T = V, n \geq 3 \quad \text{and} \quad T = W, n \geq 2.$$

The N-system consists of the identities $G_n = V_n$ for $n > 1$, $G_n = W_n$ for $n \geq 1$.

If there is a need to state the variables of the above functions, we shall write

$$T_n = T_n(x_1, x_2, \ldots, x_n) \qquad \text{for } T \in \{G, H, I, V, W\}.$$

We may extend the GP-system by letting $G_1 = x_1$ and $I_1 = 1$, the empty word.

The next two lemmas give G_n and T_n without induction.

LEMMA 2.2. *For $n \geq 2$, we have*

$$G_n = \begin{cases} (x_n x_{n-2} \cdots x_3 x_1)(x_2 x_4 \cdots x_{n-1}) & \text{if } n \text{ is odd} \\ (x_n x_{n-2} \cdots x_4 x_2)(x_1 x_3 \cdots x_{n-1}) & \text{if } n \text{ is even} \end{cases}.$$

Proof. The inductive argument is straightforward and is omitted.

LEMMA 2.3. *For $n > 2$ and $T \in \{G, H, I, V, W\}$, let $T_n = G_n x_n \overline{T_{n-1}}$. Then*

$$T_n = \begin{cases} G_n x_n G_{n-2} x_{n-2} \cdots G_3 x_3 \overline{T_2} x_4 \overline{G_4} x_6 \overline{G_6} \cdots x_{n-1} \overline{G_{n-1}} & \text{if } n \text{ is odd} \\ G_n x_n G_{n-2} x_{n-2} \cdots G_4 x_4 T_2 x_3 \overline{G_3} x_5 \overline{G_5} \cdots x_{n-1} \overline{G_{n-1}} & \text{if } n \text{ is even} \end{cases}.$$

Proof. The inductive argument is straightforward and is omitted.

3. The equivalence of the N- and the A-systems

First we identify which words of the N-system are equal to which words of the A-system.

LEMMA 3.1. *For $n > 1$, $\overline{G_n} = R_{2n-1} = R_{2n-2}$.*

Proof. The argument is by induction on n. For $n = 2$, we have $\overline{G_2} = x_1 x_2 = R_2 = R_{4-2} = R_3 = R_{4-1}$. Assume the formula valid for n. By the induction hypothesis, on the one hand,

$$G_{n+1} = x_{n+1}\overline{G_n} = x_{n+1}R_{2n-1} = x_{n+1}R_{2n-2}$$

so that

$$\overline{G_{n+1}} = \overline{R_{2n-1}}x_{n+1} = \overline{R_{2n-2}}x_{n+1}. \tag{1}$$

On the other hand,

$$R_{2(n+1)-1} = R_{2n+1} = \overline{R_{2n-1}}x_{n+1}, \tag{2}$$

$$R_{2(n+1)-2} = R_{2n} = \overline{R_{2n-2}}x_{n+1} \tag{3}$$

and the required equality follows from (1), (2) and (3).

LEMMA 3.2. *For* $n \geq 1$, $\overline{W_n} = S_{2n-1}$.

Proof. For $n = 1$, we have $\overline{W_1} = x_1^* x_1 = S_1$. Assume that the claim is valid for n. On the one hand,

$$\begin{aligned} \overline{W_{n+1}} &= \overline{G_{n+1}x_{n+1}\overline{W_n}} = \overline{W_n x_{n+1} G_{n+1}} = \overline{W_n x_{n+1} x_{n+1}\overline{G_n}} \\ &= \overline{W_n x_{n+1} G_n x_{n+1}} = \overline{S_{2n-1} x_{n+1} G_n x_{n+1}} \end{aligned} \tag{4}$$

and on the other hand, by Lemma 3.1,

$$S_{2(n+1)-1} = S_{2n+1} = \overline{S_{2n-1}x_{n+1}\overline{R_{2n-1}}x_{n+1}} = \overline{S_{2n-1}x_{n+1}G_n x_{n+1}} \tag{5}$$

and the assertion follows from (4) and (5).

LEMMA 3.3. *For* $n > 1$, $\overline{V_n} = S_{2n-2}$.

Proof. For $n = 2$, we have $\overline{V_2} = x_1 x_2^* x_1 x_2 = S_2$. Assume that the claim is valid for n. On the one hand,

$$\begin{aligned} \overline{V_{n+1}} &= \overline{G_{n+1}x_{n+1}\overline{V_n}} = \overline{V_n x_{n+1} G_{n+1}} = \overline{V_n x_{n+1} x_{n+1}\overline{G_n}} \\ &= \overline{V_n x_{n+1} G_n x_{n+1}} = \overline{S_{2n-2} x_{n+1} G_n x_{n+1}} \end{aligned} \tag{6}$$

and on the other hand, by Lemma 3.1,

$$S_{2(n+1)-2} = S_{2n} = \overline{S_{2n-2}}x_{n+1}\overline{R_{2n-2}}x_{n+1} = \overline{S_{2n-2}}x_{n+1}G_nx_{n+1} \tag{7}$$

and the assertion follows from (6) and (7).

From the above three lemmas and the definitions of the N- and A-systems we conclude that they are equivalent. See Diagram 1 and the comments at the end of Section 6.

4. The first three layers

We shall use the following notation for varieties of ∗-bands:

$\mathcal{S} = [xy = yx]$ – semilattices,

$\mathcal{N}\mathcal{B} = [axya = ayxa]$ – normal bands,

$\mathcal{R}e\mathcal{B} = [axya = axaya]$ – regular bands.

Using the ∗-operation, by ([1], Lemmas 4.1, 4.4 and 4.5, Theorem 4.2 and the diagram from [1]), we have

$\mathcal{S} = [x = x^*] = [x = x^*x] = [R_1 = S_1]$,

$\mathcal{N}\mathcal{B} = [axa = ax^*a] = [ax = ax^*ax] = [R_2 = S_2]$.

The above three varieties appear in the context of completely regular ∗-semigroups in [6]. By ([6], Propositions 6.2 and 6.3(i) and Lemma 7.3), we have

$\mathcal{S} = [x = xx^*]$,

$\mathcal{N}\mathcal{B} = [axa = axx^*a]$,

$\mathcal{R}e\mathcal{B} = [ax = aa^*xax] = [axa^*a = aa^*xa]$.

By Lemmas 3.2 and 3.3, $\mathcal{S} = [R_1 = S_1]$ and $\mathcal{N}\mathcal{B} = [R_2 = S_2]$ yield $\mathcal{S} = [G_1 = W_1]$ and $\mathcal{N}\mathcal{B} = [G_2 = V_2]$. By the change of notation $a \to x_1$ and $x \to x_2$ and using Lemma 2.1, we see that $ax = aa^*xax$ is equivalent to $R_3 = S_3$ which by Lemma 3.2 gives $\mathcal{R}e\mathcal{B} = [G_2 = W_2]$. Therefore the five varieties on the bottom of the Adair lattice are the varieties of $*$-bands: $\mathcal{T} = [x = y]$, $\mathcal{RB} = [x = xyx]$, $\mathcal{S}$, $\mathcal{NB}$ and $\mathcal{ReB}$, each of which can be determined by a single identity not involving starred variables.

Considering the words G_n, H_n, I_n, V_n and W_n, we shall prove in Lemma 5.7 that $G_{n+1} = H_{n+1} \Rightarrow G_n = V_n$ and $G_{n+1} = I_{n+1} \Rightarrow G_n = W_n$ for all relevant n. For the reverse implication, we shall apply a somewhat oblique argument strongly using the form of the lattice of varieties of $*$-bands. We shall now illustrate on the example of the three varieties considered above, the complexities involved in proving the reverse implication directly by considering the words involved.

LEMMA 4.1.
(i) $G_1 = W_1 \Rightarrow G_2 = I_2$.
(ii) $G_2 = V_2 \Rightarrow G_3 = H_3$.
(iii) $G_2 = W_2 \Rightarrow G_3 = I_3$.

Proof.
(i) First note that by $G_1 = W_1$, $x_2^* = (x_2 x_2^*)^* = x_2 x_2^* = x_2$ and substitute $x_1 \to x_2 x_1$ into $G_1 = W_1$ getting

$$x_2 x_1 = x_2 x_1 (x_2 x_1)^* = x_2 x_1 x_1^* x_2^* = x_2 x_1 x_1 x_2 = x_2 x_1 x_2.$$

(ii) In $G_2 = V_2$ substitute $x_2 \to x_3 x_1$, $x_1 \to x_2$ getting $x_3 x_1 x_2 = x_3 x_1 x_2 x_1^* x_3^* x_2$ which in turn equals $x_3 x_1 x_2 x_1 x_3^* x_1^* x_3^* x_2$, by using $G_2 = V_2$ on $x_1 x_3^*$. Then we have

$$\begin{aligned} x_3 x_1 x_2 &= (x_3 x_1 x_2) x_1 x_3^* (x_1^* x_3^* x_2) = (x_3 x_1 x_2)(x_1 x_3^* x_1^* x_3^*) x_2 \\ &= (x_3 x_1 x_2) x_1 x_3^* x_2 \qquad \text{by hypothesis} \\ &= (x_3 x_1 x_2 x_1) x_3^* x_2 x_3 x_2 \qquad \text{by hypothesis} \end{aligned}$$

and substituting $x_1 \to x_1 x_2$ we obtain

$$\begin{aligned} x_3 x_1 x_2 &= x_3 x_1 x_2 x_1 x_2 x_3^* x_2 x_3 x_2 = x_3 x_1 (x_2 x_3^* x_2 x_3) x_2 \\ &= x_3 x_1 x_2 x_3 x_2 \qquad \text{by the mirror of the hypothesis.} \end{aligned}$$

(iii) In $G_2 = W_2$ substitute $x_1 \to x_3 x_4$ getting $x_2 x_3 x_4 = x_2 x_3 x_4 x_2 x_4^* x_3^* x_3 x_4$; then substitute $x_2 \to x_3$, $x_3 \to x_1$ and $x_4 \to x_2$, obtaining $x_3 x_1 x_2 = x_3 x_1 x_2 x_3 x_2^* x_2^* x_1^* x_1 x_2$; then substitute $x_2 \to x_2 x_2^*$ getting $x_3 x_1 x_2 x_2^* = x_3 x_1 x_2 x_2^* x_3 x_2 x_2^* x_1^* x_1 x_2 x_2^*$. Multiplying on the right by x_2 gives $x_3 x_1 x_2 = (x_3 x_1 x_2) x_2^* (x_3 x_2 x_2^* x_1^* x_1 x_2)$ and substituting $x_2 \to x_2 x_1$ gives $x_3 x_1 x_2 x_1 = (x_3 x_1 x_2 x_1) x_2 (x_3 x_2 x_1 x_1^* x_2^* x_1^* x_1 x_2 x_1)$ and multiplying on the right by x_2 yields

$$x_3 x_1 x_2 = x_3 x_1 x_2 x_3 x_2 x_1 (x_1^* x_2^* x_1^*) x_1 x_2 = x_3 x_1 x_2 x_3 x_2 x_1 x_2.$$

5. Lemmas

The following sequence of lemmas will be used in the proof of the main result of the paper established in the next section. Recall that $\sim$ stands for the least band congruence on $(X \cup X^*)^+$. The first lemma relates the values of G_{n+1} and G_n, $\overline{G_{n+1}}$ and $\overline{G_n}$ evaluated when x_2 or x_3 take on some special values.

LEMMA 5.1. *Let* $n > 1$.

(i) $G_{n+1}(x_1, x_2, x_1 x_1^*, x_4, \ldots, x_{n+1}) \sim G_n(x_2, x_1, x_4, \ldots, x_{n+1})$ *if n is even.*
(ii) $G_{n+1}(x_1, x_2, x_1^* x_1, x_4, \ldots, x_{n+1}) \sim G_n(x_2, x_1, x_4, \ldots, x_{n+1})$ *if n is odd.*
(iii) $G_{n+1}(x_1, x_1^* x_1, x_3, \ldots, x_{n+1}) \sim G_n(x_1, x_3, \ldots, x_{n+1})$ *if n is even.*
(iv) $G_{n+1}(x_1, x_1 x_1^*, x_3, \ldots, x_{n+1}) \sim G_n(x_1, x_3, \ldots, x_{n+1})$ *if n is odd.*
(v) $\overline{G_{n+1}}(x_1, x_2, x_1^* x_1, x_4, \ldots, x_{n+1}) \sim \overline{G_n}(x_2, x_1, x_4, \ldots, x_{n+1})$ *if n is even.*
(vi) $\overline{G_{n+1}}(x_1, x_2, x_1 x_1^*, x_4, \ldots, x_{n+1}) \sim \overline{G_n}(x_2, x_1, x_4, \ldots, x_{n+1})$ *if n is odd.*
(vii) $\overline{G_{n+1}}(x_1, x_1 x_1^*, x_3, \ldots, x_{n+1}) \sim \overline{G_n}(x_1, x_3, \ldots, x_{n+1})$ *if n is even.*
(viii) $\overline{G_{n+1}}(x_1, x_1^* x_1, x_3, \ldots, x_{n+1}) \sim \overline{G_n}(x_1, x_3, \ldots, x_{n+1})$ *if n is odd.*

Proof. We shall use Lemma 2.2 repeatedly.

(i) $G_{n+1}(x_1, x_2, x_1 x_1^*, x_4, \ldots, x_{n+1}) = x_{n+1} x_{n-1} \cdots x_5 (x_1 x_1^*) x_1 x_2 x_4 \cdots x_n$
$\sim (x_{n+1} x_{n-1} \cdots x_5 x_1)(x_2 x_4 \cdots x_n) = G_n(x_2, x_1, x_4, \ldots, x_{n+1})$.
(ii) $G_{n+1}(x_1, x_2, x_1^* x_1, x_4, \ldots, x_{n+1}) = x_{n+1} x_{n-1} \cdots x_4 x_2 x_1 (x_1^* x_1) x_5 \cdots x_n$
$\sim (x_{n+1} x_{n-1} \cdots x_4 x_2)(x_1 x_5 \cdots x_n) = G_n(x_2, x_1, x_4, \ldots, x_{n+1})$.
(iii) $G_{n+1}(x_1, x_1^* x_1, x_3, \ldots, x_{n+1}) = x_{n+1} x_{n-1} \cdots x_3 x_1 (x_1^* x_1) x_4 \cdots x_n$
$\sim (x_{n+1} x_{n-1} \cdots x_3)(x_1 x_4 \cdots x_n) = G_n(x_1, x_3, \ldots, x_{n+1})$.
(iv) $G_{n+1}(x_1, x_1 x_1^*, x_3, \ldots, x_{n+1}) = x_{n+1} x_{n-1} \cdots x_4 (x_1 x_1^*) x_1 x_3 \cdots x_n$
$\sim (x_{n+1} x_{n-1} \cdots x_4 x_1)(x_3 \cdots x_n) = G_n(x_1, x_3, \ldots, x_{n+1})$.

The proof of parts (v)–(viii) is similar and is omitted.

The next lemma has a similar character as the first one. Only in it we consider the words H_{n+1} and I_{n+1} with similar substitutions. In this way they relate H_{n+1} with V_n and I_{n+1} with W_n.

LEMMA 5.2.

(i) *For $n > 1$ and even, we have*

$$H_{n+1}(x_1, x_2, x_1 x_1^*, x_4, \ldots, x_{n+1}) \sim V_n(x_2, x_1, x_4, \ldots, x_{n+1}).$$

(ii) *For $n > 1$ and odd, we have*

$$H_{n+1}(x_1, x_2, x_1^* x_1, x_4, \ldots, x_{n+1}) \sim V_n(x_2, x_1, x_4, \ldots, x_{n+1}).$$

(iii) *For $n > 1$ and even, we have*

$$I_{n+1}(x_1, x_1^* x_1, x_3, \ldots, x_{n+1}) \sim W_n(x_1, x_3, \ldots, x_{n+1}).$$

(iv) *For $n \geq 1$ and odd, we have*

$$I_{n+1}(x_1, x_1 x_1^*, x_3, \ldots, x_{n+1}) \sim W_n(x_1, x_3, \ldots, x_{n+1}).$$

Proof. By Lemmas 2.3 and 5.1, we obtain

$$\begin{aligned}
&H_{n+1}(x_1, x_2, x_1 x_1^*, x_4, \ldots, x_{n+1}) \\
&\quad = G_{n+1}(x_1, x_2, x_1 x_1^*, x_4, \ldots, x_{n+1}) x_{n+1} \\
&\quad \bullet\ G_{n-1}(x_1, x_2, x_1 x_1^*, x_4, \ldots, x_{n-1}) x_{n-1} \\
&\qquad\qquad \ldots \\
&\quad \bullet\ G_5(x_1, x_2, x_1 x_1^*, x_4, x_5) x_5 \\
&\quad \bullet\ G_3(x_1, x_2, x_1 x_1^*) x_1 x_1^* x_2 \\
&\quad \bullet\ x_4 \overline{G_4}(x_1, x_2, x_1 x_1^*, x_4) \\
&\quad \bullet\ x_6 \overline{G_6}(x_1, x_2, x_1 x_1^*, x_4, x_5, x_6) \\
&\qquad\qquad \ldots \\
&\quad \bullet\ x_n \overline{G_n}(x_1, x_2, x_1 x_1^*, x_4, \ldots, x_n) \\
&\quad \sim G_n(x_2, x_1, x_4, \ldots, x_{n+1}) x_{n+1} \\
&\quad \bullet\ G_{n-2}(x_2, x_1, x_4, \ldots, x_{n-1}) x_{n-1} \\
&\qquad\qquad \ldots \\
&\quad \bullet\ G_4(x_2, x_1, x_4, x_5) x_5 \\
&\quad \bullet\ (x_1 x_2) x_1 (x_1^* x_2) \qquad (= V_2(x_2, x_1)) \\
&\quad \bullet\ x_4 \overline{G_3}(x_2, x_1, x_4) \\
&\quad \bullet\ x_6 \overline{G_5}(x_2, x_1, x_4, x_5, x_6) \\
&\qquad\qquad \ldots \\
&\quad \bullet\ x_n \overline{G_{n-1}}(x_2, x_1, x_4, \ldots, x_n) \\
&\quad = V_n(x_2, x_1, x_4, \ldots, x_{n+1}).
\end{aligned}$$

The proofs of parts (ii)–(iv) are similar and are omitted.

Using the preceding lemmas, we establish now the following implications.

LEMMA 5.3. *We have*
(i) $G_{n+1} = H_{n+1} \Rightarrow G_n = V_n$ *for* $n > 1$,
(ii) $G_{n+1} = I_{n+1} \Rightarrow G_n = W_n$ *for* $n \geq 1$.

Proof. (i) For n even, from Lemmas 5.1 and 5.2(i), with the substitution $x_3 \to x_1 x_1^*$ in G_{n+1} and H_{n+1}, we get $G_n = V_n$. For n odd, from Lemmas 5.1 and 5.2(ii), with the substitution from $x_3 \to x_1^* x_1$ in G_{n+1} and H_{n+1}, we get $G_n = V_n$.

(ii) For n even, from Lemmas 5.1 and 5.2(iii), with the substitution $x_2 \to x_1^* x_1$ in G_{n+1} and I_{n+1}, we get $G_n = W_n$. For n odd, from Lemmas 5.1 and 5.2(iv), with the substitution $x_2 \to x_1 x_1^*$, we get $G_n = W_n$.

6. The equivalence of the GP- and the N-systems

To this end, it remains to establish the reverse implications in Lemma 5.3. We shall use Lemmas 3.1, 3.2 and 3.3, which together give the equivalence of the N- and the A-systems, from Adair's result that the latter together with $x_1 = x_1 x_2 x_1$ provide a complete set of inequivalent identities for $*$-bands, including the form of the lattice of varieties of $*$-bands constructed by Adair. The following is our main result.

THEOREM 6.1. *We have*

$$G_{n+1} = H_{n+1} \Leftrightarrow G_n = V_n \qquad \textit{for } n > 1,$$

$$G_{n+1} = I_{n+1} \Leftrightarrow G_n = W_n \qquad \textit{for } n \geq 1.$$

Proof. We have seen in Lemma 4.1 that

$$G_2 = I_2 \Leftrightarrow G_1 = W_1,$$

$$G_3 = H_3 \Leftrightarrow G_2 = V_2,$$

$$G_3 = I_3 \Leftrightarrow G_2 = W_2.$$

The proof of the rest is by induction. We consider only the case of the transition $G_{n+1} = H_{n+1}$ to $G_{n+1} = I_{n+1}$; the proof of the case $G_{n+1} = I_{n+1}$ to $G_{n+2} = H_{n+2}$ is analogous. Hence assume that $G_{n+1} = H_{n+1} \Leftrightarrow G_n = V_n$ for some $n > 2$. By Lemma 5.3(ii), we have $G_{n+1} = I_{n+1} \Rightarrow G_n = W_n$ and thus

$$[G_{n+1} = I_{n+1}] \subseteq [G_n = W_n]$$

for $*$-bands. By the induction hypothesis, and monotonicity,

$$\text{either } [G_{n+1} = I_{n+1}] = [G_n = W_n] \tag{8}$$

$$\text{or } [G_{n+1} = I_{n+1}] = [G_n = V_n] \quad (=[G_{n+1} = H_{n+1}]). \tag{9}$$

By [5], we know that for bands,

$$[G_{n+1} = H_{n+1}] \subset [G_{n+1} = I_{n+1}]$$

so there exists a band $B \in [G_{n+1} = I_{n+1}]$ and $B \notin [G_{n+1} = H_{n+1}]$. We shall use B to show that $[G_{n+1} = H_{n+1}] \neq [G_{n+1} = I_{n+1}]$ for $*$-bands which will exclude the alternative (9) thereby establishing the required equality (8).

We let B^* be the spined product of B and B^{opp} relative to the Green relation $\mathscr{D}$, where $x \mathscr{D} y$ if and only if $x = xyx$ and $y = yxy$. This means that

$$B^* = \{(x, y) \in B \times B \mid x \mathscr{D} y\}$$

with the multiplication $(x, y)(w, z) = (xw, zy)$. We next define a unary operation on B^* by $(x, y)^* = (y, x)$. Straightforward verification shows that B^* satisfies the axioms of a $*$-band. In fact, the only place that this relation is needed, in the proof, is to make the identity $x = xx^*x$ hold, so that B^* is indeed a $*$-band.

Let $(s_1, t_1), (s_2, t_2), \ldots, (x_{n+1}, t_{n+1}) \in B^*$ and

$$g_{n+1} = \begin{cases} (s_{n+1}, t_{n+1}) \cdots (s_1, t_1)(s_2, t_2) \cdots (s_n, t_n) & \text{if } n \text{ is even} \\ (s_{n+1}, t_{n+1}) \cdots (s_2, t_2)(s_1, t_1) \cdots (s_n, t_n) & \text{if } n \text{ is odd,} \end{cases}$$

following the pattern of Lemma 2.2 for G_{n+1} with the analogous formula for g_k and for $\overline{g_k}$, by reversing the order of factors. For $k \leq n$, let

$$i_{n+1} = \begin{cases} g_{n+1}(s_{n+1}, t_{n+1}) \cdots g_3(s_3, t_3)(s_2, t_2)(s_1, t_1)(s_2, t_2)(s_4, t_4)\overline{g_k} \cdots (s_n, t_n)\overline{g_n} \\ \hfill \text{if } n \text{ is even} \\ g_{n+1}(s_{n+1}, t_{n+1}) \cdots g_4(s_4, t_4)(s_2, t_2)(s_1, t_1)(s_2, t_2)(s_3, t_3)\overline{g_3} \cdots (s_n, t_n)\overline{g_n} \\ \hfill \text{if } n \text{ is odd} \end{cases}$$

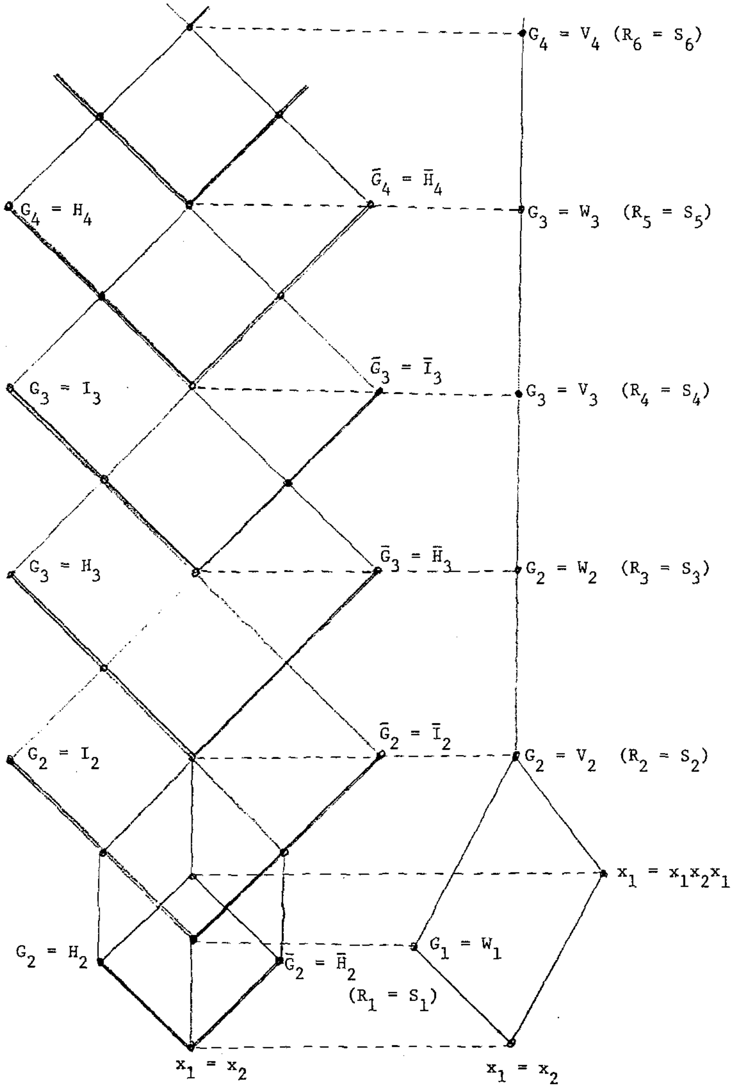

Diagram 1. Proper varieties of *-bands.

following the pattern of Lemma 2.3 for I_{n+1}. Since B satisfies the identity $G_{n+1} = I_{n+1}$, the first entries of g_{n+1} and i_{n+1} are equal. In view of Lemma 2.1, B also satisfies the identity $\overline{G_{n+1}} = \overline{I_{n+1}}$ which evidently implies that the second entries of g_{n+1} and i_{n+1} are equal. Now Lemmas 2.2 and 2.3 yield that B^* satisfies the identity $G_{n+1} = H_{n+1}$.

Since B does not satisfy the identity $G_{n+1} = H_{n+1}$, there exists a choice of elements $s_1, s_2, \ldots, s_{n+1} \in B$ for which $G_{n+1}(s_1, s_2, \ldots, s_{n+1}) \neq H_{n+1}(s_1, s_2, \ldots, s_{n+1})$. The elements $(s_1, s_1), (s_2, s_2), \ldots, (s_{n+1}, s_{n+1}) \in B^*$ evidently have the property

$$G_{n+1}((s_1, t_1), (s_2, t_2), \ldots, (s_{n+1}, t_{n+1}))$$
$$\neq H_{n+1}((s_1, t_1), (s_2, t_2), \ldots, (s_{n+1}, t_{n+1})).$$

Therefore B^* does not satisfy the identity $G_{n+1} = H_{n+1}$.

The left part of Diagram 1 represents the lattice of all proper varieties of bands as constructed by Birjukov [2], Fennemore [3] and Gerhard [4]. Its left column is labelled by the notation of the GP-system. The right part of Diagram 1 represents the lattice of all proper varieties of $*$-bands as constructed by Adair [1] with the labels of the N-system and in parentheses of the A-system. The broken lines between the two diagrams indicate the correspondence of varieties of bands and the varieties of $*$-bands. The heavy lines in the left part indicate the classes of the equivalence relation induced on the first lattice by its mapping onto the second lattice in the given correspondence of varieties. In view of Lemma 2.1, we conclude that the supports of $*$-bands, that is the underlying bands without the $*$-operators, all lie in the varieties situated in the middle column of the diagram on the left.

7. The lattice of varieties of band monoids and $*$-band monoids

Both of these have been determined by Wismath, the first in [7] and the second in [8].

In order to extend the GP-system to the case of varieties of band monoids, referring to Wismath [7], we introduce yet another system of identities.

F-*system* (Fennemore [3]); let

$$R_3 = x_1 x_2 x_3, \qquad S_3 = x_1 x_2 x_3 x_1 x_3 x_2 x_3,$$

and for $n > 3$, by induction define

$$R_n = \begin{cases} R_{n-1} x_n & \text{if } n \text{ is even} \\ x_n R_{n-1} & \text{if } n \text{ is odd} \end{cases};$$

$$S_n = \begin{cases} S_{n-1} x_n R_n & \text{if } n \text{ is even} \\ R_n x_n S_{n-1} & \text{if } n \text{ is odd} \end{cases}.$$

The F-system consists of the identities, for $n > 2$,

$$\overline{R_n} = \overline{S_n} \qquad \text{if } n \text{ is even,}$$
$$R_n = S_n \qquad \text{if } n \text{ is odd.}$$

For $n > 2$, we define the permutation φ_n by

$$\varphi_n = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \cdots x_n \\ x_2 & x_3 & x_1 & x_4 \cdots x_n \end{pmatrix}.$$

With this notation, we shall now see that the F-system is very close to a part of the GP-system, see also ([5], Proposition 9.3).

LEMMA 7.1. *For $n > 2$, we have*

$$G_n \varphi_n = \begin{cases} R_n & \text{if } n \text{ is odd} \\ \overline{R_n} & \text{if } n \text{ is even} \end{cases},$$
$$I_n \varphi_n = \begin{cases} S_n & \text{if } n \text{ is odd} \\ \overline{S_n} & \text{if } n \text{ is even} \end{cases}.$$

Proof. The proof is by induction. For the first step, that is $n = 3$, we have

$$G_3 \varphi_3 = (x_3 x_1 x_2)\varphi_3 = x_1 x_2 x_3 = R_3,$$
$$I_3 \varphi_3 = (x_3 x_1 x_2 x_3 x_2 x_1 x_2)\varphi_3 = x_1 x_2 x_3 x_1 x_3 x_2 x_3 = S_3.$$

Now assume that the claim is true for $n > 2$.

If n is odd, then

$$G_{n+1}\varphi_{n+1} = (x_{n+1}\overline{G_n})\varphi_{n+1} = x_{n+1}(\overline{G_n}\varphi_n) = x_{n+1}\overline{G_n\varphi_n} = x_{n+1}\overline{R_n} = \overline{R_{n+1}},$$
$$I_{n+1}\varphi_{n+1} = (G_{n+1}x_{n+1}\overline{I_n})\varphi_{n+1} = (G_{n+1}\varphi_{n+1})x_{n+1}(\overline{I_n}\varphi_{n+1})$$
$$= \overline{R_{n+1}}x_{n+1}\overline{I_n\varphi_n} = \overline{R_{n+1}}x_{n+1}\overline{S_n} = \overline{S_n x_{n+1} R_{n+1}} = \overline{S_{n+1}}.$$

The case when n is even is similar.

COROLLARY 7.2. *F-system is equivalent to $\{G_n = I_n \mid n > 2\}$.*

In view of ([7], Proposition 4.7), Lemma 7.1 implies that the left hand column of the lattice of proper band monoids in ([7], Figure 3) may be labelled with the

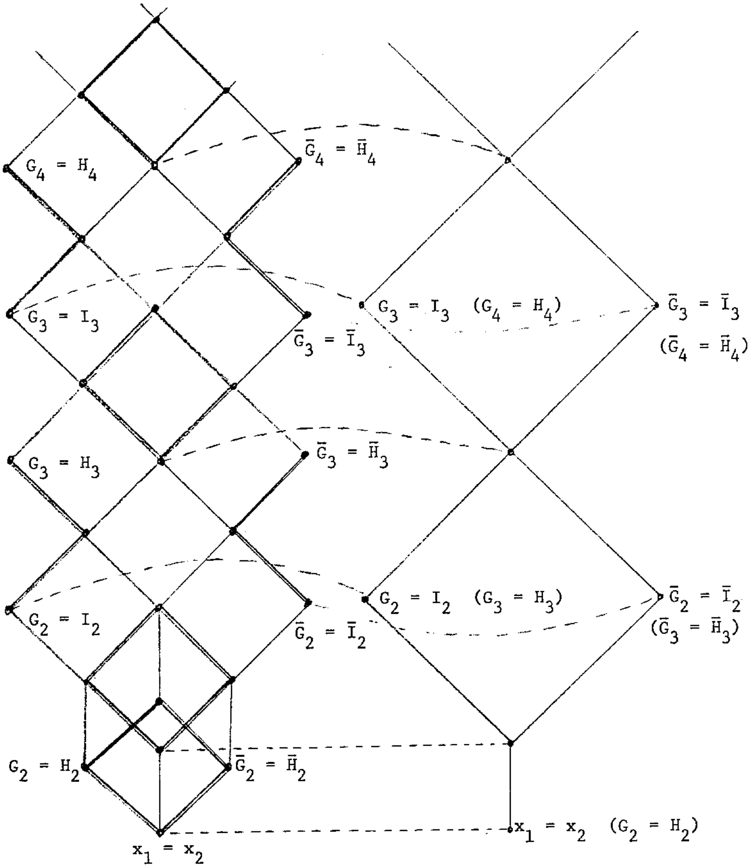

Diagram 2. Proper varieties of band monoids.

identities $G_n = I_n$, $n > 1$. This leads to Diagram 2. The left part of Diagram 2 again represents the lattice of proper varieties of bands, the right part the lattice of proper varieties of band monoids. The broken lines indicate the correspondence of the varieties of bands and the varieties of band monoids. In the left part the heavy lines indicate the classes of the (lattice) congruence induced by the correspondence. The labels are those of the GP-system. The labels in the parentheses represent an alternative set of identities.

Furthermore, ([8], Theorem) together with Lemma 7.1 imply that the lattice of proper varieties of $*$-band monoids in ([8], Figure 2) may be labelled, except for the trivial variety, by the same system $G_n = I_n$, $n > 1$. This leads to Diagram 3 for which the explanation of Diagram 2 is valid with obvious modifications.

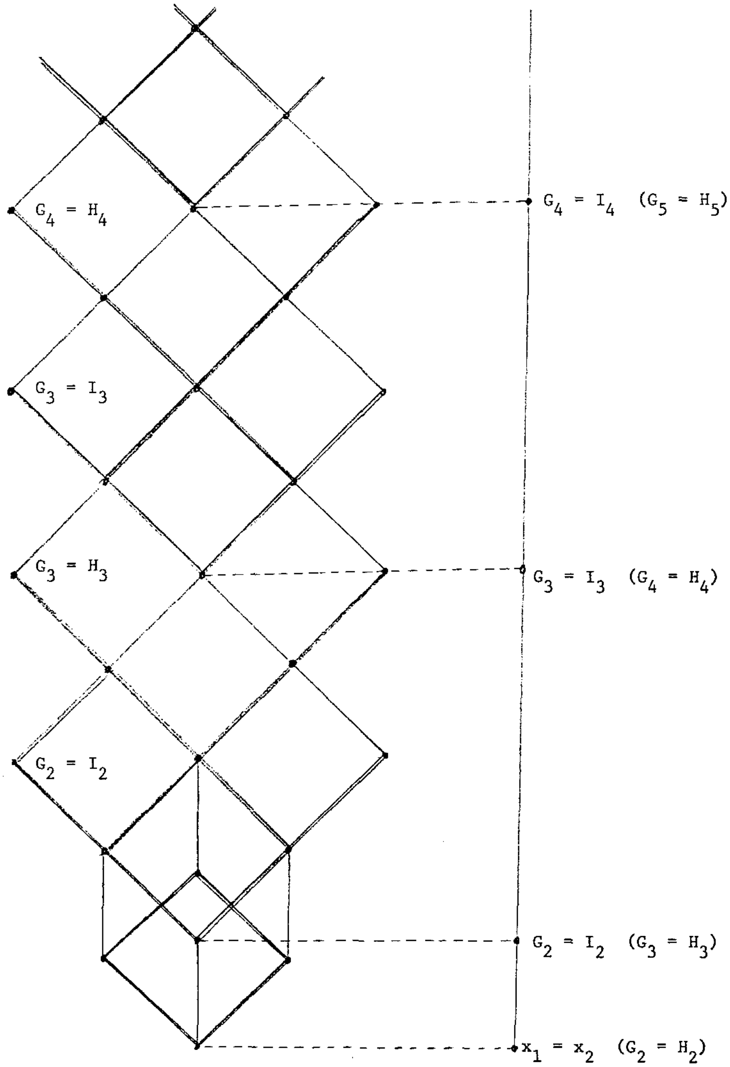

Diagram 3. Varieties of ∗-band monoids.

8. Monoid identities

Any word w over the alphabet X consists of a sequence of elements of X. If w is to be considered for monoids, we must admit also 1, the empty word as a possibility for any set of variables occurring in w. We explore here briefly what happens to some of the words $G_n, H_n, \ldots$ when we set the first or the second variable equal to 1.

In the first lemma, we set $x_1 = 1$ both in G_{n+1} and V_{n+1}, in the second we set $x_2 = 1$ both in G_{n+1} and H_{n+1} and draw some consequences. Since we shall be giving diverse values to the entries of these words, we shall write $T_n(x_1, x_2, \ldots, x_n)$ instead of T_n.

LEMMA 8.1. *Let* $n \geq 1$.

(i) $G_{n+1}(1, x_2, x_3, \ldots, x_{n+1}) = G_n(x_2, x_3, \ldots, x_{n+1})$.

(ii) $V_{n+1}(1, x_2, x_3, \ldots, x_{n+1}) = W_n(x_2, x_3, \ldots, x_{n+1})$.

Proof. (i) By Lemma 2.2, on the one hand

$$G_{n+1}(1, x_2, x_3, \ldots, x_{n+1}) = \begin{cases} (x_{n+1}x_{n-1} \cdots x_3)(x_2x_4 \cdots x_n) & \text{if } n \text{ is even} \\ (x_{n+1}x_{n-1} \cdots x_2)(x_3x_5 \cdots x_n) & \text{if } n \text{ is odd} \end{cases}$$

and on the other hand,

$$G_n(x_2, x_3, \ldots, x_{n+1}) = \begin{cases} (x_{n+1}x_{n-1} \cdots x_2)(x_3x_5 \cdots x_n) & \text{if } n \text{ is odd} \\ (x_{n+1}x_{n-1} \cdots x_3)(x_2x_4 \cdots x_n) & \text{if } n \text{ is even} \end{cases}$$

whence the desired equality.

(ii) For $n = 1$, we have $V_2(1, x_2) = x_2x_2^* = W_1(x_2)$. Assume the formula valid for n. Then by part (i) and the induction hypothesis, we get

$$\begin{aligned} &V_{n+2}(1, x_2, x_3, \ldots, x_{n+2}) \\ &\quad = G_{n+2}(1, x_2, x_3, \ldots, x_{n+2})x_{n+2}\overline{V_{n+1}}(1, x_2, x_3, \ldots, x_{n+1}) \\ &\quad = G_{n+1}(x_2, x_3, \ldots, x_{n+2})x_{n+2}\overline{W_n}(x_2, x_3, \ldots, x_{n+1}) \\ &\quad = W_{n+1}(x_2, x_3, \ldots, x_{n+2}). \end{aligned}$$

By Lemma 8.1(ii), $W_n = W_n(x_1, x_2, \ldots, x_n) = V_{n+1}(1, x_1, x_2, \ldots, x_n)$ which can, alternatively, be used as a definition of W_n, in which case only the words G_n and V_n need be defined.

LEMMA 8.2. *Let* $n \geq 1$.

(i) $G_{n+1}(x_1, 1, x_3, \ldots, x_{n+1}) = G_n(x_1, x_3, \ldots, x_{n+1})$.

(ii) $H_{n+1}(x_1, 1, x_3, \ldots, x_{n+1}) = I_n(x_1, x_3, \ldots, x_{n+1})$.

Proof. (i) By Lemma 2.2 on the one hand

$$G_{n+1}(x_1, 1, x_3, \ldots, x_{n+1}) = \begin{cases} (x_{n+1}x_{n-1} \cdots x_3x_1(x_4x_5 \cdots x_n) & \text{if } n \text{ is even} \\ (x_{n+1}x_{n-1} \cdots x_4)(x_1x_3 \cdots x_n) & \text{if } n \text{ is odd} \end{cases}$$

and on the other hand,

$$G_n(x_1, x_3, \ldots, x_{n+1}) = \begin{cases} (x_{n+1}x_{n-1} \cdots x_4x_1)(x_3x_5 \cdots x_n) & \text{if } n \text{ is odd} \\ (x_{n+1}x_{n-1} \cdots x_5x_3)(x_1x_4 \cdots x_n) & \text{if } n \text{ is even} \end{cases}$$

whence the desired equality.

(ii) For $n = 1$, we have $H_2(x_1, 1) = 1 = I_1(x_1)$. Assume the formula valid for n. Then by part (i) and the induction hypothesis, we get

$$\begin{aligned} &H_{n+2}(x_1, 1, x_3, \ldots, x_{n+2}) \\ &\quad = G_{n+2}(x_1, 1, x_3, \ldots, x_{n+2})x_{n+2}\overline{H_{n+1}}(x_1, 1, x_3, \ldots, x_{n+1}) \\ &\quad = G_{n+1}(x_1, x_3, \ldots, x_{n+2})x_{n+2}\overline{I_n}(x_1, x_3, \ldots, x_{n+1}) \\ &\quad = I_n(x_1, x_3, \ldots, x_{n+2}). \end{aligned}$$

We see from Lemma 8.2(ii) that I_n may be defined in terms of H_{n+1} similarly as above for W_n vs. V_{n+1}. The effect on the identity $G_n = I_n$ when we set one of the variables x_i equal to 1 is given next.

LEMMA 8.3. *For $n > 1$, $1 \leq i \leq n$ and $T \in \{G, I\}$, let*

$$T_n^i = T_n(x_1, x_2, \ldots, 1, \ldots, x_n)$$

where 1 *is in the i-th position. Then $G_n^i \sim I_n^i$.*

Proof. The assertion is trivial for $n = 2$ since $G_2 = x_2x_1$ and $I_2 = x_2x_1x_2$. Assume that the statement holds for $n > 1$. Using the induction hypothesis, we obtain

$$\begin{aligned} \overline{G_{n+1}^i} &= x_{n+1}\overline{G_n^i} = G_n^i x_{n+1} \sim (G_n^i x_{n+1})(G_n^i x_{n+1}) \\ &\sim I_n^i x_{n+1} G_n^i x_{n+1} = I_n^i x_{n+1} \overline{G_{n+1}^i} \end{aligned}$$

whence

$$G_{n+1}^i \sim \overline{I_n^i x_{n+1} \overline{G_{n+1}^i}} = G_{n+1}^i x_{n+1} \overline{I_n^i} = I_{n+1}^i.$$

Lemma 8.3 has a curious consequence.

COROLLARY 8.4. *For $n > 1$, if $S \in [\overline{G_n} = \overline{I_n}]$, then $S^1 \in [G_{n+1} = I_{n+1}]$.*

Proof. This is trivial if $S^1 = S$. Assume that $S^1 \neq S$. For any substitution $x_i \to 1$ with $i \leq n$, by Lemma 8.3, we have $G^i_{n+1} \sim I^i_{n+1}$ so the identity $G^i_{n+1} = I^i_{n+1}$ is trivial. For the substitution $x_{n+1} \to 1$ and x_i unchanged for $i \leq n$, we arrive at the identity $\overline{G_n} = \overline{I_n}$. If $S \in [\overline{G_n} = \overline{I_n}]$, then by these two possible cases, we obtain $S^1 \in [G_{n+1} = I_{n+1}]$.

9. Concluding remarks

We have introduced the N-system of identities for $*$-bands and proved its equivalence both with the A-system and the GP-system. It follows that the GP- and the A-systems are equivalent. Since the GP-system has no starred variables and the A-system produces essentially all varieties of $*$-bands, we conclude that all proper $*$-band varieties are determined by identities without starred variables. We can paraphrase this by saying that "the star is not needed to describe proper varieties of $*$-bands".

As a consequence of this situation, every identity on $*$-bands is equivalent to some in the GP-system or to $x_1 = x_1$, $x_1 = x_1x_2x_1$ or $x_1 = x_2$. This means that any identity with star on $*$-bands can be transformed onto an equivalent one without star. But how we are unable to say.

In paper [6], the author considered a number of identities with star and for each established that it is equivalent to one without star in a roundabout way by considering the structure of the completely regular semigroups belonging to the variety determined by that identity. And this has happened without exception! As a result, we have no example of a variety of completely regular $*$-semigroups which is not determined by an identity without star except for group varieties which cannot be defined by identities without inverses.

We have referred to the known determination of varieties of $*$-bands, band monoids and $*$-band monoids. It would be desirable to have a direct conceptual proof of these results using the GP-system as well as a solution of the word problems for these varieties.

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