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On the semimartingale representation of reflecting Brownian motion in a cusp

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Summary. Let C be the symmetric cusp $\{(x, y) \in \mathbb{R}^2 \colon -x^\beta \leqq y \leqq x^\beta, x \geqq 0\}$ where $\beta > 1$. In this paper we decide whether or not reflecting Brownian motion in C has a semimartingale representation. Here the reflecting Brownian motion has directions of reflection that make constant angles with the unit inward normals to the boundary. Our results carry through for a wide class of asymmetric cusps too.

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1 Introduction

Consider the cusp

$$C = \{(x, y) \colon x \geqq 0, \ -x^\beta \leqq y \leqq x^\beta\}$$

where $\beta > 1$. Let $\partial C_1 = \{(x, y) \colon x \geqq 0, y = -x^\beta\}$ and $\partial C_2 = \{(x, y) \colon x \geqq 0, y = x^\beta\}$. For each $z \in \partial C_j \setminus \{0\}, j = 1, 2$, let $n_j(z)$ be the inward unit normal to ∂C_j and let $v_j(z)$ make constant angle $\theta_j \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with $n_j(z)$. We take $\theta_j > 0$ iff $v_j(z)$ has a negative first component in sufficiently small neighborhoods of the origin. We normalize via $v_j(z) \cdot n_j(z) = 1, z \in \partial C_j \setminus \{0\}$. All vectors are column vectors.

Let Ω_C be the space of continuous paths from $[0, \infty)$ into C endowed with the topology of uniform convergence on compacta. For each $t \geqq 0$ let $\mathscr{M}_t$ be the σ-algebra of subsets of Ω_C generated by the coordinate maps

$$X_s(\omega) = \omega(s), \quad \omega \in \Omega_C$$

for $0 \leqq s \leqq t$. We use $\mathscr{M}$ to denote $\sigma(X_t \colon t \geqq 0)$.

Reflecting Brownian motion in C, starting at x, is a probability measure P_x on $(\Omega_C, \mathscr{M})$ that solves the following submartingale problem:

$$P_x(X_0 = x) = 1 \; ; \tag{1.1}$$

for each $f \in C_b^2(C)$,

$$f(X_t) - \frac{1}{2}\int_0^t \Delta f(X_s)ds \tag{1.2}$$

is a P_x-submartingale on $(\Omega_C, \mathscr{M}, \{\mathscr{M}_t\})$ whenever f is constant on a neighborhood of 0 and $v_j \cdot \nabla f \geqq 0$ on $\partial C_j, j = 1, 2$;

$$E^{P_x}\left[\int_0^\infty I_{\{0\}}(X_s)ds\right] = 0 . \tag{1.3}$$

In [2] we showed RBM in C starting at x exists and is unique (as a law on $(\Omega_C, \mathscr{M})$) if $\theta_1 + \theta_2 \leqq 0$. It does not exist when $\theta_1 + \theta_2 > 0$. We call $\{X_t: t \geqq 0\}$ on $(\Omega_C, \mathscr{M}, \{\mathscr{M}_t\}, P_x)$ the *canonical realization* of RBM in C starting at x.

In this paper we decide whether or not such a process is a semimartingale, in which case we identify the local martingale and describe the finite variation part. Our main results are the following theorems. We take $R(z)$ to be the 2×2 matrix whose first and second columns are the directions of reflection $v_1(z)$ and $v_2(z)$ respectively:

$$R(z) = (v_1(z), v_2(z)) . \tag{1.4}$$

Theorem 1.1 *If $\theta_1 + \theta_2 < 0$, or if $\theta_1 + \theta_2 = 0$ with $\beta < 2$, then reflecting Brownian motion in C starting at $z \in C$ has a semimartingale realization. More precisely, on some filtered probability space $(\Omega, \mathscr{S}, \{\mathscr{S}_t\}, P)$ there is a triple (Z_t, k_t, B_t) of continuous $\mathscr{S}_t$-adapted processes satisfying*

(i) *the P-law of Z on $(\Omega_C, \mathscr{M})$ is* RBM *in C starting at z;*
(ii) $k_t \in [0, \infty) \times [0, \infty)$, $k_0 = 0$;
(iii) *B_t is 2-dimensional $\mathscr{S}_t$-Brownian motion starting at* 0;
(iv) $Z_t = z + B_t + \int_0^t R(Z_u)dk_u$;
(v) *the components of k are nondecreasing and can change only when $Z \in \partial C \backslash \{0\}$:*

$$\int_0^t I_{\partial C_j \backslash 0}(Z(u))dk_j(u) = k_j(t) \quad j = 1, 2 .$$

If the starting point $z \neq 0$, then RBM *in C is a semimartingale if $\theta_1 + \theta_2 = 0$ and $\beta \geqq 2$.*

Theorem 1.2 *If $\theta_1 + \theta_2 = 0$ and $\beta \geqq 2$, then reflecting Brownian motion in C starting at 0 is not a semimartingale.*

Our method also works for a wide class of asymmetric cusps.

Theorem 1.3 *Consider the asymmetric cusp*

$$\tilde{C} = \{(x, y): x \geqq 0, \ -x^\delta \leqq y \leqq x^\beta\}$$

where $\beta > 1$ and $\delta > 2\beta - 1$. Then the conclusions of Theorems 1.1 *and* 1.2 *hold for $\tilde{C}$ instead of C.*

The proof is the same and is omitted. The condition $\delta > 2\beta - 1$ is a purely technical assumption needed for the proof of *existence* of Z.

In [2] we constructed RBM in C by conformally mapping RBM in the upper half plane to the cusp and then time changing. In [1] an explicit semimartingale representation of RBM in the upper half plane was given. We use these representations to get the results in this paper.

The paper is organized as follows. In Sect. 2 we describe RBM in the upper half space and give some preliminary results. Sections 3 and 4 are devoted to the proof of Theorems 1.1 and 1.2 respectively.

2 Preliminaries and RBM in the upper half plane

To prove Theorem 1.1 we make the following reductions. For RBM in C starting away from 0, we know from [2] that the process will never hit 0. Since $\partial C\backslash\{0\}$ is smooth and the direction of reflection at the boundary varies smoothly along $\partial C\backslash\{0\}$, the theorem follows from standard results [3, 6].

Thus we need to verify Theorem 1.1 only for RBM in C starting at 0 when $\theta_1+\theta_2<0$ or when $\theta_1+\theta_2=0$ with $\beta<2$. By localization, it suffices to prove (i)–(v) in Theorem 1.1 up to the first exit time from a small neighborhood of 0 in C.

We now give a detailed definition of RBM in the upper half plane. Let

$$S=\{(x,y)\colon y\geqq 0\}$$

with $\partial S_1=\{(x,y)\colon x\geqq 0, y=0\}$ and $\partial S_2=\{(x,y)\colon x\leqq 0, y=0\}$. The direction of reflection at $\partial S_j\backslash\{0\}$ is given by the constant vector

$$V_j=\begin{pmatrix}\gamma_j\\ 1\end{pmatrix}$$

$j=1,2$. The angle of reflection $\theta_j\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ has the same value as above and represents the angle V_j makes with the inward pointing unit normal N_j to $\partial S_j\backslash\{0\}$, $j=1,2$. Here $\theta_j>0$ means V_j points toward $\{0\}$.

Reflecting Brownian motion in S starting at x is defined as a law on the space of continuous paths in S analogously to RBM in C. Varadhan and Williams [5] proved the existence and uniqueness in law of such a process for any θ_1 and θ_2.

In [1] the following semimartingale representation of RBM in S starting at 0 was given. On some filtered space $(\Omega,\mathscr{S},\{\mathscr{S}_t\},P)$, there is a triple (G,β,Y) of continuous $\mathscr{S}_t$-adapted processes such that the P-law of

$$G_t=\beta_t+MY_t \tag{2.1}$$

on the space of continuous paths in S is RBM in S starting at 0, where β_t is two-dimensional $\mathscr{S}_t$-Brownian motion and $M=(V_1,V_2)$. Moreover

$$Y_j(t)=\int_0^t I_{\partial S_j\backslash 0}(G(u))d\varphi(u),\quad j=1,2 \tag{2.2}$$

where $\varphi_0=0$, $t\to\varphi(t)$ is continuous and nondecreasing, and φ can change only on $\partial S\backslash\{0\}$:

$$\int_0^t I_{\partial S\backslash 0}(G_u)d\varphi_u=\varphi_t\,. \tag{2.3}$$

In fact, φ is the local time at 0 of $\beta_2(t)$.

For each $r > 0$ and $z_0 \in \mathbb{R}^2$, let

$$B_r(z_0) = \{z: |z - z_0| < r\} .$$

In [2] we showed there exist $\varepsilon \in (0, 1)$, a closed set $H \subseteq \overline{B_1(0) \cap C}$, and a homeomorphism $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2)$ from $S \cap \overline{B_\varepsilon(0)}$ onto H, where $0 \in \partial H$ and $H \cap B_s(0) = C \cap B_s(0)$ for s small enough. Moreover, $\mathscr{F}: S \cap \overline{B_\varepsilon(0)} \backslash \{0\} \to H \backslash \{0\}$ is conformal with conformal inverse F, $\mathscr{F}(0) = 0$ and for some $k, K > 0$

$$k|\zeta|^{-1}(-\ln|\zeta|)^{-\beta/\beta-1} \leqq |\mathscr{F}'(\zeta)| \leqq K|\zeta|^{-1}(-\ln|\zeta|)^{-\beta/\beta-1}, \; \zeta \in S \cap \overline{B_\varepsilon(0)} \backslash \{0\}. \tag{2.4}$$

Let

$$\eta_\varepsilon = \inf\{t > 0: |G_t| = \varepsilon\} , \tag{2.5}$$

$$A_t = A(t) = \int_0^{t \wedge \eta_\varepsilon} |\mathscr{F}'(G_u)|^2 I_{S \backslash 0}(G_u) du, \quad t \geqq 0 . \tag{2.6}$$

In [2] we showed $t \to A(t)$ is continuous and strictly increasing on $[0, \eta_\varepsilon]$, with continuous strictly increasing inverse $\tau_t = \tau(t)$, and we have

$$EA(\eta_\varepsilon) < \infty \text{ if } \theta_1 + \theta_2 \leqq 0 . \tag{2.7}$$

In this case the P-law of

$$Z_t = \mathscr{F}(G(\tau(t \wedge A(\eta_\varepsilon)))), \quad t \geqq 0 \tag{2.8}$$

on $(\Omega_C, \mathscr{M})$ is RBM in C starting at 0 stopped at the first exit time $A(\eta_\varepsilon)$ of Z from $\mathscr{F}(B_\varepsilon(0))$.

Our goal is to use (2.1) and Itô's formula to calculate the stochastic differential $d\mathscr{F}(G_t)$. Since $\mathscr{F}$ is singular at $\{0\}$, we cannot do this directly. Therefore we make use of the remark in [1] after the proof of Lemma 2.7. Paraphrased, this remark is the following theorem.

Theorem 2.1 *Assume* $h \in C^2(S \cap \overline{B_\varepsilon(0)} \backslash \{0\}) \cap C(S \cap \overline{B_\varepsilon(0)})$, $h(0) = 0$, $\Delta h = 0$ *on* $S \cap \overline{B_\varepsilon(0)} \backslash \{0\}$,

$$\left[E \int_0^{t \wedge \eta_\varepsilon} |\nabla h(G_u)|^2 I_{\mathbb{R} \backslash 0}(h(G_u)) du\right] < \infty , \tag{2.9}$$

and

$$E\left[\int_0^{t \wedge \eta_\varepsilon} |\nabla h(G_u)| I_{\mathbb{R} \backslash 0}(h(G_u)) d\varphi_u\right] < \infty . \tag{2.10}$$

Then

$$\begin{aligned} h(G_{t \wedge \eta_\varepsilon}) = &\int_0^{t \wedge \eta_\varepsilon} I_{\mathbb{R} \backslash 0}(h(G_u))(\nabla h)^*(G_u) d\beta_u \\ &+ \int_0^{t \wedge \eta_\varepsilon} I_{\mathbb{R} \backslash 0}(h(G_u))\left[\sum_{j=1}^2 V_j \cdot \nabla h(G_u) dY_j(u)\right] \\ &+ a(h, t \wedge \eta_\varepsilon) , \end{aligned}$$

where $a(h, t \wedge \eta_\varepsilon) = a^+(h, t \wedge \eta_\varepsilon) - a^-(h, t \wedge \eta_\varepsilon)$, $t \to a^\pm(h, t \wedge \eta_\varepsilon)$ *are continuous, nondecreasing and can change only when* $h(G) = 0$:

$$\int_0^{t \wedge \eta_\varepsilon} I_{\{0\}}(h(G_u))d_u a^\pm(h, u) = a^\pm(h, t \wedge \eta_\varepsilon) .$$

Suppose $g_n\colon \mathbb{R} \to [0, \infty)$ is continuous with $\operatorname{supp} g_n \subseteq \left[\frac{1}{3n}, \frac{1}{n}\right]$ and $\int g_n(r)dr = 1$. Set

$$k_n(t) = \int_0^t \int_0^s g_n(r)dr ds . \tag{2.11}$$

Then $k_n \in C^2(\mathbb{R})$, $k_n \equiv 0$ in a neighborhood of 0 and for some $A > 0$, $|k_n(t) - t \vee 0| \leqq \frac{A}{n}$. Also, $0 \leqq k_n' \leqq 1$, $k_n'' \geqq 0$ and $k_n'(t) \to I_{(0,\infty)}(t)$ as $n \to \infty$.

We use k_n to prove the next result of this section.

Theorem 2.2 *Let* $h \in C(S \cap \overline{B_\varepsilon(0)}) \cap C^2(S \cap \overline{B_\varepsilon(0)} \setminus \{0\})$ *satisfy*

$$h(0) = 0$$

$$\Delta h = f_1 + f_2 \text{ on } S \cap \overline{B_\varepsilon(0)} \setminus \{0\} \text{ where } f_2 \geqq 0$$

$$\nabla h = f_3 + f_4 \text{ on } S \cap \overline{B_\varepsilon(0)} \setminus \{0\} \text{ where } V_j \cdot f_4 \geqq 0 \text{ on } (\partial S_j) \cap \overline{B_\varepsilon(0)} \setminus \{0\} \text{ for } j = 1, 2$$

and

$$E\left[\int_0^{\eta_\varepsilon} |f_1(G_u)| I_{(0,\infty)}(h(G_u))du\right] < \infty$$

$$E\left[\int_0^{\eta_\varepsilon} |V_j \cdot f_3(G_u)| I_{(0,\infty)}(h(G_u))dY_j(u)\right] < \infty, \quad j = 1, 2 .$$

Then

$$E\left[\int_0^{\eta_\varepsilon} f_2(G_u) I_{(0,\infty)}(h(G_u))du\right] < \infty$$

and

$$E\left[\int_0^{\eta_\varepsilon} V_j \cdot f_4(G_u) I_{(0,\infty)}(h(G_u))dY_j(u)\right] < \infty, \quad j = 1, 2 .$$

Proof. Since $h(0) = 0$, $k_n \circ h \in C^2(S \cap \overline{B_\varepsilon(0)})$. Hence by Itô's formula and (2.1), upon taking expectations, we get

$$E[(k_n \circ h(G_{t \wedge \eta_\varepsilon}))] - E\left[\int_0^{t \wedge \eta_\varepsilon} \frac{1}{2} f_1(G_u) k_n' \circ h(G_u)du\right]$$
$$- \sum_{j=1}^2 E\left[\int_0^{t \wedge \eta_\varepsilon} V_j \cdot f_3(G_u) k_n' \circ h(G_u)dY_j(u)\right]$$

$$= E\left[\int_0^{t \wedge \eta_\varepsilon} \frac{1}{2}[f_2(G_u)k_n' \circ h(G_u) + |\nabla h(G_u)|^2 k_n'' \circ h(G_u)]du\right]$$

$$+ \sum_{j=1}^{2} E\left[\int_0^{t \wedge \eta_\varepsilon} V_j \cdot f_4(G_u)k_n' \circ h(G_u)dY_j(u)\right].$$

All integrands on the right are nonnegative. Hence by dominated convergence on the left and Fatou's lemma on the right, we can replace $t \wedge \eta_\varepsilon$ by η_ε and let $n \to \infty$ to get

$$\infty > E[0 \vee h(G_{\eta_\varepsilon})] - E\left[\int_0^{\eta_\varepsilon} \frac{1}{2} f_1(G_u)I_{(0,\infty)}(h(G_u))du\right]$$

$$- \sum_{j=1}^{2} E\left[\int_0^{\eta_\varepsilon} V_j \cdot f_3(G_u)I_{(0,\infty)}(h(G_u))dY_j(u)\right]$$

$$\geqq E\left[\int_0^{\eta_\varepsilon} \frac{1}{2} f_2(G_u)I_{(0,\infty)}(h(G_u))du\right]$$

$$+ \sum_{j=1}^{2} E\left[\int_0^{\eta_\varepsilon} V_j \cdot f_4(G_u)I_{(0,\infty)}(h(G_u))dY_j(u)\right].$$

The desired conclusion follows from this. □

A first application of this theorem is the following result.

Theorem 2.3 *Let $p > 0$. Then for $\theta_1 + \theta_2 \leqq 0$*

$$E\left[\int_0^{\eta_\varepsilon} |G_u|^{-2}(-\ln|G_u|)^{-p-2} I_{S\setminus 0}(G_u)du\right] < \infty .$$

Proof. Define for $z = re^{i\theta}$

$$h(z) = \begin{cases} [-\ln r + f(\theta)]^{-p} & z \in S \cap \overline{B_\varepsilon(0)} \setminus \{0\} \\ 0 & z = 0 \end{cases} \tag{2.12}$$

where $f \in C^2([0, \pi])$. Then $h \in C(S \cap \overline{B_\varepsilon(0)}) \cap C^2(S \cap \overline{B_\varepsilon(0)} \setminus \{0\})$ with $h(0) = 0$. Moreover, in polar coordinates $\langle r, \theta \rangle$ for $z \neq 0$

$$\nabla h = pr^{-1}[-\ln r + f(\theta)]^{-p-1}\langle 1, -f'(\theta)\rangle , \tag{2.13}$$

$$\Delta h = pr^{-2}[-\ln r + f(\theta)]^{-p-2}\{(p+1)(1 + [f'(\theta)]^2) - [-\ln r + f(\theta)]f''(\theta)\} . \tag{2.14}$$

In polar coordinates, $V_1 = \langle -\tan\theta_1, 1\rangle$, $V_2 = \langle -\tan\theta_2, -1\rangle$ so that

$$V_1 \cdot \nabla h = pr^{-1}[-\ln r + f(\theta)]^{-p-1}\{-\tan\theta_1 - f'(\theta)\} \tag{2.15}$$

$$V_2 \cdot \nabla h = pr^{-1}[-\ln r + f(\theta)]^{-p-1}\{-\tan\theta_2 + f'(\theta)\}. \tag{2.16}$$

Now specialize: let $f(\theta) = -(\tan\theta_1)\theta$. Then for $0 < |z| \leqq \varepsilon$, $V_1 \cdot \nabla h(z) = 0$ and since $\theta_1 + \theta_2 \leqq 0$, $V_2 \cdot \nabla h(z) \geqq 0$. Moreover, for some constant $K > 0$ (making

ε smaller if necessary) $\Delta h \geqq Kr^{-2}(-\ln r)^{-p-2}$ on $S \cap \overline{B_\varepsilon(0)}\backslash\{0\}$. Hence by Theorem 2.2 with $f_1 \equiv 0$ and $f_3 \equiv 0$,

$$E\left[\int_0^{\eta_\varepsilon} |G_s|^{-2}(-\ln|G_s|)^{-p-2} I_{S\backslash 0}(G_s)ds\right]$$
$$\leqq K^{-1}E\left[\int_0^{\eta_\varepsilon} \Delta h(G_u) I_{(0,\infty)}(h(G_u))du\right]$$
$$< \infty . \quad \square$$

The final result in this section will be used to prove Theorem 1.2.

Theorem 2.4 *Let* $h \in C(S \cap \overline{B_\varepsilon(0)}) \cap C^2(S \cap \overline{B_\varepsilon(0)}\backslash\{0\})$ *with* $h(0) = 0$,

$$\Delta h = f_1 + f_2 \text{ on } S \cap \overline{B_\varepsilon(0)}\backslash\{0\} \text{ where } f_2 \geqq 0 ,$$
$$E\left[\int_0^{\eta_\varepsilon} |\nabla h(G_u)|^2 I_{(0,\infty)}(h(G_u))du\right] < \infty ,$$
$$\int_0^{\eta_\varepsilon} |f_1(G_u)| I_{(0,\infty)}(h(G_u))du < \infty \text{ a.s., and}$$
$$\int_0^{\eta_\varepsilon} |V_j \cdot \nabla h(G_u)| I_{(0,\infty)}(h(G_u))dY_j(u) < \infty \text{ a.s.,} \quad j = 1, 2 .$$

Then

$$\int_0^{\eta_\varepsilon} f_2(G_u) I_{(0,\infty)}(h(G_u))du < \infty \text{ a.s.}$$

Proof. Since $h(0) = 0$, $k_n \circ h \in C^2(S \cap \overline{B_\varepsilon(0)})$. By Itô's formula and (2.1)

$$\begin{aligned} k_n \circ h(G_{\eta_\varepsilon}) &- \int_0^{\eta_\varepsilon} k_n' \circ h(G_u)(\nabla h(G_u))^* dB_u - \int_0^{\eta_\varepsilon} \frac{1}{2} f_1(G_u) k_n' \circ h(G_u)du \\ &- \sum_{j=1}^{2} \int_0^{\eta_\varepsilon} V_j \cdot \nabla h(G_u) k_n' \circ h(G_u) dY_j(u) \\ &= \frac{1}{2} \int_0^{\eta_\varepsilon} [f_2(G_u) k_n' \circ h(G_u) + |\nabla h(G_u)|^2 k_n'' \circ h(G_u)]du . \end{aligned} \tag{2.17}$$

By hypotheses and dominated convergence

$$E\left[\int_0^{\eta_\varepsilon} k_n' \circ h(G_u)(\nabla h(G_u))^* dB_u - \int_0^{\eta_\varepsilon} I_{(0,\infty)}(h(G_u))(\nabla h(G_u))^* dB_u\right]^2$$
$$= E\left[\int_0^{\eta_\varepsilon} [k_n' \circ h(G_u) - I_{(0,\infty)}(h(G_u))]^2 |\nabla h(G_u)|^2 du\right] \to 0 \quad \text{as } n \to \infty .$$

Hence by passing to a subsequence n_m,

$$\int_0^{\eta_\varepsilon} k_{n_m}' \circ h(G_u)(\nabla h(G_u))^* dB_u \xrightarrow{m \to \infty} \int_0^{\eta_\varepsilon} I_{(0,\infty)}(h(G_u))(\nabla h(G_u))^* dB_u \quad \text{a.s.}$$

Take the limit on the left-hand side of (2.17) along this subsequence; by hypotheses and dominated convergence this limit exists and is finite a.s. On the other hand, in the right side of (2.17), all integrands are nonnegative. Hence by Fatou's lemma a.s. we have

$$\infty > \lim_{m\to\infty} \mathrm{LHS}(n = n_m) \geqq \varliminf_{m\to\infty} \frac{1}{2} \int_0^{\eta_\varepsilon} f_2(G_u) k'_{n_m} \circ h(G_u)du$$
$$\geqq \frac{1}{2} \int_0^{\eta_\varepsilon} f_2(G_u) I_{(0,\infty)}(h(G_u))du$$

as desired. □

3 Proof of Theorem 1.1

3.1

The method is to apply Theorem 2.1 to $h = \mathscr{F}_j$, $j = 1, 2$. Hence we verify (2.9) and (2.10) for these choices of h. Theorem 2.2 is the tool for this. First we consider (2.9) for $h = \mathscr{F}_j, j = 1, 2$.

Lemma 3.1 *For* $j = 1, 2$

$$E\left[\int_0^{\eta_\varepsilon} |\nabla \mathscr{F}_j(G_u)|^2 I_{\mathbb{R}\setminus 0}(\mathscr{F}_j(G_u))du\right] < \infty .$$

Proof. Since $G_u = 0 \Rightarrow \mathscr{F}_j(G_u) = 0, j = 1, 2$, and since $|\mathscr{F}'(z)|^2 = |\nabla \mathscr{F}_j(z)|^2$ for $z \neq 0, j = 1, 2$, (by conformality), it suffices to show

$$E\left[\int_0^{\eta_\varepsilon} |\mathscr{F}'(G_u)|^2 I_{S\setminus 0}(G_u)du\right] < \infty .$$

By (2.4), this follows from

$$E\left[\int_0^{\eta_\varepsilon} |G_u|^{-2}(-\ln|G_u|)^{-2\beta/(\beta-1)} I_{S\setminus 0}(G_u)du\right] < \infty ,$$

which in turn follows from Theorem 2.3 with $p = \dfrac{2}{\beta - 1}$. □

The next lemma verifies (2.10) for $h = \mathscr{F}_j, j = 1, 2$.

Lemma 3.2 *If* $\theta_1 + \theta_2 < 0$, *or if* $\theta_1 + \theta_2 = 0$ *with* $\beta < 2$, *then*

$$E\left[\int_0^{\eta_\varepsilon} |\nabla \mathscr{F}_j(G_u)| I_{\mathbb{R}\setminus 0}(\mathscr{F}_j(G_u))d\varphi_u\right] < \infty , \quad j = 1, 2 .$$

Proof. As in the proof of Lemma 3.1, it suffices to show

$$E\left[\int_0^{\eta_\varepsilon} |G_u|^{-1}(-\ln|G_u|)^{-\beta/(\beta-1)} I_{S\setminus 0}(G_u)d\varphi_u\right] < \infty . \tag{3.1}$$

First consider the case $\theta_1 + \theta_2 < 0$. Define for $z = re^{i\theta} \in S \cap \overline{B_\varepsilon(0)}$,

$$h_3(z) = \begin{cases} [-\ln r + c_1\theta]^{-1/(\beta-1)} & z \in S \cap \overline{B_\varepsilon(0)} \backslash \{0\} \\ 0 & z = 0 \end{cases}$$

where $c_1 \in (\tan\theta_2, -\tan\theta_1)$. This choice of c_1 is possible because $\theta_1 + \theta_2 < 0$. Then $h_3 \in C(S \cap \overline{B_\varepsilon(0)}) \cap C^2(S \cap \overline{B_\varepsilon(0)} \backslash \{0\})$, $h_3(0) = 0$ and by making ε smaller if necessary, $h_3(z) > 0$ for $z \in S \cap \overline{B_\varepsilon(0)} \backslash \{0\}$. By (2.12) and (2.14)–(2.16), on $S \cap \overline{B_\varepsilon(0)} \backslash \{0\}$, $\Delta h_3 \geqq 0$ and

$$V_1 \cdot \nabla h_3 = \frac{1}{\beta - 1} r^{-1}[-\ln r + c_1\theta]^{-\beta/(\beta-1)}\{-\tan\theta_1 - c_1\},$$

$$V_2 \cdot \nabla h_3 = \frac{1}{\beta - 1} r^{-1}[-\ln r + c_1\theta]^{-\beta/(\beta-1)}\{-\tan\theta_2 + c_1\}.$$

Since $\tan\theta_2 < c_1 < -\tan\theta_1$, we get for some $K > 0$

$$V_j \cdot \nabla h_3 \geqq K r^{-1}[-\ln r]^{-\beta/(\beta-1)}, \quad j = 1, 2, z \in S \cap \overline{B_\varepsilon(0)} \backslash \{0\}.$$

Applying Theorem 2.2 with $h = h_3, f_1 = 0$, and $f_3 = 0$, by (2.2) we get

$$E\left[\int_0^{\eta_\varepsilon} |G_u|^{-1}(-\ln|G_u|)^{-\beta/(\beta-1)} I_{S\backslash 0}(G_u)\, d\varphi_u\right] < \infty,$$

as desired.

Next we consider $\theta_1 + \theta_2 = 0$ with $\beta < 2$. By Theorem 2.3 $\left(\text{with } p = \frac{2-\beta}{\beta-1}\right)$ and (2.2)

$$E\left[\int_0^{\eta_\varepsilon} |G_u|^{-2}(-\ln|G_u|)^{-\beta/(\beta-1)} I_{S\backslash 0}(G_u)\, du\right] < \infty. \tag{3.2}$$

Choose $f \in C^\infty([0, \pi])$ such that $\inf f > 0$,

$$-\tan\theta_1 - f'(0) > 0, \text{ and}$$
$$-\tan\theta_2 + f'(\pi) > 0.$$

Define for $z = re^{i\theta} \in S \cap \overline{B_\varepsilon(0)}$

$$h_4(z) = \begin{cases} [-\ln r + f(\theta)]^{-1/(\beta-1)} & z \in S \cap \overline{B_\varepsilon(0)} \backslash \{0\} \\ 0 & z = 0. \end{cases}$$

Then $h_4 \in C(S \cap \overline{B_\varepsilon(0)}) \cap C^2(S \cap \overline{B_\varepsilon(0)} \backslash \{0\})$ and $h(0) = 0$. By (2.12)–(2.16) for some constant $k > 0$,

$$V_j \cdot \nabla h_4 \geqq k r^{-1}(-\ln r)^{-\beta/(\beta-1)}, z \in (\partial S_j) \cap \overline{B_\varepsilon(0)} \backslash \{0\}, |z| \leqq \varepsilon, j = 1, 2 \tag{3.3}$$

(note $z \in \partial S_1 \backslash \{0\} \Leftrightarrow \theta = 0$; $z \in \partial S_2 \backslash \{0\} \Leftrightarrow \theta = \pi$) and

$$\Delta h_4 = h_5 + h_6 \text{ on } S \cap \overline{B_\varepsilon(0)} \backslash \{0\}$$

where for $z = re^{i\theta} \in S \cap \overline{B_\varepsilon(0)} \setminus \{0\}$

$$h_5(z) = -\frac{1}{\beta - 1} r^{-2}[-\ln r + f(\theta)]^{-\frac{\beta}{\beta-1}} f''(\theta) \text{ and}$$

$$h_6(z) = \frac{\beta}{(\beta-1)^2} r^{-2}[-\ln r + f(\theta)]^{-\frac{1}{\beta-1}-2}(1 + [f'(\theta)]^2) \geqq 0 .$$

By (3.2)

$$E\left[\int_0^{\eta_\varepsilon} |h_5(G_u)| I_{(0,\infty)}(h_4(G_u)) du\right] < \infty$$

so by Theorem 2.2 (with $h = h_4, f_3 = 0$) and (2.2)

$$E\left[\int_0^{\eta_\varepsilon} V_j \cdot \nabla h_4(G_u) I_{(0,\infty)}(h_4(G_u)) I_{\partial S_j \setminus 0}(G_u) d\varphi_u\right] < \infty .$$

The desired bound (3.1) follows from this and (3.3). □

Corollary 3.3 *If $\theta_1 + \theta_2 < 0$ or if $\theta_1 + \theta_2 = 0$ with $\beta < 2$, then*

$$E\left[\int_0^{\eta_\varepsilon} |\mathscr{F}'(G_u)| dY_k\right] < \infty, \quad j = 1, 2 . \quad \square$$

By Lemmas 3.1 and 3.2, the hypotheses (2.9) and (2.10) of Theorem 2.1 hold for $h = \mathscr{F}_j, j = 1, 2$. Hence by that theorem,

$$\begin{aligned} \mathscr{F}_j(G_{t \wedge \eta_\varepsilon}) = & \int_0^{t \wedge \eta_\varepsilon} I_{\mathbb{R} \setminus 0}(\mathscr{F}_j(G_u))(\nabla \mathscr{F}_j)^*(G_u) d\beta_u \\ & + \int_0^{t \wedge \eta_\varepsilon} I_{\mathbb{R} \setminus 0}(\mathscr{F}_j(G_u))\left[\sum_{k=1}^{2} V_k \cdot \nabla \mathscr{F}_j(G_u) dY_k(u)\right] \\ & + a(\mathscr{F}_j, t \wedge \eta_\varepsilon), \quad j = 1, 2 . \end{aligned} \tag{3.4}$$

3.2

The next step is to replace the $I_{\mathbb{R} \setminus 0}(\mathscr{F}_j(G_u))$ factor in the martingale part of (3.4) by $I_{S \setminus 0}(G_u)$. By Theorem 2.1 with $h = \mathscr{F}_1 + \mathscr{F}_2$,

$$\begin{aligned} \mathscr{F}_1(G_{t \wedge \eta_\varepsilon}) + \mathscr{F}_2(G_{t \wedge \eta_\varepsilon}) = & \int_0^{t \wedge \eta_\varepsilon} I_{\mathbb{R} \setminus 0}(\mathscr{F}_1 + \mathscr{F}_2)(\nabla \mathscr{F}_1 + \nabla \mathscr{F}_2)^* d\beta \\ & + \int_0^{t \wedge \eta_\varepsilon} I_{\mathbb{R} \setminus 0}(\mathscr{F}_1 + \mathscr{F}_2) \sum_{k=1}^{2} V_k \cdot (\nabla \mathscr{F}_1 + \nabla \mathscr{F}_2) dY_k \\ & + a(\mathscr{F}_1 + \mathscr{F}_2, t \wedge \eta_\varepsilon) . \end{aligned} \tag{3.5}$$

Here we have written $\mathscr{F}_1 + \mathscr{F}_2$ for $\mathscr{F}_1(G_u) + \mathscr{F}_2(G_u)$, etc., in the integrals. On the other hand, by (3.4),

$$\begin{aligned}\mathscr{F}_1(G_{t \wedge \eta_\varepsilon}) + \mathscr{F}_2(G_{t \wedge \eta_\varepsilon}) = & \int_0^{t \wedge \eta_\varepsilon} (I_{\mathbb{R}\backslash 0}(\mathscr{F}_1)(\nabla \mathscr{F}_1)^* + I_{\mathbb{R}\backslash 0}(\mathscr{F}_2)(\nabla \mathscr{F}_2)^*)d\beta \\ & + \int_0^{t \wedge \eta_\varepsilon} \sum_{k=1}^{2} V_k \cdot (I_{\mathbb{R}\backslash 0}(\mathscr{F}_1)\nabla \mathscr{F}_1 + I_{\mathbb{R}\backslash 0}(\mathscr{F}_2)\nabla \mathscr{F}_2) dY_k \\ & + a(\mathscr{F}_1, t \wedge \eta_\varepsilon) + a(\mathscr{F}_2, t \wedge \eta_\varepsilon) \,. \end{aligned} \tag{3.6}$$

Thus the martingale parts of (3.5) and (3.6) are the same, so their quadratic variations are the same: (for $t \leqq \eta_\varepsilon$)

$$I_{\mathbb{R}\backslash 0}(\mathscr{F}_1 + \mathscr{F}_2) 2 |\mathscr{F}'|^2 \, dt = (I_{\mathbb{R}\backslash 0}(\mathscr{F}_1) + I_{\mathbb{R}\backslash 0}(\mathscr{F}_2)) |\mathscr{F}'|^2 \, dt \,, \tag{3.7}$$

where we have used conformality of $\mathscr{F}$ once again. Since $\mathscr{F}' \neq 0$ away from 0, (3.7) is the same as $2I_{\mathbb{R}\backslash 0}(\mathscr{F}_1 + \mathscr{F}_2)dt = (I_{\mathbb{R}\backslash 0}(\mathscr{F}_1) + I_{\mathbb{R}\backslash 0}(\mathscr{F}_2))dt$. Multiplying through by $I_{\{0\}}(\mathscr{F}_2)$ yields

$$2I_{\{0\}}(\mathscr{F}_2) I_{\mathbb{R}\backslash 0}(\mathscr{F}_1) dt = I_{\mathbb{R}\backslash 0}(\mathscr{F}_1) I_{\{0\}}(\mathscr{F}_2) dt \,.$$

Hence

$$I_{\{0\}}(\mathscr{F}_2) I_{\mathbb{R}\backslash 0}(\mathscr{F}_1) dt \equiv 0 \,.$$

But $\mathscr{F}_1(G_u) = 0 \Leftrightarrow G_u = 0$ and we know $I_{\{0\}}(G_t)dt \equiv 0$. Thus

$$I_{\{0\}}(\mathscr{F}_1(G_t)) dt \equiv 0$$

and so addition of the last 2 equalities gives

$$I_{\{0\}}(\mathscr{F}_2(G_t)) dt \equiv 0 \,.$$

Therefore (3.4) becomes

$$\begin{aligned}\mathscr{F}_j(G_{t \wedge \eta_\varepsilon}) = & \int_0^{t \wedge \eta_\varepsilon} I_{S\backslash 0}(G_u)(\nabla \mathscr{F}_j)^* d\beta_u \\ & + \int_0^{t \wedge \eta_\varepsilon} I_{\mathbb{R}\backslash 0}(\mathscr{F}_j(G_u)) \left[\sum_{k=1}^{2} V_k \cdot \nabla \mathscr{F}_j(G_u) dY_k(u) \right] \\ & + a(\mathscr{F}_j, t \wedge \eta_\varepsilon), \quad j = 1, 2 \,. \end{aligned} \tag{3.8}$$

3.3

Next consider the finite variation parts in (3.5) and (3.6). They are the same: for $t \leqq \eta_\varepsilon$

$$\begin{aligned} & I_{\mathbb{R}\backslash 0}(\mathscr{F}_1 + \mathscr{F}_2) \sum_{k=1}^{2} V_k \cdot (\nabla \mathscr{F}_1 + \nabla \mathscr{F}_2) dY_k + da(\mathscr{F}_1 + \mathscr{F}_2) \\ & \quad = \sum_{k=1}^{2} V_k \cdot (I_{\mathbb{R}\backslash 0}(\mathscr{F}_1)\nabla \mathscr{F}_1 + I_{\mathbb{R}\backslash 0}(\mathscr{F}_2)\nabla \mathscr{F}_2) dY_k + da(\mathscr{F}_1) + da(\mathscr{F}_2) \,. \end{aligned} \tag{3.9}$$

Since Y_k changes only when $G \in \partial S_k \backslash 0$ we have for $t \leqq \eta_\varepsilon$

$$\begin{aligned} dY_k(t) &= I_{\partial S_k \backslash 0}(G_t) dY_k(t) \\ &= I_{\partial C_k \backslash 0}(\mathscr{F}(G_t)) dY_k(t) \end{aligned}$$

and consequently (making ε smaller if necessary) for $j, k \in \{1, 2\}$ and $t \leqq \eta_\varepsilon$,

$$\begin{aligned} &[I_{\mathbb{R}\backslash 0}(\mathscr{F}_1 + \mathscr{F}_2) - I_{\mathbb{R}\backslash 0}(\mathscr{F}_j)] dY_k(t) \\ &= [I_{\mathbb{R}\backslash 0}(\mathscr{F}_1 + \mathscr{F}_2) - I_{\mathbb{R}\backslash 0}(\mathscr{F}_j)] I_{\partial C_k \backslash 0}(\mathscr{F}) dY_k(t) \\ &= [1 - 1] I_{\partial C_k \backslash 0}(\mathscr{F}) dY_k(t) \\ &\equiv 0 \,. \end{aligned}$$

Using these in (3.9) gives for $t \leqq \eta_\varepsilon$,

$$da(\mathscr{F}_1 + \mathscr{F}_2) = da(\mathscr{F}_1) + da(\mathscr{F}_2) \,. \tag{3.10}$$

Recalling that $a(\mathscr{F}_1 + \mathscr{F}_2)$, $a(\mathscr{F}_1)$ and $a(\mathscr{F}_2)$ are carried on $\mathscr{F}_1 + \mathscr{F}_2 = 0$, $\mathscr{F}_1 = 0$ and $\mathscr{F}_2 = 0$ respectively, we see (3.10) implies each is actually carried on $\mathscr{F} = 0$. In particular, for $t \leqq \eta_\varepsilon$

$$a(\mathscr{F}_j, t) = \int_0^t I_{\{0\}}(\mathscr{F}(G_u)) d_u a(\mathscr{F}_j, u) \quad j = 1, 2 \,. \tag{3.11}$$

The following argument to show $a(\mathscr{F}_j, \cdot) \equiv 0$ is due to Ruth Williams. Since $G_0 = 0$ and $\theta_1 + \theta_2 \leqq 0$, by (4.14) in [1]

$$1 = P(G_t \in S \backslash \{0\} \quad \text{for all} \quad t > 0) \,.$$

Thus

$$1 = P(\mathscr{F}(G_t) \in C \backslash \{0\} \quad \text{for all} \quad t \in (0, \eta_\varepsilon]) \,. \tag{3.12}$$

By (3.11) $a(\mathscr{F}_j, t)$ can change only when $\mathscr{F}(G)$ is at $\{0\}$; hence by (3.12) $a(\mathscr{F}_j, \cdot)$ must be constant (a.s.) on the time interval $(0, \eta_\varepsilon]$. By continuity of $a(\mathscr{F}_j, \cdot)$ we get (a.s.)

$$0 = a(\mathscr{F}_j, 0) = a(\mathscr{F}_j, t), \quad t \leqq \eta_\varepsilon \,.$$

Thus (3.8) becomes for $j = 1, 2$,

$$\begin{aligned} \mathscr{F}_j(G_{t \wedge \eta_\varepsilon}) = &\int_0^{t \wedge \eta_\varepsilon} I_{S \backslash 0}(G_u)(\nabla \mathscr{F}_j)^* d\beta_u \\ &+ \int_0^{t \wedge \eta_\varepsilon} I_{\mathbb{R}\backslash 0}(\mathscr{F}_j(G_u)) \sum_{k=1}^2 V_k \cdot \nabla \mathscr{F}_j(G_u) dY_k(u) \,. \end{aligned} \tag{3.13}$$

3.4

Now we examine the dY_k terms in (3.13). For $j, k \in \{1, 2\}$ and $|\zeta| \leqq \varepsilon$,

$$\mathscr{F}_j(\zeta) \neq 0 \text{ and } \zeta \in \partial S_k \backslash 0 \Leftrightarrow \mathscr{F}_j(\zeta) \neq 0 \text{ and } \mathscr{F}(\zeta) \in \partial C_k \backslash 0 \;\Leftrightarrow\; \mathscr{F}(\zeta) \in \partial C_k \backslash 0 \,.$$

Using this in (3.13) yields for $j = 1, 2$,

$$\mathscr{F}_j(G_{t \wedge \eta_\varepsilon}) = \int_0^{t \wedge \eta_\varepsilon} I_{S \setminus 0}(G_u)(\nabla \mathscr{F}_j)^* d\beta_u + \sum_{k=1}^{2} \int_0^{t \wedge \eta_\varepsilon} I_{\partial C_k \setminus 0}(\mathscr{F}(G_u)) V_k \cdot \nabla \mathscr{F}_j(G_u) dY_k(u) . \tag{3.14}$$

If

$$\mathscr{J}(\zeta) = \begin{pmatrix} \dfrac{\partial \mathscr{F}_1}{\partial \zeta_1} & \dfrac{\partial \mathscr{F}_1}{\partial \zeta_2} \\ \dfrac{\partial \mathscr{F}_2}{\partial \zeta_1} & \dfrac{\partial \mathscr{F}_2}{\partial \zeta_2} \end{pmatrix}, \qquad \zeta = (\zeta_1, \zeta_2) \in S \cap \overline{B_\varepsilon(0)} \setminus \{0\}$$

is the Jacobian of $\mathscr{F}$, then by conformality

$$v_j(\mathscr{F}(\zeta)) = |\mathscr{F}'(\zeta)|^{-1} \mathscr{J}(\zeta) V_j, \quad \zeta \in \partial S_j \cap \overline{B_\varepsilon(0)} \setminus \{0\}, \quad j = 1, 2 .$$

Thus

$$R(\mathscr{F}(\zeta)) = |\mathscr{F}'(\zeta)|^{-1} \mathscr{J}(\zeta) M, \quad \zeta \in S \cap \overline{B_\varepsilon(0)} \setminus \{0\} . \tag{3.15}$$

Now define

$$\ell_k(t \wedge \eta_\varepsilon) = \int_0^{t \wedge \eta_\varepsilon} |\mathscr{F}'(G(u))| I_{\partial C_k \setminus 0}(\mathscr{F}(G(u))) dY_k(u), \quad k = 1, 2 . \tag{3.16}$$

This is well defined by Corollary 3.3. Then it is easy to see (3.14) is equivalent to

$$\mathscr{F}(G_{t \wedge \eta_\varepsilon}) = \mathscr{N}_{t \wedge \eta_\varepsilon} + \int_0^{t \wedge \eta_\varepsilon} R \circ \mathscr{F}(G_u) d\ell_u , \tag{3.17}$$

where

$$\mathscr{N}^*_{t \wedge \eta_\varepsilon} = \left(\int_0^{t \wedge \eta_\varepsilon} I_{S \setminus 0}(G_u)(\nabla \mathscr{F}_1)^*(G_u) d\beta_u, \int_0^{t \wedge \eta_\varepsilon} I_{S \setminus 0}(G_u)(\nabla \mathscr{F}_2)^*(G_u) d\beta_u \right) .$$

3.5

Now we complete the proof of Theorem 1.1. We set (recall τ_t is from (2.8))

$$B_t = \mathscr{N}(\tau_{t \wedge A(\eta_\varepsilon)})$$
$$k_t = l \circ \tau_{t \wedge A(\eta_\varepsilon)} .$$

Then by (2.8), (3.15)–(3.17)

$$Z_t = \mathscr{F}(G(\tau(t \wedge A(\eta_\varepsilon)))) = B_t + \int_0^t R(Z_u) dk_u .$$

Since $\int_0^\infty I_{\{0\}}(G_u) du = 0$, standard theorems show B_t is two-dimensional Brownian motion stopped at time $A(\eta_\varepsilon)$. We know from [2] that the law of Z on $(\Omega_C, \mathscr{M})$ is

RBM in C starting at 0. The proofs of the remaining statements in Theorem 1.1 are left to the reader. □

4 Not a semimartingale when starting at 0 with $\beta \geqq 2$, $\theta_1 + \theta_2 = 0$

4.1

In this section always assume $\beta \geqq 2$ and $\theta_1 + \theta_2 = 0$. Our characterization of RBM in C is as a law on $(\Omega_C, \mathscr{M})$. It is possible to have realizations of this law on different filtered spaces. So to prove RBM in C starting at 0 is *not* a semimartingale, we must show no realization is a semimartingale. To be precise, a *realization* of RBM in C starting at z is a continuous adapted process Z_t on some filtered space $(\Omega, \mathscr{S}, \{\mathscr{S}_t\}, P)$ satisfying the following properties.

(i) $Z_t(\omega) \in C \quad \forall (\omega, t) \in \Omega \times [0, \infty)$;

(ii) $Z_0(\omega) = z$ a.s.;

(iii) the law of Z on $(\Omega_C, \mathscr{M})$ is RBM in C starting at z.

The next lemma and uniqueness in law simplify our job by showing that it suffices to pick our favorite realization of RBM in C starting at 0 and prove it is not a semimartingale.

Lemma 4.1 *Suppose the process Z_t on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ is a realization of RBM in C starting at* 0. *Let $P \circ Z^{-1}$ be the law of Z on $(\Omega_C, \mathscr{M})$:*

$$P \circ Z^{-1}(A) = P(\omega\colon Z(\omega) \in A), \quad A \in \mathscr{M} .$$

Then Z is a semimartingale on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ iff the coordinate process $X_t(\omega) = \omega(t)$ is a semimartingale on $(\Omega_C, \mathscr{M}, \{\mathscr{M}_t\}, P \circ Z^{-1})$.

Proof. This was done in Williams [8] (Lemma 2) for C replaced by a wedge S, but the proof carries through in the present context. Let us also point out that the proof is also an almost immediate consequence of Theorem 3.1 in Stricker [5]. □

The proof of Theorem 1.2 is by contradiction. Hence assume RBM in C starting at 0 is a semimartingale. By Lemma 4.1 this means that any realization is a semimartingale. We use the realization (2.8). Thus we assume

$$Z_t = \tilde{M}_{t \wedge A(\eta_\varepsilon)} + \tilde{A}_{t \wedge A(\eta_\varepsilon)} \tag{4.1}$$

where $\tilde{M}_t$ is a continuous local martingale with respect to the filtration $\mathscr{H}_t = \{\mathscr{S}_{\tau_t}\}$ (recall G_t from (2.1) is defined on the filtered space $(\Omega, \mathscr{S}, \{\mathscr{S}_t\}, P)$), $\tilde{A}_t$ is a continuous $\mathscr{H}_t$-adapted finite variation process, and $\tilde{A}_0 = \tilde{M}_0 = 0$. (Here we use the notation and terminology of Rogers and Williams [4]). Below in Sect. 4.3 we show (4.1) implies

$$\int_0^{\eta_\varepsilon} |G_u|^{-1}(-\ln|G_u|)^{-\frac{\beta}{\beta-1}} I_{S\setminus 0}(G_u)\,dY_j(u) < \infty \text{ a.s.}, \quad j = 1, 2 , \tag{4.2}$$

where G is RBM in S given by (2.1).

Taking this for granted, choose $f \in C^\infty([0, \pi])$ such that $\sup f'' < 0$ and $\inf f > 0$. Define for $z = re^{i\theta}$

$$h(z) = \begin{cases} [-\ln r + f(\theta)]^{-\frac{1}{\beta-1}} & z \in S \cap \overline{B_\varepsilon(0)} \setminus \{0\} \\ 0 & z = 0 \,. \end{cases}$$

Then $h \in C(S \cap \overline{B_\varepsilon(0)}) \cap C^2(S \cap \overline{B_\varepsilon(0)} \setminus \{0\})$ and $h(0) = 0$. By (2.12)–(2.16)

$$|\nabla h|^2 \leqq Cr^{-2}[-\ln r]^{-\frac{2}{\beta-1}-2},$$

$$|V_j \cdot \nabla h| \leqq Cr^{-1}[-\ln r]^{-\frac{\beta}{\beta-1}} \text{ on } (\partial S_j) \cap \overline{B_\varepsilon(0)} \setminus \{0\},$$

and $\Delta h = f_1 + f_2$, where

$$f_1 = \frac{\beta}{(\beta-1)^2} r^{-2}[-\ln r + f(\theta)]^{-\frac{1}{\beta-1}-2}(1 + [f'(\theta)]^2),$$

$$f_2 = \frac{1}{\beta-1} r^{-2}[-\ln r + f(\theta)]^{-\frac{\beta}{\beta-1}}(-f''(\theta)) \geqq 0 \,.$$

By Theorem 2.3 and (4.2) the hypotheses of Theorem 2.4 hold. Then by that theorem we have

$$\int_0^{\eta_\varepsilon} f_2(G_u) I_{S\setminus 0}(G_u)\,du < \infty \text{ a.s.}$$

By choice of $f(\theta)$,

$$r^{-2}(-\ln r)^{-\frac{\beta}{\beta-1}} \leqq cf_2 \,,$$

and so

$$\int_0^{\eta_\varepsilon} |G_u|^{-2}(-\ln|G_u|)^{-\frac{\beta}{\beta-1}} I_{S\setminus 0}(G_u)\,du < \infty \text{ a.s.} \tag{4.3}$$

We will show this is impossible, giving the desired contradiction.

For $0 < \gamma < \delta < \varepsilon$ and $n \geqq 1$, define

$$T_\gamma = \inf\{t > \eta_\delta : |G_u| = \gamma\}$$

$$S_n = \inf\left\{t \geqq 0 : \int_0^{\eta_\varepsilon \wedge t} |G_u|^{-2}(-\ln|G_u|)^{-\frac{\beta}{\beta-1}} I_{S\setminus 0}(G_u)\,du \geqq n\right\}.$$

By (4.3) $S_n \uparrow \infty$. Let $p > 0$ be small and define for $z = re^{i\theta}$

$$h_1(z) = \begin{cases} [-\ln r - \theta \tan\theta_1]^{-p} & z \in S \cap \overline{B_\varepsilon(0)} \setminus \{0\} \\ 0 & z = 0 \,. \end{cases}$$

Then $h_1 \in C(S \cap \overline{B_\varepsilon(0)}) \cap C^2(S \cap \overline{B_\varepsilon(0)} \setminus \{0\})$ and $h(0) = 0$. Since $\theta_1 = -\theta_2$, by (2.12)–(2.16),

$$V_j \cdot \nabla h_1 = 0 \text{ on } (\partial S_j) \cap \overline{B_\varepsilon(0)} \setminus \{0\}, j = 1, 2 \,.$$

By (2.12) and (2.14), for some $C > 0$ independent of p,

$$0 \leqq \Delta h_1 \leqq Cp(p+1)r^{-2}[-\ln r - \theta \tan \theta_1]^{-p-2}\,.$$

But $\beta \geqq 2$ so $\frac{\beta}{\beta-1} - 2 = \frac{2-\beta}{\beta-1} \leqq 0 < p$; that is, $-p-2 \leqq -\frac{\beta}{\beta-1}$. Hence by making ε smaller if necessary (independently of p), for $0 < r \leqq \varepsilon$,

$$\begin{aligned} 0 &\leqq \Delta h_1 \leqq Cp(p+1)r^{-2}[-\ln r - \theta \tan \theta_1]^{-\frac{\beta}{\beta-1}} \\ &\leqq Kp(p+1)r^{-2}[-\ln r]^{-\frac{\beta}{\beta-1}} \end{aligned} \tag{4.4}$$

where $K > 0$ is independent of p.

By Itô's formula and (2.1)

$$\begin{aligned} k_m \circ h_1(G_{t \wedge T_\gamma \wedge \eta_\varepsilon \wedge S_n}) - k_m \circ h_1(G_{t \wedge \eta_\delta \wedge S_n}) = \int_{t \wedge \eta_\delta \wedge S_n}^{t \wedge T_\gamma \wedge \eta_\varepsilon \wedge S_n} k'_m \circ h_1(G_u)(\nabla h_1(G_u))^* d\beta_u \\ + \frac{1}{2} \int_{t \wedge \eta_\delta \wedge S_n}^{t \wedge T_\gamma \wedge \eta_\varepsilon \wedge S_n} [k'_m \circ h_1(G_u) \Delta h_1(G_u) \\ + k''_m \circ h_1(G_u)|\nabla h_1(G_u)|^2] du\,. \end{aligned}$$

Since $|\nabla h_1|^2 \leqq C(p) r^{-2}[-\ln r]^{-2p-2}$, by Theorem 2.3 the martingale part converges in L^2 as $t \to \infty$ to

$$\int_{\eta_\delta \wedge S_n}^{T_\gamma \wedge \eta_\varepsilon \wedge S_n} k'_m \circ h_1(G_u)(\nabla h_1(G_u))^* d\beta_u\,.$$

By passing to a subsequence $t_\ell \to \infty$ we get a.s.

$$\begin{aligned} k_m \circ h_1(G_{T_\gamma \wedge \eta_\varepsilon \wedge S_n}) - k_m \circ h_1(G_{\eta_\delta \wedge S_n}) = \int_{\eta_\delta \wedge S_n}^{T_\gamma \wedge \eta_\varepsilon \wedge S_n} k'_m \circ h_1(G_u)(\nabla h_1(G_u))^* d\beta_u \\ + \frac{1}{2} \int_{\eta_\delta \wedge S_n}^{T_\gamma \wedge \eta_\varepsilon \wedge S_n} [k'_m \circ h_1(G_u) \Delta h_1(G_u) + k''_m \circ h_1(G_u)|\nabla h_1(G_u)|^2] du\,. \end{aligned}$$

If m is large enough, supp $k''_m \cap \{h_1(G_u): \eta_\delta \wedge S_n \leqq u \leqq T_\gamma \wedge \eta_\varepsilon \wedge S_n\} = \emptyset$; also, as $m \to \infty$ the martingale part converges in L^2 to $\int_{\eta_\delta \wedge S_n}^{T_\gamma \wedge \eta_\varepsilon \wedge S_n} I_{S\backslash 0}(G_u)(\nabla h_1(G_u))^* d\beta_u$ (using Theorem 2.3 as above) and the k'_m part of the du integral a.s. converges to $\frac{1}{2}\int_{\eta_\delta \wedge S_n}^{T_\gamma \wedge \eta_\varepsilon \wedge S_n} I_{S\backslash 0}(G_u)\Delta h_1(G_u) du$ by dominated convergence and (4.3)–(4.4). Thus we have (a.s.)

$$\begin{aligned} h_1(G_{T_\gamma \wedge \eta_\varepsilon \wedge S_n}) - h_1(G_{\eta_\delta \wedge S_n}) = \int_{\eta_\delta \wedge S_n}^{T_\gamma \wedge \eta_\varepsilon \wedge S_n} I_{S\backslash 0}(G_u)(\nabla h_1(G_u))^* d\beta_u \\ + \frac{1}{2} \int_{\eta_\delta \wedge S_n}^{T_\gamma \wedge \eta_\varepsilon \wedge S_n} I_{S\backslash 0}(G_u)\Delta h_1(G_u) du\,. \end{aligned}$$

Upon taking expectations and using (4.4) this becomes

$$E[h_1(G_{T_\gamma \wedge \eta_\varepsilon \wedge S_n})] - E[h_1(G_{\eta_\delta \wedge S_n})]$$
$$\leqq \frac{K}{2} p(p+1) E\left[\int_{\eta_\delta \wedge S_n}^{T_\gamma \wedge \eta_\varepsilon \wedge S_n} |G_u|^{-2}[-\ln|G_u|]^{-\frac{\beta}{\beta-1}} du\right].$$

Since $T_\gamma \to \infty$ as $\gamma \to 0$, by dominated convergence and (4.3) we can let $\gamma \to 0$ and omit T_γ. Another application of (4.3) and dominated convergence as $\delta \downarrow 0$ yields

$$E[h_1(G_{\eta_\varepsilon \wedge S_n})] - 0 \leqq \frac{K}{2} p(p+1) E\left[\int_0^{\eta_\varepsilon \wedge S_n} |G_u|^{-2}[-\ln|G_u|]^{-\frac{\beta}{\beta-1}} I_{S\setminus 0}(G_u) du\right].$$

By making ε smaller – independently of p – if necessary,

$$0 < -\ln r - \theta \tan\theta_1 \leqq C(-\ln r).$$

Then we get

$$\begin{aligned} C^{-p} E[(-\ln|G_{\eta_\varepsilon \wedge S_n}|)^{-p}] &\leqq E[h_1(G_{\eta_\varepsilon \wedge S_n})] \\ &\leqq \frac{K}{2} p(p+1) \int_0^{\eta_\varepsilon \wedge S_n} |G_u|^{-2}(-\ln|G_u|)^{-\frac{\beta}{\beta-1}} I_{S\setminus 0}(G_u) du \\ &\leqq \frac{K}{2} p(p+1)(n+1) \end{aligned}$$

by definition of S_n. Since G_t never hits 0 (a.s.) for $t > 0$, by Fatou's lemma

$$1 = E\left\{\lim_{p \to 0} C^{-p}[-\ln|G_{\eta_\varepsilon \wedge S_n}|]^{-p}\right\} \leqq \varliminf_{p \to 0} \frac{K}{2} p(p+1)(n+1) = 0.$$

Contradiction. Thus our assumption that RBM in C starting at 0 for $\beta \geqq 2$ is a semimartingale is false and Theorem 1.2 is proved.

All that remains is to show (4.2) holds assuming (4.1). Below we will see it is easy to identify $\tilde{M}_{t \wedge A(\eta_\varepsilon)}$ as two-dimensional Brownian motion stopped at time $A(\eta_\varepsilon)$ and see that $\tilde{A}_{t \wedge A(\eta_\varepsilon)}$ is supported on ∂C. The trouble lies in getting an explicit representation of $\tilde{A}_t$. The trick is to use (2.8) to get an explicit semimartingale representation of $F(Z_{t \wedge A(\eta_\varepsilon)})$ (recall $F = \mathscr{F}^{-1}$) and compare it with the one obtained by using (4.1) to compute $dF(Z_{t \wedge A(\eta_\varepsilon)})$. Unfortunately matters are complicated because F is not C^2.

4.2

Since $\theta_1 + \theta_2 = 0$ RBM G_t in S never hits 0 once it is away from 0. Hence by (2.8) RBM Z_t in C stopped at time $A(\eta_\varepsilon)$ never hits 0 once it is away from 0. Since $\partial C\setminus\{0\}$ is smooth, standard results show that for each $\delta > 0$, $\{\tilde{M}_{t \wedge A(\eta_\varepsilon)}: t \geqq \delta\}$ is two-dimensional Brownian motion starting from $\tilde{M}_{\delta \wedge A(\eta_\varepsilon)}$ and $t \in [\delta, \infty) \to \tilde{A}_{t \wedge A(\eta_\varepsilon)}$ can change only when Z_t is on $\partial C\setminus\{0\}$. Thus taking the limit as $\delta \to 0$, by continuity and a.s. uniqueness of semimartingale representations, we see $\tilde{M}_{t \wedge A(\eta_\varepsilon)}$ must be two-dimensional Brownian motion stopped at time $A(\eta_\varepsilon)$ and $\tilde{A}_{t \wedge A(\eta_\varepsilon)}$ can change only when Z is in ∂C.

Thus we have for $\sigma_\varepsilon = \inf\{t > 0: |F(Z_t)| = \varepsilon\} = A(\eta_\varepsilon)$,

$$\begin{pmatrix} d[\tilde{M}_1, \tilde{M}_1] \\ d[\tilde{M}_1, \tilde{M}_2] \\ d[\tilde{M}_2, \tilde{M}_2] \end{pmatrix} = \begin{pmatrix} dt \\ 0 \\ dt \end{pmatrix}, \quad t \leqq \sigma_\varepsilon . \tag{4.5}$$

4.3 Proof of (4.2)

Using Itô's formula and (4.1), by (4.5) for any real a, b, and $t \leqq \sigma_\varepsilon$

$$\begin{aligned} d[k_n(aF_1(Z_t) + bF_2(Z_t))] = {} & k_n'(aF_1(Z_t) + bF_2(Z_t))(a\nabla F_1(Z_t) + b\nabla F_2(Z_t))^* \\ & \times [d\tilde{M}_t + d\tilde{A}_t] + \frac{1}{2} k_n''(aF_1(Z_t) \\ & + bF_2(Z_t))[a^2 + b^2]|\nabla F_1(Z_t)|^2\, dt . \end{aligned} \tag{4.6}$$

On the other hand, by (4.5) and (2.2)–(2.3), for $t \leqq \sigma_\varepsilon$,

$$\begin{aligned} d[k_n(aF_1(Z_t) + bF_2(Z_t))] = {} & k_n'(aF_1(Z_t) + bF_2(Z_t))[a\{d(\beta_1(\tau_t)) + \gamma d(\varphi(\tau_t))\} \\ & + b\{d(\beta_2(\tau_t)) + d(\varphi(\tau_t))\}] \\ & + \tfrac{1}{2} k_n''(aF_1(Z_t) + bF_2(Z_t))[a^2 + b^2]|\nabla F_1(Z_t)|^2 dt . \end{aligned} \tag{4.7}$$

Here we have used that $\theta_1 = -\theta_2 \Rightarrow M = \begin{pmatrix} \gamma & \gamma \\ 1 & 1 \end{pmatrix}$ for some real γ. Similar expressions hold for k_n replaced by h_n.

For typographical clarity we write

$$\begin{aligned} H_n(a, b) &= [k_n' + h_n'](aF_1(Z_t) + bF_2(Z_t)) , \\ (u, v) &= F(Z_t) , \\ \nabla u &= \nabla F_1(Z_t), \text{ etc.} \end{aligned}$$

Then we have two ways to compute

$$d(k_n + h_n)(aF_1(Z_t) + bF_2(Z_t)) :$$

from (4.6) with its analogue for h_n and from (4.7) with its analogue for h_n. In particular, first taking $a = b = 1$ and then $a = 1 = -b$, upon comparing finite variation parts we get

$$\begin{pmatrix} H_n(1, 1)[\nabla u + \nabla v]^* d\tilde{A}_t \\ H_n(1, -1)[\nabla u - \nabla v]^* d\tilde{A}_t \end{pmatrix} = \begin{pmatrix} H_n(1, 1)(\gamma + 1) d\varphi(\tau_t) \\ H_n(1, -1)(\gamma - 1) d\varphi(\tau_t) \end{pmatrix}, \quad t \leqq \sigma_\varepsilon .$$

Replace the first and second rows by

$\tfrac{1}{2}H_n(1, -1)$ Row 1 $+ \tfrac{1}{2}H_n(1, 1)$ Row 2 and $\tfrac{1}{2}H_n(1, -1)$ Row 1 $- \tfrac{1}{2}H_n(1, 1)$ Row 2 ,

respectively, to end up with

$$H_n(1, 1)H_n(1, -1)\begin{pmatrix} (\nabla u)^* d\tilde{A} \\ (\nabla v)^* d\tilde{A} \end{pmatrix} = H_n(1, 1)H_n(1, -1)\begin{pmatrix} \gamma \\ 1 \end{pmatrix} d\varphi(\tau_t), \quad t \leqq \sigma_\varepsilon .$$

Since $M = \begin{pmatrix} \gamma & \gamma \\ 1 & 1 \end{pmatrix}$, if we set

$$J(z) = \begin{pmatrix} \frac{\partial F_1}{\partial z_1} & \frac{\partial F_1}{\partial z_2} \\ \frac{\partial F_2}{\partial z_1} & \frac{\partial F_2}{\partial z_2} \end{pmatrix}$$

We have (by (2.2))

$$H_n(1, 1)H_n(1, -1)J(Z_t)d\tilde{A} = H_n(1, 1)H_n(1, -1)M dY(\tau_t) .$$

By (3.15)

$$\begin{aligned} J^{-1}(z)M &= \mathscr{J}(F(z))M \\ &= |\mathscr{F}'(F(z))|R(z) . \end{aligned}$$

Thus for $j = 1, 2$

$$\begin{aligned} &H_n(1, 1)H_n(1, -1)I_{\partial C_j \setminus 0}(Z_t)d\tilde{A}_t \\ &= H_n(1, 1)H_n(1, -1)I_{\partial C_j \setminus 0}(Z_t)|\mathscr{F}'(F(Z_t))|R(Z_t)dY(\tau_t) . \end{aligned}$$

In particular,

$$\begin{aligned} &\int_0^{\sigma_\varepsilon} H_n(1, 1)H_n(1, -1)I_{\partial C_j \setminus 0}(Z(t))(R_{2j}(Z(t)))^{-1}d\tilde{A}_2(t) \\ &= \int_0^{\sigma_\varepsilon} H_n(1, 1)H_n(1, -1)I_{\partial C_j \setminus 0}(Z_t)|\mathscr{F}'(F(Z_t))|dY_j(\tau_t), \quad j = 1, 2 . \end{aligned} \tag{4.8}$$

By making ε smaller if necessary, our normalizations $v_j \cdot n_j = 1, j = 1, 2$ force

$$\inf\{|R_{2j}(z)|: \quad j = 1, 2, \ z \in B_\varepsilon(0) \cap \partial C_j \setminus 0\} > 0 .$$

Hence for some constant $C > 0$,

$$\begin{aligned} \overline{\lim_{n \to \infty}} &\left| \int_0^{\sigma_\varepsilon} H_n(1, 1)H_n(1, -1)I_{\partial C_j \setminus 0}(Z(t))[R_{2j}(Z(t))]^{-1}d\tilde{A}_2(t) \right| \\ &\leqq C\|\tilde{A}_2\|_{\sigma_\varepsilon} \\ &< \infty \ \text{ a.s.}, \quad j = 1, 2 , \end{aligned}$$

where $\|\cdot\|_T$ is the total variation over $[0, T]$. By (4.8) and Fatou's lemma this yields (since $H_n(a, b) \to I_{\mathbb{R} \setminus 0}(au + bv)$)

$$\int_0^{\sigma_\varepsilon} I_{\mathbb{R} \setminus 0}(u + v)I_{\mathbb{R} \setminus 0}(u - v)I_{\partial C_j \setminus 0}(Z_t)|\mathscr{F}'(F(Z_t))|dY_j(\tau_t) < \infty \ \text{ a.s.} ,$$

$j = 1, 2$. But

$$\begin{aligned} &I_{\mathbb{R} \setminus 0}(u + v)I_{\mathbb{R} \setminus 0}(u - v)I_{\partial C_j \setminus 0}(Z_t) \\ &= I_{\mathbb{R} \setminus 0}(F_1(Z_t) + F_2(Z_t))I_{\mathbb{R} \setminus 0}(F_1(Z_t) - F_2(Z_t))I_{\partial S_j \setminus 0}(F(Z_t)) \\ &= I_{\partial S_j \setminus 0}(F(Z_t)) \end{aligned}$$

and so (recall $\sigma_\varepsilon = A(\eta_\varepsilon)$)

$$\int_0^{A(\eta_\varepsilon)} I_{\partial S_j \setminus 0}(F(Z_t))|\mathscr{F}'(F(Z_t))|dY_j(\tau_t) < \infty \text{ a.s. } j = 1, 2 .$$

Making the change of variables $u = \tau_t$ and using (2.8) we end up with

$$\int_0^{\eta_\varepsilon} I_{\partial S_j \setminus 0}(G_u)|\mathscr{F}'(G_u)|dY_j(u) < \infty \text{ a.s. } j = 1, 2 .$$

Formula (4.2) follows from this and (2.4). □

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