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Singular perturbations as a selection criterion for periodic minimizing sequences

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Summary. Minimizers of functionals like $\int_0^1 \epsilon^2 u_{xx}^2 + (u_x^2 - 1)^2 + u^2 \mathrm{d}x$ subject to periodic (or Dirichlet) boundary conditions are investigated. While for $\epsilon = 0$ the infimum is not attained it is shown that for sufficiently small $\epsilon > 0$, all minimizers are periodic with period $\sim \epsilon^{1/3}$. Connections with solid-solid phase transformations are indicated.

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1. Introduction

Motivations and results

In this paper we study minimizers of singularly perturbed functionals like

$$\int_0^1 (\epsilon^2 u_{xx}^2 + (u_x^2 - 1)^2 + u^2)\mathrm{d}x$$

subject to suitable boundary conditions. We show that as $\epsilon \longrightarrow 0$ the minimizers develop a periodic structure with period $\sim \epsilon^{1/3}$.

The motivation for this work stems from the study of coherent solid-solid phase transformations. These transformations often lead to a fine mixture of different phases with a characteristic geometric structure (see e.g. [Ar 90], [ATL 85], [BC 54], [Wa 64]). The existence of such a fine structure and certain of its features have been explained by the minimization of elastic energy by Khachaturyan, Roitburd and Shatalov ([KH 67], [KS 69], [Ro 69], [Ro 78]; see [Kh 83] for a comprehensive treatment) in the context of geometrically linear elasticity and more recently by Ball and James ([BJ 87], [BJ 92]) using fully nonlinear elasticity. These approaches recover the predictions of the more phenomenologically crystallographic theory of martensite (see e.g. [BMK 54], [WLR 53]) but fail to predict finer details such as the length scale of the structure and its frequently observed periodicity. It is generally believed that such details can be obtained by the inclusion of surface energy (see [KS 69] for a heuristic explanation of periodicity).

In the analogous, but analytically simpler situation of the liquid-vapour transition in a van der Waals gas the inclusion of surface energy terms (or more precisely density gradient terms) has been studied extensively in the mathematical literature (see [Ba 91], [FT 89], [KS 89], [Mo 87], [OS 91]) and has led to the expected results. In the context of solid-solid transformations serious difficulties appear due to the fact that one is dealing with gradients (like the deformation gradient) rather than with scalar functions (like the density of a gas); see [AG 87], [AG 89] and [Fo 89] for first attempts to address some of the issues involved.

Mathematically minimization of the elastic energy typically leads to functionals which are *not* weakly lower semicontinuous in the appropriate Sobolev spaces. Thus minimizing sequences often develop increasingly faster oscillations (the mathematical counterpart of the observed fine structure). Inclusion of surface energy introduces higher derivatives which make the functional well-behaved. The simplest model problem which shares these features is

$$\text{Minimize} \int_0^1 (u_x^2 - 1)^2 + u^2 \mathrm{d}x \tag{1.1}$$

subject to periodic boundary conditions.

The infimum of the functional is zero and minimizing sequences try to achieve simultaneously $u_x \approx \pm 1$ and $u \approx 0$, resulting in a sawtooth pattern in which locally the "phases" $u_x \sim 1$ and $u_x \sim -1$ appear in a ratio of (approximately) $1:1$.

Inclusion of surface energy penalizes aprupt changes in u_x and one is lead to the functional

$$\int_0^1 \epsilon^2 u_{xx}^2 + (u_x^2 - 1)^2 + u^2 \mathrm{d}x \, .$$

To state the main result of this paper we consider slightly more general double-well potentials W which will be required to satisfy

$$W(-1) = W(1) = 0, W > 0 \text{ else }, \tag{H1}$$

$$W(-z) = W(z) \, , \tag{H2}$$

$$W \in C^3, \quad W''(\pm 1) > 0 \quad W''(0) < 0 \, , \tag{H3}$$

$$W' < 0 \text{ on } (-\infty, -1) \cup (0,1), \; W' > 0 \text{ on } (-1,0) \cup (1, \infty) \, , \tag{H4}$$

$$\pm \liminf_{z \to \pm\infty} W'(z) > 0 \, . \tag{H5}$$

We constantly write

$$\sigma = W' \, .$$

Set

$$I^\epsilon(u) = \int_0^1 \epsilon^2 u_{xx}^2 + W(u_x) + u^2 \mathrm{d}x \, ,$$

and denote the space of periodic H^2-functions by

$$H^2_\#(0,1) = \{u \in H^2(0,1) : u(0) = u(1),\ u_x(0) = u_x(1)\}\ .$$

Let

$$A_0 = 2\int_{-1}^{1} W^{1/2}(z)\mathrm{d}z\ . \tag{1.2}$$

Theorem 1.1 *Assume that W satisfies (H1) to (H5). Then there exists a constant $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0)$ the following conclusions hold.*
(i) If u^ϵ is a minimizer of I^ϵ in $H^2_\#$ then u^ϵ is periodic with minimal period P^ϵ,

$$P^\epsilon = 2(6A_0\epsilon)^{1/3} + \mathcal{O}(\epsilon^{2/3})\ . \tag{1.3}$$

Moreover

$$u^\epsilon\left(x + \frac{\epsilon}{2}\right) = -u^\epsilon(x),\ \textit{for}\ x \in (0,1)\ , \tag{1.4}$$

$$I^\epsilon(u^\epsilon) = \frac{1}{4}(6A_0\epsilon)^{2/3} + \mathcal{O}(\epsilon^{4/3})\ .$$

(ii) There are at most two distinct (up to translations) minimizers of I^ϵ.
(iii) There is a minimizer u^ϵ of I^ϵ satisfying

$$u^\epsilon(0) = u^\epsilon\left(\frac{1}{2}\right) = 0\ ,$$

$$u^\epsilon\left(\frac{1}{2} - x\right) = -u^\epsilon\left(\frac{1}{2} + x\right),\quad \textit{for}\ x \in \left(0, \frac{1}{2}\right)\ .$$

Remarks

1. The theorem implies in particular that there is a separation of scales. The scale on which u_x changes from approximately -1 to approximately 1 is of order ϵ (see (1.14) and (3.9)) and thus is much shorter than the period of u^ϵ.
2. Similar results can be shown for other boundary conditions, see Corollary 1.2 below.
3. The symmetry hypothesis (H2) is important in the proof. I believe, however, that the conclusion (i) (with the exception of (1.4)) remains valid without it. Hypotheses (H4) and (H5) and the C^3 regularity of W simplify the proof but are most likely not essential.
4. With more work one can show that that the minimizer is unique (up to translations) for all but countably many values of $\epsilon \in (0, \epsilon_0)$ (see Sect. 6, in particular (6.13)).

One may view I^ϵ as a phenomenological model for the situation at an austenite/twinned-martensite interface. The states $u_x \sim \pm 1$ correspond to the finely mixed martensite phases, the penalty term $\int_0^1 u^2$ corresponds to compatibility with the austenite phase and enforces the martensitic phases to appear in a certain ratio (which in the real situation is determined by a geometric compatibility condition).

A more careful analysis reveals that in the above context the penalty should be the square of the $H^{1/2}$ rather than the L^2-norm (see [BHK 87], [HBK 91], [KM 92]). It is highly plausible that similar results hold in this case (with $P^\epsilon \sim \epsilon^{1/2}$) but the nonlocal character of the $H^{1/2}$ norm leads to technical difficulties. Very recently interesting new phenomena have been observed for a two-dimensional model in the same spirit (see [KM 92], [KM 93]).

The following corollary improves the result in [Mu 89].

Corollary 1.2 *Let W and ϵ be as in Theorem 1.1. Let u^ϵ be a minimizer of I^ϵ subject to*

$$u(0) = u(1) = 0\ .$$

Then (the antiperiodic extension of) u^ϵ is periodic with minimal period P^ϵ and

$$P^\epsilon = 2(6A_0\epsilon)^{1/3} + \mathcal{O}(\epsilon^{2/3})\ .$$

Moreover

$$u^\epsilon\left(x + \frac{P^\epsilon}{2}\right) = -u^\epsilon(x), \quad \textit{for } x \in (0,1)$$

and

$$I^\epsilon(u^\epsilon) = \frac{1}{4}(6A_0\epsilon)^{2/3} + \mathcal{O}(\epsilon^{4/3})\ .$$

To understand any dynamics related to I^ϵ (e.g. its gradient flow) it would be important to analyze not only the minizers of I^ϵ but also its stationary points. I conjecture that at least the stationary points with sufficiently low energy can be classified as follows.

Conjecture. Let W satisfy (H1) and (H3), let ϵ be sufficiently small and let u^ϵ be a stationary point of I^ϵ subject to periodic boundary conditions. Assume that $I^\epsilon(u^\epsilon)$ is sufficiently small (e.g. $o(|\ln\epsilon|^{-1})$). Then either (i) or (ii) below holds.

(i) The function u^ϵ is a stable stationary point and is periodic with minimal period P^ϵ, where $A_0\frac{2}{P^\epsilon} + \frac{1}{12}(\frac{P^\epsilon}{2})^2 \approx I^\epsilon(u^\epsilon)$.

(ii) The function u^ϵ is an unstable stationary point and there are two zeros of u^ϵ_x whose distance (identifying the endpoints of $[0,1]$) is of order $\epsilon|\ln\epsilon|$.

Indeed it seems natural that each pair of zeros as in (ii) contributes one unstable eigendirection.

Sketch of the proof of Theorem 1.1. The idea is to construct a periodic candidate for the minimizer and then to show that it is optimal and that all other minimizers have to be periodic as well.

To construct a candidate consider the functional

$$I_l(u) = \int_{-l/2}^{l/2} (\epsilon^2 u_{xx}^2 + W(u_x) + u^2)\mathrm{d}x$$

Here as in the following we suppress dependence on ϵ to avoid clumsy notation. Let u_l be a minimizer of I_l subject to

$$u_x\left(\pm\frac{l}{2}\right) = 0\ , \quad u_x \geq 0\ , \tag{1.6}$$

$$u(-x) = -u(x) \quad \text{for } x \in \left(-\frac{l}{2}, \frac{l}{2}\right)\ . \tag{1.7}$$

Let

$$\tilde{E}(l) = I_l(u_l)\ . \tag{1.8}$$

For $N \in \mathbb{N}$, let $l = \frac{1}{2N}$ and denote by u_l also the antiperiodic extension of u_l to $[0,1]$. Then $u_l \in H^2_\#(0,1)$ and $I(u_l) = 2N\,\tilde{E}(\frac{1}{2N})$. Let

$$E_0 = \min\left\{2N\tilde{E}(\frac{1}{2N}) : N \in \mathbb{N}\right\}. \tag{1.9}$$

We can thus construct a candidate $\overline{u} \in H^2_\#(0,1)$ with $I(\overline{u}) = E_0$. It remains to show that

$$I(u) \geq E_0 \quad \text{for all } u \in H^2_\#(0,1)\ .$$

Let

$$E(l) = \min\left\{I_l(u) : u_x\left(\pm\frac{l}{2}\right) = 0,\ u_x \geq 0\right\}\ . \tag{1.10}$$

Let $x_1, \ldots, x_{2N}$ be the points where u_x changes sign (there has to be an even number by periodicity; we will show that the set where $u_x = 0$ is discrete if u is a minimizer).

Let $l_i = x_{i+1} - x_i$, where $x_{2N+1} = x_1 + 1$. Thus

$$I(u) \geq \sum_{i=1}^{2N} E(l_i)\ .$$

Assume for a moment that we could show that for all l

$$\tilde{E}(l) = E(l)\ , \tag{1.11}$$

$$E''(l) > 0\ . \tag{1.12}$$

Then the strict convexity of E would imply

$$I(u) \geq 2NE\left(\frac{1}{2N}\right) = 2N\tilde{E}\left(\frac{1}{2N}\right) \geq E_0\ ,$$

and equality could only hold if $l_i = \overline{l}$ for all i. It would follow that $\overline{u}$ is minimizing. If we show in addition a uniqueness result for minimizers of I_l we could infer periodicity of minimizers of I.

The rigorous argument is slightly more complicated since (1.11) and (1.12) only hold if

$$l \ll 1\ , \quad \epsilon \ll \frac{l}{|\ln l|}\ . \tag{1.13}$$

A simple separate argument (see Sect. 6, in particular Proposition 6.1) is needed to ensure that for minimizers of I, the l_i always lie in the above range.

The main technical work goes into proving (1.12). A boundary layer construction suggests that

$$E(l) \approx \epsilon A_0 + \frac{1}{12}l^3\ ,$$

where A_0 is given by (1.2). Formal differentiation would give (1.12).
To make this rigorous we consider rescaled functions

$$v(x) = l^{-1}u(lx) \tag{1.14}$$

and let

$$J_l(v) = \int_{-1/2}^{1/2} (\epsilon^2 l^{-1} v_{xx}^2 + lW(v_x) + l^3 v^2)\mathrm{d}x \ . \tag{1.15}$$

One sees that

$$E(l) = \min\left\{ J_l(v) : v_x(\pm 1) = 0,\ v_x \geq 0 \right\} , \tag{1.16}$$

$$\tilde{E}(l) = \min\left\{ J_l(v) : v_x(\pm 1) = 0,\ v_x \geq 0,\ v(-x) = -v(x) \text{ for } x \in \left(-\frac{1}{2}, \frac{1}{2}\right)\right\}. \tag{1.17}$$

We first show (assuming (1.13)) that the minima of J_l are unique, antisymmetric and behave like the boundary layer construction suggests (see Theorems 3.1 and 4.2). Subsequently we obtain estimates on the eigenvalues of the linearized problem and employ the implicit function theorem to see that the minimizers v_l of J_l depend smoothly on l. We finally compute a good approximation of $\frac{d}{dl} v_l$ which enables us to estimate (see Theorem 5.1) the quantity

$$E''(l) = \frac{d^2}{dl^2} J_l(v_l).$$

The above approach was partially inspired by the work of Carr and Pego [CP 89], although both the focus of their analysis (a very careful study of slow evolution) and the actual estimates involved are quite different.

Other approaches

The strategy just sketched gives a rather sharp result but requires very precise estimates. It would be interesting to have a method which would at least show that u^ϵ is close to a periodic function and which could also handle more general functionals of the type

$$\int_0^1 \epsilon^2 a^2(x) u_{xx}^2 + W(u_x) + u^2 \mathrm{d}x \ , \tag{1.18}$$

where a is smooth and $a(x) \geq c > 0$. In this case one would expect the minimizers to be "locally" close to a periodic function with period $\approx 2(6A_0 a(x)\epsilon)^{1/3}$. One possible approach is to show that the functionals I^ϵ converge in a suitable sense to a limit functional whose minimizers have the desired periodicity properties. The natural notion of convergence is Γ-convergence (see [At 84], [DG 79], [DD 83]) which was introduced by De Giorgi and is essentially equivalent to convergence of minimizers.

The Γ-limit of I^ϵ is, however, not of much interest, as all minimizing sequences converge to zero (strongly in L^2 and weakly in a Sobolev space determined by the growth conditions imposed on W) and hence no information can be recovered from the limit. Indeed, Theorem 1.1 suggests to study instead the rescaled functions $w(x) = \epsilon^{-1/3} u(\epsilon^{1/3} x)$ and the functionals

$$I^\epsilon(w) = \int_{-\epsilon^{-1/3}/2}^{\epsilon^{-1/3}/2} (\epsilon^{2/3} w_{xx}^2 + \epsilon^{-2/3} W(w_x) + w^2)\mathrm{d}x \ .$$

Formal reasoning (inspired by [CGS 84], [Mo 87]; see the arguments following (2.8), (2.9) for some details) suggest that the limit problem should be to "minimize"

(1.19) $$I^\infty(w) = \int_{-\infty}^{\infty} \frac{1}{2} A_0 |w_{xx}| + w^2 dx ,$$

subject to

(1.20) $$|w_x| = 1 .$$

Note that this is a discrete problem as by (1.20) the minimization is carried out over sawtooth functions. Therefore $\int_a^b \frac{1}{2}|w_{xx}|$ is just the number of discontinuities of w_x in (a, b). Continuing the formal argument one sees that it is best to distribute these discontinuities equidistantly with distance $(6A_0)^{1/3}$ (one could subtract $-\frac{1}{4}(6A_0\epsilon)^{2/3}$ from the integrand in (1.19) to keep the integral finite).

When trying to make the above approach rigorous one faces two problems. First one deals with functionals defined on different spaces. Secondly the limit functional will be defined on the whole real line and it becomes difficult to define a suitable notion of minimizer as the energy is typically ∞, see [LM 89] for interesting progress.

Considering for variety (1.18) a closely related approach would be to try to show that minimizers of (1.18) are close (e.g. in $\|u_x\|_{L^p} + \epsilon^{-1/3}\|u\|_{L^2}$) to those of

$$\int_0^1 A_0 \ a(x)|u_{xx}| + u^2 dx ,$$

subject to

$$|u_x| = 1 \ \text{ a.e.}$$

If $a \equiv 1$ this follows of course from Theorem 1.1 (or even from the weaker estimates in [Mu 89]) but it would be desirable to have a simpler and more flexible method.

A completely different strategy would be to start from the Euler-Lagrange equation

(1.21) $$\epsilon^2 u_{xxxx} - \frac{1}{2}\sigma(u_x)_x + u = 0 ,$$

which upon the rescaling $t = \frac{x}{\epsilon}$, $v(t) = \epsilon^{-1}u(\epsilon t)$, becomes

(1.22) $$\overset{....}{v} - \frac{1}{2}\sigma(\dot{v})^{\cdot} + \epsilon^2 v = 0 .$$

The latter equation is equivalent to the Hamiltonian system

(1.23) $$\dot{q}_i = \frac{\partial H}{\partial p_i} , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} , i = 1, 2 ,$$

with

$$\begin{aligned} H(q,p) &= \frac{1}{2}p_1^2 - \frac{1}{2}W(q_1) - \frac{1}{2}p_2^2 - \epsilon q_1 q_2 \\ &= H_1(p_1, q_1) + H_2(p_2, q_2) + \epsilon H_3(p, q) . \end{aligned}$$

Indeed, if (p, q) is a solution of (1.23) then $v = -\epsilon^{-1}\dot{q}_2$ solves (1.22). Conversely if v solves (1.22) one obtains a solution of (1.23) by letting $q_2 = -\epsilon \int_0^x v(\xi)d\xi + \epsilon^{-1}(\dddot{v}(0) - \frac{1}{2}\sigma(\dot{v}(0)))$, $q_1 = \dot{v}$, $p_1 = \dot{q}_1$, $p_2 = -\dot{q}_2$.

The phase plane corresponding to $\dot{q}_1 = \frac{\partial H_1}{\partial p_1}$, $\dot{p}_1 = -\frac{\partial H_1}{\partial q_1}$ contains a heteroclinic loop (cf. (3.3), (3.4)). One could thus hope to apply perturbation methods such as Melnikov's method (see e.g. [GH 86]) to study the periodicity properties of solutions to the full system. In view of the rescaling and Theorem 1.1 solutions v which arise from minimizers have a period $\approx \epsilon^{-2/3}$. A difficulty arises from the fact that the Hamiltonian H_2 is independent of q_2 (so that lines of constant energy are not bounded in the phase plane) and standard estimates do not apply. It thus seems that finer estimates are required. I do not know whether the difficulty could be circumvened by using a different scaling.

Outline

The key estimates are contained in Theorems 3.1, 4.2 and 5.1. In Sect. 2 we derive preliminary estimates on the minimizers of J_l and show that the minimizer satisfies $v_x > 0$ on $(-\frac{1}{2}, \frac{1}{2})$ so that the Euler-Lagrange equations hold. Section 3 contains the main technical work and gives precise estimates on the difference between a minimizer and its approximation by the boundary layer construction. In Sect. 4 we prove uniqueness of minimizers and derive a lower bound for the eigenvalues of the linearized Euler-Lagrange operator. Section 5 is devoted to the computation of E'' by means of the implicit function theorem. Finally in Sects. 6 and 7, Theorem 1.1 and Corollary 1.2, respectively, are proved.

Notation

The space of weakly differentiable functions on $(0,1)$ with derivatives up to order k in L^2 is denoted $H^k(0,1)$. The subspace $H^k_0(0,1)$ consists of functions for which $u^{(j)}(0) = u^{(j)}(1) = 0$, $j = 0, \dots, k-1$; $H^k_\#(0,1)$ are the 1-periodic H^k functions. The space of k-times continuously differentiable functions is denoted $C^k(0,1)$ and $C^k([0,1])$ is the subspace of functions whose derivatives up to order k have continuous extensions at 0 and 1. Throughout this paper we always assume that ϵ is chosen sufficiently small and we often suppress dependence on ϵ. The letters C, c, etc. denote generic constants (independent of ϵ and l) whose value may change from line to line.

2. Preliminary estimates and Euler-Lagrange equations

In this section we collect some simple estimates for minimizers of the functional

$$J_l(v) = \int_{-1/2}^{1/2} (\epsilon^2 l^{-1} v_{xx}^2 + lW(v_x) + l^3 v^2) \mathrm{d}x \,, \tag{2.1}$$

subject to the conditions

$$v_x\left(\pm\frac{1}{2}\right) = 0, \ v_x \geq 0 \text{ on } \left[-\frac{1}{2}, \frac{1}{2}\right], \tag{2.2}$$

and we derive the Euler-Lagrange equations showing in particular that $v_x > 0$ on $(-\frac{1}{2}, \frac{1}{2})$ for sufficiently small ϵ and l.

Note that J_l attains its infimum on functions satisfying (2.2) since J_l is convex and coercive in v_{xx} and since the imbedding $H^2(-\frac{1}{2}; \frac{1}{2}) \hookrightarrow C^1([-\frac{1}{2}, \frac{1}{2}])$ is compact. Recall that

$$E(l) = \min\{J_l(v) : v \text{ satisfies } (2.2)\} \,, \tag{2.3}$$

$$\tilde{E}(l) = \min\left\{J_l(v) : v \text{ satisfies (2.2), and } v(-x) = v(x) \text{ for } x \in \left(\frac{1}{2}, \frac{1}{2}\right)\right\} \,. \tag{2.4}$$

Let

$$A_0 = 2\int_{-1}^{1} W^{1/2}(t)\mathrm{d}t\ , \quad B_0 = 2\int_{-1/2}^{1/2} W^{1/2}(t)\mathrm{d}t\ .$$

Proposition 2.1 *Let* $0 < \epsilon \le 1$, $0 < l \le 1$. *Then*

$$E(l) \le \tilde{E}(l) \le A_0\epsilon + \frac{1}{12}l^3, \tag{2.5}$$

$$\tilde{E}(l) \ge E(l) \ge \min\left(B_0\epsilon, lW\left(\frac{1}{2}\right)\right), \tag{2.6}$$

$$\tilde{E}(l) \ge E(l) \ge cl^3\ , \tag{2.7}$$

where

$$c = \min\left\{\frac{1}{48}\tau^3 + (1-\tau)W\left(\frac{1}{2}\right) : \tau \in [0,1]\right\} > 0\ .$$

Remark. The lower bounds are rather crude. Under slightly more restrictive hypotheses sharper bounds are obtained in Corollary 3.2 below (see also Theorem 5.1).

Proof. We first show (2.6). Following Modica [Mo 87] we let

$$H(t) := \int_0^t W^{1/2}(\tau)\mathrm{d}\tau = \frac{1}{2}\int_{-t}^{t} W^{1/2}(\tau)\mathrm{d}\tau\ .$$

Then for any interval $(a,b) \subset (-\frac{1}{2},\frac{1}{2})$

$$\begin{aligned}\int_a^b (\epsilon^2 l^{-1} v_{xx}^2 + lW(v_x))\mathrm{d}x &\ge \int_a^b 2\epsilon|W^{1/2}(v_x)v_{xx}|\mathrm{d}x \\ &\ge \int_a^b 2\epsilon|H(v_x)_x|\mathrm{d}x \quad \ge 2\epsilon H(v_x)|_a^b\ .\end{aligned} \tag{2.8}$$

Let $\overline{x}$ be the point where v_x attains its maximum, let $M = v_x(\overline{x})$. If $M \le \frac{1}{2}$ one has $J_l(v) \ge lW(\frac{1}{2})$ by (2.1) and (H4). If $M > \frac{1}{2}$ apply (2.8) to $(-\frac{1}{2},\overline{x})$ and $(\overline{x},\frac{1}{2})$ to obtain $J_l(v) \ge 4\epsilon H(\frac{1}{2}) = B_0\epsilon$. Thus (2.6) is proved. To show (2.5) recall the notation

$$\sigma = W'$$

and let q be the solution of

$$\begin{aligned}-2q'' + \sigma(q) &= 0\ ,\\ q(0) = 0, \quad q'(0) &= (W(0))^{1/2}\ .\end{aligned}$$

One verifies easily that

$$q' = W^{1/2}(q) > 0\ , \tag{2.9}$$

and $0 \le q(z) < 1$ for $z \ge 0$. Let

$$v = \int_0^x q\left(\frac{l}{\epsilon}(\frac{1}{2} - |\xi|)\right) d\xi\ .$$

Then $v_x(\pm\frac{1}{2}) = 0,\ v(-x) = -v(x)$ and $v_x(x) \geq 0,\ |v(x)| \leq x$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$ and thus

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} l^3 v^2 \mathrm{d}x \leq \frac{1}{12} l^3 .$$

Moreover, for $(a,b) = (-\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ the inequalities in (2.8) become identities (because of (2.9) which indeed was the reason for the choice of q). Hence

$$\tilde{E}(l) \leq J_l(v) \leq 4\epsilon H\left(q\left(\frac{l}{2\epsilon}\right)\right) + \frac{1}{12} l^3 \leq A_0 \epsilon + \frac{1}{12} l^3 .$$

To show (2.7) observe that we may assume $\int_{-1/2}^{1/2} v = 0$ since $J_l(v)$ only increases if $\int_{-1/2}^{1/2} v \neq 0$. Let x_0 be a point where $v(x_0) = 0$ and let

$$m^{\pm} = \operatorname{meas}\left\{x : \pm(x - x_0) > 0,\ v_x(x) \geq \frac{1}{2}\right\}, \quad m = m^+ + m^- .$$

Note that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} l W(v_x) \geq (1-m) W(\frac{1}{2}) l \geq (1-m) W\left(\frac{1}{2}\right) l^3 . \tag{2.10}$$

Moreover using $v_x \geq 0$ one easily verifies (e.g. by rearrangement) that

$$\int_{x_0}^{\frac{1}{2}} v^2 \mathrm{d}x \geq \left(\frac{1}{2}\right)^2 \frac{(m^+)^3}{3} .$$

With the analogous estimate for $\int_{-1/2}^{x_0} v^2$ one has, using the convexity of $z \mapsto z^3$,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} l^3 v^2 \mathrm{d}x \geq \frac{1}{4}\left(\frac{(m^+)^3}{3} + \frac{(m^-)^3}{3}\right) l^3 \geq \frac{1}{48} m^3 l^3 . \tag{2.11}$$

Now (2.7) follows from (2.10) and (2.11). □

In the following we let

$$V(x) = \int_{-\frac{1}{2}}^{x} v(\xi) \mathrm{d}\xi$$

Lemma 2.2 (Euler-Lagrange equations) *There exist positive constants ϵ_0, l_0, c, C with the following property. If $\epsilon \leq \epsilon_0$, $l \leq l_0$, $\epsilon \leq cl$ and if v is a minimizer of J subject to $v_x \geq 0$ then*

$$v_x > 0 \quad on \quad \left(-\frac{1}{2}, \frac{1}{2}\right) . \tag{2.12}$$

Moreover $v \in C^4(-\frac{1}{2}, \frac{1}{2})$ and v satisfies the (integrated) Euler-Lagrange equation

$$-2\epsilon^2 l^{-2} v_{xxx} + \sigma(v_x) = 2l^2 V , \tag{2.13}$$

with boundary conditions

$$v_x\left(\pm\frac{1}{2}\right) = V\left(\pm\frac{1}{2}\right) = 0\,, \tag{2.14}$$

as well as the energy equation

$$-\epsilon^2 l^{-2} v_{xx}^2 + W(v_x) - 2l^2 V v_x + l^2 v^2 \Big|_{-\frac{1}{2}}^{x} = 0\,. \tag{2.15}$$

In addition one has the following a priori estimates

$$|v_x| \le 1 + Cl^2 \le 2\,, \quad |v| \le 1\,, \quad |V| \le 1\,. \tag{2.16}$$

Remark. We will see that without the hypothesis $\epsilon \le cl$ one can show that either (2.12) holds or $v \equiv 0$.

Proof. We first show that $v \in H^3(-\frac{1}{2}, \frac{1}{2})$ by reduction to a standard second order obstacle problem. Let

$$K = \left\{x \in \left(-\frac{1}{2}, \frac{1}{2}\right) : v_x = 0\right\}, \qquad U = \left(-\frac{1}{2}, \frac{1}{2}\right) \setminus K\,.$$

As $v \in H^2 \subset C^1$, K is (relatively) closed. Since v is minimizing one has

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (2\epsilon^2 l^{-2} v_{xx}\varphi_{xx} + \sigma(v_x)\varphi_x + 2l^2 v\varphi)\mathrm{d}x \ge 0\,,$$

for all $\varphi \in H^2(-\frac{1}{2}, \frac{1}{2})$ satisfying $\varphi_x \ge 0$ on K and $\varphi_x(\pm\frac{1}{2}) = 0$. Choosing $\varphi \equiv 1$ one sees that $V(\frac{1}{2}) = 0$ which proves (2.14). Let $w = v_x$, $f = \sigma(v_x) - 2l^2 V$. Integration by parts yields

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (2\epsilon^2 l^{-2} w_x \psi_x + f\psi)\mathrm{d}x \ge 0\,,$$

for all $\psi \in H_0^1(-\frac{1}{2}, \frac{1}{2})$ with $\psi \ge 0$ on $K = \{x \in (-\frac{1}{2}, \frac{1}{2}) : w = 0\}$.

By standard regularity results $w \in H^2(-\frac{1}{2}, \frac{1}{2})$ (see e.g. [Li 69], Chapter 2.8). Hence $v \in H^3$ and upon resubstitution

$$-2\epsilon^2 l^{-2} v_{xxx} + \sigma(v_x) - 2l^2 V = 0 \text{ on } U\,. \tag{2.17}$$

One easily deduces that v is C^4 on U.

Since $v_{xx} = 0$ on K, multiplication of (2.17) by v_{xx} yields (2.15).

We proceed with the proof of the *a priori* estimates (2.16). Assume first $\epsilon \le l$. By (2.5)

$$\sup |V| \le \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} v^2\right)^{\frac{1}{2}} \le C(l^{-1} + 1) \le Cl^{-1}\,.$$

Let $\overline{x}$ be the point where v_x attains its maximum. If $v_x(\overline{x}) = 0$ then $v_x \equiv 0$ and by (2.14) $v \equiv V \equiv 0$. If $v_x(\overline{x}) > 0$ then (2.17) applies and as $\overline{x}$ is maximizing,

$v_{xxx}(\overline{x}) \le 0$. It follows that $\sigma(v_x)(\overline{x}) \le 2l^2 V(\overline{x}) \le Cl$. Thus by hypotheses (H4) and (H5) in Sect. 1, $v_x \le 1 + Cl$ and hence $|v| \le C$, $|V| \le C$ since by (2.14)

$$\sup |v| \le \frac{1}{2} \sup |v_x|, \quad \sup |V| \le \frac{1}{8} \sup |v_x| \, . \tag{2.18}$$

Therefore $\sigma(v_x)(\overline{x}) \le Cl^2$, $v_x \le 1 + Cl^2$ and (2.16) follows.

If $\epsilon > l$ one finds that $\|v_{xx}\|_{L^2} \le C$ and hence, taking into account (2.14), $|v_x| + |v| + |V| \le C$ and the argument is finished as above.

In view of (2.17) it only remains to show

$$K = \emptyset \, .$$

Assume otherwise and let $x_0 \in K$. Then $v_x(x_0) = 0$ and since $v \in H^3$ and $v_x \ge 0$ also $v_{xx}(x_0) = 0$. Application of (2.15), (2.16) and (2.18) yields for all $x \in (-\frac{1}{2}, \frac{1}{2})$

$$\begin{aligned} W(v_x)(x) &\ge -\epsilon^2 l^{-2} v_{xx}^2(x) + W(v_x)(x) \\ &\ge -\epsilon^2 l^{-2} v_{xx}^2(x_0) + W(v_x(x_0)) - Cl^2 (\sup |v_x|)^2 \\ &\ge W(0) - Cl^2 (\sup |v_x|)^2 \, . \end{aligned}$$

If l_0 is chosen sufficiently small one deduces in particular that by (H1) and (H4) $\sup |v_x|$ is small. As $W'(0) = 0$ by (H2) one has, using (H3),

$$W(v_x)(x) \le W(0) - C(\sup |v_x|)^2 .$$

Thus $v_x \equiv 0$ if l_0 is chosen sufficiently small. Hence $J(v) = lW(0)$. This contradicts (2.5) if c and l_0 are chosen sufficiently small. Thus $K = \emptyset$ and the proof is finished. □

3. Main estimates for the minimizer of J_l

This section contains the key estimates of the paper. We show that the minimizers of J_l behave just as one would expect from formal asymptotic expansions. As before let

$$J_l(v) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \{\epsilon^2 l^{-1} v_{xx}^2 + lW(v_x) + l^3 v^2\} \, dx \, . \tag{3.1}$$

We are interested in minimizers of J_l subject to

$$v_x\left(\pm\frac{1}{2}\right) = 0, \quad v_x \ge 0 \quad \text{on} \quad \left[-\frac{1}{2}; \frac{1}{2}\right] \, . \tag{3.2}$$

Recall that $q : \mathbb{R} \to \mathbb{R}$ denotes the solution of

$$-2q'' + \sigma(q) = 0 \, , \tag{3.3}$$

$$q(0) = 0, \quad q'(0) = (W(0))^{1/2} \, . \tag{3.4}$$

One verifies easily that

$$\lim_{z \to \pm\infty} q(z) = \pm 1 \tag{3.5}$$

with exponential convergence. For the model energy $W(t) = (t^2 - 1)^2$ one has $q(t) = \tanh t$. Let

$$q_l(x) = q\left(\frac{l}{\epsilon}\left(\frac{1}{2} - |x|\right)\right) . \tag{3.6}$$

Theorem 3.1 *Assume that W satisfies hypotheses (H1) to (H5) of Sect. 1. Then there exist constants $C > 0$, $c_0 > 0$ with the following property. If*

$$l \leq c_0 , \quad \epsilon \leq c_0 \frac{l}{|\ln l|} \tag{3.7}$$

and if v is a minimizer of J_l subject to (3.2) then

$$|v_x(x) - 1| + \frac{\epsilon}{l}|v_{xx}(x)| + \left(\frac{\epsilon}{l}\right)^2 |v_{xxx}(x)| \leq Cl^2, \text{ if } |x| \leq \frac{1}{2} - C|\ln l|\frac{\epsilon}{l}, \tag{3.8}$$

$$\left|v_x(x) - q\left(\frac{l}{\epsilon}\left(\frac{1}{2} - |x|\right)\right)\right| \leq Cl^2, \text{ for } x \in \left[-\frac{1}{2}, \frac{1}{2}\right], \tag{3.9}$$

$$\left|\frac{\epsilon}{l} v_{xx}(x) + \operatorname{sgn} x \, q'\left(\frac{l}{\epsilon}\left(\frac{1}{2} - |x|\right)\right)\right| \leq Cl^2, \text{ for } x \in \left[-\frac{1}{2}, \frac{1}{2}\right] . \tag{3.10}$$

Corollary 3.2 *Under the assumptions of Theorem 3.1 one also has the following estimates.*

$$\|v - x\|_{L^2} + \|v_x - 1\|_{L^2} + \frac{\epsilon}{l}\|v_{xx}\|_{L^2} \leq C\left(\frac{\epsilon}{l} + l^2\right) , \tag{3.11}$$

$$\|\sigma(v_x)\|_{L^2} \leq C\left(\frac{\epsilon}{l} + l^2\right) , \tag{3.12}$$

$$\left|E(l) - \left(A_0\epsilon + \frac{1}{12}l^3\right)\right| \leq C\left(\frac{\epsilon}{l} + l^2\right) l^3 . \tag{3.13}$$

Proof of Theorem 3.1. The proof which is broken up into four steps relies mainly on standard techniques, in particular a "time-map" estimate (see (3.26)) and a comparison principle (see Proposition 3.3). Both arguments only apply in the region where v_x is sufficiently large (as otherwise $\sigma'(v_x)$ has the wrong sign). A perturbation argument near the boundary of $[-\frac{1}{2}, \frac{1}{2}]$ is used to compensate for this (see Claim #1 below). In the following we assume without further comment that c_0 is chosen sufficiently small and we let

$$V(x) = \int_{-\frac{1}{2}}^{x} v(\xi)d\xi .$$

Claim #1

$$\left|W(0) - \epsilon^2 l^{-2} v_{xx}^2\left(\pm\frac{1}{2}\right)\right| \leq C\left(\frac{\epsilon}{l} + l^2\right), \tag{3.14}$$

$$v_x\left(\pm\left(\frac{\epsilon}{l} - \frac{1}{2}\right)\right) \geq \frac{q(1)}{2} , \tag{3.15}$$

$$\pm v_{xx}(x) > 0 , \quad \text{if } \left|x \pm \frac{1}{2}\right| \leq \frac{\epsilon}{l} . \tag{3.16}$$

Proof. By (2.15)

$$-\epsilon^2 l^{-2} v_{xx}^2 + W(v_x) - 2l^2 V v_x + l^2 v^2 \Big|_{-\frac{1}{2}}^{x} = 0 . \tag{3.17}$$

Let $\overline{x}$ be the point where v_x attains its maximum. Then $v_{xx}(\overline{x}) = 0$, $W(v_x(\overline{x})) \geq 0$ which by (2.16) implies the lower bound for $W(0) - \epsilon^2 l^{-2} v_{xx}^2(\pm\frac{1}{2})$. Moreover (2.16) and (3.17) yield

$$W(v_x) \geq \left(W(0) - \epsilon^2 l^{-2} v_{xx}^2 \left(-\frac{1}{2}\right)\right) - Cl^2 .$$

By (2.3), (2.5)

$$A_0 \frac{\epsilon}{l} + \frac{1}{12} l^2 \geq \int_{-\frac{1}{2}}^{\frac{1}{2}} W(v_x)\, \mathrm{d}x \geq \left(W(0) - \epsilon l^{-2} v_{xx}^2 \left(-\frac{1}{2}\right)\right) - Cl^2 .$$

This proves (3.14).

To show (3.15) and (3.16) introduce the rescaled quantities

$$z = \frac{l}{\epsilon}\left(\frac{1}{2} + x\right), \tag{3.18}$$

$$p(z) = v_x(x) = v_x\left(\frac{\epsilon z}{l} - \frac{1}{2}\right) . \tag{3.19}$$

Let

$$f(z) = l^2 V\left(\frac{\epsilon z}{l} - \frac{1}{2}\right) . \tag{3.20}$$

The Euler-Lagrange equation (2.13) becomes

$$-2p'' + \sigma(p) = f, \tag{3.21}$$

with initial condition

$$p(0) = 0,\ p'(0) = \epsilon l^{-1} v_{xx}\left(-\frac{1}{2}\right) . \tag{3.22}$$

Now by (2.16)

$$|f| \leq Cl^2 , \tag{3.23}$$

and by (3.14)

$$\left|p'(0) - W^{1/2}(0)\right| \leq C\left(l^2 + \frac{\epsilon}{l}\right) .$$

Comparison with (3.3), (3.4) gives

$$|p(z) - q(z)| + |p'(z) - q'(z)| \leq C\left(l^2 + \frac{\epsilon}{l}\right) ,$$

for $z \in [0, 1]$ and (3.15), (3.16) follow since $q(1) > 0$, $q' \geq c > 0$ on $[0, 1]$.

Claim #2

(i) There exists a constant $C_1 > 0$ with the following property. If v_x has an interior minimum at $\overline{x}$ then

$$v_x(\overline{x}) \geq 1 - C_1 l^2 .$$

(ii) There exist constants $C_2, C_3 > 0$ and points $x_1, x_3 \in [-\frac{1}{2}, \frac{1}{2}]$ such that

$$\begin{aligned} v_x(x_i) &\geq 1 - C_2 l^2 , \quad i = 1, 2 , \\ x_1 &\leq -\frac{1}{2} + C_3 |\ln l| \frac{\epsilon}{l} , \\ x_2 &\geq \ \frac{1}{2} - C_3 |\ln l| \frac{\epsilon}{l} . \end{aligned}$$

Moreover v_x is monotone increasing (decreasing) on $(-\frac{1}{2}; x_1)$ (resp. $(x_2; \frac{1}{2})$).

(iii) There exists a constant C such that

$$v_x(x) \geq 1 - Cl^2 \quad \text{if } |x| \leq \frac{1}{2} - C|\ln l| \frac{\epsilon}{l} . \tag{3.24}$$

Proof. Assertion (iii) follows immediately from (i) and (ii) if one chooses $C = \max(C_1, C_2, C_3)$. To prove (i) note that $v_{xxx}(\overline{x}) \geq 0$ as v_x has a minimum at $\overline{x}$. From (2.13) and (2.16) one obtains

$$\sigma(v_x(\overline{x})) \geq 2l^2 V \geq -2l^2 .$$

From the behaviour of σ we see that either $v_x(\overline{x}) \geq 1 - Cl^2$ in which case we are done or $v_x(\overline{x}) \leq Cl^2$. Assume the latter. Since $v_{xx}(\overline{x}) = 0$ equation (2.15) yields

$$\left| W(v_x(\overline{x})) - \left(W(0) - \epsilon^2 l^{-2} v_{xx}^2 \left(-\frac{1}{2}\right)\right) \right| \leq Cl^2$$

and thus

$$\left| \epsilon^2 l^{-2} v_{xx}^2 \left(-\frac{1}{2}\right) \right| \leq Cl^2 ,$$

contradicting (3.14). Thus (i) is proved.

To prove (ii) it suffices to construct x_1, as x_2 can be obtained in an analogous fashion. Set $\delta_0 = \frac{q(1)}{2}$ and choose $x_0 \in [-\frac{1}{2}, -\frac{1}{2} + \frac{\epsilon}{l}]$ with $v_x(x_0) = \delta_0$ (cf. (3.15)). Let $\delta_1 = 1 - C_2 l^2$ with $C_2 > C_1$ sufficiently large. Then (cf.(H4)) for sufficiently small l

$$\sigma(t) \leq -4l^2 \quad \text{for } t \in [\delta_0, \delta_1].$$

Let (x_0, x_1) be the maximal interval on which

$$v_x(x) \in (\delta_0, \delta_1) .$$

Then by (2.13) and (2.16)

$$2\epsilon^2 l^{-2} v_{xxx} = \sigma(v_x) - 2l^2 V \leq \frac{1}{2}\sigma(v_x) \quad \text{on } (x_0, x_1) . \tag{3.25}$$

1^{st} *case*. Assume that v_x does not have a local maximum in (x_0, x_1). Then $v_x(x_1) = \delta_1$. Moreover multiplication of (3.25) by v_{xx} and integration gives

$$\left(\epsilon^2 l^{-2} v_{xx}^2 - \frac{1}{2} W(v_x)\right)(x) \geq \left(\epsilon^2 l^{-2} v_{xx}^2 - \frac{1}{2} W(v_x)\right)(x_1)$$
$$\geq -\frac{1}{2} W(v_x)(x_1) .$$

Using the change of variables (3.18), (3.19) this becomes

$$p_z^2(z) \geq \frac{1}{2}(W(p(z)) - W(\delta_1)) .$$

Thus

$$z_1 - z_0 \leq \int_{\delta_0}^{\delta_1} \frac{\sqrt{2}dy}{(W(y) - W(\delta_1))^{1/2}} . \tag{3.26}$$

The integral is easily estimated (see below) and one has

$$z_1 - z_0 \leq C \ln(1 - \delta_1)^{-1} \leq C |\ln l| , \tag{3.27}$$

so that

$$x_1 + \frac{1}{2} \leq \frac{\epsilon}{l}(z_1 - z_0) + \frac{\epsilon}{l} \leq C_3 |\ln l| \frac{\epsilon}{l}$$

and

$$v_x(x_1) = \delta_1 = 1 - C_2 l^2 .$$

To verify (3.27) choose $\delta \in (0, 1)$ such that $W'' \geq c > 0$ on $[\delta, 1]$. Since $W'(1) = 0$

$$-W'(\xi) \geq c\,(1 - \xi) \qquad \text{for } \xi \in [\delta, 1) ,$$
$$W(y) - W(\delta_1) = -\int_y^{\delta_1} W'(\xi) d\xi \geq \frac{c}{2}[(1 - y)^2 - (1 - \delta_1)^2] \quad \text{for } y \in [\delta, \delta_1].$$

Thus

$$\begin{aligned}\int_{\delta}^{\delta_1} \frac{dy}{(W(y) - W(\delta_1))^{1/2}} &\leq \left(\frac{2}{c}\right)^{1/2} \int_{1-\delta_1}^{1-\delta} \frac{dz}{[z^2 - (1 - \delta_1)^2]^{1/2}} \\ &= \left(\frac{2}{c}\right)^{1/2} \int_1^{\frac{1-\delta}{1-\delta_1}} \frac{d\xi}{(\xi^2 - 1)^{1/2}} \\ &= \left(\frac{2}{c}\right)^{1/2} \operatorname{arccosh}\zeta \Big|_{\zeta=1}^{\frac{1-\delta}{1-\delta_1}} \\ &\leq \left(\frac{2}{c}\right)^{1/2} \operatorname{arccosh}\frac{1}{1 - \delta_1} \\ &\leq C \ln(1 - \delta_1)^{-1} .\end{aligned}$$

Moreover

$$\int_{\delta_0}^{\delta} \frac{dy}{(W(y) - W(\delta_1))^{1/2}} \leq C$$

and (3.27) follows.

2^{nd} *case.* Assume that v_x has a local maximum in (x_0, x_1). We claim that $v_x(x_1) = \delta_0$. Indeed either $v_x(x_1) = \delta_1$ or $v_x(x_1) = \delta_0$ by maximality of (x_0, x_1). But if $v_x(x_1) =$

δ_1, v_x would also attain an interior minimum $\leq \delta_1 < 1 - Cl^2$ which contradicts (i). Thus $v_x(x_1) = \delta_0$. By the same argument v_x can have at most one local maximum in (x_0, x_1). Assume this was attained at $x = x_2$. Applying the consideration in the first case to the intervals (x_0, x_2) and (x_2, x_1) one sees that

$$x_1 - x_0 \leq C|\ln l|\frac{\epsilon}{l} .$$

Let x_3 be a point in $[\frac{1}{2} - \frac{\epsilon}{l}, \frac{1}{2}]$ with $v_x(x_3) = \delta_0$, $v_{xx}(x_3) < 0$ (cf. (3.15), (3.16)). Then $x_1 \leq x_3$. By (i) indeed $x_1 = x_3$ since otherwise v_x would have a local minimum $\leq \delta_0$ on (x_1, x_3). Thus

$$1 - 2\frac{\epsilon}{l} \leq x_1 - x_0 \leq C|\ln l|\frac{\epsilon}{l} .$$

This contradicts the choice of ϵ and l. Thus the 2^{nd} case cannot occur and (ii) is proved.

Claim #3 There exists $C > 0$ such that for $|x| \leq \frac{1}{2} - C|\ln l|\frac{\epsilon}{l}$

$$|v_x(x) - 1| + \epsilon l^{-1}|v_{xx}(x)| + \epsilon^2 l^{-2}[v_{xxx}(x)| \leq Cl^2 . \tag{3.28}$$

Proof. From (2.16) and (3.24) one has $|v_x(x) - 1| \leq Cl^2$. This implies the estimate on v_{xxx} by the Euler-Lagrange equation (2.13) and (2.16). Finally the estimate on v_{xx} follows by interpolation. Indeed, using the rescaling (3.18), (3.19) one has (for $t \in [a, b]$; $a = C|\ln l|$, $b = l/\epsilon - C|\ln l|$)

$$|p(z) - 1| \leq Cl^2 , \quad |p_{zz}(z)| \leq Cl^2 .$$

Thus for $z \in [a + 1, b]$

$$\left|p_z(z) - \int_{z-1}^{z} p_z(\zeta)d\zeta\right| \leq \max_{[a,b]} |p_{zz}| \leq Cl^2 ,$$

$$\left|\int_{z-1}^{z} p_z(\zeta)d\zeta\right| \leq \left|(p(\zeta) - 1)|_{\zeta=z-1}^{z}\right| \leq Cl^2 .$$

A similar estimate holds for $z \in [a, b-1]$. Thus $|p_z| \leq C$ on $[a, b]$ and (3.28) follows.

Claim #4 For all $x \in [-\frac{1}{2}, \frac{1}{2}]$,

$$\left|v_x(x) - q\left(\frac{l}{\epsilon}\left(\frac{1}{2} - |x|\right)\right)\right| \leq Cl^2,$$

$$\left|\frac{\epsilon}{l}v_{xx}(x) + (\text{sgn}x)q'\left(\frac{l}{\epsilon}\left(\frac{1}{2} - |x|\right)\right)\right| \leq Cl^2 .$$

Proof. We prove the estimates for $x \in [-\frac{1}{2}, 0]$ the other case being analogous. With the rescaling (3.18), (3.19) we have to show

$$|p(z) - q(z)| + |p'(z) - q'(z)| \leq Cl^2 \quad \text{for } z \in \left[0, \frac{l}{2\epsilon}\right] .$$

Concentrating on the first term for the time being we choose $\eta > 0$ such that $\sigma'(t) \geq c$ for $t \in [1 - 2\eta, 1 + 2\eta]$. Recall that q given by (3.3), (3.4) is strictly increasing with $\lim_{t\to\infty} q(z) = 1$ and define $\overline{z}$ by

$$q(\overline{z}) = 1 - \eta \, .$$

Improving (3.14) we first show that

$$|p_z(0) - q_z(0)| \leq Cl^2 \, . \tag{3.29}$$

To this end let $\overline{x}$ be the point where v_x attains its maximum. Then $v_{xx}(\overline{x}) = 0$ and, by Claim #2(ii), $v_x(\overline{x}) \geq 1 - Cl^2$. In view of (2.15) and (2.16) this gives

$$|\epsilon^2 l^{-2} v_{xx}^2 \left(-\frac{1}{2}\right) - W(0)| \leq Cl^2 \, ,$$

and hence (3.29). Now compare (3.3), (3.4) with (3.21), (3.22). Taking into account (3.23) and (3.29), we deduce that

$$|p(z) - q(z)| \leq Cl^2 \, , \quad \text{for } z \in [0, \overline{z}]. \tag{3.30}$$

Let C' be large enough and let $z_1 = C'|\ln l|$. Then by (Claim #3) and exponential convergence of $q(z)$ as $z \to \infty$

$$|p(z) - q(z)| \leq Cl^2 \quad \text{for } z \in \left[z_1, \frac{l}{2\epsilon}\right] \, . \tag{3.31}$$

In addition we claim that

$$p(z) \in [1 - 2\eta; 1 + 2\eta], \quad q(z) \in [1 - 2\eta; 1 + 2\eta], \quad \text{if } z \in [\overline{z}; z_1] \, .$$

For $q(z)$ this follows from monotonicity. Regarding $p(z)$ note that by (3.30), $p(\overline{z}) \geq 1 - 2\eta$. By (3.31) and Claim #2(i), thus $p(z) \geq 1 - 2\eta$ on $[\overline{z}, z_1]$. The upper bound $p(z) \leq 1 + 2\eta$ follows from (2.16).

As $\sigma'(t) \geq c$ for $t \in [1 - 2\eta, 1 + 2\eta]$ and $|p(z) - q(z)| \leq Cl^2$ for $z \in \{\overline{z}, z_1\}$ a standard comparison argument (see Proposition 3.3 below) in connection with (3.23) gives $|p(z) - q(z)| \leq Cl^2$ on $[\overline{z}, z_1]$.

Summarizing we have

$$|p(z) - q(z)| \leq Cl^2 \text{ on } \left[0, \frac{l}{2\epsilon}\right] \, .$$

Now (3.3), (3.21) and (3.23) yield

$$|p''(z) - q''(z)| \leq Cl^2 \text{ on } \left[0, \frac{l}{2\epsilon}\right] \, ,$$

and hence

$$|p'(z) - q'(z)| \leq Cl^2 \text{ on } \left[0, \frac{l}{2\epsilon}\right]$$

by interpolation (cf. the proof of Claim #3).

Proof of the Theorem. This follows from Claims #3 and #4. □

Proposition 3.3 *Let* $\sigma : \mathbb{R} \to \mathbb{R}$ *be a* C^1 *function with* $0 < \alpha \leq \sigma' \leq \beta$, *let* $[a, b] \subset \mathbb{R}$, *let* $f \in L^\infty(a, b)$ *and let* y, y_0 *be solutions of*

$$-y'' + \sigma(y) = f\ ,$$
$$-y_0'' + \sigma(y_0) = 0\ ,$$

with

$$|(y - y_0)(a)| \le \delta\ , \quad |(y - y_0)(b)| \le \delta\ .$$

Then

$$\|y - y_0\|_\infty \le \frac{1}{\alpha}\|f\|_\infty + \delta\ .$$

Proof. This follows from the maximum principle. Indeed, let $y_+ = y_0 + \frac{1}{\alpha}\|f\|_\infty + \delta$. Then $y_+(x) \ge y(x)$ if $x \in \{a, b\}$. Since $\sigma' \ge \alpha$ moreover $\sigma(y_+) \ge \sigma(y_0) + \|f\|_\infty$. Thus

$$-y_+'' + \sigma(y_+) \ge -y_0'' + \sigma(y_0) + \|f\|_\infty \ge -y + \sigma(y)$$

and hence $y_+ \ge y$ on $[a, b]$ by the maximum principle. The lower bound is shown analogously.

Proof of Corollary 3.2. Since $\int_{-\frac{1}{2}}^{\frac{1}{2}}(v - x)\,\mathrm{d}x = 0$ (cf. (2.14)) one has $\|v - x\|_{L^2} \le \|v_x - 1\|_{L^2}$. To estimate $\|v_x - 1\|_{L^2}$ we first apply (3.9). It then suffices to show that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|1 - q\left(\frac{l}{\epsilon}\left(\frac{1}{2} - |x|\right)\right)\right|^2 \mathrm{d}x \le C\frac{\epsilon}{l}\ .$$

After splitting the integral into two parts this is easily verified by use of the substitutions $z = \frac{l}{\epsilon}(\frac{1}{2} \mp x)$ for $\pm x > 0$ and the exponential convergence of q. The estimate for $\|v_{xx}\|_{L^2}$ is proved similarly using (3.10). Inequality (3.12) follows from (3.11) since, by (2.16), $0 \le v_x \le 2$, so that $\sigma(v_x) \le C|v_x - 1|$. Finally (3.11) implies that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} v^2\,\mathrm{d}x \ge \frac{1}{12} - C\left(\frac{\epsilon}{l} + l^2\right)\ .$$

By Claim #2(iii) in the proof of Theorem 3.1

$$m := \max\{v_x(x) : x \in \left[\frac{1}{2}; \frac{1}{2}\right]\} \ge 1 - Cl^2.$$

Thus by (2.8)

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (\epsilon^2 l^{-1} v_{xx}^2 + lW(v_x)) \ge 4\epsilon H(m) \ge 4\epsilon\left[H(1) - C(1 - m)^2\right]$$
$$= \epsilon A_0(1 - Cl^2),$$

since $H'(1) = W^{1/2}(1) = 0$. Thus $E(l) = J_l(v) \ge \epsilon A_0 + \frac{1}{12}l^3 - C(\frac{\epsilon}{l} + l^2)l^3$ and (3.13) follows from (2.5). □

4. Stability, uniqueness and eigenvalues

In this section we derive a lower bound for the first eigenvalue of the linearized Euler-Lagrange operator associated with the functional

$$J_l(v) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \epsilon^2 l^{-1} v_{xx}^2 + lW(v_x) + l^3 v^2 \, dx \, .$$

The bound is then used to establish uniqueness of minimizers. In the following section it will be employed to study the dependence of minimizers on l. Let

$$X = \left\{ v \in H^2\left(-\frac{1}{2}; \frac{1}{2}\right) : v_x(\pm\frac{1}{2}) = 0 \right\} , \tag{4.1}$$

recall the definition of q from (3.3), (3.4) and define

$$q_l(x) = q\left(\frac{l}{\epsilon}\left(\frac{1}{2} - |x|\right)\right) . \tag{4.2}$$

Lemma 4.1 *There exist positive constants c_0, λ_0 with the following property. If $\epsilon/l \leq c_0$ then*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (2\epsilon^2 l^{-2} \varphi_{xx}^2 + \sigma'(q_l)\varphi_x^2) \, dx \geq \lambda_0 \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi_x^2 \, dx \, , \quad \textit{for all } \varphi \in X \, .$$

We first indicate some consequences of the lemma postponing its proof until the end of this section.

Theorem 4.2 *There exist positive constants c_0, c_1, c_2 with the following properties. If*

$$l \leq c_0 \, , \quad \epsilon \leq c_0 \frac{l}{|\ln l|}$$

then one has

(i) J_l has a unique minimizer v_l in X and $v_l(-x) = -v_l(x)$.

(ii) If $v \in X$ is a weak solution of the Euler-Lagrange equations, i.e.

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} 2\epsilon^2 l^{-2} v_{xx}\varphi_{xx} + \sigma(v_x)\varphi_x + 2l^2 v\varphi = 0 \, , \quad \textit{for all } \varphi \in X, \tag{4.3}$$

if

$$\sup |v_x - q_l| < c_1 \tag{4.4}$$

and if v_l is a minimizer of J_l in X then $v = v_l$.

(iii) For all $\varphi \in X$ one has

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (2\epsilon^2 l^{-2} \varphi_{xx}^2 + \sigma'(v_{lx})\varphi_x^2 + 2l^2\varphi^2) dx$$
$$\geq c_2 \left(\epsilon^2 l^{-2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi_{xx}^2 dx + \lambda_0 \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi_x^2 dx + l^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi^2 dx \right) , \tag{4.5}$$

where λ_0 is the constant in Lemma 4.1 .

Proof. First note that (ii) implies (i). Indeed, if v is a minimizer of J_l, then v satisfies (4.3) and, by Theorem 3.1, also (4.4) for a suitable choice of c_0 and c_1. Thus (ii) implies uniqueness of minimizers. Moreover the minimizer has to be antisymmetric since $J(v_l) = J_l(\tilde{v}_l)$ where $\tilde{v}_l(x) = -v_l(-x)$.

To prove (ii) let $\varphi = v_l - v$. Using Taylor expansion around v one finds

$$\begin{aligned}\frac{1}{l}(J_l(v_l) - J_l(v)) = &\int_{-\frac{1}{2}}^{\frac{1}{2}} 2\epsilon^2 l^2 v_{xx}\varphi_{xx} + \sigma(v_x)\varphi_x + 2l^2 v\varphi \\ &+ \int_{-\frac{1}{2}}^{\frac{1}{2}} (\epsilon^2 l^{-2}\varphi_{xx}^2 + l^2\varphi^2)\mathrm{d}x \\ &+ \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_0^1 (1-s)\sigma'(v_x + s\varphi_x)\varphi_x^2 \,\mathrm{d}s\,\mathrm{d}x\,.\end{aligned}$$

The first integral vanishes since v solves the Euler-Lagrange equations. Moreover (3.9) and (4.4) imply that $\sigma'(v_x + s\varphi_x) \geq \sigma'(q_l) - \lambda_0/4$ if c_1 is chosen sufficiently small. Thus by Lemma 4.1

$$\begin{aligned}0 &\geq \frac{1}{l}(J_l(v_l) - J_l(v)) \\ &\geq \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\epsilon^2 l^{-2}\varphi_{xx}^2 + \frac{1}{2}(\sigma'(q_l) - \frac{\lambda_0}{4})\varphi_x^2 + l^2\varphi^2\right)\mathrm{d}x \\ &\geq \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2}\epsilon^2 l^{-2}\varphi_{xx}^2 + \frac{\lambda_0}{8}\varphi_x^2 + l^2\varphi^2\right)\mathrm{d}x\,.\end{aligned}$$

Hence $\varphi \equiv 0$ and (ii) follows. Finally (iii) is an easy consequence of (3.9) and Lemma 4.1.

Proof of Lemma 4.1. Let $L = \frac{l}{\epsilon}$. With the rescaling $\psi(x) = \varphi_x(\frac{\epsilon}{l}x - \frac{1}{2})$ we have to show that $\lambda_L \geq \lambda_0 > 0$ for $L \geq c_0^{-1}$ where

$$\lambda_L := \inf\left\{\int_0^L 2\psi_x^2 + \sigma'(q)\psi^2 : \psi \in H_0^1(0,L), \|\psi\|_{L^2} = 1\right\}\,.$$

Extending ψ by zero one sees that λ_L is decreasing in L. It thus suffices to show that

$$\lambda_\infty > 0\,.$$

Note that $q(z) \to 1$ exponentially as $z \to \infty$. If $\lambda_\infty \geq \sigma'(1)$ there is nothing to show. Assume $\lambda_\infty < \sigma'(1)$. Then the infimum is attained as for a minimizing sequence "no mass can escape to infinity" (see e.g. Agmon [Ag 82]). It thus suffices to show

$$\int_0^\infty 2\psi_x^2 + \sigma'(q)\psi^2\mathrm{d}x > 0 \quad \text{if } \psi \in H^1(0,\infty), \quad \psi(0) = 0, \quad \psi \not\equiv 0\,. \tag{4.6}$$

Dropping the condition $\psi(0) = 0$ we define

$$\mu_\infty := \inf\left\{\int_0^\infty 2\psi_x^2 + \sigma'(q)\psi^2 : \psi \in H^1(0,\infty),\ \|\psi\|_{L^2} = 1\right\}\,.$$

I claim that $\mu_\infty = 0$. The choice $\psi_0 = q_x$ in connection with (3.3) shows that $\mu_\infty \le 0 < \sigma'(1)$. Thus the infimum is attained and μ_∞ is the lowest eigenvalue of

$$-2\psi_{xx} + \sigma'(q)\psi = \mu\psi \ .$$

The corresponding eigenspace is one-dimensional and the eigenfunction does not change sign (see [RS 78], Theorem XIII.44 for a very general result in this direction; for the case at hand it suffices to note that with ψ also $|\psi|$ is minimizing so that by regularity $\psi(x_0) = 0$ implies $\psi_x(x_0) = 0$ and hence $\psi \equiv 0$ by uniqueness for ode's). By L^2- orthogonality all the other eigenfunctions must change sign. Now $q_x > 0$ on $(0, \infty)$ and q_x is an eigenfunction for $\mu = 0$. Hence $\mu_\infty = 0$ and the infimum is attained for $\psi = \pm q_x / \|q_x\|_{L^2}$. As $q_x(0) \neq 0$ uniqueness of ψ implies (4.6). □

5. Estimates of $E''(l)$ via the implicit function theorem

We show that the minimizers v_l of J_l depend differentiably on l and that $\frac{d}{dl} v_l$ is the unique solution of a problem involving the linearized Euler-Lagrange operator. We subsequently obtain an approximation for $\frac{d}{dl} v_l$ which is used to estimate $E''(l)$ (recall the definition of E and $\tilde{E}$ from (2.3), (2.4)). The main result is

Theorem 5.1 *Let $E(l)$, $\tilde{E}(l)$ be given by (2.3), (2.4). There exist positive constants c_0, C with the following property. If*

$$l \le c_0 \ , \quad \epsilon \le c_0 \frac{l}{|\ln l|} \ , \tag{5.1}$$

then

$$\tilde{E}(l) = E(l) \ , \tag{5.2}$$

$$\left|E(l) - \left(\epsilon A_0 + \frac{1}{12} l^3\right)\right| \le C \left(\frac{\epsilon}{l} + l^2\right) l^3 \ , \tag{5.3}$$

$$\left|E'(l) - \frac{1}{4} l^2\right| \le C \left(\frac{\epsilon}{l} + l^2\right) l^2 \ , \tag{5.4}$$

$$\left|E''(l) - \frac{1}{2} l\right| \le C \left(\frac{\epsilon}{l} + l^2\right) l \ . \tag{5.5}$$

We first show that the minimizer v_l (which is unique by Theorem 4.2) depends differentiably on l. As before let X be the Hilbert space

$$X = \left\{ v \in H^2 \left(-\frac{1}{2}; \frac{1}{2}\right) : v_x(\pm\frac{1}{2}) = 0 \right\} \subset L^2 \left(-\frac{1}{2}; \frac{1}{2}\right) \ , \tag{5.6}$$

equipped with $\|\cdot\|_X = \|\cdot\|_{H^2}$ and let X' be its dual, i.e. the completion of $L^2(-\frac{1}{2}; \frac{1}{2})$ under the norm

$$\|f\|_{X'} = \inf \left\{ \int_{-\frac{1}{2}}^{\frac{1}{2}} f\varphi : \varphi \in X, \|\varphi\|_X = 1 \right\} \ .$$

Let ϵ be sufficiently small and let

$$U = \left\{ l \in (0,1) : l < c_0,\ \frac{l}{|\ln l|} > c_0^{-1}\epsilon \right\}, \tag{5.7}$$

where c_0 is chosen so small that the conclusions of Theorems 3.1 and 4.2 apply. Let v_l be the (unique) minimizer of J_l in X and let $L_l : X \to X'$ denote the linearized (around v_l) Euler-Lagrange operator, i.e. for all $w,\ \varphi \in X$ let

$$\langle L_l w, \varphi \rangle_{X',X} = \int_{-\frac{1}{2}}^{\frac{1}{2}} (2\epsilon^2 l^{-2} w_{xx}\varphi_{xx} + \sigma'(v_{lx}) w_x \varphi_x + 2l^2 w\varphi) \mathrm{d}x\ , \tag{5.8}$$

and let finally $f_l \in X'$ be given by

$$\langle f_l, \varphi \rangle_{X',X} = \int_{-\frac{1}{2}}^{\frac{1}{2}} (-4\epsilon^2 l^{-3} v_{lxx}\varphi_{xx} + 4l v_l \varphi) \mathrm{d}x\ . \tag{5.9}$$

Theorem 5.2 *The map $\Phi : U \to X$ given by $\Phi(l) = v_l$ satisfies $\Phi \in C^1(U; X)$ and the derivative*

$$w_l = \frac{d}{dl} v_l \in X$$

is given as the unique solution of the equation

$$L_l w_l + f_l = 0\ . \tag{5.10}$$

The proof of the theorem will follow from the implicit function theorem which for our purposes may be stated as follows (see e.g. [MH 83], [Ze 86]).

Implicit function theorem *Let X, Y be Banach spaces, let $U \subset \mathbb{R}$ be open and let $A : U \times X \to Y$ be a C^1 map. Let $(l_0, v_0) \in U \times X$ be such that*

$$A(l_0,\ v_0) = 0\ .$$

and assume that the linear operator $L_0 = \frac{\partial A}{\partial v}|_{(l_0,v_0)} : X \to Y$ is bounded and invertible with bounded inverse. Then there exists a neighbourhood U_0 of l_0 and a C^1 map $\phi : U_0 \to X$ such that $\phi(l_0) = v_0$,

$$A(l, \phi(l)) = 0 \textit{ for } l \in U_0$$

and such that $w = \frac{d}{dl}|_{l=l_0}$ is the unique solution of

$$L_0 w + \frac{\partial A}{\partial l}|_{(l_0,v_0)} = 0\ .$$

Proof of Theorem 5.2 . Let X, U as in (5.6), (5.7), let $Y = X'$ and let $A : U \times X \to X'$ be the Euler-Lagrange operator, i.e. for all $\varphi \in X$

$$\langle A(l,v), \varphi \rangle_{X',X} = \int_{-\frac{1}{2}}^{\frac{1}{2}} (2\epsilon^2 l^{-2} v_{xx}^2 \varphi_{xx} + \sigma(v_x)\varphi_x + 2l^2 v\varphi) \mathrm{d}x\ .$$

One easily verifies that $A \in C^1(U \times X, Y)$ – it suffices to note that the map $v \mapsto \sigma(v_x)$ is a C^1 map from C^1 (and hence H^2) to C^0, (see e.g. [MH 83], Chapter 3, Theorem 1.13). Moreover for the minimizer v_l

$$A(l, v_l) = 0\,,$$
$$\frac{\partial A}{\partial v}|_{(l,v_l)} = L_l \quad , \quad \frac{\partial A}{\partial l}|_{(l,v)} = f_l\,.$$

The quadratic form $\varphi \mapsto \langle L_l\varphi, \varphi\rangle$ is bounded from above and by (4.5) one also has the lower bound

$$\langle L_l\varphi, \varphi\rangle \geq c_{\epsilon,l}\|\varphi\|^2_{H^2}\,.$$

In view of the Lax-Milgram theorem $L_l : X \to X'$ is invertible with bounded inverse. Let $l_0 \in U$. By the implicit function theorem there is a neighbourhood U_0 of l_0 and a C^1 function $\phi : U_0 \to X$ such that $\phi(l_0) = v_0$ and

$$A(l, \phi(l)) = 0\,.$$

Let $\tilde{v}_l = \phi(l)$. It only remains to show that $\tilde{v}_l = v_l$, i.e. that the $\tilde{v}_l$ are actually minimizers of J_l. By the Sobolev embedding theorem

$$\sup|\tilde{v}_{lx} - v_{l_0x}| \leq C\|\tilde{v}_l - v_{l_0}\|_X \to 0 \quad \text{as } l \to l_0\,,$$

and by direct inspection of (4.2)

$$\sup|q_l - q_{l_0}| \to 0 \quad \text{as } l \to l_0\,.$$

Combining this with (3.9) (applied to $l = l_0$) one finds that

$$\sup|\tilde{v}_{lx} - q_l| < c_1\,,$$

if $|l - l_0|$ is sufficiently small. Hence $\tilde{v}_l = v_l$ by Theorem 4.2(ii) and the theorem is proved. □

In order to compute $E''(l)$ we need to compute a sufficiently good approximation of $w_l = \frac{d}{dl}v_l$. Formal differentiation of (3.9) suggests that

$$(w_l)_x \approx \frac{1}{\epsilon}\left(\frac{1}{2} - |x|\right) q'\left(\frac{l}{\epsilon}\left(\frac{1}{2} - |x|\right)\right) \approx \frac{1}{l}(x - \frac{\text{sgn}x}{2})v_{xx}(x).$$

To avoid the nondifferentiability of the sgn function we choose $\psi \in C^2([-\frac{1}{2}, \frac{1}{2}])$ with $\psi(-x) = -\psi(x)$ for all $x \in [-\frac{1}{2}, \frac{1}{2}]$, $\psi(\frac{1}{2}) = \frac{1}{2}$ and $\psi'(x) = \psi''(x) = 0$ if $|x| > \frac{1}{4}$. From now on we will write w, v and f instead of w_l, v_l and f_l (see (5.9)) and we let

$$w^0(x) = l^{-1}\left[(x - \psi(x))v_x - \int_0^x (1 - \psi'(\xi))v_x(\xi)d\xi\right]\,, \tag{5.11}$$

so that

$$w^0_x(x) = l^{-1}(x - \psi(x))v_{xx}\,, \tag{5.12}$$
$$w^0_{xx}(x) = l^{-1}[(1 - \psi'(x))v_{xx} + (x - \psi(x))v_{xxx}]\,. \tag{5.13}$$

Lemma 5.3 *Assume that (5.1) holds with sufficiently small c_0 and let w be the solution of (5.10). Then*

$$\epsilon l^{-1}\|(w - w^0)_{xx}\|_{L^2} + \|(w - w^0)_x\|_{L^2} + \|w - w^0\|_{L^2} \leq Cl\,. \tag{5.14}$$

Proof. Let

$$W^0(x) = \int_{-\frac{1}{2}}^{x} w^0(\xi)\mathrm{d}\xi \,; \quad V(x) = \int_{-\frac{1}{2}}^{x} v(\xi)\mathrm{d}\xi \,.$$

Note that $W^0(\frac{1}{2}) = W^0(-\frac{1}{2}) = 0$ since w^0 is antisymmetric by Theorem 4.2(i). Moreover

$$|W^0| \le Cl^{-1} \,. \tag{5.15}$$

The definition of w^0 and integration by parts yield

$$\begin{aligned} T &:= \int_{-\frac{1}{2}}^{\frac{1}{2}} (2\epsilon^2 l^{-2} w^0_{xx}\varphi_{xx} + \sigma'(v_x) w^0_x \varphi_x)\mathrm{d}x \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \{2\epsilon^2 l^{-3}(1-\psi')v_{xx}\varphi_{xx} + 2\epsilon^2 l^{-3}(x-\psi(x))v_{xxx}\varphi_{xx} \\ &\quad + l^{-1}(x-\psi(x))\sigma'(v_x)\varphi_x\}\mathrm{d}x \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \{2\epsilon^2 l^{-3}(1-\psi')v_{xx}\varphi_{xx} - 2\epsilon^2 l^{-3}(1-\psi')v_{xxx}\varphi_x \\ &\quad - 2\epsilon^2 l^{-3}(x-\psi(x))v_{xxxx}\varphi_x + l^{-1}(x-\psi(x))\sigma'(v_x)\varphi_x\}\mathrm{d}x \,. \end{aligned}$$

The Euler-Lagrange equation for v (differentiate (2.13)) and integration by parts in the second term give

$$\begin{aligned} T = \int_{-\frac{1}{2}}^{\frac{1}{2}} &\{4\epsilon^2 l^{-3}(1-\psi')v_{xx}\varphi_{xx} - 2\epsilon^2 l^{-3}\psi'' v_{xx}\varphi_x \\ &+ 2l(x-\psi(x))v\varphi_x\}\mathrm{d}x \,. \end{aligned}$$

Hence by (5.8), (5.9) (dropping the subscript l)

$$\langle Lw_0 + f, \varphi\rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} g\varphi_x \,,$$

where

$$\begin{aligned} g &= \left(4\epsilon^2 l^{-3}\psi' v_{xx}\right)_x - 2\epsilon^2 l^{-3}\psi'' v_{xx} + 2l\left(x-\psi(x)\right)v - 2l^2 W^0 - 4lV \\ &= 4\epsilon^2 l^{-3}\psi' v_{xxx} + 2\epsilon^2 l^{-2}\psi'' v_{xx} + 2l(x-\psi(x))v - 2l^2 W^0 - 4lV \,. \end{aligned}$$

Since $\langle Lw + f, \varphi\rangle = 0$ by Theorem 5.2

$$\langle L(w-w^0), \varphi\rangle = -\int_{-\frac{1}{2}}^{\frac{1}{2}} g\varphi_x \,.$$

Now $\psi'(x) = 0$ for $|x| > \frac{1}{4}$ and therefore by (2.13), (2.16), (3.9), (3.10) and (5.15)

$$|g| \le Cl \,.$$

Choosing $\varphi = (w - w_0)$ and applying Theorem 4.2(iii) it follows that

$$\epsilon^2 l^{-2}\|(w-w_0)_{xx}\|_{L^2}^2 + \|(w-w_0)_x\|_{L^2}^2 \le Cl\|(w-w_0)_x\|_{L^2} \le \frac{1}{2}\|(w-w_0)_x\|_{L^2}^2 + Cl^2 \,.$$

This gives the estimates for $(w-w_0)_{xx}$ and $(w-w_0)_x$. Finally choosing $\varphi = \text{const}$ one finds that $w-w_0$ has average zero on $(-\frac{1}{2},\frac{1}{2})$. Hence

$$\|w-w_0\|_{L^2} \le C\|(w-w_0)_x\|_{L^2}$$

and the proof is finished. □

Proof of Theorem 5.1. Identity (5.2) follows from Theorem 4.2(i) while the estimate (5.3) is contained in Corollary 3.2. To prove the remaining inequalities let

$$w_l = \frac{d}{dl}v_l \,.$$

Then

$$E'(l) = \frac{d}{dl}J_l(v_l) = (\frac{\partial}{\partial l}J_l)(v_l) + (\delta J_l)(v_l)w_l = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(-\epsilon^2 l^{-2} v_{lxx}^2 + W(v_{lx}) + 3l^2 v_l^2\right)\,\mathrm{d}x,$$

as δJ_l vanishes since v_l is minimizing. Moreover

$$E''(l) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(2\epsilon^2 l^{-3} v_{lxx}^2 + 6lv_l^2\right)\,\mathrm{d}x + \int_{-\frac{1}{2}}^{\frac{1}{2}} (-2\epsilon^2 l^{-2} v_{lxx} w_{lxx} + \sigma(v_{lx})w_{lx} + 6l^2 v_l w_l)\mathrm{d}x \,. \tag{5.16}$$

In the sequel we write v and w instead of v_l and w_l. To estimate $E'(l)$ observe that $\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 = \frac{1}{12}$ so that by (3.11)

$$\left|3l^2\int_{-\frac{1}{2}}^{\frac{1}{2}} v^2\mathrm{d}x - \frac{1}{4}l^2\right| \le Cl^2\left(\frac{\epsilon}{l}+l^2\right) \,. \tag{5.17}$$

Let $q_l(x) = q(\frac{l}{\epsilon}(\frac{1}{2}-|x|))$. Since $-\epsilon^2 l^{-2}(q_l')^2 + W(q_l) = 0$ for $x \neq 0$ (see (2.9)) one has in view of (3.9), (3.10), Taylor expansion around q_l and (3.11), (3.12)

$$\left|\int_{-\frac{1}{2}}^{\frac{1}{2}} (-\epsilon^2 l^{-2} v_{xx}^2 + W(v_x))\,\mathrm{d}x\right| \le Cl^2\int_{-\frac{1}{2}}^{\frac{1}{2}} \epsilon l^{-1}|v_{xx}| + |\sigma(v_x)|\,\mathrm{d}x + Cl^4 \le Cl^2(\frac{\epsilon}{l}+l^2) \,.$$

In conjunction with (5.17) this gives (5.4).

To estimate $E''(l)$ let $w^1 = w - w^0$, where w^0 is given by (5.11). Then

$$E''(l) = T_1 + T_2 + T_3 + T_4 + \frac{1}{2}l,$$

where

$$T_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} (-2\epsilon^2 l^{-2} v_{xx} w^1_{xx} + \sigma(v_x) w^1_x + 6l^2 v w^1) \mathrm{d}x,$$

$$T_2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} 6l^2 v w^0 \mathrm{d}x,$$

$$T_3 = \int_{-\frac{1}{2}}^{\frac{1}{2}} (2\epsilon^2 l^{-3} v_{xx}^2 - 2\epsilon^2 l^{-2} v_{xx} w^0_{xx} + \sigma(v_x) w^0_x)\ \mathrm{d}x,$$

$$T_4 = \int_{-\frac{1}{2}}^{\frac{1}{2}} 6 l v^2 \mathrm{d}x - \frac{1}{2} l\ .$$

In view of Lemma 5.3, (2.16), (3.11) and (3.12) one has

$$|T_1| \le C\left(\frac{\epsilon}{l} + l^2\right) l\ .$$

Taking into account that $V(x)$ and $x - \psi(x)$ vanish at $x = \pm\frac{1}{2}$ (see (2.14)) integration by parts and (3.11) and (5.12) yield

$$\begin{aligned} |T_2| &= \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} 6l^2 V w^0_x \mathrm{d}x \right| \\ &= \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} 6lV(x - \psi(x)) v_{xx} \mathrm{d}x \right| \\ &= \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} 6l(v(x - \psi(x)) + V(1 - \psi'(x)))(v_x - 1) \mathrm{d}x \right| \\ &\le C\left(\frac{\epsilon}{l} + l^2\right) l\ . \end{aligned} \tag{5.18}$$

Using (2.13), (5.12) and (5.13) the term T_3 is estimated as follows

$$\begin{aligned} T_3 &= \int_{-\frac{1}{2}}^{\frac{1}{2}} (2\epsilon^2 l^{-3} v_{xx}^2 - 2\epsilon^2 l^{-2} v_{xx} w^0_{xx} + \sigma(v_x) w^0_x)\ \mathrm{d}x \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} (2\epsilon^2 l^{-3} \psi' v_{xx}^2 - 2\epsilon^2 l^{-3} v_{xx}(x - \psi(x)) v_{xxx} + l^{-1}\sigma(v_x)(x - \psi(x)) v_{xx}) \mathrm{d}x \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} 2\epsilon^2 l^{-3} \psi' v_{xx}^2\ \mathrm{d}x + \int_{-\frac{1}{2}}^{\frac{1}{2}} 2lV(x - \psi(x)) v_{xx}\ \mathrm{d}x\ . \end{aligned}$$

The second term was already estimated in (5.18). As regards the first term, (3.10) implies that $|\psi' v_{xx}^2| \le l^6 \epsilon^{-2}$ since $\psi'(x)$ vanishes if $|x| > \frac{1}{4}$. Hence the first term is bounded by Cl^3 and thus

$$|T_3| \le C(\frac{\epsilon}{l} + l^2) l.$$

Finally by (5.17)

$$|T_4| \le C(\frac{\epsilon}{l} + l^2)l\ ,$$

and the estimate (5.5) follows. The theorem is proved. □

6. Periodic boundary conditions

Proof of Theorem 1.1. As outlined in the introduction the theorem would follow from an easy convexity argument if we knew $E''(l) > 0$ for all $l \in (0,1)$. This is, however, not true and we need a separate argument to rule out exceptionally large or small values of l (see proof of Claim #3 below). Before we address that issue we first show that estimates for minimizers of J_l lead to an upper bound for the minimum of I_l by a straight forward (anti-)periodic extension (see Claims #1 and #2 below). Finally we address the uniqueness question for minimizers of I_l (see Claim #4).

Recall that

$$E_0 = \min\left\{2N\tilde{E}\left(\frac{1}{2N}\right) : N \in \mathbb{N}\right\}$$

where $\tilde{E}(l)$ is given by (2.4).

Claim #1 Let

$$E_1 = \min\left\{2NE\left(\frac{1}{2N}\right) : N \in \mathbb{N}\right\}\ . \tag{6.1}$$

Then $E_1 = E_0$.

Proof. Note that $E_1 \le E_0$ and that the minimum in (6.1) is attained. Indeed if $N \ge \frac{1}{\epsilon}$ then (2.6) implies that the right hand side of (6.1) is larger than $c > 0$ (independent of ϵ) while the choice $N \sim \epsilon^{-1/3}$ in conjunction with (2.5) gives $E_1 \le C\epsilon^{2/3}$. Let N_0 be a value where the minimum in (6.1) is attained. By the above argument $N_0 < \frac{1}{\epsilon}$ so that (2.6), (2.7) give $E_0 \ge c(N_0\epsilon + N_0^{-2})$. In view of the upper bound one obtains

$$c\epsilon^{-1/3} \le N_0 \le C\epsilon^{-1/3}\ .$$

Hence Theorem 5.1 applies with $l = (2N_0)^{-1}$. The claim follows from (5.2).

Claim #2 $\min\{I(u) : u \in H^2_\#\} \le E_0$.

Proof. Let N_0 be as above and let $l_0 = (2N_0)^{-1} \sim \epsilon^{1/3}$. Let v_0 be the corresponding minimizer of J_{l_0} (see (1.15)). By Theorem 4.2, v_0 is antisymmetric. Extend v_0 antiperiodically to the whole line and let

$$u(x) = l_0 v_0(l_0^{-1}x)\ .$$

Then one has

$$u\left(x + \frac{1}{2N_0}\right) = u(x + l_0) = -u(x)\ ,$$

and, using the antisymmetry of v_0,

$$\begin{aligned} I(u) &= 2N_0 \int_0^{\frac{1}{2N_0}} \epsilon^2 u_{xx}^2 + W(u_x) + u^2 = 2N_0 E\left(\frac{1}{2N_0}\right) \\ &= E_0\ . \end{aligned} \tag{6.2}$$

The claim is proved. For future reference note that by the antisymmetry of v_0

$$u(0) = u\left(\frac{1}{2}\right) = 0\ , \tag{6.3}$$

$$u\left(\frac{1}{2} - x\right) = -u\left(\frac{1}{2} + x\right)\ . \tag{6.4}$$

Claim #3 $\min\{I(u) : u \in H^2_\#\} \geq E_0$.

Proof. Let

$$\hat{E}_0 = \min\{l^{-1}E(l) : l \in (0,1)\}\ .$$

As in the proof of Claim #1 we see that $\hat{E}_0 \sim \epsilon^{2/3}$ and that a minimizing l_0 satisfies $l_0 \sim \epsilon^{1/3}$. Let

$$f(l) = l^{-1}E(l)\ .$$

It follows from (5.3) to (5.5) that

$$f''(l) \geq \frac{1}{12} \quad \text{if} \quad c\epsilon^{1/3} \leq l \leq C\epsilon^{1/3}. \tag{6.5}$$

Thus l_0 is unique and moreover

$$E_0 - \hat{E}_0 \leq C \min\left\{\left|l_0 - \frac{1}{2N}\right|^2 : N \in \mathbb{N}\right\} \leq C\epsilon^{4/3}\ . \tag{6.6}$$

Let u be a minimizer of I in $H^2_\#$. Then u is C^4 and solves the Euler-Lagrange equation

$$2\epsilon^2 u_{xxxx} - \sigma(u_x)_x + 2u = 0\ .$$

Thus the set $\{x \in [0,1] : u_x(x) = 0\}$ consists of a finite number of points. Otherwise it would have an accumulation point a at which all derivatives of order ≤ 4 vanish. The equation would imply $u \equiv 0$ and u would not be minimizing. Let $0 < x_1 < \ldots < x_{2N} < 1$ denote those zeros of u_x where u_x changes sign (we may assume $u_x(0) \neq 0$ by translation invariance). The number of those zeros has to be even as $u_x(0) = u_x(1)$ by periodicity. Let

$$l_i = x_{i+1} - x_i\ ,$$

where formally $x_{2n+1} = 1 + x_1$. By definition of $E(l)$ one has (using the symmetry of W)

$$F_i := \int_{x_i}^{x_{i+1}} (\epsilon^2 u_{xx}^2 + W(u_x) + u^2)\mathrm{d}x \geq E(l_i)\ . \tag{6.7}$$

Moreover by Proposition 6.1 below

$$F_i \geq c\epsilon\ .$$

Thus (by periodicity)

$$I(u) = \sum_{i=1}^{2N} F_i \geq \sum_{i=1}^{2N} \max(E(l_i), c\epsilon)\ . \tag{6.8}$$

If we knew already that (5.1) holds for all l_i then we would be finished by convexity of E (see (6.9) below). Now we may always assume $l_0 \le c_0$, in view of (2.7) and the estimate $I(u) \le E_0 \le C\epsilon^{2/3}$. Similarly, by (2.6), $2N \le C\epsilon^{-1/3}$. Let

$$I_1 = \left\{ i : \frac{l_i}{|\ln l_i|} \ge c_0^{-1}\epsilon,\ 1 \le i \le 2N \right\} ,$$
$$I_2 = \left\{ i : \frac{l_i}{|\ln l_i|} < c_0^{-1}\epsilon,\ 1 \le i \le 2N \right\} ,$$

where c_0 is chosen sufficiently small for Theorems 3.1, 4.2 and 5.1 to hold.

To show that $I_2 = \emptyset$, we argue by contradiction. Assume that $I_2 \ne \emptyset$ and let

$$m = \# I_2 \ ; \qquad d = \sum_{i \in I_2} l_i \ .$$

By (2.6)

$$E(l_i) \ge c \frac{l_i}{|\ln l_i|} \quad \text{for } i \in I_2 \ .$$

By (5.5), $E''(l) > 0$ if $\frac{l}{|\ln l|} \ge c_0^{-1}\epsilon$. Using in addition the convexity of $z \mapsto z/|\ln z|$ for $z \in (0,1)$ and Proposition 6.1 one finds

$$\begin{aligned} I(u) &\ge \sum_{i \in I_1} E(l_i) + \sum_{i \in I_2} \frac{1}{2}(E(l_i) + c\epsilon) \\ &\ge (2N - m) E\left(\frac{1-d}{2N - m} \right) + cm \frac{d/m}{|\ln(d/m)|} + c\epsilon m \\ &\ge (1-d)\hat{E}_0 + c \frac{d}{|\ln d| + |\ln m|} + c\epsilon m \ . \end{aligned}$$

As $m \le 2N \le C\epsilon^{-1/3}$ it follows that

$$E_0 - \hat{E}_0 \ge \left(\frac{c}{|\ln d| + |\ln \epsilon|} - \hat{E}_0 \right) d + c\epsilon m \ .$$

As $\hat{E}_0 \sim \epsilon^{2/3}$ the first term on the right hand side is positive for sufficiently small ϵ if $d \ge \epsilon$ and is thus always larger than $-\hat{E}_0\epsilon \ge -C\epsilon^{5/3}$. The term on the left is bounded by $C\epsilon^{4/3}$ (see (6.6)). Thus

$$m \le C\epsilon^{1/3}$$

and hence $m = 0$ and $I_2 = \emptyset$ for sufficiently small ϵ. Thus $\# I_1 = 2N$ and by (5.5)

$$I(u) \ge \sum_{i \in I_1} E(l_i) \ge 2NE\left(\frac{1}{2N}\right) \ge E_0 \ . \tag{6.9}$$

Claim #3 is proved.

Claim #4 Any minimizer of I^ϵ in $H^2_\#$ is periodic with period $2(6A_0)^{1/3}\epsilon^{1/3} + \mathcal{O}(\epsilon^{2/3})$. Moreover for given $\epsilon > 0$ there are at most two distinct minimizers (up to translation).

Proof. As above let u be a minimizer and let $0 < x_1 < \dots x_{2N} < 1$ be the points where u_x changes sign. In view of (6.7), (6.9) and Claims #2 and #3 one has

$$E_0 \geq I(u) \geq \sum_{i=1}^{2N} F_i \geq \sum_{i=1}^{2N} E(l_i) \geq 2NE\left(\frac{1}{2N}\right) \geq E_0 \,. \tag{6.10}$$

By the strict convexity of E in the interesting range and (6.7) one deduces that $l_i = \bar{l} = 1/(2N)$ for $i = 1, \dots, 2N$ and

$$f\left(\frac{1}{2N}\right) = 2NE\left(\frac{1}{2N}\right) = E_0, \quad F_i = E\left(\frac{1}{2N}\right) \,. \tag{6.11}$$

In view of the rescaling $v(x) = l_i^{-1}u(l_i x + \frac{x_i + x_{i+1}}{2})$ Theorem 4.2 shows that $u|_{[x_i, x_{i+1}]}$ is uniquely determined (up to sign) and that u is antiperiodic with period $1/N$. Moreover for given N (satisfying (6.11)) u is uniquely determined up to translation and change of sign. Now the latter is equivalent to a translation by $1/(2N)$. As (6.11) can at most have two solutions in view of (6.5), there can be at most two distinct minimizers (up to translation).

To estimate the period $1/N$ observe that the value l_0 which minimizes $f(l) = l^{-1}E(l)$ satisfies

$$\begin{aligned} 0 &= f'(l_0) = l_0^{-1}E'(l_0) - l_0^{-2}E(l_0) \\ &= \frac{1}{6}l_0 - \frac{\epsilon}{l_0^2}A_0 + \mathcal{O}(\epsilon) \,, \end{aligned}$$

where we used (5.3), (5.4) and the fact that $l_0 \sim \epsilon^{1/3}$. By (6.5)

$$l_0 = (6A_0\epsilon)^{1/3} + \mathcal{O}(\epsilon) \,. \tag{6.12}$$

Now again in view (6.5), $N = [\frac{1}{2l_0}]$ (the largest integer $\leq \frac{1}{2l_0}$), or $N = [\frac{1}{2l_0}] + 1$. Thus

$$\frac{1}{N} = 2l_0 + \mathcal{O}(\epsilon^{2/3}) = 2(6A_0\epsilon)^{1/3} + \mathcal{O}(\epsilon^{2/3}) \,.$$

The claim is proved.

To prove Theorem 1.1 note that by (5.3), (6.6) and (6.12)

$$E_0 = f(l_0) + \mathcal{O}(\epsilon^{4/3}) = \frac{1}{4}(6A_0\epsilon)^{2/3} + \mathcal{O}(\epsilon^{4/3}) \,.$$

The theorem now follows from Claims #2 to #4 and (6.3), (6.4).

Uniqueness as a generic property. As before let $f(\epsilon, l) = l^{-1}E(\epsilon, l)$. It suffices to show that

$$\frac{\partial^2}{\partial\epsilon\partial l} f(\epsilon, l) < 0 \tag{6.13}$$

in the relevant range for ϵ and l. This will imply that for each N there is at most one ϵ such that

$$0 = f(\epsilon, \frac{1}{N}) - f\left(\epsilon, \frac{1}{N+1}\right) = \int_{1/(N+1)}^{1/N} \frac{\partial f}{\partial l}(\epsilon, l)\mathrm{d}l \,.$$

Thus there is at most a countable number of values of ϵ where the minimizer of I^ϵ is not unique (up to tanslations).

To show (6.13) one needs to estimate $\partial E/\partial\epsilon$ and $\partial^2 E/\partial\epsilon\partial l$. This can be done analogously to the estimates for $\partial^2 E/\partial l^2$, the details are left to the courageous reader.

Proposition 6.1 *There exist positive constants* ϵ_0, c_1 *with the following properties. If* $\epsilon < \epsilon_0$, *if* u *is a minimizer of* I^ϵ *in* $H^2_\#$ *and if* $u_x(x_1) = u_x(x_2) = 0$, $\pm u_x \geq 0$ *on* $[x_1, x_2]$ *then*

$$\int_{x_1}^{x_2} \epsilon^2 u_{xx}^2 + W(u_x) + u^2 \mathrm{d}x \geq c_1 \epsilon \, .$$

Proof. In view of (2.6) the estimate is obvious if $x_2 - x_1 \geq \epsilon$. Thus we only have to deal with extremly short intervals. We will show that it is always favourable to remove these alltogether. We first need some preparations.

Assume for convenience $u_x \geq 0$ on $[x_1, x_2]$. Let

$$E = \int_{x_1}^{x_2} (\epsilon^2 u_{xx}^2 + W(u_x) + u^2) \mathrm{d}x \, .$$

Let c_1 be a small constant (to be fixed later) and assume for a contradiction that

$$E \leq c_1 \epsilon \, .$$

By Modica's energy estimate (see (2.8))

$$u_x \leq CH(u_x) \leq \epsilon^{-1} CE \leq Cc_1 \, .$$

Choose c_1 so small that

$$Cc_1 \leq \frac{1}{2} \, .$$

Then $W(u_x) \geq W(\frac{1}{2})$ and hence

$$l := x_2 - x_1 \leq CE \leq C\epsilon \, .$$

We next consider competing functions where the interval $[x_1, x_2]$ is "removed". By translation we may assume $x_1 > 0, \ x_2 < 1$. Let

$$\eta := \int_{x_1}^{x_2} u_x \mathrm{d}x \, ,$$

and define $w : [0, 1 - l] \to \mathbb{R}$ by

$$w(x) = \begin{cases} u_x(x) + \eta(1 - l)^{-1} \, , & \text{if } x \leq x_1 \, , \\ u_x(x + l) + \eta(1 - l)^{-1} \, , & \text{if } x > x_1 \, . \end{cases} \tag{6.14}$$

Note that w is continuous at $x = x_1$, that $w(1 - l) - w(0) = u_x(1) - u_x(0) = 0$ and

$$|\eta| \leq \frac{1}{2} l \leq CE \, .$$

For $x \in [0, 1 - l]$ let

$$\hat{u}(x) = u(0) + \int_0^x w(\xi)\mathrm{d}\xi \, .$$

Then $\hat{u}_x = w$ and by definition of η, $\hat{u}(1-l) = u(1) = u(0) = \hat{u}(0)$. Moreover

$$\begin{aligned} |\hat{u}(x) - u(x)| &\le \eta(1-l)^{-1}x &&\le \eta\,, \quad \text{if } x \le x_1\,,\\ |\hat{u}(x) - u(x+l)| &\le \eta(1-l)^{-1}(1-l-x) &&\le \eta\,, \quad \text{if } x > x_1\,. \end{aligned}$$

Taking into account $\sigma(t) \le CW^{1/2}(t)$ if $|t| \le 2$ one has

$$\begin{aligned} &\int_0^{1-l} \epsilon^2\hat{u}_{xx}^2 + W(\hat{u}_x) + \hat{u}^2\mathrm{d}x \\ &\le \int_0^{x_1} \epsilon^2 u_{xx}^2 + W(u_x) + u^2\mathrm{d}x + \int_{x_2}^1 \epsilon^2 u_{xx}^2 + W(u_x) + u^2\mathrm{d}x \\ &\quad + 2\eta\int_0^1 |u|\mathrm{d}x + \eta^2 + \int_0^1 2|\sigma(u_x)|\frac{\eta}{(1-l)}\mathrm{d}x + C\left(\frac{\eta}{1-l}\right)^2 \\ &\le I(u) - E + C\eta(I(u))^{1/2} + C\eta^2 \le I(u) - \frac{E}{2}\,, \end{aligned}$$

since $I(u) \le C\epsilon^{2/3} \ll 1$ and $\eta \le CE$.

Finally let

$$\tilde{u}(x) = (1-l)^{-1}\hat{u}((1-l)x)\,.$$

Then $\tilde{u} \in H^2_{\#}$ and

$$\begin{aligned} I(\tilde{u}) &= \int_0^{1-l} (\epsilon^2(1-l)\hat{u}_{xx}^2 + (1-l)^{-1}W(\hat{u}_x) + (1-l)^{-3}\hat{u}^2)\mathrm{d}x \\ &\le (1+Cl)\int_0^{1-l} \epsilon^2\hat{u}_{xx}^2 + W(\hat{u}_x) + \hat{u}^2\mathrm{d}x \\ &\le I(u) + ClI(u) - \frac{E}{2} \\ &\le I(u) + E(CI(u) - \frac{1}{2}) < I(u)\,. \end{aligned}$$

This is the desired contradiction. □

7. Dirichlet boundary conditions

We prove Corollary 1.2 by reduction to the periodic case. First consider the following situation. Let $u \in H^2(0,1) \cap H^1_0(0,1)$. Extend u antisymmetrically to $(0,2)$, i.e. $u(2-x) = -u(x)$ for $x \in (0,1)$. Then $u(0) = u(2) = 0$, $u_x(0) = u_x(2)$. Define

$$\begin{aligned} u_{\#}(x) &= \frac{1}{2}u(2x)\,, \\ I^{\epsilon}_{\#}(v) &= 4\int_0^1 \left(\frac{\epsilon}{4}\right)^2 v_{xx}^2 + \frac{1}{4}W(v_x) + v^2\mathrm{d}x\,. \end{aligned}$$

Then $u_\# \in H^2_\#$ and

$$\begin{aligned} I^\epsilon(u) &= \int_0^1 \epsilon^2 u_{xx}^2 + W(u_x) + u^2 \mathrm{d}x \\ &= \frac{1}{2}\int_0^2 \epsilon^2 u_{xx}^2 + W(u_x) + u^2 \mathrm{d}x \\ &= \frac{1}{2}\int_0^1 \left(\frac{1}{4}\epsilon^2 (u_\#)_{xx}^2 + W((u_\#)_x) + 4u_\#^2\right) 2\ \mathrm{d}x \\ &= I^\epsilon_\#(u_\#)\ . \end{aligned}$$

Lemma 7.1 *For sufficiently small ϵ one has*

$$\min_{u\in H^2\cap H^1_0} I^\epsilon(u) = \min_{v\in H^2_\#} I^\epsilon_\#(v)\ .$$

Moreover $\overline{u}$ is a minimizer of I^ϵ in $H^2 \cap H^1_0$ if and only if $\overline{u}_\#$ is a minimizer of $I^\epsilon_\#$.

Proof. From the above $I^\epsilon_\#(u_\#) = I^\epsilon(u)$ and $u_\# \in H^2_\#$ if $u \in H^2 \cap H^1_0$. Thus

$$\min_{H^2_\#} I^\epsilon_\# \le \min_{H^2\cap H^1_0} I^\epsilon\ .$$

To show the converse inequality let w be a minimizer of $I^\epsilon_\#$. By Theorem 1.1(iii) we may assume that $w(0) = w(\frac{1}{2}) = 0$ and $w(\frac{1}{2} - x) = -w(\frac{1}{2} + x)$. Define $u : (0, 1) \to \mathbb{R}$ by $u(x) = 2w(\frac{1}{2}x)$. Then $u \in H^2 \cap H^1_0$ and $u_\# = w$. Thus

$$\min_{H^2\cap H^1_0} I^\epsilon \le \min_{H^2_\#} I^\epsilon_\#\ ,$$

and the lemma is proved. □

Proof of Corollary 1.2. Let u be a minimizer of I subject to $u(0) = u(1) = 0$. Let $u_\#$ be the corresponding minimizer of $I^\epsilon_\#$, let $\epsilon_\# = \frac{\epsilon}{4}$ and $A_\# = 2\int_{-1}^1 (\frac{1}{4}W)^{1/2}(z)dz = A_0/2$. By Theorem 1.1, $u_\#$ is periodic with period $P_\# = 2(6A_\#\epsilon_\#)^{1/3} + \mathcal{O}(\epsilon^{2/3}) = (6A_0\epsilon)^{1/3} + \mathcal{O}(\epsilon^{2/3})$. Thus u is the restriction of a periodic function with period $2P_\#$. This proves (i).

If u and v are minimizers of I in $H^2(0, 1) \cap H^1_0(0, 1)$ and if neither $u \equiv v$ nor $u \equiv -v$ then $u_\#$ and $v_\#$ cannot just differ by a translation. Indeed, using Theorem 4.2 one easily sees that minimizers of $I_\#$ contain exactly one zero in each half-period. Thus the condition $u_\#(\frac{1}{2}) = v_\#(\frac{1}{2}) = 0$ implies that if $u_\#$ and $v_\#$ differ by a translation the translation must be multiple of a half-period. In this case $u_\# \equiv \pm v_\#$ and hence $u \equiv \pm v$, a contradiction. Now (ii) easily follows. □

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