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Minimal relation algebras

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Dedicated to Bjarni Jónsson on his 70th birthday

Abstract. This paper is concerned with the covers of the atoms in the lattice of varieties of relation algebras. A *minimal* relation algebra is one that is simple and generates such a subvariety. The main result we prove is that there are exactly three finite minimal relation algebras that are *totally symmetric* (i.e., satisfy the identities $x = x^{\smile}$ and $x \leq x ; x$). We also give an example of an infinite minimal totally symmetric relation algebra, and some results about other subvarieties.

1. Preliminaries

For a very readable introduction to relation algebras (and the notation used here) we refer the reader to [4]. Here we recall some standard results. The variety of all relation algebras will be denoted by $\mathscr{RA}$ and the lattice of its subvarieties by Λ. $\mathscr{RA}$ is a discriminator variety, whence the notions of simple, subdirectly irreducible and directly indecomposable coincide. Furthermore, the simplicity of a relation algebra can be characterized by the property that $1 ; x ; 1 = 1$ holds for every nonzero element x.

A relation algebra is said to be

(i) *integral* if $x ; y \neq 0$ for all nonzero elements x and y,
(ii) *commutative* if it satisfies the identity $x ; y = y ; x$,
(iii) *symmetric* if it satisfies the identity $x^{\smile} = x$, and
(iv) *totally symmetric* if it satisfies the identities $x^{\smile} = x$ and $x \leq x ; x$.

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Recall that a relation algebra is integral if and only if any of the following conditions holds:

(i_1) 1' is an atom;
(i_2) $x ; 1 = 1$ for any nonzero element x;
(i_3) for any elements x, y, the inclusion $1' \leq x ; y$ holds if and only if $xy^\smile \neq 0$.

The variety generated by all integral relation algebras is denoted by $\mathscr{IRA}$ and the varieties of all commutative, all symmetric and all totally symmetric relation algebras are denoted by $\mathscr{CRA}$, $\mathscr{SRA}$ and $\mathscr{TRA}$ respectively. Clearly $\mathscr{TRA} \subseteq \mathscr{SRA} \subseteq \mathscr{CRA}$, and $\mathscr{CRA} \subseteq \mathscr{IRA}$ follows from (i_2) and from the fact that $1 ; x ; 1 = 1$ holds for any nonzero element of a subdirectly irreducible relation algebra.

We use the notation $Re(X)$ for the full relation algebra over the set X and the notation $R(G)$ for the relation algebra over the group G.

Finally, let $\mathscr{O}$ be the variety of one-element relation algebras and let $\mathscr{V}(A)$ denote the variety generated by A. The following results of B. Jónsson and A. Tarski describe the atoms and the covers of two of the atoms in Λ.

THEOREM [6] [8].
(i) *There are exactly three varieties* $\mathscr{A}_i = \mathscr{V}(A_i)$ ($i = 1, 2, 3$, *see Table* 1) *in the lattice of varieties of relation algebras that cover the variety* $\mathscr{O}$ *and in every simple relation algebra the smallest subalgebra is isomorphic to one of* A_1, A_2 *or* A_3.
(ii) $\mathscr{A}_1$ *has no join irreducible covers, while* $\mathscr{A}_2$ *has exactly one join ireducible cover*, $\mathscr{V}(Re(2))$.

Since relation algebras have Boolean algebras as reducts, $\mathscr{RA}$ is congruence distributive and hence Λ is a distributive lattice. So, in order to complete the description of the covers of the atoms of Λ, we only have to find the join irreducible

Table 1. Simple relation algebras generated by the constants

Name – Representation	0' ; 0'
$A_1 = R(\mathbb{Z}_1) = Re(1)$	0
$A_2 = R(\mathbb{Z}_2) < Re(2)$	1'
$A_3 < R(\mathbb{Z}_3) < Re(3)$	1

covers of $\mathscr{A}_3$. At present even the question whether the number of these is finite or infinite is unsolved. However we show (Theorem 7) that in the lattice of totally symmetric subvarieties there are only three finitely generated join irreducible covers.

Clearly a join irreducible cover of $\mathscr{A}_i$ ($i = 2$ or 3) can be generated by one subdirectly irreducible (hence simple) algebra, containing A_i as a subalgebra. We call such an algebra A *minimal.* Since relation algebras are discriminator algebras, the subdirectly irreducible elements of $\mathscr{V}(A)$ are subalgebras of an ultrapower of A. In case A is finite it follows that A is minimal if and only if A_i is the only proper subalgebra. Thus, for example, all simple 8 element relation algebras are minimal. However, we will give an example of an infinite simple relation algebra whose subalgebras are all isomorphic to the algebra itself or to A_3 yet it does not generate a cover of $\mathscr{A}_3$ (see Theorem 11).

In order to list all the known minimal relation algebras together with a nice representation (where possible), we introduce the following notation. Let $x^s = x + x^\smile$ for an element x of a relation algebra. For a group G define $S(G) = \{x^s: x \in R(G)\}$. It is easy to see that if G is abelian then $S(G)$ is in fact a subalgebra of $R(G)$. In this notation we have that $A_i \cong S(\mathbb{Z}_i)$ ($i = 1, 2, 3$) where $\mathbb{Z}_i$ denotes the cyclic group of order i. We also introduce the notation x^n which is defined as

$$x^0 = 1' \quad \text{and for } n > 0 \qquad x^n = x^{n-1} ; x.$$

In Table 2 we list the known symmetric relation algebras that are minimal. In [2] it is shown that the finite representations given here are the smallest possible, except in the case of B_7, which has a smallest representation as the subalgebra of $S(\mathbb{Z}_3 \times \mathbb{Z}_3)$ with atoms $\{(1,0),(0,1)\}^s$, $\{(1,1),(1,-1)\}^s$ and $\{(0,0)\}$.

We also need the notion of an *equivalence element*, i.e., an element e of a relation algebra such that $e ; e = e$ and $e^\smile = e$. An equivalence element is *nontrivial* if $0 < e0' < 0'$. Note that the first four algebras in Table 2 each contain a nontrivial equivalence element $b + 1'$. The following theorem of Jónsson [4] implies that no other minimal integral algebras contain nontrivial equivalence elements.

THEOREM 1 ([4]). *If A is a relation algebra and e is an equivalence element in A then e generates a finite subalgebra of A. If A is integral and e is a nontrivial equivalence element then e generates a subalgebra isomorphic to B_1, B_2, B_3 or B_4.*

For the sake of completeness we also include a list of the 5 known integral nonsymmetric minimal relation algebras $C_1, \ldots, C_5$ in Table 3 and all nonintegral

Table 2. Minimal symmetric relation algebras ($x^s = x + x^{\smile}$)

Name – Repr.	a	b	$a ; a$	$b ; b$	$a ; b$
$B_1 = S(\mathbb{Z}_4)$	$\{1\}^s$	$\{2\}$	$b + 1'$	$1'$	a
$B_2 < S(\mathbb{Z}_6)$	$\{1, 3\}^s$	$\{2\}^s$	$b + 1'$	$b + 1'$	a
$B_3 < S(\mathbb{Z}_6)$	$\{1, 2\}^s$	$\{3\}$	1	$1'$	a
$B_4 < S(\mathbb{Z}_9)$	$\{1, 2, 4\}^s$	$\{3\}^s$	1	$b + 1'$	a
$B_5 = S(\mathbb{Z}_5)$	$\{1\}^s$	$\{2\}^s$	$b + 1'$	$a + 1'$	$0'$
$B_6 < S(\mathbb{Z}_8)$	$\{2, 3\}^s$	$\{1, 4\}^s$	1	$a + 1'$	$0'$
$B_7 < S(\mathbb{Z}_{12})$	$\{1, 2, 5\}^s$	$\{3, 4, 6\}^s$	1	1	$0'$

Name – Repr.	$a ; a$	$b ; b$	$c ; c$	$a ; b$	$a ; c$	$b ; c$
$B_8 < S(\mathbb{Z})$	1	c^-	$a + 1'$	$0'$	$0'$	a
$B_9 = S(\mathbb{Z}_7)$	$b + 1'$	$c + 1'$	$a + 1'$	$a + c$	$b + c$	$a + b$
B_{10} Nonrepr.	c^-	$c + 1'$	$a + 1'$	$a + c$	$b + c$	$a + b$
B_{11} Nonrepr.	c^-	a^-	$a + 1'$	$a + c$	$b + c$	$a + b$
B_{12} Nonrepr.	c^-	a^-	b^-	$a + c$	$b + c$	$a + b$

Name – Repr.	generator x	atoms a_n	$a_n ; a_m$
$B_\infty < S(\mathbb{Z} \times \mathbb{Z})$	$(\{(1, 0), (0, 1)\}^s)^2$	$x^n(x^{n-1})^-$	$\sum_{i=\lvert n-m\rvert}^{n+m} a_i$

Table 3. Minimal nonsymmetric integral relation algebras

Name – Repr.	a	$a ; a$	$a ; a^{\smile}$	$a^{\smile} ; a$
$C_1 < R(\mathbb{Z}_7)$	$\{1, 2, -3\}$	$0'$	1	1
$C_2 < R(\mathbb{Q})$	$\{q \in \mathbb{Q}: q > 0\}$	a	1	1
$C_3 = R(\mathbb{Z}_3)$	$\{1\}$	$a^{\smile}$	$1'$	$1'$

Name	$a ; a$	$a ; a^{\smile}$	$a^{\smile} ; a$	$b ; b$	$a ; b$	$a^{\smile} ; b$
C_4	a	1	b^-	1	b	$a^{\smile} + b$

Name	$a ; a$	$a ; a^{\smile}$	$a^{\smile} ; a$	$b ; b$	$a ; b$	$a^{\smile} ; b$	$c ; c$	$a ; c$	$a^{\smile} ; c$	$b ; c$
C_5	$0'$	1	1	$(b + c)^-$	$0'$	c^-0'	b^-	b^-0'	$0'$	a

$b = b^{\smile}$ and $c = c^{\smile}$

Table 4. Minimal nonintegral relation algebras.

Name – Repr.	e_1	e_2
$N_{11} = \mathrm{Re}(2)$	$\{(0,0)\}$	$\{(1,1)\}$
$N_{12} < \mathrm{Re}(3)$	$\{(0,0)\}$	$\{(1,1),(2,2)\}$
$N_{13} < \mathrm{Re}(4)$	$\{(0,0)\}$	$\{(1,1),(2,2),(3,3)\}$
$N_{23} < \mathrm{Re}(5)$	$\{(0,0),(1,1)\}$	$\{(2,2),(3,3),(4,4)\}$

$e_1, e_2, c, c^{\smile}, d_1$ and d_2 are atoms or zero, where
$1' = e_1 + e_2$, $c = e_1 ; 1 ; e_2$ and $d_i = (e_i ; 1 ; e_i)0'$

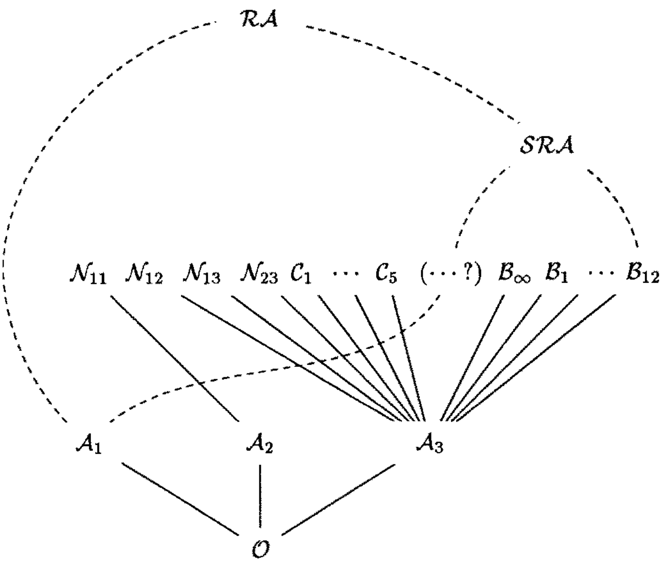

Figure 1. Join irreducibles on the bottom levels of A.

minimal relation algebras in Table 4. In fact, Maddux [7] proves that every nonintegral simple (semiassociative) relation algebra has a subalgebra isomorphic to N_{11}, N_{12}, N_{13}, N_{23}, B_2 or B_4.

2. Totally symmetric minimal relation algebras

We will show in Theorem 7 that all the totally symmetric finite minimal relation algebras occur in Table 2. We begin with a few lemmas that will be useful when looking for minimal algebras.

LEMMA 2. *For any* x, $(x^{\cup} ; x^{-})^{-}(x^{\cup} ; x^{-})^{-\cup}$ *is an equivalence element containing* 1'.

Proof. Note that the conditions

$$z \leq (x^{\cup} ; x^{-})^{-} \qquad z(x^{\cup} ; x^{-}) = 0 \qquad x^{-}(x ; z) = 0 \qquad x ; z \leq x \qquad (*)$$

are equivalent, hence for $y = (x^{\cup} ; x^{-})^{-}$ we have $x ; y \leq x$ and therefore $x ; y ; y \leq x ; y \leq x$. Applying $(*)$ in the reverse direction for $z = y ; y$, we obtain $y ; y \leq y$. Also $(*)$ with $z = 1'$ shows that $1' \leq y$. Taking converses, we get $y^{\cup} ; y^{\cup} \leq y^{\cup}$ and $1' \leq y^{\cup}$ hence $yy^{\cup}$ is an equivalence element containing the identity. □

In the subsequent results we frequently make use of the fact that every simple symmetric relation algebra is integral.

LEMMA 3. *Let A be a simple symmetric relation algebra. Then the following are equivalent*:

(i) *A has no nontrivial equivalence elements*;

(ii) *for any* $x \in A$, $0 < x < 0'$ *implies* $x ; x^{-}0' = 0'$.

Proof. Assume (i) and let $x \in A$ satisfy $0 < x < 0'$. Let us define $y = x^{-}0'$. By symmetry and Lemma 2, $(x ; x^{-})^{-}$ is an equivalence element including 1'. Since A is integral, $x ; x^{-} \neq 0$, i.e. $(x ; x^{-})^{-} \neq 1$, so by (i) we have $(x ; x^{-}) = 1'$, thus $x ; x^{-} = 0'$. Now $0' = x ; x^{-} = x ; (y + 1') = x ; y + x$ implies that $x ; y \geq y$. By exchanging the role of x and y in the above argument, we also get that $x ; y \geq x$. Therefore $x ; y \geq 0'$ and, since $xy = 0$, we have $(x ; y)1' = 0$ whence $x ; y = 0'$.

Conversely, suppose e is a nontrivial equivalence element. Since A is integral, $e \geq 1'$. Then $0 < e0' < 0'$, and $(e0' ; e)e^{-} \leq ee^{-} = 0$ implies $(e0' ; e^{-})e = 0$, hence $e0 ; e^{-} \leq e^{-} < 0'$. □

LEMMA 4. *If A is a simple symmetric relation algebra that has no nontrivial equivalence elements then* $x ; x + x^{-} ; x^{-} = 1$ *for every element* $0 < x < 0'$.

Proof. Let $u = (x ; x + x^{-} ; x^{-})^{-}$. Since $(x ; x)u = 0$, it follows that $(x ; u)x = 0$ and hence $x ; u \leq x^{-}$. Now $(x^{-} ; x ; u)u \leq (x^{-} ; x^{-})u = 0$, which implies that $(u ; u)(x ; x^{-}) = 0$. By Lemma 3 we get $0' \leq x ; x^{-}$, so $u ; u \leq 1'$. Therefore $u + 1'$ is an equivalence element. By assumption $u + 1' = 1$ or $u + 1' = 1'$. Since $u \leq 0'$, the first case implies $u = 0'$. Hence $0' ; 0' = u ; u \leq 1'$. But then $x ; 1 = x ; 0' + x ; 1' \leq 0' ; 0' + x \leq 1' + x < 1$ contradicts the integrality of A. Therefore $u + 1' = 1'$, which implies $u = 0$ and hence $x ; x + x^{-} ; x^{-} = 1$. □

LEMMA 5. *For any x in a simple symmetric relation algebra, $x ; x + x^- ; x^- = 1$ implies $x ; x = 1$ or $x^- ; x^- = 1$.*

Proof. Suppose $x^- ; x^- \neq 1$ and let $z = (x^- ; x^-)^-$. If $x ; x + x^- ; x^- = 1$ then we have $z \leq x ; x$, hence $(x ; x)z \neq 0$ and therefore $(z ; z)x \neq 0$. It follows that

$$1 = (z ; x)x ; 1 = (z ; x)x ; x + (z ; x)x ; x^- \leq x ; x + z ; x ; x^-.$$

Now $z(x^- ; x^-) = 0$ implies $(z ; x^-)x^- = 0$ whence $z ; x^- \leq x$, so $1 \leq x ; x + z ; x ; x^- \leq x ; x$. □

THEOREM 6. *Let A be a simple symmetric relation algebra. If $x ; x < 1$ and $x^-0' ; x^-0' < 1$ for some $0 < x < 0'$ then $A \cong B_5$ or A contains a nontrivial equivalence element.*

Proof. Let $0 < x < 0'$, $y = x^-0'$ and suppose A does not contain any nontrivial equivalence elements. We will show that if $x ; x < 1$ and $y ; y < 1$ then x and y are in fact atoms. The result then follows since B_5 is the only 8 element relation algebra that satisfies the preceding conditions.

By Lemmas 4 and 5, we have that $1 = x^- ; x^- = (y + 1') ; (y + 1') = y ; y + y + 1' = y ; y + y$, and similarly $1 = x ; x + x$. The product of the two equations gives that $x ; x + y ; y = 1$ and we also obtain that $0 \neq (x ; x)^- \leq x$ and $0 \neq (y ; y)^- \leq y$. Let us assume that one of x and y, say, x is not an atom. Then there exist disjoint nonzero elements x_1, x_2 such that $x = x_1 + x_2$ and $x_1 \leq (x ; x)^-$. This implies that $x_1(x ; x) = 0$, i.e. $(x ; x_1)x = 0$, consequently $x ; x_1 \leq y + 1'$. In particular, $x_2 ; x_1 \leq 0'(y + 1') = y$ and thus $x_1 ; x_1 \leq x_1 ; 1' ; x_1 \leq x_1 ; x_2 ; x_2 ; x_1 \leq y ; y$ and similarly $x_2 ; x_2 \leq y ; y$. So $(y ; y)^- \leq x ; x = x_1 ; x_1 + x_2 ; x_2 + x_1 ; x_2$ implies that $(y ; y)^- \leq x_1 ; x_2$. Hence $(x_1 ; x_2)(y ; y)^- \neq 0$ and then $x_1((y ; y)^- ; x_2) \neq 0$. Let $u = x_1((y ; y)^- ; x_2)$. We will show that $u + 1'$ gives a nontrivial equivalence element, thus reaching a contradiction.

$(u ; u)x \leq (x_1 ; x_1)x = 0$ by the choice of x_1, so $u ; u \leq 1' + y$. But $(u ; u)y \leq (((y ; y)^- ; x_2) ; u)y \leq ((y ; y)^- ; x_2 ; x_1)y \leq ((y ; y)^- ; y)y = 0$, since $(y ; y)(y ; y)^- = 0$. This shows that $u ; u = 1'$ thus $(u + 1')^2 = u + 1'$, so by symmetry, $u + 1'$ is indeed an equivalence element. □

Now we have all the necessary tools to prove that Table 2 contains every finite minimal totally symmetric relation algebra.

THEOREM 7. *Let A be a finite nontrivial simple totally symmetric relation algebra. Then $A \cong B_{12}$ or A contains a subalgebra isomorphic to B_4 or B_7.*

Proof. Suppose A does not contain a subalgebra isomorphic to B_4 or B_7. Then Theorem 1 implies that A does not contain any nontrivial equivalence elements (B_1, B_2 and B_3 are not totally symmetric). Since the algebra B_5 is not totally symmetric, Lemma 3 and Theorem 6 imply that for any disjoint nonzero x, y that satisfy $x + y = 0'$ we have

$$x \,;\, y = 0' \qquad (*)$$

and

$$\text{either } x \,;\, x = 1 \text{ or } y \,;\, y = 1, \text{ but not both.} \qquad (**)$$

The 'or' in $(**)$ is exclusive since otherwise B_7 would be a subalgebra.

We will call a triple (u, v, w) of disjoint nonzero elements in A a *principal triple* if $u + v + w = 0'$, the element $z = u + v$ is minimal with respect to the condition $z \,;\, z = 1$ and for u and v we have $u = z(w \,;\, w)^-$ and $v = z(w \,;\, w)$. Such a triple exists since, first of all, the finiteness of A and $(**)$ imply that we can find $0 < z < 0'$ such that z is minimal with respect to $z \,;\, z = 1$. Then we define $w = z^- 0'$, and u, v as described in the conditions for principal triples. If $u = 0$ then $z \leq w \,;\, w$, thus by total symmetry $w \,;\, w = 1$, contradicting $(**)$. If $v = 0$ then $w \,;\, w = 1' + w$, so $1' + w$ would be a nontrivial equivalence element.

CLAIM 1. $w \leq u \,;\, v$ for any principal triple (u, v, w). By $(*)$, $w \leq u \,;\, (v + w) = u \,;\, v + u \,;\, w$ and $w(u \,;\, w) = 0$ follows from the condition on u in the definition of a principal triple.

CLAIM 2. $w \leq u \,;\, u$ for any principal triple (u, v, w). Suppose $w_2 = w(u \,;\, u)^- \neq 0$ and let $w_1 = w(u \,;\, u)$. Then $(u \,;\, u)w_2 = 0$ implies $(u \,;\, w_2)u = 0$, and $(w \,;\, w)u = 0$ implies $(u \,;\, w)w = 0$, therefore $u \,;\, w_2 \leq v$. Now $u \,;\, u = u \,;\, 1' \,;\, u \leq u \,;\, w_2 \,;\, w_2 \,;\, u \leq v \,;\, v$ and similarly $w_2 \,;\, w_2 \leq v \,;\, v$. Since $u + w_1 \leq u \,;\, u$ and $w_2 \leq w_2 \,;\, w_2$, we get that $v \,;\, v \geq 1' + v + u + w_1 + w_2 = 1$. But this contradicts the minimality of $z = u + v$.

CLAIM 3. $v_1(u + v_2)^2 = 0$ if (u, v, w) is a principal triple and v_1 and v_2 are disjoint elements such that $v = v_1 + v_2$. Suppose that $v_1' = v_1(u + v_2)^2 \neq 0$ and let $v_2' = v_1'^- v$. Then $v_1' \leq (u + v_2)^2 \leq (u + v_2')^2$. Now $1' + u + v_2' \leq (u + v_2')^2$ follows from the total symmetry. Finally $w \leq u^2 \leq (u + v_2')^2$ by Claim 2, thus $(u + v_2')^2 = 1$ contrary to the assumption on the minimality of $z = u + v$.

CLAIM 4. For any principal triple (u, v, w), the element v is an atom. Suppose that v_1, v_2 are disjoint elements such that $v = v_1 + v_2$ and assume $v_1 \neq 0$. Then $v_1 = vv_1 \leq (w \,;\, w)v_1 \leq ((u \,;\, u) \,;\, w)v_1$ by Claim 2. Furthermore, since $u(w \,;\, w) = 0$,

$$(u \,;\, (u \,;\, w))v_1 \leq (u \,;\, (u + v))v_1 = (u \,;\, v_1)v_1 + (u \,;\, (u + v_2))v_1 = (u \,;\, v_1)v_1 \leq u \,;\, v_1$$

where the second equality follows from $(u ; (u + v_2))v_1 \leq v_1(u + v_2)^2 = 0$ by Claim 3. Therefore $v_1 \leq u ; v_1$. By $(*)$ we have

$$v_2 = v_2 0' = v_2(v_1 ; (u + v_2 + w)) = v_2(v_1 ; w) + v_2(v_1 ; (u + v_2)) = v_2(v_1 ; w) \leq v_1 ; w$$

because $v_1(v_2 ; (u + v_2)) \leq v_1(u + v_2)^2 = 0$ by Claim 3 and this implies that $v_2(v_1 ; (u + v_2)) = 0$. Thus $v_2 = (v_1 ; w)v_2 \leq ((v_1 ; u) ; w)v_2 = 0$ since $(v_1 ; v_2)(u ; w) \leq (v_1 ; v_2)(u + v) = (v_1 ; v_2)(u + v_1) + (v_1 ; v_2)v_2 = 0$ again follows from Claim 3.

CLAIM 5. If (u, v, w) is a principal triple then (v, w, u) and (w, u, v) are also principal triples. Obviously, it is enough to prove that (v, w, u) is a principal triple. Let $z = u + v$ and $\tilde{z} = v + w$. First we show that $\tilde{z}$ is minimal with respect to $\tilde{z} ; \tilde{z} = 1$. Note that $u ; u < 1$ follows from the minimality of z, so $(**)$ gives that $\tilde{z} ; \tilde{z} = 1$. Also by $(**)$, $z ; z = 1$ implies that $w ; w < 1$. Since v is an atom by Claim 4, in order to prove the minimality of $\tilde{z}$, it is now enough to show that $(v + w')^2 < 1$, or equivalently according to $(**)$, $(u + w'^- w)^2 = 1$ for every $w' < w$. By total symmetry and Claim 2, $u ; u \geq 1' + u + w$. On the other hand, Claim 1 implies that $w'^- w(u ; v) \neq 0$, so for the atom v we get $v \leq u ; w'^- w$. Thus $(u + w'^- w)^2 \geq u ; u + u ; w'^- w \geq 1$. To prove the principality of (v, w, u), we still need to show that $v = \tilde{z}(u ; u)^-$ and $w = \tilde{z}(u ; u)$. We have $\tilde{z}(u ; u) = (v + w)(u ; u) \geq w$ by Claim 2, on the other hand, $\tilde{z}(u ; u) < \tilde{z} = v + w$ because otherwise we would get $u ; u = 1$, contradicting the minimality of z. Since v is an atom, the only possibility is that $\tilde{z}(u ; u) = w$. Finally, $\tilde{z}(u ; u)^- = (\tilde{z}^- + \tilde{z}(u ; u))^- = (1' + u + w)^- = v$.

CLAIM 6. If (u, v, w) is a principal triple then u, v and w are all atoms. By Claim 4, the second component of any principal triple must be an atom, so using Claim 5 we obtain that v, w and u are each atoms.

CLAIM 7. $A \cong B_{12}$. Total symmetry and the conditions on u and v in the definition of the principal triple (u, v, w) clearly imply that $w ; w = 1' + w + v$, and similarly the principal triples (v, w, u) and (w, u, v) give $u ; u = 1' + u + w$ and $v ; v = 1' + v + u$. In particular, $v(u ; u) = 0$ and $u \leq v ; v$ hold for the atoms u, v. These, together with $w \leq u ; v$ from Claim 1, will yield that $u ; v = v + w$ for any principal triple (u, v, w), thus Claim 5 ensures that we also have $v ; w = w + u$ and $w ; u = u + v$ as required in the definition of B_{12}. □

3. Infinite minimal relation algebras

In this section we show that the finiteness condition in Theorem 7 cannot be removed. The first lemma implies that the collection of all finite and cofinite elements of $S(\mathbb{Z})$ is a subalgebra of $S(\mathbb{Z})$ that has no finite nontrivial subalgebras. This example was constructed independently by B. Jónsson and S. Givant. A simple infinite relation algebra that has no finite nontrivial subalgebras generates a variety that has no finite nontrivial members. In [1] it is shown that there are uncountably many such varieties.

B. Jónsson also constructed the totally symmetric algebra B_∞. In Theorem 10 we prove that this algebra is minimal, and in Theorem 11 we show that the first example is not minimal.

LEMMA 8. *Let A be an integral relation algebra and suppose that $B \subseteq A$ is closed for the operations*: $1'$, $\cdot$, $+$, $;$, $^\smile$ *and for all $x, y \in B$ we have $xy^- \in B$. If B has no largest element then the subalgebra generated by B consists of the elements of B and their complements only and, in case B has no nontrivial equivalence elements, this subalgebra has no nontrivial finite subalgebra.*

Furthermore, if B' is a subset of an integral relation algebra A' satisfying the above conditions and $\varphi : B \to B'$ is a lattice isomorphism preserving the operations $;$ *and* $^\smile$ *then φ can be extended to an isomorphism between the generated subalgebras.*

Proof. Let $B^* = B \cup \{a^- : a \in B\}$.

CLAIM. If $a \in B$ then there is a non-zero element a_1 of B such that $a_1 \le a^-$. Since $a^- \ge a^- b \in B$ for any $b \in B$, the claim holds unless $a^- b = 0$ for all b, in which case $b \le a$ for every $b \in B$. But then a is the largest element of B, contrary to the assumptions about B.

Clearly, B^* is a Boolean subalgebra, and it is just as easy to see that B^* is closed for $^\smile$. So it remains to show that $a ; b^- \in B^*$ and $a^- ; b^- \in B^*$ for any $a, b \in B$. Since this is certainly true if $a = 0$ or $b = 0$, we may assume that both are nonzero.

$a ; b^- \in B^*$: Consider the elements $x = a ; b$, $y = (a^\smile ; x)b^-$ and $z = (a ; y)^- x$, which by assumption are in B. We show that $a ; b^- = z^-$, which is in B^*. Firstly $(a ; b^-)z = 0$ since

$$(a^\smile ; z)b^- = (a^\smile ; z)(a^\smile ; x)b^- = (a^\smile ; z)y = 0,$$

where the last equality follows because $(a ; y)z = 0$. Secondly, $y \le b^-$ implies $a ; y \le a ; b^-$, so

$$a ; b^- + z = a ; b^- + (a ; y)x + (a ; y)^- x = a ; b^- + x = a ; b^- + a ; b = a ; 1 = 1$$

and this time the last equality follows from the assumption of integrality.

$a^- ; b^- \in B^*$: Let $0 \ne x \le a^-$ where $x \in B$, $y = x ; b$ and $0 \ne z \le (y ; b^\smile + a)^-$ where $z \in B$ (the existence of x and z follows from the claim above). Then $z(y ; b^\smile) = 0$ implies that $y(z ; b) = 0$ and this together with $z ; b + z ; b^- = z ; 1 = 1$ gives that $y \le z ; b^-$, so $x ; b = y \le z ; b^- \le a^- ; b^-$. Since $x \le a^-$ implies that $x ; b^- \le a^- ; b^-$, we get $1 = x ; 1 = x ; b + x ; b^- \le a^- ; b^-$.

Assume that B^* has no nontrivial equivalence elements. Let C be a finite subalgebra of B^*. Then defining e to be the sum of the elements of $B \cap C$, we get that $e ; e \in B \cap C$, so $e ; e \leq e$, and similarly, $e^{\smile} \leq e$ and $1' \leq e$. Thus e is an equivalence element in B, consequently $e = 1'$ and C is the trivial subalgebra.

Finally, if for the bijection $\varphi : B \to B'$ we define $\bar{\varphi}$ so that $\bar{\varphi}(a^-) = \bar{\varphi}(a)^-$ for any element a of B then the formulas $a^{-\smile} = a^{\smile -}$, $a ; b^- = ((a ; ((a^{\smile} ; a ; b)b^-))^-(a ; b))^-$ and $a^- ; b^- = 1$ derived in the proof immediately imply that $\bar{\varphi}$ is a relation algebra isomorphism. □

COROLLARY 9. *Let C be a set of infinitely many pairwise disjoint elements in an integral relation algebra A, and suppose that the set B of all finite sums of elements of C is closed under conversion, relative products and contains $1'$. Then the subalgebra generated by C consists of the elements of B and the complements of these, and, in case B contains no nontrivial equivalence elements, this subalgebra has no nontrivial finite subalgebras.*

Furthermore, if $\varphi : C \to C'$ is a bijection onto a set C' of pairwise disjoint elements of an integral relation algebra A' so that the natural extension of φ to B preserves the operations ; and $^{\smile}$ for the elements of C then φ can be extended to an isomorphism between the subalgebras generated by C and C', respectively.

THEOREM 10. *B_∞ defined in Table 2 is a totally symmetric minimal relation algebra.*

Proof. Define a norm on $\mathbb{Z} \times \mathbb{Z}$ by $\|(k, l)\| = |k| + |l|$. Then for the disjoint elements $a_i = \{x \in \mathbb{Z} \times \mathbb{Z}\colon \|x\| = 2i\}$ for $i = 0, 1, 2, \ldots$ in $S(\mathbb{Z} \times \mathbb{Z})$, the relative multiplication rule $a_i ; a_j = \sum_{t=|i-j|}^{i+j} a_t$ can be proved in the following way. Since $\|x + y\| \leq \|x\| + \|y\|$ and since $\|x + y\| - \|x\| - \|y\|$ is always divisible by 2, the inclusion $a_i ; a_j \leq \sum_{t=|i-j|}^{i+j} a_t$ obviously holds.

On the other hand, let $w = (k, l) \in a_t$ for some t such that $|i - j| \leq t \leq i + j$. Without loss of generality, we may assume that $i \geq j$ and $k \geq l \geq 0$. Then $u = ((k - t) + (i - j),\ l + ((i + j) - t))$ and $v = (t - (i - j),\ -((i + j) - t))$ shows that $w = u + v \in a_i ; a_j$. Using Corollary 9 and the fact that $a_i^{\smile} = a_i$ for every i, we get that the algebra generated by $\{a_i\colon i = 0, 1, 2, \ldots\}$ consists of the finite sums of these elements and of their complements. Since the atoms of this algebra satisfy the identities $x = x^{\smile}$ and $x \leq x ; x$ and for the complements of finite sums of atoms $x ; x = 1$, the algebra is totally symmetric.

Notice also that for $x = (\{(1, 0), (0, 1)\}^s)^2$ we have $a_1 = x$ and then the relative multiplication rule gives that $a_n = x^n(x^{n-1})^-$. Hence the algebra defined above is indeed the same as B_∞ defined in Table 2.

For any $y \in B_\infty$, the following statements are true: if $y \,;\, y = 1$ then $y^- \,;\, y^- < 1$ and if $y \notin \{0, 1'\}$ and $y \,;\, y < 1$ then $y \,;\, y$ generates a subalgebra isomorphic to B_∞. Since these statements must hold in any subalgebra of any ultrapower of B_∞, we obtain that every nontrivial subalgebra of an ultrapower of B_∞ contains a subalgebra isomorphic to B_∞, so B_∞ is minimal. □

We now give an example of a relation algebra N in which every nontrivial subalgebra is isomorphic to N, yet this algebra is not minimal. Let N be the subalgebra of $S(\mathbb{Z})$ generated by the element $\{1\}^s$. Note that this is the subalgebra of all finite and cofinite elements of $S(\mathbb{Z})$ mentioned at the beginning of this section.

THEOREM 11. *For the relation algebra N defined above*
(i) *every nontrivial subalgebra of N is isomorphic to A_3 or N, but*
(ii) *$\mathscr{V}(N)$ is strictly greater than $\mathscr{V}(B_\infty)$.*

Proof. (i) Observe first that every atom $\{n\}^s$ of N generates a subalgebra N_n isomorphic to N. By Corollary 9 the generated subalgebra consists of the finite unions of $\{kn\}^s$ $(k = 0, 1, \ldots)$ and the complements of those, and $\varphi : \{k\}^s \mapsto \{kn\}^s$ induces an isomorphism between N and N_n. We are going to prove that every subalgebra is isomorphic to one of these N_n.

Let A be a proper subalgebra of N different from A_3, and let $G \subseteq \mathbb{Z}$ be the set of those group elements that occur in a finite element of N. Then G is nonempty since A is nontrivial, furthermore G is a subgroup of $\mathbb{Z}$, since the relative products and converses of finite elements are finite. Thus there is a positive integer n such that $G = n\mathbb{Z}$ and every cofinite element of A contains $\mathbb{Z}\backslash n\mathbb{Z}$. This shows that A is a subalgebra of $N_n \cong N$. Changing N to N_n allows us to assume that $G = \mathbb{Z}$.

Let us denote by x^* the greatest number in a finite element $x \in A$ and for an integer k let $\bar{k} = \{0, 1, \ldots, k\}^s$. Take an element $x > 1'$ in A that satisfies the following three conditions:

(1) $\overline{\lceil x^*/2 \rceil} \leq x$ (where $\lceil x^*/2 \rceil$ is the least integer $\geq x^*/2$);
(2) $|\overline{x^*}x^-|$ is minimal (where $|\,|$ denotes the cardinality of the element as a subset of $\mathbb{Z}$);
(3) among those satisfying the above conditions, x^* is minimal.

We will show that $x = \{0, 1\}^s$, whence $A = N$ and we are done. First we need to prove that such an x exists, i.e. there is an element above 1' satisfying condition (1). Take a finite element y of A such that $0, 1 \in y$, which exists since we assumed $G = \mathbb{Z}$. Notice that for $k \geq y^*$, the inclusions $y^k \geq \overline{y^*}$ and $y^k \geq \{0, y^*, 2y^*, \ldots, ky^*\}^s$ hold, consequently $y^{2k} \geq \overline{ky^*} = \overline{(y^{2k})^*/2}$, so y^{2k} satisfies condition (1).

Let $n = x^*$. Then clearly, $x^2 \geq \bar{n} + \{n + |t| : 0 \neq t \in x\}^s$ (where the latter two sets are disjoint), so $|x^2| \geq 2n + 1 + |x| - 1$. This gives that $|\overline{x^* x^-}| = 2n + 1 - |x| = (4n + 1) - (2n + |x|) \geq |\overline{(x^2)^*(x^2)^-}|$, hence condition (2) gives that $x^2 = \bar{n} + \{n + |t| : 0 \neq t \in x\}^s$.

Now consider $y = x(x^2x^-)^2$. We can see that $t \in (x^2x^-)^2$ holds for $0 \leq t \leq n - 1$, since for $t < n/2$, we have $t = (n + \lceil n/2 \rceil) - (n + (\lceil n/2 \rceil - t)) \in (x^2x^-)^2$ and for $t \geq n/2$, we have $t = (2n) - (n + (n - t)) \in (x^2x^-)^2$. On the other hand, $n \notin (x^2x^-)^2$, because $(x^2x^-)\overline{\lceil n/2 \rceil} = 0$ implies that n cannot be obtained as the sum of two nonnegative elements of x^2x^- and, in order to be the difference of two positive elements, n should be of the form $(n + |t|) - |t|$ but $|t| \notin x^2x^-$.

Thus we obtained that $y = x(\overline{n - 1})$. This y clearly satisfies condition (1) and we also have $|\overline{y^*y^-}| \leq |\overline{x^*x^-}|$ and $y^* < x^*$. Because of the choice of x, this means that $y = 1'$, consequently $x = \{0, n\}^s$. On the other hand, $x \geq \overline{\lceil n/2 \rceil}$ holds, so $n = 1$. Thus $x = \{0, 1\}^s$ generates N.

(ii) We will show that B_∞ is a subalgebra of an ultrapower of N. This implies that $\mathcal{V}(B_\infty) \subseteq \mathcal{V}(N)$, and the two varieties are obviously not equal since N is not totally symmetric. It is easy to check that the map $\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}^\omega/F$ (where F is the congruence defined by a nonprincipal ultrafilter on ω) defined by $\varphi((1, 0)) = (1, 1, 1, \ldots)/F$ and $\varphi((0, 1)) = (1, 2, 3, \ldots)/F$ is an embedding and it defines a one-to-one map ψ from the atoms of $R(\mathbb{Z} \times \mathbb{Z})$ into $R(\mathbb{Z})^\omega/F$ in a natural way: $\psi(\{(k, l)\}) = (\{k + l\}, \{k + 2l\}, \{k + 3l\}, \ldots)/F$. In this way we obtain a map satisfying the conditions of Corollary 9, and thus ψ has an extension $\bar{\psi}$ to the subalgebra generated by the atoms of $R(\mathbb{Z} \times \mathbb{Z})$. The only thing we have to check is that the image of B_∞ is in the subalgebra N^ω/F, and this follows from the fact that B_∞ is generated by the element $x = (\{(1, 0), (0, 1)\}^s)^2$ and $\bar{\psi}(\{(1, 0), (0, 1)\}^s) = (\{1\}^s + \{1\}^s, \{1\}^s + \{2\}^s, \{1\}^s + \{3\}^s, \ldots)/F \in N^\omega/F$. □

4. Further results about symmetric minimal relation algebras

The first result is a generalization of Theorem 7. It has a similar though longer proof which can be found in [3].

THEOREM 12. *Let A be a finite simple symmetric relation algebra. If for all $x \in A$ either $x \leq x ; x$ or $x(x ; x) = 0$ then A has a subalgebra isomorphic to one of $B_1, \ldots, B_7, B_9, \ldots, B_{12}$.*

B_8 is the only finite minimal relation algebra in Table 2 that does not satisfy the universal sentence $x \leq x ; x$ or $x(x ; x) = 0$.

LEMMA 13. *Let A be a simple symmetric relation algebra, and for $u \leq 0'$ define $a = u^{-}0'$, $b = u(u\ ;u)$ and $c = u(u\ ;u)^{-}$. If $a\ ;a = 1$ and $a \leq (b\ ;b)(c\ ;c)(b\ ;c)$ then a generates a subalgebra isomorphic to B_8.*

Proof. From the definition of b and c it follows that $u = b + c$ and $c(u\ ;u) = 0$, hence $0 = c(b\ ;b) = c(b\ ;c) = c(c\ ;c)$ and consequently $0 = b(b\ ;c) = b(c\ ;c)$. Also $b \leq u\ ;u = b\ ;b + b\ ;c + c\ ;c$ and therefore $b \leq b\ ;b$. By assumption a is below each of $b\ ;b$, $c\ ;c$ and $b\ ;c$, so we obtain $b\ ;b = a + b + 1'$, $c\ ;c = a + 1'$ and $b\ ;c = a$.
It remains to show that $a\ ;b = 0' = a\ ;c$. By integrality we have $b = b\ ;1' \leq b\ ;(c\ ;c) = (b\ ;c)\ ;c \leq a\ ;c$ and similarly $c \leq a\ ;b$. Now

$$a = c\ ;b \leq c\ ;(b\ ;b) = (c\ ;b)\ ;b = a\ ;b$$

$$b \leq c\ ;a \leq c\ ;(b\ ;b) = (c\ ;b)\ ;b = a\ ;b$$

and

$$a + c \leq a\ ;b \leq (c\ ;c)\ ;b = c\ ;(c\ ;b) = c\ ;a$$

hence $a\ ;b = 0' = a\ ;c$. □

THEOREM 14. *Let A be a simple symmetric relation algebra, and suppose a is an atom of A such that $a \leq 0'$ and a satisfies $a\ ;a = 1$. Then either $A \cong A_3$ or A has a subalgebra isomorphic to B_3, B_4, B_6, B_7 or B_8.*

Proof. If A has a nontrivial equivalence element, then Theorem 1 implies that A has a subalgebra isomorphic to B_3 or B_4. On the other hand, if A has no nontrivial equivalence element, then it follows from Lemma 3 that $x\ ;x^{-}0' = 0'$ for all $0 < x < 0'$. So if we let $u = a^{-}0'$ then either $u = 0$, in which case $A \cong A_3$, or $u \leq 0' = a\ ;u$. In the latter case, since a is an atom, it follows that $a \leq u\ ;u$. If $u\ ;u = a + 1'$ or $u\ ;u = 1$ then we have a subalgebra isomorphic to B_6 or B_7 respectively. Hence we may assume that $u = b + c$, where $b, c \neq 0$, $b \leq u\ ;u$ and $c(u\ ;u) = 0$. Note that $0' = a + b + c$, and a, b, c are disjoint and nonzero, so $0' = b\ ;(a + c) = b\ ;a + b\ ;c$ by Lemma 3. Then $u(b\ ;c) \leq u(u\ ;c) = 0$ implies that $b + c = u \leq b\ ;a$. Since a is an atom we have $a \leq b\ ;b$ and $a \leq b\ ;c$. Similarly $c \leq 0' = c\ ;(a + b) = c\ ;a + c\ ;b$ and $c(c\ ;b) \leq c(u\ ;u) = 0$, hence $c \leq c\ ;a$ and therefore $a \leq c\ ;c$. Now we have satisfied all the assumptions of Lemma 13, so a generates a subalgebra isomorphic to B_8. □

By a similar, though much longer argument, we can show that if a simple symmetric relation algebra has two atoms a, b such that $(a + b)\ ;(a + b) = 1$ then it has a subalgebra isomorphic to one of B_i ($i = 1, 2, \ldots, 12$). Note that stating this result for the sum of any finite number of atoms in the assumption instead of two would yield a full classification of the finite minimal symmetric relation algebras. However no generalization along these lines has emerged so far. Thus we conclude this paper with the following two open problems:

(i) Are there other infinite minimal simple relation algebras? How many?
(ii) Is the list of finite minimal relation algebras complete?

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