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Remarks on the Stability of Minimal Submanifolds of $\mathbb{R}^n$

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1. Statement of Results

Let M^m be an m-dimensional compact orientable C^∞ manifold with boundary ∂M and let $x\colon M\to\mathbb{R}^n$ be a minimal immersion of M into Euclidean n space. It is well known that M is stationary with respect to m-dimensional volume. That is, if E is a normal vector field on M vanishing on ∂M and if φ_t denotes the flow generated by E, then setting $A(t)=\text{volume } \varphi_t(M)$, $\left.\frac{dA}{dt}\right|_{t=0}=0$. We say that M *is* (infinitesimally) *stable* if $\left.\frac{d^2A}{dt^2}\right|_{t=0}>0$, i.e. $A(M)$ is a strict minimum for all such variations.

Recently Barbosa and do Carmo [1] have shown that if M is a minimal surface in $\mathbb{R}^3$ ($m=2$, $n=3$) such that the area of the spherical image (without multiplicity) of M is less than 2π, then M is stable. The constant 2π is sharp. It is the purpose of this note to prove similar but weaker stability results in the general case.

Theorem 1. *Let $x\colon M^2\to\mathbb{R}^n$, $n>3$ be a minimal immersion. There is a constant c_1 depending only on n such that if $\int_M |K|\,dV<c_1$, then M is stable.*

Theorem 2. *Let $x\colon M^m\to\mathbb{R}^n$, $m\geqq 3$ be a minimal immersion. There is a constant c_2 depending only on m such that if $\left(\int_M \|B\|^m dV\right)^{1/m}<c_2$, then M is stable.*

Here K is the Gauss curvature of M and $\|B\|$ is the norm of the second funcamental from of M and is defined in Section 2. We point out that although these two results formally look the same, the proof of Theorem 1 is of a different character than that of Theorem 2 and it is best to separate the two cases.

As an application of Theorem 1 we obtain information about solutions of Plateau's problem for a C^2 Jordan curve Γ in $\mathbb{R}^n$. Let $x\colon \Delta\to\mathbb{R}^n$, $\Delta=$the unit disk in $\mathbb{R}^2$ be a $C^2(\bar{\Delta})$ mapping such that x is harmonic and conformal in Δ and $x|\partial D$ is a homeomorphism. Denote by $\int_\Gamma k(s)\,ds$ the total curvature of Γ.

Corollary 3. *Let $x\colon \Delta\to\mathbb{R}^n$ be a solution to Plateau's problem for a C^2 Jordan curve Γ. If $\int_\Gamma k(s)\,ds<2\pi+c_1$ (c_1 as in Theorem 1) then $x(\Delta)$ is stable.*

Corollary 3 follows from Theorem 1, since the generalized Gauss Bonnet formula [7] and the assumption $\int_{\Gamma} k(s)\,ds < 2\pi + c_1$ imply that x is an immersion and $\int_{x(\Delta)} |K|\,dV < c_1$.

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2. The Second Variation and Sobolev's Inequality

Let $x\colon M^m \to \mathbb{R}^n$ be a minimal immersion. We denote by $\bar{\nabla}$ the standard connection on $\mathbb{R}^n$ and by ∇ the induced Riemannian connection on M. The tangent and normal bundle of M are denoted by TM and NM respectively, and X^T, X^N denote the projection of a Euclidean vector field X along the mapping onto TM, NM respectively. The second fundamental form of the immersion $B\colon TM \times TM \to NM$ is given by

$$B(X, Y) = \bar{\nabla}_X Y - [\bar{\nabla}_X Y]^T = \bar{\nabla}_X Y - \nabla_X Y = [\bar{\nabla}_X Y]^N.$$

Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of TM_p and let $E \in NM_p$. We define

$$\|B \cdot E\|^2 = \sum (B(e_i, e_j) \cdot E)^2.$$

It is easy to see that if $|E| = 1$, $\|B \cdot E\|^2$ is the sum of the squares of the principal curvatures of M at p with respect to the unit normal direction E. Let $E_1, \ldots, E_{n-m}$ be an orthonormal basis of NM_p. Then the quantity

$$\|B\|^2 = \sum_k \|B \cdot E_k\|^2$$

is the square of the length of the second fundamental form. From the equation of Gauss we find that for a minimal immersion, $-\|B\|^2$ is just the intrinsic scalar curvature of M. In particular for $m = 2$, $-\|B\|^2 = 2K$, where K is the Gauss curvature of M.

We next define the Laplace operator $\Delta\colon \Gamma(N(M)) \to \Gamma(N(M))$, where $\Gamma(N(M))$ denotes the space of C^∞ normal vector fields on M. Let ∇_X define the connection in NM given by $\nabla_X v = (\bar{\nabla}_X v)^N$, $X \in \Gamma(TM)$. In terms of this connection

$$\Delta v(p) = \sum_{j=1}^{m} (\nabla_{e_j} \nabla_{e_j} v - \nabla_{\nabla_{e_j}} v)(p)$$

where $e_1, \ldots, e_m$ are an orthonormal basis of TM_p.

We can now state the second variational formula ([5], p. 48):

$$\left.\frac{d^2 A}{dt^2}\right|_{t=0} = -\int_M (E \cdot \Delta E + \|B \cdot E\|^2)\,dV \tag{1}$$

for a variation vector field $E \in \Gamma(N(M))$, $E|\partial M = 0$.

We analyze this formula further by writing $E=uv$, $|v|=1$ and $u|\partial M=0$. Then

$$\begin{aligned}\Delta E &= \sum_{j=1}^{m} (\nabla_{e_j} \nabla_{e_j}(uv) - \nabla_{\nabla_{e_j} e_j}(uv)) \\ &= \sum_{j=1}^{m} (\nabla_{e_j}(e_j(u)v + u\nabla_{e_j} v) - (\nabla_{e_j} e_j)(u)v - u\nabla_{\nabla_{e_j} e_j} v) \\ &= \sum_{j=1}^{m} \{(e_j(e_j(u)) - (\nabla_{e_j} e_j)(u))v + u(\nabla_{e_j} \nabla_{e_j} v - \nabla_{\nabla_{e_j} e_j} v) + 2e_j(u)\nabla_{e_j} v\} \\ &= (\Delta u)v + u\Delta v + 2\sum_{j=1}^{m} e_j(u)\nabla_{e_j} v.\end{aligned}$$

Hence

$$E \cdot \Delta E = u\Delta u + u^2 v \cdot \Delta v \tag{2}$$

since $v \cdot \nabla_{e_j} v = 0$. Combining (1) and (2) we have the formula

$$\left.\frac{d^2 A}{dt^2}\right|_{t=0} = -\int_M (u\Delta u + (v \cdot \Delta v + \|B \cdot v\|^2)u^2\, dV \tag{3}$$

for a variation vector field $E=uv$.

Lemma 4.

$$\left.\frac{d^2 A}{dt^2}\right|_{t=0} \geqq -\int_M (u\Delta u + \|B\|^2 u^2)\, dV = \int_M (|\nabla u|^2 - \|B\|^2 u^2)\, dV.$$

Proof. Since $\|B \cdot v\|^2 \leqq \|B\|^2$ we need only show $v \cdot \Delta v \leqq 0$. For

$$v \cdot \Delta v = \sum_{j=1}^{m} v \cdot (\nabla_{e_j} \nabla_{e_j} v - \nabla_{\nabla_{e_j} e_j} v) = \sum_{j=1}^{m} v \cdot \nabla_{e_j} \nabla_{e_j} v = -\sum_{j=1}^{m} (\nabla_{e_j} v)^2.$$

Here we have used the identities $v\nabla_X v = 0$ and $0 = X(v \cdot \nabla_X v) = (\nabla_X v)^2 + v \cdot \nabla_X \nabla_X v$. Substitution in (3) completes the proof.

We see from the lemma that a sufficient condition for stability of M is that the principal eigenvalue of the linear eigenvalue problem

$$\begin{aligned}\Delta u + \lambda \|B\|^2 u &= 0 \quad \text{on } M \\ u &= 0 \quad \text{on } \partial M\end{aligned}$$

is greater than one. Therefore it suffices to show that

$$\int_M \|B\|^2 u^2\, dV < \int_M |\nabla u|^2\, dV \tag{4}$$

for all $u \not\equiv 0$ in the Sobolev space $\mathring{H}_1(M)$.

Proof of Theorem 2. As we have just remarked, we need only show that the hypothesis $(\int_M \|B\|^m dV)^{1/m} < c_2$ implies (4). Suppose for contradiction that (4) is false, namely

$$\int_M |\nabla u|^2\, dV \leqq \int_M \|B\|^2 u^2\, dV \tag{5}$$

for some $u \in \mathring{H}_1(M)$, $u \not\equiv 0$. Since M is minimal we can apply the Sobolev inequality of Michael and Simon [6] to u to obtain

$$\Big(\int_M u^{\frac{2m}{m-2}} dV\Big)^{\frac{m-2}{2m}} \leqq c_2^{-1}(m)\Big(\int_M |\nabla u|^2 dV\Big)^{\frac{1}{2}}. \tag{6}$$

Then from (5) we have

$$\Big(\int_M u^{\frac{2m}{m-2}} dV\Big)^{\frac{m-2}{2m}} \leqq c_2^{-1}\Big(\int_M \|B\|^2 u^2 dV\Big)^{\frac{1}{2}}. \tag{7}$$

But by Hölder's inequality

$$\int_M \|B\|^2 u^2 dV \leqq \Big(\int_M \|B\|^m dV\Big)^{\frac{2}{m}} \Big(\int_M u^{\frac{2m}{m-2}} dV\Big)^{\frac{m-2}{m}}. \tag{8}$$

Combining (7) and (8) gives

$$\Big(\int_M u^{\frac{2m}{m-2}} dV\Big)^{\frac{m-2}{2m}} \leqq c_2^{-1}\Big(\int_M \|B\|^m dV\Big)^{\frac{1}{m}} \Big(\int_M u^{\frac{2m}{m-2}} dV\Big)^{\frac{m-2}{2m}}.$$

Hence $\big(\int_M \|B\|^m dV\big)^{1/m} \geqq c_2$ contradicting our assumption.

3. The Isoperimetric Inequality on the Gaussian Image

In order to prove Theorem 1, it suffices to show that

$$\int_M -2Ku^2 dV < \int_M |\nabla u|^2 dV \tag{9}$$

for all $u \in \mathring{H}_1(M)$, $u \not\equiv 0$. To establish this inequality we make use of the generalized Gauss map [2, 5], the necessary properties of which we now briefly describe.

If we think of M^2 as a Riemann surface, the generalized Gauss map is an antiholomorphic mapping, $g\colon M^2 \to Q_{n-2}(\mathbb{C}) \subset P_{n-1}(\mathbb{C})$ into complex projective space, such that the image lies in the algebraic subvariety Q_{n-2} which is given in homogeneous coordinates by the equation $Z_1^2 + \cdots + Z_n^2 = 0$. The Gauss map induces a new metric on M by setting $\langle U, V\rangle_S = \langle g_* U, g_* V\rangle$, $U, V \in TM$ where $\langle\, ,\rangle$ denotes the (renormalized) Fubini-Study metric on $P_{n-1}(\mathbb{C})$. As in the classical case

$$\langle U, V\rangle_S = -K\, U \cdot V \quad \text{([5], pp. 116–117)}. \tag{10}$$

Since K is nonpositive and vanishes only at isolated points of M (branch points of g), $\langle\, ,\rangle_S$ defines a Riemannian metric away from these points. Also we have the important formula for the area $A(g(M))$ of the Gaussian image of M counting multiplicites:

$$A(g(M)) = \int_M |K|\, dV.$$

In proving (9) we will first establish a Sobolev inequality for smooth functions with respect to the metric $\langle\, ,\rangle_S$.

Lemma 5. *Let D be a relatively compact subdomain of M which contains no branch points of g. Then if $u \in C_0^\infty(D)$*

$$\left(\int_D u^2\, dS\right)^{\frac{1}{2}} \leqq c(n) \int_D |\nabla_S u|\, dS \tag{11}$$

provided $\int_D |K|\, dV < \varepsilon(n)$. Here ε is a positive constant which depends on the geometry of Q_{n-2} and hence only on n; $c(n)$ is an absolute constant.

Proof. Let $D_t = \{p \in D \mid u(p) > t\}$. We write $A_S(D_t)$ for the area of D_t and $L_S(\partial D_t)$ for the length of ∂D_t with respect to the metric $\langle\ ,\ \rangle_S$. Since D contains no branch points of g, $g\colon D \to Q_{n-2}$ is an immersion. In fact g is a *minimal immersion* since g is antiholomorphic and Q_{n-2} is a Kähler manifold ([5], pp. 33). It follows from the isoperimetric inequality in the Gaussian image $g(D_t)$ (Theorem 2.2 of [4]) that for almost all t

$$A_S^{\frac{1}{2}}(D_t) \leqq c(n)\, L_S(\partial D_t) \tag{12}$$

provided $\int_D |K|\, dV < \varepsilon(n)$. It is now a well-known fact that (12) is equivalent to (11). For we have the formulas

$$\int_D u^2\, dS = \int_0^\infty 2t A_S(D_t) \quad \text{and} \quad \int_D |\nabla_S u|\, dS = \int_0^\infty L_S(\partial D_t)\, dt. \tag{13}$$

Since $\int_0^\infty 2t A_S(D_t)\, dt \leqq \left(\int_0^\infty A_S^{\frac{1}{2}}(D_t)\, dt\right)^2$ (12) and (13) imply (11).

(The proof of this inequality is elementary; the point is that $A_S^{\frac{1}{2}}(D_t)$ is non-nonincreasing and $\int_0^\infty A_S^{\frac{1}{2}}(D_t)\, dt < \infty$.)

Proof of Theorem 1. Let $u \in \mathring{H}_1(M)$, $u \not\equiv 0$. Since u can be approximated in $H_1(M)$ by $C_0^\infty(M)$ functions, we can assume that $u \in C_0^\infty(M)$ and $D = \text{support } u$ is a relatively compact subdomain of M. In particular D can contain only a finite number of branch points of g. Let D' denote D with these branch points deleted. Then if $\int_M |K|\, dV < \varepsilon(n)$

$$\left(\int_D u^2\, dS\right)^{\frac{1}{2}} \leqq c(n) \int_D |\nabla_S u|\, dS \tag{14}$$

since we can approximate u by functions with compact support in D' each satisfying (11) (by Lemma 5) and obtain (14) in the limit. Applying Schwarz's inequality to (14) we obtain

$$\int_D u^2\, dS \leqq c^2(n) \int_D |\nabla_S u|^2\, dS \int_D dS$$

or equivalently using relation (10)

$$\int_D -K u^2\, dV \leqq c^2(n) \int_D |K|\, dV \int_D |\nabla u|^2\, dV. \tag{15}$$

Finally choosing the constant c_1 in the statement of Theorem 1 as $c_1 = \min(\varepsilon(n), \frac{1}{2}c^{-2}(n))$ we have $\int_D -Ku^2\, dV < \frac{1}{2}\int_D |\nabla u|^2\, dV$ establishing (9) and completing the proof of Theorem 1.

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