Algebra Universalis, 33 (1995) 216–236 0002-5240/95/020216-21$01.50 + 0.20/0

Monotone Jónsson operations and near unanimity functions

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Abstract. We define nonextendible colored posets and zigzags of a poset. These notions are related to the earlier notions of gaps, holes, obstructions and zigzags considered by Duffus, Nevermann, Rival, Tardos and Wille. We establish some properties of zigzags. By using these properties we give a proof of the well known conjecture that states that any finite bounded poset which admits Jónsson operations, also admits a near unanimity function. We also provide an infinite poset that shows that we cannot drop the finiteness in this conjecture.

1. Introduction

A *clone* on a set A is a set of finitary operations on A that contains the projection operations and is closed under composition of functions. A *monotone clone* consists of all monotone operations on a partially ordered set. Monotone clones have received a great deal of attention. A poset is called *bounded* if it has a largest and a smallest element. Martynjuk proved in [8] that the monotone clone of a finite bounded poset is maximal, i.e., a dual atom, in the lattice of all clones on the underlying set. Later, in [15] Rosenberg showed that there are six classes of maximal clones on an arbitrary finite set. One of them is the class of monotone clones of finite bounded posets. The clones of the other five types were shown to be finitely generated; see [7]. The problem remained: Is the monotone clone of a finite bounded poset finitely generated? The answer is yes if the poset is in one of the following classes: finite lattice ordered sets [7], finite bounded posets with at most seven elements [7], posets obtained from some finite lattice ordered set by cancelling a convex subset of it [4] and finite posets with the strong selection property [14]. But in 1986 Tardos answered the problem negatively in [17] by showing that the monotone clone of the eight element poset **T** shown in Figure 1 is not finitely generated.

Each monotone clone of a finite bounded poset that is known to be finitely generated contains a special, n-ary operation, called a near unanimity function. For

Presented by R. McKenzie.
Received February 12, 1992; accepted in final form February 24, 1994.

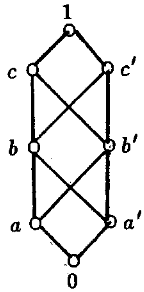


Figure 1. Poset **T**.

$n \geq 3$ an n-ary function f is called a *near unanimity function*, briefly a *nuf*, if it obeys the identity $f(x, \ldots, x, \underset{i}{y}, x, \ldots, x) = x$ for every $1 \leq i \leq n$. If $n = 3$, then f is called a *majority function*. So the next question that occurs for finite bounded posets is whether a monotone clone is finitely generated if and only if it contains a near unanimity function. For any algebra that has a near unanimity function among its term operations, the clone of term operations is finitely generated, which proves one direction of the claim; see [1] and for an explicit proof see [16]. The remaining part, to prove or disprove the other direction, is mentioned as an open problem in [2], [3] and [9]. In [5] Demetrovics and Rónyai proved that the monotone clone of any crown is finitely generated, but it has no near unanimity function. As a crown is not a bounded poset the previous problem is still open.

In [17] Tardos uses certain special objects, zigzags, to prove that the clone of **T** is not finitely generated. In the same paper there is a remark that characterizes the finite posets having a monotone near unanimity function, in terms of their zigzags. Objects similar to zigzags called gaps [6] and [13], holes [12], and obstructions [11], have been studied from an order theoretical point of view. In [12] and [14] there is a characterization of finite posets having a monotone majority function. The proof establishes a connection, similar to Tardos's remark, between holes and ternary near unanimity functions. The ideas in the papers cited in this paragraph led us to study the zigzags of arbitrary posets.

A variety of algebras is called *congruence modular* if the congruence lattice of every algebra in the variety satisfies the modular law. A variety of algebras is called *congruence distributive* if the congruence lattice of every algebra in the variety is distributive. It is a basic observation (see [10]) that if an algebra has a near unanimity term operation, then the variety generated by the algebra is congruence distributive and so it is congruence modular. We call an algebra a *monotone algebra* of a poset **P** if the set of term operations of the algebra coincides with the set of all monotone operations of **P**. Davey showed in [2] that a monotone algebra of a bounded poset generates a congruence modular variety if and only if it generates a congruence distributive variety. In [9] McKenzie gave a useful characterization of finite bounded posets for which the corresponding monotone algebras generate congruence distributive varieties. It was conjectured in [2], [3] and [9] that if a monotone algebra of a finite bounded poset generates a congruence distributive variety, then the algebra has a near unanimity term operation. Since the congruence distributivity of the variety generated by an algebra is equivalent to the algebra having some special term operations, called Jónsson operations, satisfying certain identities, we can rephrase the preceding conjecture as follows. A finite bounded poset **P** has monotone Jónsson operations if and only if there exists a monotone near unanimity operation on **P**. By the use of zigzags we settle this conjecture. So, as an immediate corollary, we get that the monotone clone of a finite bounded poset **P** is finitely generated, whenever **P** has monotone Jónsson operations.

The outline of the paper is as follows. In Section 2 we give the basic definitions. Some examples of zigzags are presented. Finite posets with the strong selection property and finite posets with near unanimity functions are characterized in terms of zigzags. In Section 3 we prove some claims about the shape of zigzags. In Section 4 we give a proof of the above mentioned conjecture that states that for every finite bounded poset **P** there exist monotone Jónsson operations if and only if there exists a monotone near unanimity operation on **P**. We also give an example of an infinite bounded poset that has monotone Jónsson terms but has no monotone near unanimity function. So in the infinite case that conjecture does not hold.

2. Zigzags and related concepts

The main concept of this paper is the zigzag. To define it we need to clarify some basic concepts involving partially ordered sets. After defining a zigzag we present some examples. Then, in Proposition 2.3 we characterize via zigzags the finite posets with the strong selection property. In 2.4 we give a proof of Tardos's remark in [17] that describes, via zigzags, the finite posets with monotone near unanimity functions.

A *partially ordered set*, briefly, a *poset* **P** is a nonempty set P with a reflexive, transitive, antisymmetric relation $\leq_{\mathbf{P}}$ on it, i.e., $\mathbf{P} = (P, \leq_{\mathbf{P}})$. A poset with a largest and a smallest element is called *bounded*. For an arbitrary poset **P** we define $<_{\mathbf{P}} = \leq_{\mathbf{P}} \setminus \{(p, p): p \in P\}$. We use boldface capital letters to denote a poset throughout this paper and when it is possible we leave off the subscript from the relational symbol. In a poset **P**, $b \in P$ *covers* $a \in P$, i.e., $a \prec_{\mathbf{P}} b$ if $a <_{\mathbf{P}} b$ and there is no $c \in P$ such that $a <_{\mathbf{P}} c <_{\mathbf{P}} b$.

Let $\mathbf{P}$ be a poset and let T be a subset of $P \cup \prec_{\mathbf{P}}$ with $P \nsubseteq T$. We denote the poset $(P \backslash T, (\leq_{\mathbf{P}} |_{P \backslash T}) \backslash T)$ by $\mathbf{P} \backslash T$ and we say that T is *cancelled* from $\mathbf{P}$. For two posets $\mathbf{P}$ and $\mathbf{Q}$ with $P \cap Q = \varnothing$ let $\mathbf{P} + \mathbf{Q}$ denote the poset $(P \cup Q, \leq_{\mathbf{P}} \cup \leq_{\mathbf{Q}} \cup \{(p, q): p \in P, q \in Q\})$. Let I be an index set and let $\mathbf{P}_i$, $i \in I$, be posets. Then the *product* $\prod_{i \in I} \mathbf{P}_i$ is a poset with the base set $\prod_{i \in I} P_i$ and the ordering $(a_i)_{i \in I} \leq (b_i)_{i \in I}$ if and only if $a_i \leq_{\mathbf{P}_i} b_i$ for every $i \in I$.

A poset $\mathbf{Q}$ is a *subposet* of $\mathbf{P}$, i.e., $\mathbf{Q} \leq \mathbf{P}$ if $Q \subseteq P$ and $\leq_{\mathbf{Q}} = \leq_{\mathbf{P}} |_Q$. Mainly the following weaker notion of comparing posets is used throughout the paper. We say that a poset $\mathbf{Q}$ is *contained in* $\mathbf{P}$ if $Q \subseteq P$ and $\leq_{\mathbf{Q}} \subseteq \leq_{\mathbf{P}} |_Q$. If $\mathbf{Q}$ is contained in $\mathbf{P}$ we write $\mathbf{Q} \subseteq \mathbf{P}$. We say that $\mathbf{Q}$ is *properly contained* in $\mathbf{P}$ if $\mathbf{Q} \subseteq \mathbf{P}$ and $\mathbf{Q} \neq \mathbf{P}$. We note that each poset $\mathbf{P}$ contains any antichain defined on any nonempty subset of $\mathbf{P}$. An *up set* of $\mathbf{P}$ is a subset S of P for which $s \in S$ and $s \leq p \in P$ imply $p \in S$. A *down set* of $\mathbf{P}$ is defined dually. Let $S \subseteq P$. Then S^* denotes the set of all elements of P which are greater than or equal to every element of S in $\mathbf{P}$. S_* is defined dually. A map $f: Q \to P$ is called *monotone* with respect to $\mathbf{Q}$ and $\mathbf{P}$ if for every $a \leq_{\mathbf{Q}} b$ we have $f(a) \leq_{\mathbf{P}} f(b)$. For such a map we use the abbreviation that $f: \mathbf{Q} \to \mathbf{P}$ is monotone. We say that an n-ary operation f on P *preserves* $\mathbf{P}$ or $\mathbf{P}$ *admits* f if and only if $f: \mathbf{P}^n \to \mathbf{P}$ is monotone.

Let $\mathbf{P}$ and $\mathbf{Q}$ be posets. A pair $(\mathbf{Q}, f)$ is called a $\mathbf{P}$-*colored poset* if f is a partially defined map from Q to P. If f can be extended to a fully defined monotone map $f': \mathbf{Q} \to \mathbf{P}$ on Q then f and $(\mathbf{Q}, f)$ are called $\mathbf{P}$-*extendible*, otherwise f and $(\mathbf{Q}, f)$ are called $\mathbf{P}$-*nonextendible*. A $\mathbf{P}$-*zigzag* is a $\mathbf{P}$-nonextendible, $\mathbf{P}$-colored poset $(\mathbf{H}, f)$, where H is finite and for every $\mathbf{K}$ properly contaned in $\mathbf{H}$, the $\mathbf{P}$-colored poset $(\mathbf{K}, f|_K)$ is $\mathbf{P}$-extendible. Roughly speaking, the $\mathbf{P}$-zigzags are the finite, minimal, nonextendible $\mathbf{P}$-colored posets. The notion of a zigzag is related to that of the gap [6], hole [12], obstruction [11] and the zigzags defined in [17]. When it is clear what $\mathbf{P}$ is we leave it off from the terms like $\mathbf{P}$-zigzags, $\mathbf{P}$-extendible, etc.

For two $\mathbf{P}$-colored posets $(\mathbf{H}, f)$ and $(\mathbf{Q}, g)$ we say that $(\mathbf{H}, f)$ is *contained* in $(\mathbf{Q}, g)$ and we write $(\mathbf{H}, f) \subseteq (\mathbf{Q}, g)$ if $\mathbf{H} \subseteq \mathbf{Q}$ and $f = g|_H$. Observe that every finite nonextendible colored poset contains a zigzag. Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-colored poset and let T be a subset of $H \cup \prec_{\mathbf{H}}$ with $H \nsubseteq T$. We denote the $\mathbf{P}$-colored poset $(\mathbf{H} \backslash T, f|_{H \backslash T})$ by $(\mathbf{H}, f) \backslash T$ and we say that T is *cancelled* from $(\mathbf{H}, f)$. For a $\mathbf{P}$-*colored poset* $(\mathbf{H}, f)$ we define $C(\mathbf{H}, f) = \{h \in H : f(h) \text{ exists}\}$ and $N(\mathbf{H}, f) = H \backslash C(\mathbf{H}, f)$. We call the elements of $C(\mathbf{H}, f)$ *colored elements* and the elements of $N(\mathbf{H}, f)$ *noncolored elements*. If $C(\mathbf{H}, f)$ and $N(\mathbf{H}, f)$ are nonempty we define the posets $\mathbf{C}(\mathbf{H}, f)$ and $\mathbf{N}(\mathbf{H}, f)$ by the restriction of $\leq_{\mathbf{H}}$ to $C(\mathbf{H}, f)$ and $N(\mathbf{H}, f)$, respectively.

Each poset $\mathbf{P}$ is associated with two undirected graphs on P. One is called the *comparability graph* of $\mathbf{P}$ that has an edge between a and b if $a <_{\mathbf{P}} b$. The other is the *covering graph* of $\mathbf{P}$ that has an edge between a and b if $a \prec_{\mathbf{P}} b$. Sometimes, as an example, we shall draw a picture of a **P**-colored poset $(\mathbf{H}, f)$ for some particular **P**. A picture like this consists of the covering graph of **H** and an element of **H** is drawn as a small shaded circle or a small empty circle according to whether f is defined or not defined on the given point. Every shaded point is labelled by the value of f.

EXAMPLE 2.1. Let **P** be the poset shown in Figure 2. Then the **P**-colored poset $(\mathbf{H}, f)$ shown in Figure 2 is a **P**-zigzag.

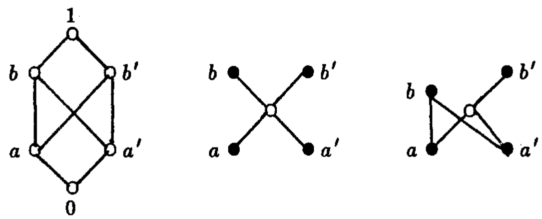


Figure 2. Poset **P**, a **P**-zigzag $(\mathbf{H}, f)$ and $(\mathbf{H}, f)\backslash\{(h_1, h_2)\}$.

Proof. The **P**-colored poset $(\mathbf{H}, f)$ clearly is **P**-nonextendible. Let h_1 be the noncolored element and let h_2 be the element colored by b in $(\mathbf{H}, f)$. Then $(\mathbf{H}, f)\backslash\{(h_1, h_2)\}$, which is a maximal **P**-colored poset properly contained in $(\mathbf{H}, f)$, is extendible by coloring h_1 by b'. The other three maximal **P**-colored posets properly contained in $(\mathbf{H}, f)$ are **P**-extendible by symmetric arguments. Thus, by definition, $(\mathbf{H}, f)$ is a **P**-zigzag. □

A **P**-colored poset $(\mathbf{H}, f)$ is called *monotone* if f is a monotone map on its domain; otherwise $(\mathbf{H}, f)$ is *nonmonotone*. A monotone **P**-colored poset $(\mathbf{H}, f)$ is called an *extension* of the **P**-colored poset $(\mathbf{H}, g)$ if f is an extension of g. Observe that for any poset **P** the **P**-colored two element chain in which the top is colored by a and the bottom is colored by b where $b \nleq a \in P$, is a nonmonotone **P**-zigzag and every nonmonotone **P**-zigzag is of this form. So for every monotone zigzag $(\mathbf{H}, f)$, f is monotone on its domain and there is at least one element of H where f is not defined.

A poset determined by the ordering of a complete lattice is called a *complete lattice ordered set*. Observe that there are no monotone **L**-zigzags for a complete lattice ordered set **L**. The reason is that every monotone **L**-colored poset $(\mathbf{Q}, f)$ is extendible by $f' : a \mapsto \bigwedge \{f(b) : b \in Q, a \leq b\}$. In particular, no finite lattice ordered set possesses monotone zigzags.

EXAMPLE 2.2. Let **T** be the poset shown in Figure 3. Then the **T**-colored posets shown in Figure 3 are **T**-zigzags.

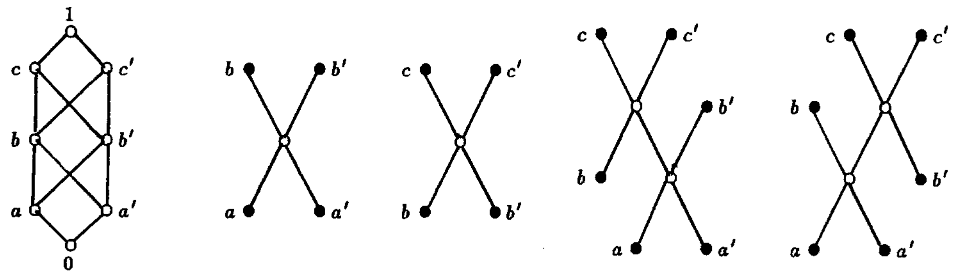


Figure 3. Poset **T** and some **T**-zigzags.

Proof. Argue as in Example 2.1 or see [17]. □

We note that it is easy to prove that the only monotone **P**-zigzag is $(\mathbf{H}, f)$ for **P** in Figure 2. It is also easy to prove that the poset **T** in Figure 3 has infinitely many zigzags. All the monotone **T**-zigzags are described by Tardos in [17].

Let us fix a finite poset **P**. Let $B_{\mathbf{P}} = \{(D, U): D$ is a down set of $\mathbf{P}$, U is an up set of $\mathbf{P}$ and $D^* \cap U_* \neq \varnothing\}$ and let $\mathbf{B_P}$ be the poset on $B_{\mathbf{P}}$ given by the ordering $(D_1, U_1) \leq (D_2, U_2)$ if and only if $D_1 \subseteq D_2$ and $U_2 \subseteq U_1$. We say **P** has the *strong selection property* if and only if there exists a monotone map $g: \mathbf{B_P} \to \mathbf{P}$ such that for every $(D, U) \in B_{\mathbf{P}}$, $g((D, U)) \in D^* \cap U_*$.

In the following proposition we describe via zigzags the finite posets with the strong selection property. Their monotone zigzags turn out to be the ones with exactly one noncolored element. It is not hard to get a description of the zigzags with one noncolored element; see Proposition 3.11.

PROPOSITION 2.3. *A finite poset* **P** *has the strong selection property if and only if every* **P***-zigzag has at most one noncolored element.*

Proof. Let $(\mathbf{H}, f)$ be a **P**-zigzag, where **P** has the strong selection property. With every $h \in H$ we can associate a pair (D_h, U_h) defining $D_h = \{a \in P :$ there exists $h' \in C(\mathbf{H}, f)$, $h' \leq h$ and $a \leq f(h')\}$ and $U_h = \{a \in P:$ there exists $h' \in C(\mathbf{H}, f)$, $h \leq h'$ and $f(h') \leq a\}$. We claim that $(\mathbf{H}, f)$ cannot contain two or more noncolored elements. For otherwise if $h \in C(\mathbf{H}, f)$ then $f(h) \in D_h^* \cap U_{h*}$ and if $h \in N(\mathbf{H}, f)$ then by cancelling a noncolored point $h_0 \neq h$ from $(\mathbf{H}, f)$ the resulting colored poset will be extendible, so $D_h^* \cap U_{h*} \neq \varnothing$. Thus, by setting $f'(h) = g(D_h, U_h)$, where g is obtained from the definition of the strong selection property, we would get a monotone extension of f to **H**.

Now, let us suppose that **P** is a finite poset such that **P**-zigzags have at most one noncolored element. We want to show that the above map g exists. Let h be the partial map from $B_{\mathbf{P}}$ to P defined by $h((\{p\}_*, \{p\}^*)) = p, p \in P$. The map h is, clearly, monotone from its domain to **P**. We claim that the **P**-colored poset $(\mathbf{B_P}, h)$ is **P**-extendible. For otherwise there is a **P**-zigzag $(\mathbf{H}, f)$ contained in $(\mathbf{B_P}, h)$. Since every monotone **P**-zigzag has exactly one noncolored element, by (2) of Claim 3.11, $(\mathbf{H}, f)$ is of the form shown in Figure 4. Let (D, U) be the only noncolored element of $(\mathbf{H}, f)$. Obviously, $(D, U) \in \mathbf{B_P}$. On the other hand, (D, U) cannot be in $B_\mathbf{P}$ since $a_1, \ldots, a_n \in D$ and $b_1, \ldots, b_m \in U$ and so $D^* \cap U_*$ is empty. Thus, $(\mathbf{B_P}, h)$ is an extendible colored poset. Let g be a monotone extension of h to $\mathbf{B_P}$. The map g clearly satisfies the requirements in the definition of the strong selection property. □

An n-ary function $f: P^n \to P$, $n \geq 3$, is called a *near unanimity function*, briefly an n-nuf if and only if $f(a, \ldots, a, \underset{i}{b}, a, \ldots, a) =$ for every $a, b \in P$ and for every $1 \leq i \leq n$.

As we will see in the next remark the number of colored elements in the zigzags of a finite poset has a great importance in dealing with near unanimity functions preserving the poset. This remark is mentioned by Tardos in [17] without proof.

Remark 2.4. Let $n \geq 3$. A finite poset **P** admits an n-ary near unanimity function if and only if in every **P**-zigzag the number of colored elements is at most $n - 1$.

Proof. Let us suppose **P** admits an n-nuf. Let $(\mathbf{H}, f)$ be a **P**-colored poset and let $C(\mathbf{H}, f) = \{h_1, \ldots, h_l\}$ such that $n \leq l = |C(\mathbf{H}, f)|$. Furthermore, let us suppose that for every $\mathbf{H}'$ properly contained in $\mathbf{H}$, $(\mathbf{H}', f|_{H'})$ is extendible. So for any $h \in C(\mathbf{H}, f)$ the colored poset $(\mathbf{H}_h, f|_{H_h})$, where $\mathbf{H}_h = (H \backslash \{h\}, \leq_\mathbf{H}|_{H\backslash\{h\}})$, is extendible. Let us take a function $f_h : H \to P$ for each $h \in C(\mathbf{H}, f)$ such that $f_h|_{H_h}$ is a **P**-extension of $f|_{H_h}$ to $\mathbf{H}_h$. By the hypothesis there is an l-nuf M_l preserving **P**. Hence $M_l(f_{h_1}, \ldots, f_{h_l})$ is a fully defined montone map from **H** to **P** that extends f. Thus, every **P**-zigzag must have at most $n - 1$ colored elements.

Now, let us suppose that in every **P**-zigzag the number of colored elements is at most $n - 1$. We look at the partial map $M_n : P^n \to P$ defined by $M_n(a, b, \ldots, b) = \cdots = M_n(b, \ldots, b, a) = b$. Let us suppose M_n is not extendible to $\mathbf{P}^n$ as a monotone map from $\mathbf{P}^n$ to **P**. Then the colored poset $(\mathbf{P}^n, M_n)$ contains a **P**-zigzag $(\mathbf{H}, f)$. We know that $|C(\mathbf{H}, f)| \leq n - 1$. Hence there exists an i with $1 \leq i \leq n$ such that f takes on the i-th component for each element of $C(\mathbf{H}, f)$. But then the i-th projection from H to P is a **P**-extension of f to **H**, which is a contradiction. □

We note that the "only if" part of the proof is valid for any poset **P**.

3. Properties of zigzags

In the previous section we defined zigzags. Now, we will explore the properties of these objects. In Proposition 3.1 we give a useful characterization of zigzags. Then in 3.2 through 3.11 we prove some claims concerning the shape of zigzags. In Proposition 3.12 we show that every zigzag of a finite poset **P** is a monotone image of a zigzag of height less than the height of **P**.

A poset is called *connected* if its comparability graph is connected.

PROPOSITION 3.1. *Let* $(\mathbf{H}, f)$ *be a* **P***-colored poset, where* H *is finite. Then* $(\mathbf{H}, f)$ *is a* **P***-zigzag if and only if* **H** *is connected,* $(\mathbf{H}, f)$ *is not* **P***-extendible and by cancelling any covering pair of* $(\mathbf{H}, f)$ *the resulting colored poset is* **P***-extendible.*

Proof. The "only if" part should be clear by the definition of a zigzag. To show the "if" part, let $\mathbf{H}' \subseteq \mathbf{H}$ and let us suppose $(\mathbf{H}', f|_{H'})$ is not **P**-extendible. We will show $\mathbf{H}' = \mathbf{H}$. By the assumption $\mathbf{H}'$ has to contain all covering pairs of **H** otherwise $\mathbf{H}'$ would be extendible. Then H' has to contain every point of H which is in a connected component C of the comparability graph of **H**, where C has at least two elements. By the assumption **H** has only one component and this component has more than one element since $(\mathbf{H}, f)$ is not **P**-extendible. So $\mathbf{H} \subseteq \mathbf{H}'$; hence $\mathbf{H} = \mathbf{H}'$. □

Now, we list some facts concerning the shape of a **P**-zigzag $(\mathbf{H}, f)$.

CLAIM 3.2. *Let* $(\mathbf{H}, f)$ *be a* **P***-zigzag. The subgraph spanned by* $N(\mathbf{H}, f)$ *in the covering graph of* **H** *is connected.*

Proof. Let us suppose the claim is not true. Then if we cancel the elements of one component of the subgraph spanned by $N(\mathbf{H}, f)$ from $(\mathbf{H}, f)$ we get a colored poset that is extendible. If we cancel all the other elements of $N(\mathbf{H}, f)$ from $(\mathbf{H}, f)$ we get another extendible colored poset. Because of the assumption, by taking the union of two extensions which extend the above two colored posets we would get an extension of f to **H**. □

CLAIM 3.3. *Let* $(\mathbf{H}, f)$ *be a monotone zigzag and let* $a \in C(\mathbf{H}, f)$*. For every* $b \in H$ *which satisfies* $a \prec b$ *or* $b \prec a$ *we have* $b \in N(\mathbf{H}, f)$*.*

Proof. Without loss of generality we can assume $b \prec a$. If $b \in C(\mathbf{H}, f)$ then cancelling (b, a) in $(\mathbf{H}, f)$ we get an extendible colored poset and since $(\mathbf{H}, f)$ is monotone putting back (b, a) we still have an extendible colored poset. This contradicts the fact that $(\mathbf{H}, f)$ is not extendible. □

A *monotone map* between two **P**-colored posets means a monotone map between the two base posets which maps each a-colored element to an a-colored element and each noncolored element to a noncolored element. We say that a **P**-colored poset $(\mathbf{H}, f)$ is a *monotone image* of a **P**-colored poset $(\mathbf{H}', f')$ if there exists a monotone map from $(\mathbf{H}', f')$ onto $(\mathbf{H}, f)$.

CLAIM 3.4. *For every* **P**-*zigzag* $(\mathbf{H}, f)$ *there exists a* **P**-*zigzag* $(\mathbf{H}', f')$ *such that* $N(\mathbf{H}, f) = N(\mathbf{H}', f')$, $(\mathbf{H}, f)$ *is a monotone image of* $(\mathbf{H}', f')$ *and every colored element of* $(\mathbf{H}', f')$ *occurs in exactly one covering pair of* $\mathbf{H}'$.

Proof. For nonmonotone **P**-zigzags the claim is obvious. For a monotone **P**-zigzag $(\mathbf{H}, f)$ the **P**-colored poset $(\mathbf{H}', f')$ is defined as follows. The poset $\mathbf{N}(\mathbf{H}', f')$ is contained in $\mathbf{N}(\mathbf{H}, f)$ in such a way that the covering graph of $\mathbf{N}(\mathbf{H}', f')$ is the subgraph spanned by $N(\mathbf{H}, f)$ in the covering graph of $(\mathbf{H}, f)$. For every $s \in C(\mathbf{H}, f)$ and $h \in N(\mathbf{H}, f)$ with $s \prec_{\mathbf{H}} h$ there is a single element $s' \in C(\mathbf{H}', f')$ such that h is the unique element covering s' in $(\mathbf{H}', f')$ and $f'(s') = f(s)$. By Proposition 3.1 the so defined colored poset $(\mathbf{H}', f')$ is a zigzag which obviously satisfies the requirements of the claim. □

A colored poset in which every colored element occurs in exactly one covering pair is called a *standard colored poset*.

CLAIM 3.5. *Let* $(\mathbf{H}, f)$ *be a* **P**-*zigzag and let* a *and* b *be two different elements of* $C(\mathbf{H}, f)$. *Let us suppose that there exists* $c \in N(\mathbf{H}, f)$ *with* $c \prec a, b$. *Then* $f(a) \nleq f(b)$.

Proof. Let us suppose that the claim is not true. Then $f(a) \leq f(b)$. If we cancel (c, b) we get an extendible colored poset for which any extension extends $(\mathbf{H}, f)$, too. □

CLAIM 3.6. *Let* $(\mathbf{H}, f)$ *be a* **P**-*zigzag and let* $a, b \in C(\mathbf{H}, f)$, *where* $a < b$. *Then* $f(a) \neq f(b)$.

Proof. The claim is obvious for a nonmonotone zigzag. Now, let us suppose $(\mathbf{H}, f)$ is a monotone zigzag. Then, by Claim 3.3, there is a $c \in N(\mathbf{H}, f)$ such that $a < c < b$. By cancelling c from $(\mathbf{H}, f)$ the resulting colored poset has an extension f'. If $f(a) = f(b)$ then f' together with the coloring of c by $f(a)$ is an extension of f to $\mathbf{H}$. □

In 3.4 we split colored points to obtain a new zigzag. In certain cases we can do the reverse.

CLAIM 3.7. *Let* $(\mathbf{H}, f)$ *be a* $\mathbf{P}$*-zigzag. Let* $a, b \in C(\mathbf{H}, f)$ *be two different maximal elements of* $\mathbf{H}$ *for which* $f(a) = f(b)$. *Then there exists a zigzag* $(\mathbf{H}', f')$ *for which* $N(\mathbf{H}', f') = N(\mathbf{H}, f)$ *and there is an onto monotone map form* $(\mathbf{H}, f)$ *to* $(\mathbf{H}', f')$ *which identifies only a and b.*

Proof. We define $\mathbf{H}'$ as follows. Above every element of $\mathbf{H}\backslash\{a, b\}$ covered by a or b in $\mathbf{H}$ we put the covering element $c \notin H$. The coloring f' of $\mathbf{H}'$ is defined by f on $\mathbf{H}\backslash\{a, b\}$ and by $f'(c) = f(a)$. By Proposition 3.1, $(\mathbf{H}', f')$ is a zigzag which satisfies the requirements of the claim. □

CLAIM 3.8. *Let* $(\mathbf{H}, f)$ *be a* $\mathbf{P}$*-zigzag. Every monotone* $g : \mathbf{H} \to \mathbf{H}$ *that is the identity map on* $C(\mathbf{H}, f)$ *has to be onto, i.e., an automorphism of* $\mathbf{H}$.

Proof. Let us suppose g is a monotone map that is the identity on $C(\mathbf{H}, f)$ and maps H into a proper subset H' of H. Since $(\mathbf{H}, f)$ is a zigzag there is a $\mathbf{P}$-extension f' of f to $\mathbf{H}'$. So $f' \circ g$ is a $\mathbf{P}$-extension of f to $\mathbf{H}$ which contradicts that $(\mathbf{H}, f)$ is not extendible. □

Let $\mathbf{Q}$ be a finite poset. Then $a \in \mathbf{Q}$ is called *retractable* if there is a non-onto monotone map on $\mathbf{Q}$ that fixes each element different from a. An element $a \in \mathbf{Q}$ is called *irreducible* if there is a unique $b \in Q$ with $a \prec b$ or $b \prec a$. Observe that every irreducible element is retractable.

CLAIM 3.9. *Let* $(\mathbf{H}, f)$ *be a* $\mathbf{P}$*-zigzag. Then* $N(\mathbf{H}, f)$ *has no retractable element of* $\mathbf{H}$.

Proof. Apply Claim 3.8. □

CLAIM 3.10. *If* $\mathbf{P} = \mathbf{Q} + 1$, *then every maximal element of a* $\mathbf{P}$*-zigzag* $(\mathbf{H}, f)$ *is colored.*

Proof. Let us suppose there exists a maximal element h of $\mathbf{H}$ that is not colored in $(\mathbf{H}, f)$. By cancelling h in $(\mathbf{H}, f)$ we get an extendible colored poset. Now, an extension of f to this colored poset together with the coloring of h by 1 extends f to $\mathbf{H}$. This is a contradiction, so we have the claim. □

CLAIM 3.11. *For a* $\mathbf{P}$*-zigzag* $(\mathbf{H}, f)$ *the following hold.*

(1) *If* $|N(\mathbf{H}, f)| = 0$, *then* $(\mathbf{H}, f)$ *is a two element nonmonotone zigzag.*

(2) *If* $|N(\mathbf{H}, f)| = 1$, *then* $(\mathbf{H}, f)$ *is the first colored poset shown in Figure* 4, *where m and n are nonnegative integers such that* $m + n > 0$ *and* $n, m \neq 1$. *Moreover, f is an order isomorphism on its domain.*

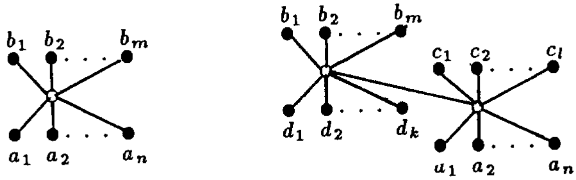


Figure 4. Monotone zigzags with one and two noncolored elements.

(3) *If* $|N(\mathbf{H}, f)| = 2$, *then* $(\mathbf{H}, f)$ *is the second colored poset shown in Figure* 4, *where* $k, l \geq 1$ *and* m *and* n *are nonnegative integers for which* $m, n \neq 1$. *Moreover, any comparable pair in Range*(f) *not shown in the picture is of the form* $d_i < c_j$, $c_j < b_s$ *or* $a_t < d_i$ *for some* $1 \leq i \leq k$, $1 \leq j \leq l$, $1 \leq s \leq m$ *and* $1 \leq t \leq n$.

Proof. We showed (1) before Example 2.2. First we prove (2). By Claim 3.3 we get the picture of $(\mathbf{H}, f)$. Obviously, $m + n > 0$. Claim 3.9 gives $n, m \neq 1$. Claim 3.5 gives that f is an order isomorphism on its domain. Next we prove (3). By using Claim 3.2 and Claim 3.3 we can see that $(\mathbf{H}, f)$ has to be a standard zigzag as shown in the picture. Claim 3.9 gives $k, l \geq 1$ and $m, n \neq 1$. To prove the last claim use Claim 3.5, Claim 3.6 and the definition of zigzag. □

Let $\mathbf{S}_i$ be the poset that we obtain by cancelling the top element from a Boolean lattice ordered set with i atoms. Let $\mathbf{S}_i^d$ be the dual of $\mathbf{S}_i$. Let $\mathbf{T}_{l,k}$ be the poset $(\mathbf{S}_l + 1 + \mathbf{S}_k^d) \times (1 + 1)$ without its top and bottom element. Then the reader can easily check the following claims. The poset $\mathbf{S}_n + \mathbf{S}_m^d$ has a zigzag of the form in (2) of Claim 3.11. The poset $\mathbf{S}_n + \mathbf{T}_{l,k} + \mathbf{S}_m^d$ has a zigzag of the form in (3) of Claim 3.11. For more on these posets see [11] and [13].

In a finite poset $\mathbf{P}$ we define the *length* between two elements $a \leq_{\mathbf{P}} b$ as the maximum cardinality of a chain between a and b. The length between a and b is denoted by $\ell_{\mathbf{P}}(a, b)$.

PROPOSITION 3.12. *Let* $\mathbf{P}$ *be a finite, bounded poset. Then for every monotone* $\mathbf{P}$*-zigzag* $(\mathbf{H}, f)$ *there is a standard zigzag* $(\mathbf{H}', f')$ *such that* $(\mathbf{H}, f)$ *is a monotone image of* $(\mathbf{H}', f')$ *and for every maximal chain* $a = a_1 < a_2 < \cdots < a_n = b$ *of* $\mathbf{H}'$, $n \leq \ell_{\mathbf{P}}(f'(a), f'(b)) + 1$.

Proof. First of all, recall Claim 3.10 to see that for every maximal chain of any $\mathbf{P}$-zigzag the bottom and top elements are colored. The proof will proceed by induction on the cardinality of the set of noncolored elements of a zigzag. By (2) in Claim 3.11, the zigzags with one noncolored element satisfy the claim. Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag with $|N(\mathbf{H}, f)| = m \geq 2$ and let us suppose that for every $\mathbf{P}$-zigzag with $m - 1$ noncolored elements we have the claim. Let $h \in N(\mathbf{H}, f)$ be a maximal element of $\mathbf{N}(\mathbf{H}, f)$. Let us color h by $p \in P$ in $(\mathbf{H}, f)$. For every $p \in P$ the resulting colored poset contains a zigzag $(\mathbf{H}_p, f_p)$ such that $h \in H_p$. We select $(\mathbf{H}_p, f_p)$ nonmonotone whenever this is possible. If $(\mathbf{H}_p, f_p)$ is monotone then by the induction hypothesis there exists a standard zigzag $(\mathbf{Q}_p, g_p)$ which has $(\mathbf{H}_p, f_p)$ as its monotone image and its maximal chains satisfy the desired property in the claim. Observe that under the monotone map from $(\mathbf{Q}_p, g_p)$ onto $(\mathbf{H}_p, f_p)$ the preimage of h contains only maximal elements of $\mathbf{Q}_p$. If $(\mathbf{H}_p, f_p)$ is nonmonotone then we make $(\mathbf{Q}_p, g_p)$ a copy of $(\mathbf{H}_p, f_p)$. Now, we can construct a standard colored poset $(\mathbf{Q}, g)$ from $(\mathbf{Q}_p, g_p), p \in P$, by gluing all elements of the preimages of h into one single noncolored point, called h', meanwhile preserving the coloring of the other points. This colored poset is not extendible since $(\mathbf{Q}_p, g_p), p \in \mathbf{P}$, is not extendible. So it contains a zigzag $(\mathbf{Q}', g')$ in which the colored elements are exactly the extremal elements. Let us construct a standard zigzag $(\mathbf{H}', f')$ for $(\mathbf{Q}', g')$ as in Claim 3.4.

Clearly, there is a monotone map from $(\mathbf{H}', f')$ to $(\mathbf{H}, f)$ and this map must be onto; otherwise $(\mathbf{H}', f')$ would be extendible. Let $a = a_1 < a_2 < \cdots < a_n = b$ be a maximal chain of $\mathbf{H}'$. If this chain does not contain h' then we are done by the induction hypothesis. Otherwise $a_{n-1} = h' \prec_{\mathbf{H}'} b$. The chain $a = a_1 < a_2 < \cdots < a_{n-1} = h'$ has to be in the preimage of H_p for some p. If $(\mathbf{H}_p, f_p)$ is monotone then by applying the induction hypothesis to $(\mathbf{Q}_p, g_p)$ we get $n - 1 \leq \ell_{\mathbf{P}}(g_p(a), p) + 1$. Since $(\mathbf{H}_p, f_p)$ is monotone our construction guarantees that if we color h by p in $(\mathbf{H}, f)$ we get a monotone colored poset. Now, $f'(b)$ must be the color of an element above h in $(\mathbf{H}, f)$. Hence, by Claim 3.9 and Claim 3.5 we have $p < f'(b)$. Since $g_p(a) = f'(a)$ we have $\ell_{\mathbf{P}}(g_p(a), p) + 1 \leq \ell_{\mathbf{P}}(f'(a), f'(b))$. By combining the preceding two inequalities we get the claim. If $(\mathbf{H}_p, f_p)$ is nonmonotone, then $n = 3$ and the claim is obvious. □

4. Finite bounded posets admitting Jónsson operations

Ternary operations $d_i, 0 \leq i \leq n$, on a set are called *Jónsson operations* if they satisfy the *Jónsson identities* given by

$$d_0(x, y, z) = d_n(z, y, x) = d_i(x, y, x) = x \qquad \text{for } 0 \leq i \leq n,$$

$$d_{2i}(x, x, y) = d_{2i+1}(x, x, y) \qquad \text{for } 0 \leq i \leq (n-1)/2$$

and

$$d_{2i+1}(x, y, y) = d_{2i+2}(x, y, y) \qquad \text{for } 0 \leq i \leq (n-2)/2.$$

As we mentioned in Section 1 an algebra has Jónsson operations among its term operations if and only if the variety generated by the algebra is congruence distributive. Moreover, if an algebra has a near unanimity function among its term operations, then it also has Jónsson operations. The converse of this claim for monotone algebras of finite bounded posets is conjectured in [2], [3] and [9]. The main result of the paper is the proof of this conjecture in Theorem 4.1. In Theorem 4.1 we give a list of equivalent properties for finite bounded posets. The first property on the list is that a finite bounded poset **P** admits a near unanimity function and the second one is that **P** admits Jónsson operations. We also provide an example of an infinite bounded poset which admits Jónsson operations but admits no nuf; see Example 4.3.

A finite poset is called a *fence* if its comparability graph is a path. In a connected poset **P** we define the *distance* $d(a, b)$ between a and b as $n-1$, where n is the smallest integer for which there exists an n-element subfence connecting a and b in **P**. So $d(a, a) = 0$. The *diameter of* **P** is the supremum of $d(a, b)$, where $a, b \in P$. The *diameter* of a colored poset is the diameter of its base poset. In **P** we define the *up distance* from a to b as the least positive integer n such that there is a subset $\{a_0, \ldots, a_n\} \subseteq P$ with $a = a_0$, $b = a_n$ and $a_0 \leq a_1 \geq a_2 \leq \cdots$. We define the *down distance* from a to b dually. We note that by the definition both the up and down distances from a to a are 1.

THEOREM 4.1. *For a finite bounded poset* **P** *the following are equivalent*:

(1) **P** *admits a near unanimity function.*

(2) **P** *admits Jónsson operations.*

(3) **P** *admits ternary operations* $D_1, \ldots, D_{n'}$, *for an* $n' \geq 1$, *satisfying*

$$D_1(x, x, y) = D_{n'}(y, x, x) = D_i(x, y, x) = x \qquad \text{for } 1 \leq i \leq n'$$

and

$$D_i(x, y, y) = D_{i+1}(x, x, y) \qquad \text{for } 1 \leq i \leq n'-1.$$

(4) *For some* n *there exists a partially defined, monotone* n-*nuf that is fully defined on the set of* n-*tuples* $A_n = \{(a, \ldots, a, \underset{i}{b}, c, \ldots, c): a, b, c \in P, 1 \leq i \leq n\}$.

(5) *There exists a finite* m *such that every* **P**-*zigzag has a diameter at most* m.

(6) *The number of* **P**-*zigzags is finite.*

Proof. (1) implies (2): One can prove it easily as follows. Let $f: \mathbf{P}^s \to \mathbf{P}$ be a monotone nuf. Then we define $d_{2i-1}(x, y, z) = f(z, \ldots, z, \underset{i+1}{y}, x, \ldots, x)$ and

$d_{2i}(x, y, z) = d_{2i-1}(x, z, z)$ for $1 \le i \le s-1$. Let $d_0(x, y, z) = x$. So the operations $d_j(x, y, z)$, $0 \le j \le 2(s-2)$, are Jónsson operations.

(2) implies (3): In [9] McKenzie proves that **P** admits Jónsson operations if and only if **P** admits operations $b_0(x, y), \ldots, b_{m'}(x, y)$ which satisfy

$$x = b_0(x, y) = b_i(x, x) = b_{m'}(y, x) \qquad \text{for } 0 \le i \le m',$$

$$b_{2i}(x, y) \le b_{2i+1}(x, y) \qquad \text{for } 0 \le i \le (m'-1)/2$$

and

$$b_{2i+1}(x, y) \ge b_{2i+2}(x, y) \qquad \text{for } 0 \le i \le (m'-2)/2.$$

In fact, in his proof, when he proves the if part (Theorem 2.3), he uses the b_i to construct monotone Jónsson operations $d_0(x, y, z), \ldots, d_{2n'-1}(x, y, z)$, where the operations with even indices do not depend on their second variable. With the help of the d_{2i-1}, $1 \le i \le n'$, we define

$$D_1(x, y, z) = d_1(x, y, z), \ldots, D_{n'}(x, y, z) = d_{2n'-1}(x, y, z).$$

For these operations the first line of identities in (3) immediately follows from the Jónsson identities, and the second line of identities in (3) follows from

$$\begin{aligned} D_i(x, y, y) &= d_{2i-1}(x, y, y) = d_{2i}(x, y, y) = d_{2i}(x, x, y) \\ &= d_{2i+1}(x, x, y) = D_{i+1}(x, x, y). \end{aligned}$$

Thus if **P** admits Jónsson operations then **P** admits operations defined in (3).

(3) implies (4): Let $B_i = \{(a, \ldots, a, \underset{i}{b}, c, \ldots, c): a, b, c \in P\} \subseteq P^{n'+2}$ for $1 \le i \le n'+2$. Note that $B_1 \subseteq B_2$ and $B_{n'+2} \subseteq B_{n'+1}$. So $A_{n'+2} = \bigcup_{i=2}^{n'+1} B_i$. Let $D_1, \ldots, D_{n'}$ be the ternary operations given in (3). We define an $(n'+2)$-nuf f on $A_{n'+2}$. Let $f(a, \ldots, a, \underset{i}{b}, c, \ldots, c) = D_{i-1}(c, b, a)$ on B_i for $2 \le i \le n'+1$. Observe that if $\mathbf{d} = (a, \ldots, a, b, c, \ldots, c) \in B_i \cap B_j$, where $2 \le i < j \le n'+1$, then either $\mathbf{d}$ is a constant vector or $a = b \ne c$ or $a \ne b = c$. In the last two cases j has to be $i+1$. Since $f(a', \ldots, \underset{i}{a'}, b', \ldots, b') = D_{i-1}(b', a', a') = D_i(b', b', a')$ for $2 \le i \le n'$, $f(\mathbf{d})$ is defined the same on B_i and B_{i+1}. Thus f is a well defined function on $A_{n'+2}$. Also, f is a nuf on $A_{n'+2}$ because $D_1(x, x, y) = D_{n'}(y, x, x) = D_i(x, y, x) = x$ for $1 \le i \le n'$.

Lastly, we show that f is a monotone on $A_{n'+2}$. Let $\mathbf{d} = (a, \ldots, a, \underset{i}{b}, c, \ldots, c) < \mathbf{e} = (a', \ldots, a', \underset{j}{b'}, c', \ldots, c')$, where $\mathbf{d}, \mathbf{e} \in A_{n'+2}$. We want to show $f(\mathbf{d}) \le f(\mathbf{e})$.

If $i = j$ the proof is obvious. The $j < i$ case is the dual of the $i < j$ case. So let $i < j$. Then

$$\begin{aligned} f(\mathbf{d}) &= D_{i-1}(c, b, a) \le D_{i-1}(c, a', a') = D_i(c, c, a') \\ &\le D_i(c, a', a') = \cdots \le D_{j-2}(c, a', a') = D_{j-1}(c, c, a') \\ &\le D_{j-1}(c', b', a') = f(\mathbf{e}), \end{aligned}$$

which proves the claim.

(4) implies (5): Let n be as in (4) and let us suppose there is a **P**-zigzag with a diameter at least $n+2$. Then, as in Proposition 3.4, from this zigzag we can construct a standard zigzag $(\mathbf{H}, f)$ which still has a diameter at least $n+2$. Hence $\mathbf{N}(\mathbf{H}, f)$ has a diameter $d \ge n$. Let us select two points $a, b \in N(\mathbf{H}, f)$ such that their distance is d in $\mathbf{N}(\mathbf{H}, f)$. We know that cancelling a in $(\mathbf{H}, f)$ leaves a **P**-colored poset that is **P**-extendible. Let f_a be such a monotone extension. Similarly, let f_b be a monotone extension corresponding to the cancellation of b in $(\mathbf{H}, f)$. Let $B_i = \{h : h \in N(\mathbf{H}, f)$ and h has down distance i from a in $\mathbf{N}(\mathbf{H}, f)\}$, where $1 \le i \le d+1$. We note that by definition $a \in B_1$. Let $d_0 = d$ if $B_{d+1} = \emptyset$ otherwise $d_0 = d + 1$.

Now, we make some observations. Since b has a distance d from a, $B_i \ne \emptyset$ for $1 \le i \le d_0$. The sets $B_1, \ldots, B_{d_0}$ give a partition of $N(\mathbf{H}, f)$ with $a \in B_1$ and $b \in B_{d_0}$. For any $1 \le i \le d_0$, B_i is a down set if i is odd and B_i is an up set if i is even. Moreover, $\bigcup_{j=1}^{i-1} B_j$ and $N(\mathbf{H}, f) \setminus \bigcup_{j=1}^{i} B_j$ span two subposets of $\mathbf{N}(\mathbf{H}, f)$ which cannot be connected in $\mathbf{N}(\mathbf{H}, f) \setminus B_i$.

We define a function g_i on H for every $1 \le i \le d_0$. Let g_i be f_b on $\bigcup_{j=1}^{i-1} B_j$ and let g_i be f_a on $N(\mathbf{H}, f) \setminus \bigcup_{j=1}^{i} B_j$. On B_i let g_i be equal to 0 if i is odd and 1 if i is even. Since $(\mathbf{H}, f)$ is standard every element of $C(\mathbf{H}, f)$ is connected to $N(\mathbf{H}, f)$ by a single covering edge. Depending on whether i is odd or i is even we define g_i to be 0 or 1 on those elements of $C(\mathbf{H}, f)$ which are connected to some element of B_i by covering edges. For the remaining elements of $C(\mathbf{H}, f)$ the function g_i is defined by the corresponding values of f. By the previous observations g_i, clearly, is a monotone function from **H** to **P**.

Since $d_0 \ge n$, by the hypothesis there exists M_{d_0}, a monotone partial d_0-nuf, which is fully defined on A_{d_0}. Now, $M_{d_0}(g_1(x), \ldots, g_{d_0}(x))$ is a monotone map from **H** to **P** which extends f to **H**. This contradicts the fact that $(\mathbf{H}, f)$ is a zigzag.

(5) implies (6): For a finite poset **Q** let $\ell(\mathbf{Q})$ denote the number of elements in a subchain of maximum cardinality. For an $a \in Q$ let $\ell_{\mathbf{Q}}(a)$ denote the maximum number of elements in a subchain with a top element a. Of course, we always have $\ell_{\mathbf{Q}}(a) \le \ell(\mathbf{Q})$ for every $a \in Q$.

Let us suppose (5) is true and **P** has infinitely many zigzags. Let k be the cardinality of **P**. Since **P** is finite there is a **P**-zigzag $(\mathbf{H}, f)$ such that $|H| \geq \sum_{i=0}^{m+1} k^{ik}$. By Proposition 3.12 we can assume that $\ell(\mathbf{H})$ is at most $k-1$. The basic idea of the proof is simple. Starting from $(\mathbf{H}, f)$ we create a sequence of zigzags $(\mathbf{H}_i, f_i)$, $1 \leq i \leq m+1$, such that each $(\mathbf{H}_i, f_i)$ has diameter at least i. The large size of $|H|$ will guarantee that we can construct these zigzags. The existence of $(\mathbf{H}_{m+1}, f_{m+1})$ contradicts (5) and so we get the claim. In order to create the $(\mathbf{H}_i, f_i)$, $1 \leq i \leq m+1$, we need to prove the following two claims.

CLAIM 1. *Let* **P** *be a finite poset of cardinality* k. *Let* $(\mathbf{H}, f)$ *be a monotone* **P**-*zigzag and let* D *be a down set of* **H**. *Then there exist a* **P**-*zigzag* $(\mathbf{H}', f')$, *a down set* D' *of* **H**′ *and a monotone map* g *from* $(\mathbf{H}', f')$ *onto* $(\mathbf{H}, f)$ *such that the following hold.*

(a) $\mathbf{H}' \backslash D' = \mathbf{H} \backslash D$, $g(u) = u$ *for every* $u \in H' \backslash D'$ *and* $g(D') = D$.

(b) $|\{d'\}_*| < k^{\ell_{\mathbf{H}}(g(d'))}$ *for every* $d' \in D'$.

(c) $\ell(\mathbf{H}') \leq \ell(\mathbf{H})$.

Proof of Claim 1. Let $(\mathbf{H}, f)$ be a **P**-zigzag and let D be a down set of **H**. We prove the existence of $(\mathbf{H}', f')$, D' and g satisfying (a), (b) and (c) by induction on $|D|$. If $|D| = 0$ there is nothing to prove. Let us suppose $|D| \geq 1$. Let $d \in D$ be maximal in the poset **D** spanned by the elements of D in **H**. We apply the induction hypothesis for $(\mathbf{H}, f)$ and $D \backslash \{d\}$. Thus, there exist a **P** zigzag $(\mathbf{H}_0, f_0)$, a down set D_0 of $\mathbf{H}_0$ and a monotone map g_0 from $(\mathbf{H}_0, f_0)$ such that the following hold.

(a′) $\mathbf{H}_0 \backslash D_0 = \mathbf{H} \backslash (D \backslash \{d\})$, $g_0(u) = u$ for every $u \in H_0 \backslash D_0$ and $g_0(D_0) = D \backslash \{d\}$.

(b′) $|\{d_0\}_*| < k^{\ell_{\mathbf{H}}(g_0(d_0))}$ for every $d_0 \in D_0$.

(c′) $\ell(\mathbf{H}_0) \leq \ell(\mathbf{H})$.

Observe that the properties of g_0 guarantee $\{d\}_* \backslash \{d\} \subseteq D_0$. Now, we create a new **P**-colored poset $(\mathbf{H}_1, f_1)$ from $(\mathbf{H}_0, f_0)$ by replacing d in $(\mathbf{H}_0, f_0)$ by elements $d_1, \ldots, d_t$ as follows. For each antichain in $\{d\}_* \backslash \{d\}$ with at most k elements we pick a new element d_i that covers those elements, and d_i itself is covered by all elements that cover d in $\mathbf{H}_0$. We leave d_i noncolored if d is noncolored in $(\mathbf{H}_0, f_0)$; otherwise $f_1(d_i) = f_0(d)$. The colored poset $(\mathbf{H}_1, f_1)$ so obtained is **P**-nonextendible. For otherwise let f_1' be a monotone extension of f_1 to $\mathbf{H}_1$. Since $(\mathbf{H}_1, f_1) \backslash \{d_1, \ldots, d_t\} = (\mathbf{H}_0, f_0) \backslash \{d\}, f_1'|_{H_0 \backslash \{d\}}$ is a monotone extension of f_0 to $\mathbf{H}_0 \backslash \{d\}$. Note that the colored poset $(\mathbf{H}_0, f_1'|_{H_0 \backslash \{d\}})$ is nonextendible since $(\mathbf{H}_0, f_0)$ is nonextendible. So it contains a zigzag $(\mathbf{Q}, g)$. Observe that $d \in Q$. Also $(\mathbf{Q}, g)$ is a nonmonotone zigzag when d is colored and d is the only noncolored element of $(\mathbf{Q}, g)$ when d is noncolored. By (1) and (2) of Claim 3.11, $(\mathbf{Q}, g)$ has its colored elements in $\{d\}_* \cup \{d\}^*$ and the elements of $(\mathbf{Q}, g)$ in $\{d\}_* \backslash \{d\}$ form an antichain with at most k elements. This is impossible; otherwise $(\mathbf{Q}, g)$ would also be contained in the colored poset $(\mathbf{H}_1, f_1'|_{H_1\setminus\{d_1,\ldots,d_t\}})$ that is assumed to be extendible. Thus $(\mathbf{H}_1, f_1)$ is nonextendible. Hence it contains a zigzag $(\mathbf{H}', f')$.

There is a monotone map g_1 from $(\mathbf{H}_1, f_1)$ to $(\mathbf{H}_0, f_0)$ that is the identity map on $H_1\setminus\{d_1, \ldots, d_t\}$ and sends the elements $d_1, \ldots, d_t$ to d. Observe that there does not exist a nonempty set T of points and covering pairs in $\mathbf{H}_0$ such that $(\mathbf{H}_0, f_0)\setminus T$ is a monotone image of $(\mathbf{H}', f')$; otherwise by composing an extension of f_0 to $(\mathbf{H}_0, f_0)\setminus T$ with $g_1|_{H'}$ we would get a monotone extension of f' to $\mathbf{H}'$. Hence $\mathbf{H}'\setminus\{d_1, \ldots, d_t\} = \mathbf{H}_1\setminus\{d_1, \ldots, d_t\}$ and H' contains at least one of $d_1, \ldots, d_t$. Let $g = g_0 \circ g_1|_{H'}$ and let $D' = g^{-1}(D)$. Now, clearly, g is a monotone map onto $(\mathbf{H}, f)$ and D' is a down set of $\mathbf{H}'$.

We want to show that $(\mathbf{H}', f')$, D' and g satisfy (a), (b) and (c). First of all, by (a′) and the fact that $\mathbf{H}'\setminus\{d_1, \ldots, d_t\} = \mathbf{H}_1\setminus\{d_1, \ldots, d_t\}$, (a) is satisfied. Let $d' \in D'$. We show (b) holds even in $\mathbf{H}_1$. If $d' \in D_0$, then by (b′) we have the claim. If $d' = d_i$ for some i and d' is minimal in $\mathbf{H}_1$, then (b) is obvious. In the remaining case $d' = d_i$ for some i, and the number of elements covered by d' in $\mathbf{H}_1$ is at least one and at most k. The elements covered by d' are in D_0. So by (b′) we have $|\{d'\}_*| < kk^{\ell_{\mathbf{H}}(g(d_0))}$ for an element $d_0 \in D_0$ covered by d'. Now, notice that $\ell_{\mathbf{H}}(g(d_0)) \leq \ell_{\mathbf{H}}(g(d')) - 1$ since $g(d_0) \in D\setminus\{d\}$ and g is monotone. Thus we have (b). Finally, (c) is obvious from (c′) and the construction of $(\mathbf{H}', f')$. □

Let $\mathbf{Q}$ be a connected poset. Let $a \in Q$ and $B \subseteq Q$. Then $d_{\mathbf{Q}}(a, B)$ denotes the minimum of $d_{\mathbf{Q}}(a, b)$, $b \in B$, where $d_{\mathbf{Q}}(a, b)$ is the distance between a and b in $\mathbf{Q}$.

CLAIM 2. *Let* $\mathbf{P}$ *be a finite poset of cardinality* k. *Let* $(\mathbf{H}, f)$ *be a* $\mathbf{P}$*-zigzag with* $w \in H$. *Let us suppose that* $(\mathbf{H}, f)$ *and* w *satisfy the following properties.*

(A) $H = A \cup B \cup C$, *where* A, B *and* C *are pairwise disjoint sets.*

(B) B *and* C *are not empty and* B *is an up set in* $\mathbf{H}$.

(C) *For every* $a \in A$ *and* $c \in C$ *we have* $a \parallel c$ *in* $\mathbf{H}$.

(D) $w \in A \cup B$.

Then there exist a $\mathbf{P}$*-zigzag* $(\mathbf{H}', f')$ *and* $w' \in H'$ *with the following properties.*

(a) $H' = A' \cup B' \cup C'$, *where* A', B' *and* C' *are pairwise disjoint sets.*

(c) B' *is a nonempty up set in* $\mathbf{H}'$, $|B'| \leq |B|$ *and* $|C| \leq |C'|$.

(c) *For every* $a' \in A'$ *and* $c' \in C'$ *we have* $a' \parallel c'$ *in* $\mathbf{H}'$.

(d) $w' \in A' \cup B'$.

(e) $d_{\mathbf{H}'}(w', B') \geq d_{\mathbf{H}}(w, B)$

(f) $\ell(\mathbf{H}') \leq \ell(\mathbf{H})$.

(g) *The number of elements* $c' \in C'$ *with* $c' < b'$ *for some* $b' \in B'$ *is at most* $kd|B|$, *where* d *is the maximum of* $|\{c\}_*|$ *for* $c \in C$.

Proof of Claim 2. We note that if the number of covering pairs between B and C is at most $k|B|$, then there is nothing to do. In any case, we construct a nonextendible colored poset $(\mathbf{Q}, g)$ from $(\mathbf{H}, f)$ as follows. Let $\mathbf{C} = \mathbf{H} \backslash (A \cup B)$. For every monotone extension t of $f|_C$ to $\mathbf{C}$ there exists a zigzag in $(\mathbf{H}, f \cup t)$. Let $(\mathbf{Q}_t, g_t)$ be such a zigzag for each t. Observe that $\varnothing \neq Q_t \cap C \subseteq C(\mathbf{Q}_t, g_t)$. If $(\mathbf{Q}_t, g_t)$ is monotone, then by Claim 3.3 every element of $Q_t \cap C$ is covered from $Q_t \cap (A \cup B)$, hence by property (C), from $Q_t \cap B$. Note that if $(\mathbf{Q}_t, g_t)$ is non-monotone we also have that every element of $Q_t \cap C$ is covered from $Q_t \cap B$. Let us take disjoint copies, one for $\mathbf{C}$ and one for each $\mathbf{Q}_t$. Then let us stick together the copies of the $\mathbf{Q}_t$ to the copy of $\mathbf{C}$ along the elements that were common in $Q_t \cap C$. In this way we get a poset $\mathbf{Q}$. We refer to the copy of $\mathbf{Q}_t$ in $\mathbf{Q}$ as $\mathbf{Q}_{t,0}$ and to the copy of $\mathbf{C}$ in $\mathbf{Q}$ as $\mathbf{C}_0$. The coloring g on $\mathbf{Q}$ inherits $f|_C$ on C_0 and $f|_{Q_t \backslash C}$ on $Q_{t,0}$ for all t. Now, $(\mathbf{Q}, g)$ is not extendible since for every monotone extension t of g to $\mathbf{C}_0$ there exists a copy of $(\mathbf{Q}_t, g_t)$ contained in $(\mathbf{Q}, g)$.

Let $(\mathbf{H}', f')$ be a zigzag contained in $(\mathbf{Q}, g)$. There is a monotone map h between the colored posets $(\mathbf{H}', f')$ and $(\mathbf{H}, f)$, where h maps an element $u \in H'$ to the element of H from which u is copied. Observe that h must be onto; otherwise $(\mathbf{H}', f')$ would be extendible. We define w', A', B' and C' as follows. Let w' be an element of H' with $h(w') = w$. So there is an s with $w' \in Q_{s,0}$. Let A' be $h^{-1}(A) \cap Q_{s,0}$. Let B' be $h^{-1}(B) \cap Q_{s,0}$. Finally, let $C' = H' \backslash (A' \cup B')$.

Then we obviously have (a). Since h is onto we have $C_0 \subseteq H'$. So $C_0 \subseteq C'$. Hence $|C| \leq |C'|$. By the definition, B' is an up set, and clearly $|B'| \leq |B|$. Since $\mathbf{H}'$ is connected w' is connected by a fence to an element of C_0 in $\mathbf{H}'$ and this fence must use a copy of an element of B by properties (C) and (D). Hence B' is not empty and (b) holds. By (C) we get (c) and by the definition of w' we get (d). Since

$$d_{\mathbf{H}'}(w', B') \geq d_{\mathbf{Q}_{s,0}}(w', B') \geq d_{\mathbf{Q}_s}(w, B \cap Q_s) \geq d_{\mathbf{H}}(w, B),$$

(e) also holds. Clearly, $\ell(\mathbf{H}') \leq \ell(\mathbf{Q}) \leq \ell(\mathbf{H})$, which gives (f). By Proposition 3.3 and Proposition 3.5 every element of a $\mathbf{P}$-zigzag covers at most k colored elements. Let us apply this to the zigzag $(\mathbf{Q}_s, g_s)$. By property (C) we get that the elements of $B \cap Q_s$ together cover at most $k|B \cap Q_s|$ colored elements of $(\mathbf{Q}_s, g_s)$. So in $\mathbf{Q}$ the copy of $B \cap Q_s$ covers at most $k|B \cap Q_s|$ elements of C_0. Hence the number of elements of C_0 dominated by some elements of B' is at most $kd|B|$, where d is the maximum of $|\{c\}_*|$ for $c \in C$. Thus (g) is satisfied in $(\mathbf{H}', f')$. □

With the help of the preceding two claims, for $0 \leq i \leq m+1$, we give a recursive definition of $(\mathbf{H}_i, f_i)$, A_i, B_i, $C_i \subseteq H_i$ and $a_i \in H_i$ that satisfy the following properties.

($\mathrm{a_i}$) $H_i = A_i \cup B_i \cup C_i$, where A_i, B_i and C_i are pairwise disjoint sets.

($\mathrm{b_i}$) If i is even then B_i is a nonempty up set of $\mathbf{H}_i$. If i is odd then B_i is a nonempty down set of $\mathbf{H}_i$. In both cases $|B_i| \leq k^{ik}$ and $|C_i| \geq |H| - \sum_{j=0}^{i} k^{jk}$.

(c_i) For every $a \in A_i$ and $c \in C_i$ we have $a \parallel c$ in $\mathbf{H}_i$.

(d_i) $a_i \in A_i \cup B_i$.

(e_i) $d_{\mathbf{H}_i}(a_i, B_i) \geq i$

(f_i) $\ell(\mathbf{H}_i) \leq k - 1$

We define $(\mathbf{H}_0, f_0) = (\mathbf{H}, f)$, $A_0 = \varnothing$, $B_0 = \{a_0\}$ and $C_0 = H \backslash \{a_0\}$, where a_0 is a maximal element of $\mathbf{H}$. Observe that $(\mathbf{H}_0, f_0)$, A_0, B_0, C_0 and a_0 satisfy (a_0)–(f_0). We define $(\mathbf{H}_i, f_i)$, A_i, B_i, C_i and a_i for $i \geq 1$. We only do this for an odd i. For an even i one can define and prove everything dually.

So let $i \geq 1$ be odd. Then $(\mathbf{H}_{i-1}, f_{i-1})$, A_{i-1}, B_{i-1}, C_{i-1} and a_{i-1} are defined already and satisfy (a_{i-1})–(f_{i-1}). Since $i - 1$ is even B_{i-1} is an up set in $\mathbf{H}_{i-1}$. Let us apply Claim 1 to $(\mathbf{H}_{i-1}, f_{i-1})$ with $D = C_{i-1}$. The resulting zigzag $(\mathbf{H}'_{i-1}, f'_{i-1})$ with A_{i-1}, B_{i-1}, $D' = C'_{i-1}$ and a_i still satisfies (a_{i-1})–(f_{i-1}), and we have gained the property that for every $c \in C'_{i-1}$, $|\{c\}_*| \leq k^{k-1}$. Now we apply Claim 2 to $(\mathbf{H}'_{i-1}, f'_{i-1})$ with $A = A_{i-1}$, $B = B_{i-1}$, $C = C'_{i-1}$ and $w = a_{i-1}$. Let $(\mathbf{H}_i, f_i)$ be the resulting zigzag. We define $A_i = A' \cup B'$. Let B_i be the set of elements in C' that are dominated by some elements of B' and let $C_i = H_i \backslash (A_i \cup B_i)$. Finally let $a_i = w'$.

Let us check the properties (a_i)–(f_i). First of all, (a_i), (c_i), (d_i) and (f_i) are obvious. Moreover, (e_i) is obvious, if we show $B_i \neq \varnothing$. So the property that really needs a proof is (b_i). By (f) and (g) of Claim 2 and (b) of Claim 1 we have $|B_i| \leq k k^{k-1} k^{(i-1)k} = k^{ik}$. By (b) of Claim 2 and (a) of Claim 1, $|C'| \geq |C'_{i-1}| \geq C_{i-1}|$. Since $C_i = C' \backslash B_i$ and $B_i \subseteq C'$, by (b_{i-1}) we have

$$|C_i| = |C'| - |B_i| \geq |C_{i-1}| - |B_i| \geq |H| - \sum_{j=0}^{i-1} k^{jk} - k^{ik} = |H| - \sum_{j=0}^{i} k^{jk}.$$

Finally, B_i is nonempty since B' and C' are nonempty and $(\mathbf{H}_i, f_i)$ is connected. So we have (5) implies (6).

(6) implies (1): Use Remark 2.4. □

We remark that (4) implies (2) for any poset $\mathbf{P}$ since by the proof of (1) implies (2) we can also obtain Jónsson operations from a partial n-nuf that is fully defined on A_n.

PROPOSITION 4.2. *Any finite poset* $\mathbf{P}$ *with the strong selection property has a partially defined, monotone* 4-*nuf that is fully defined on* A_4.

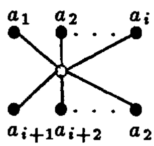
Proof. Let $\mathbf{P}$ be an arbitrary finite poset with the strong selection property. Then by Proposition 2.3 every $\mathbf{P}$-zigzag has at most one noncolored element.

We show that there is a monotone, partial 4-nuf t on $\mathbf{P}$ that is fully defined on $A_4 = \{(a, \ldots, a, \underset{j}{b}, c, \ldots, c): a, b, c \in P, 1 \leq j \leq 4\} \subseteq P^4$. Suppose this is not true. This means that the colored poset $(\mathbf{A}_4, g)$, where g is given by $g(a, \ldots, a, \underset{j}{b}, a, \ldots, a) = a$, $1 \leq j \leq 4$, $a, b \in P$, is not $\mathbf{P}$-extendible. So it contains a monotone $\mathbf{P}$-zigzag $(\mathbf{H}, f)$. Let h be the only noncolored element of $(\mathbf{H}, f)$. Recall that the zigzag $(\mathbf{H}, f)$ is of the form described in (2) of Claim 3.11. Note that $h \in A_4$ has at least two coordinates which are the same, say $a \in P$. But then $f(h') \leq a \leq f(h'')$ for any $h', h'' \in H$ with $h' < h < h''$. Hence the coloring of h by a yields a monotone extension of $(\mathbf{H}, f)$, which is a contradiction. □

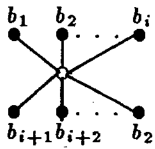
Now, we are prepared to give an example of an infinite bounded poset that admits Jónsson operations but does not admit a near unanimity function.

EXAMPLE 4.3. Let $\mathbf{P}_n = \mathbf{S}_n + \mathbf{S}_n^d$, where $\mathbf{S}_n$ is the poset given by the Boolean lattice of n atoms without its top element and $\mathbf{S}_n^d$ is the dual of $\mathbf{S}_n$. Let $\mathbf{P} = \prod_{i \in I} \mathbf{P}_i$, where $I = \{2, 3, \ldots\}$. Then $\mathbf{P}$ admits Jónsson operations and $\mathbf{P}$ admits no nuf.

Proof. By [13], each $\mathbf{P}_i$, $i \in I$, has the strong selection property. Then, by Proposition 4.2, for each $i \in I$ there exists a partial 4-nuf t_i on $\mathbf{P}_i$ that is fully defined on $B_i = \{(a, \ldots, a, \underset{j}{b}, c, \ldots, c): a, b, c \in P_i, 1 \leq j \leq 4\} \subseteq P_i^4$. With the help of the t_i, $i \in I$, we can also define a monotone, partial 4-ary near unanimity operation on $\{(a, \ldots, a, \underset{j}{b}, c, \ldots, c): a, b, c \in P, 1 \leq j \leq 4\}$ coordinatewise. So $\mathbf{P}$ admits Jónsson operations by the remark following Theorem 4.1. After the proof of Claim 3.11 we noted that each $\mathbf{P}_i, i \in I$, has a $\mathbf{P}_i$-zigzag of the form



with $2i$ colored elements. Let 1_i be the top element of P_i for $i \in I$. For each $i \in I$ let $b_j \in \mathbf{P}$, $1 \leq j \leq 2i$, be defined by $b_j(i) = a_j$ and $b_j(l) = 1_l$ for $l \in I \backslash \{i\}$. Then it is easy to see that the $\mathbf{P}$-colored poset



is a $\mathbf{P}$-zigzag with $2i$ colored elements. Hence, by the note after Remark 2.4, $\mathbf{P}$ admits no nuf. □

Acknowledgement

I would like to thank Joel Berman for his help in writing this paper.

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