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Uniqueness for the harmonic map flow in two dimensions

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Abstract. Let M be a two-dimensional Riemannian manifold with smooth (possibly empty) boundary. If u and v are weak solutions of the harmonic map flow in $H^1(M \times [0,T]; S^N)$ whose energy is non-increasing in time and having the same initial data $u_0 \in H^1(M, S^N)$ (and same boundary values $\gamma \in H^{3/2}(\partial M; S^N)$ if $\partial M \neq \emptyset$) then $u = v$.

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1. Introduction

Let M be a compact two-dimensional Riemannian manifold with smooth (possibly empty) boundary ∂M. In this paper we obtain a uniqueness result for solutions of the 'harmonic map flow' on M:

$$(1.1) \qquad \begin{cases} u_t - \Delta u = u|\nabla u|^2 \text{ on } M \times (0,T) \\ u(x,t) = \gamma(x) \text{ for } t \geq 0, x \in \partial M \\ u(x,0) = u_0(x), x \in M \end{cases}$$

where $u(x,t)$ takes values in the unit sphere $S^N \subset \mathbb{R}^{N+1}$. Time-independent solutions of (1.1) correspond to harmonic maps from M to S^N. The following existence and uniqueness theorem for weak solutions of (1.1) when $\partial M = \emptyset$ was obtained by M.Struwe. Define:

$$V_T = H^1(M \times [0,T]; S^N) \cap L^\infty([0,T]; H^1(M; S^N)) \cap L^2([0,T]; H^2(M; S^N))$$

Theorem 1.1. (M. Struwe, [1].) *Assume $\partial M = \emptyset$. For any initial value $u_0 \in H^1(M; S^N)$ there exists a number $T_0 = T_0(u_0) > 0$ and a solution $v \in \bigcap_{T' < T_0} V_{T'}$ of (1.1) with $u(.,0) = u_0$. Moreover,*

(i) v *is regular in* $M \times (0, T_0]$ *with the exception of finitely many points* (x_i, T_0), $1 \leq i \leq K$;
(ii) v *is the unique solution of (1.1) in the space* $\bigcap_{T'<T_0} V_{T'}$ *with initial data* u_0;
(iii) The energy $E_v(t) = \int_{M\times\{t\}} |\nabla v|^2 dx$ *is finite for all* $t \in [0, T_0]$ *and nonincreasing in* t.

The same conclusions hold when $\partial M \neq \emptyset$, assuming $\gamma \in H^{3/2}(\partial M)$ (K.C.Chang [2]). We have only stated some of the conclusions in [1] and [2]. In particular, M.Struwe's and K.C. Chang's results hold for arbitrary compact target manifolds. These authors also show that the solution can be continued to a weak solution v of (1.1) in $M \times [0, \infty)$ whose singular set is finite. Precisely,

$$v \in H^1(M \times [0, \infty), S^N) \cap L^\infty([0, \infty), H^1(M; S^N))$$

and one may find a finite sequence of times $0 < T_1 < \cdots < T_k = \infty$ such that

$$v \in \bigcap_{i=1}^{k-1} L^2_{loc}([T_i, T_{i+1}); H^2(M; S^N)).$$

The solution v is unique in this class of weak solutions; we will refer to it as the 'almost smooth' solution. It is natural to wonder whether, with the same initial data, any other weak solutions $u \in H^1(M \times [0, \infty), S^N)$ with bounded energy in $[0, \infty)$ may exist. In this direction the following result was recently obtained by T.Rivière([3]).

Theorem 1.2. (T. Rivière, [3].) *Assume* $\partial M \neq \emptyset$. *There exists* $\alpha > 0$ *such that, for any boundary values* $\gamma \in H^{3/2}(\partial M; S^N)$ *and any* $u_0 \in H^1(M; S^N)$ *satisfying* $E(u_0) < \alpha$, *(1.1) has a unique solution in* $H^1_{loc}([0, \infty) \times M)$ *for which* $E_u(t) \leq E(u_0)$ *for a.e. t. This solution is regular in* $[0, \infty) \times M$.

In [3] this theorem is stated for M an open set in $\mathbb{R}^2$ with smooth boundary, but it is not hard to see that the proof applies to arbitrary Riemannian surfaces with non-empty boundary. The main result in this paper states that, assuming monotonicity of the energy, we have uniqueness in H^1 without the small-energy assumption of theorem 1.2.

Theorem 1.3. *Let* M *be a two-dimensional Riemannian manifold with smooth (possibly empty) boundary. If* u *and* v *are weak solutions of (1.1) in* $H^1(M \times [0, T]; S^N)$ *satisfying* $E_u(t) \leq E_u^{(s)}, E_v(t) \leq E_v^{(s)}$ *for a.e.* $s < t$ *and having the same initial data* $u_0 \in H^1(M; S^N)$ *(and same boundary values* $\gamma \in H^{3/2}(\partial M, S^N)$ *if* $\partial M \neq \emptyset$*) then* $u = v$.

Combining this statement with theorem 1.1 we immediately obtain the following 'partial regularity' result:

Corollary 1.4. *Any weak solution* $u \in H^1(M \times [0, T]; S^N)$ *of (1.1) such that* $u_0 \in H^1(M; S^N)$ *and* $E_u(t)$ *is non-increasing in* t *(with* $\gamma \in H^{3/2}(\partial M, S^N)$ *if* $\partial M \neq \emptyset$*) is smooth in* $M \times (0, T]$ *away from finitely many points.*

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2. Preliminaries

In this section we list some well-known results that are used in the proof. We assume throughout that M is a compact Riemannian n-manifold with smooth (possibly empty) boundary.

2.1. Interpolation inequality

Assume M is two-dimensional. There exists $c_1 = c_1(M) > 0$ such that if $f \in H^1(M)$,

$$\int_M |f|^4 dx \leq c_1 \left(\int_M |\nabla f|^2 dx\right)\left(\int_M |f|^2 dx\right). \tag{2.1}$$

2.2. Hodge decomposition theorem

2.2.1 ($\partial M = \emptyset$) Denote by $\mathscr{H}^p$ $(0 \leq p \leq n)$ the space of harmonic forms in M of degree p. We have the orthogonal Hilbert space decomposition:

$$\Lambda^p L^2(M) = d\Lambda^{p-1}H^1(M) \oplus \delta\Lambda^{p+1}H^1(M) \oplus \mathscr{H}^p.$$

2.2.2 ($\partial M \neq \emptyset$)[5, chapter 4.1] Let $\theta \in \Lambda^1(M)_{|\partial M}$ be the metric dual to the unit normal ν. Any p-form $\omega \in \Lambda^p(M)$ has a unique orthogonal decomposition at points of ∂M: $\omega = \omega_t + \theta \wedge \omega_n$, where $i_\nu \omega_n = 0$. Denote by:

$\Lambda_T^{p+1}H^1(M)$- the H^1 closure of the space of smooth (p+1)-forms ω in M such that $\omega_n = 0$ on ∂M;

$\mathscr{H}_T^p$- the space of (smooth) p-forms ω in M such that $d\omega = \delta\omega = 0$ and $\omega_n = 0$ on ∂M. This is a finite dimensional vector space, isomorphic to the ('absolute') cohomology space $H^p(M, \mathbb{R})$. We have:

$$\Lambda^p L^2(M) = d\Lambda^{p-1}H^1(M) \oplus \delta\Lambda_T^{p+1}H^1(M) \oplus \mathscr{H}_T^p.$$

In the unique decomposition

$$\omega = d\alpha + \delta\beta + h \quad (\delta\alpha = d\beta = 0)$$

corresponding to either of the two splittings above, one has the bounds:

$$\|\alpha\|_{H^1} \leq c_2\|\omega\|_{L^2}, \|\beta\|_{H^1} \leq c_2\|\omega\|_{L^2},$$

for some $c_2 = c_2(M) > 0$.

2.3. Linear parabolic theory

The next two results summarize the existence and uniqueness theory in Sobolev spaces for the linear parabolic equation:

$$(2.2)\qquad \begin{cases} \Phi_t - \Delta\Phi = g & \text{in } M \times (0,T) \\ \Phi(x,.) = 0 & \text{on } \partial M \\ \Phi(.,0) = 0 & \text{in } M \end{cases}$$

We set $I = [0,T]$.

2.3.1 Theorem. (J.L. Lions–E. Magenes [7], p.89). *Assume $g \in L^2(I, H^{-1}(M))$. Then problem (2.2) has a unique solution in the space:*

$$W_0 = \{\Phi \in L^2(I,H^1) | \Phi_t \in L^2(I,H^{-1}), \Phi_{|\partial M} = 0, \Phi(.,0) = 0 \textit{ in } M\}.$$

Moreover the map $\mathscr{U} : L^2(I,H^{-1}) \to W_0$, $\mathscr{U}(g) = \Phi$ is an isomorphism with inverse $L\Phi = \Phi_t - \Delta\Phi$.

2.3.2 Theorem. (P. Grisvard [6], Theorem 9.3 and Remark 9.15). *Assume $g \in L^p(I, L^q(M))$, where $1 < p,q < \infty$ are arbitrary. Problem (2.2) has a unique solution in the space:*

$$L^p_0(I, W^{2,q}) = \{\Phi \in L^p(I, W^{2,q}) | \Phi_t \in L^p(I, L^q), \Phi_{|\partial M} = 0, \Phi(.,0) = 0 \textit{ in } M\}.$$

Moreover the map $\mathscr{L} : L^p(I, L^q) \to L^p_0(I, W^{2,q})$, $\mathscr{L}(g) = \Phi$ is an isomorphism with inverse $L\Phi = \Phi_t - \Delta\Phi$.

In the references given these results are stated for bounded domains in Euclidean space; the reader familiar with the proofs will observe that they also apply to the present context.

2.4. Wente's theorem

Let M *b*e a compact two-dimensional Riemannian manifold with (possibly empty) smooth boundary. If $\eta \in \Lambda^2 H^1(M)$ and $\theta \in H^1(M)$, then $< \delta\eta, d\theta > \in H^{-1}(M)$ and

$$||\langle\delta\eta, d\theta\rangle||_{H^{-1}} \leq c_3 ||\delta\eta||_{L^2} ||d\theta||_{L^2},$$

for some $c_3 = c_3(M) > 0$.

When M is a bounded domain in $\mathbb{R}^2$ (with the Euclidean metric) and $\eta = \eta_1 dx \wedge dy$, $d\theta = \theta_x dx + \theta_y dy$, we have:

$$\langle\delta\eta, d\theta\rangle = \theta_x(\eta_1)_y - \theta_y(\eta_1)_x,$$

and the lemma is proved in [9, lemma A.2] following Wente's original proof for $M = \mathbb{R}^2$ [8]. In the general case one takes local conformal coordinates in which the metric is written as $g_{ij} = e^{-2v}\delta_{ij}$, $1 \leq i,j \leq 2$. This implies $\delta_g \eta = e^{-2v}\delta_{eucl}\eta$, so locally we are back in the Euclidean case, and we may globalize with a simple partitions-of-unity argument. We omit the details.

2.5. Notation

We try to adhere to self-explanatory notation; the following abbreviations are often used:

$$Q = M \times [0,T];\ I = [0,T];\ M_t = M \times \{t\}.$$

$W^{k,p}(M)$ is the Sobolev space of functions (or maps to S^N) which have k distributional derivatives in L^p; $H^s, s \in \mathbb{R}$, denotes the scale of Hilbert spaces with $H^k = W^{k,2}$ for $k \in \mathbb{N}$. The domain M and target (S^N or R^{N+1}) are usually omitted from the notation, with the understanding that, as usual:

$$W^{k,p}(M, S^N) = \{u \in W^{k,p}(M, \mathbb{R}^{N+1}) | u(x) \in S^{N+1} a.e.(x)\}.$$

$$L^p(I, W^{k,q}) = L^p([0,T]; W^{k,q}(M)).$$

$\Lambda^p W^{k,p}, \Lambda^p H^s$, etc. denote spaces of differential forms of degree p with coefficients in the corresponding Sobolev spaces (smooth forms if no space is indicated); δ denotes the co-differential in the metric of M.

c denotes a generic positive constant whose value depends only on M.

3. Proof of Theorem 1.3

(i) It is enough to prove the theorem assuming v is the 'almost smooth' solution. Moreover we may take T to be the $T(u_0)$ given by theorem 1.1, and assume v is smooth in $M \times (0,T)$. For then the conclusion $u = v$ in $M \times (0,T)$ will imply u is smooth in $M \times (0,T)$, hence by the uniqueness result (ii) in theorem 1.1 $u = v$ in M_T and we may iterate.

(ii) Applying the interpolation inequality (2.1) to $\nabla v \in H^1(M_t)$ we obtain for all $t \in (0,T)$:

$$\int_{M_t} |\nabla v|^4 dx \le c_2 E_0 \int_{M_t} |\nabla^2 v|^2 dx. \tag{3.1}$$

Since, for each $T' < T$, $v \in L^2([0,T'], H^2)$, this shows the function $t \mapsto \int_{M_t} |\nabla v|^4 dx$ is in $L^1([0,T'])$. We fix an arbitrary $T' < T$ for the remainder of the proof.

(iii) Let $w = u - v$. Then $w \in H^1(Q) \cap L^\infty(I, H^1)$ and is a solution of:

$$\begin{cases} w_t - \Delta w = u|\nabla u|^2 - v|\nabla v|^2 \text{ in } Q; \\ w(x,t) = 0, x \in \partial M; \\ w(x,0) = 0, x \in M. \end{cases} \tag{3.2}$$

The main step in the proof is the following lemma:

Lemma 3.1. *Let u and v satisfy the assumptions of the main Theorem 1.3; in addition, assume v is smooth in $M \times (0,T)$. Let $w = u - v$. Then there exists $T' < T$ such that $\nabla w \in L^4([0,T'], W^{1,4}(M))$.*

Lemma 3.1 implies that $w \in H^2(M_t, \mathbb{R}^{N+1})$ for a.e. $t \in [0, T']$ (this was also observed by Rivière in [3]): it suffices to write the equation for w in the form:

$$w_t - \Delta w = u|\nabla w|^2 + 2u\nabla v.\nabla w + w|\nabla v|^2,$$

from which it follows that:

$$|w_t - \Delta w| \leq c(|\nabla w|^2 + |\nabla v|^2),$$

which is in $L^2(M \times [0, T'])$ by the lemma. Since $w_t \in L^2(M \times [0, T])$, we conclude that $w \in H^2(M_t)$ for a.e. t. Thus we may integrate by parts (as in the uniqueness proof in [1]) and obtain for a.e. $t \in [0, T']$:

$$\frac{1}{2}\frac{d}{dt}\int_{M_t} |w|^2 dx + \int_{M_t} |\nabla w|^2 dx = \int_{M_t} < u|\nabla u|^2 - v|\nabla v|^2, w > dx$$

$$\leq \int_{M_t} [|w|^2|\nabla u|^2 + |v||w||\nabla U||\nabla w|]dx$$

$$\leq c\int_{M_t} |w|^2|\nabla U|^2 dx + \frac{1}{2}\int_{M_t} |\nabla w|^2 dx,$$

where following [1] we adopt the suggestive notation $|\nabla U|^p = |\nabla u|^p + |\nabla v|^p$ ($p \geq 1$). This implies (using the interpolation inequality 2.1):

$$\frac{1}{2}\frac{d}{dt}\int_{M_t} |w|^2 dx + \frac{1}{2}\int_{M_t} |\nabla w|^2 dx \leq c\left(\int_{M_t} |w|^4 dx\right)^{\frac{1}{2}}\left(\int_{M_t} |\nabla U|^4 dx\right)^{\frac{1}{2}}$$

$$\leq c\left(\int_{M_t} |w|^2 dx\right)^{\frac{1}{2}}\left(\int_{M_t} |\nabla w|^2 dx\right)^{\frac{1}{2}}\left(\int_{M_t} |\nabla U|^4 dx\right)^{\frac{1}{2}}$$

$$(3.3)\quad \leq c\left(\int_{M_t} |w|^2 dx\right)\left(\int_{M_t} |\nabla U|^4 dx\right) + \frac{1}{2}\int_{M_t} |\nabla w|^2 dx,$$

for a.e. $t \in [0, T']$.

From Lemma 3.1 we obtain that $t \mapsto \int_{M_t} |\nabla w|^4 dx$ is in $L^1([0, T'])$; combined with (3.1), this shows that $t \mapsto \int_{M_t} |\nabla U|^4 dx$ is also in $L^1([0, T'])$. Given that $w(., 0) \equiv 0$, (3.3) and Gronwall's lemma show that $w = 0$ a.e. in $M \times [0, T']$. Now iterate the argument, using monotonicity of the energy.

Proof of Corollary 1.4. A few words have to be said, since the theorem only implies $u = v$ *a.e.* on $M \times [0, T]$. The argument following the statement of lemma 3.1 shows that w (hence u) is in $\bigcap_{T' < T_0} V_T$ (this would also imply uniqueness, by (ii) in Struwe's theorem 1.1 above). By theorem 4.1 in [1], u is smooth in $(0, T(u_0)) \times M$ and at most a finite number of singularities develop at $T(u_0)$. Iterating the argument yields the conclusion.

4. Proof of Lemma 3.1

The proof of Lemma 3.1 follows, broadly speaking, the same steps as the arguments in [3] and [4], with some changes. The main difference is that we appeal to linear parabolic existence theory in spaces of the form $L^p([0, T], L^q(M))$ and $L^2([0, T], H^{-1}(M))$, in contrast with the elliptic theory used in [3] and [4].

4.1. Use of the Hodge decomposition

We begin by applying the Hodge decomposition theorem (2.2.1 or 2.2.2) to the 1-forms:

$$a^{ij} = u^i du^j - u^j du^i \in \Lambda^1 L^\infty(I, L^2), \qquad 1 \leq i,j \leq N+1,$$

(4.1) $$||a^{ij}||_{L^\infty(I,L^2)} \leq c||du||_{L^\infty(I,L^2)} \leq cE_0^{\frac{1}{2}} .$$

In the Hodge decomposition:

(4.2) $$a^{ij} = d\alpha^{ij} + \delta\beta^{ij} + h^{ij} \text{ in } M \times (0,T)$$

we have:

(4.3) $$||\alpha^{ij}||_{L^\infty(I,H^1)} + ||\beta^{ij}||_{L^\infty(I,H^1)} \leq c||a^{ij}||_{L^\infty(I,L^2)} \leq cE_0^{\frac{1}{2}} .$$

(measurability in t is not a problem; for example we could consider the Hodge decomposition in the corresponding Hilbert spaces for $M \times (0,T)$, which coincides with the 'slice-wise' decomposition above for a.e.$t \in [0,T]$, by uniqueness). We will sometimes write 4.2 in the form:

(4.4) $$a^{ij} = \delta\beta^{ij} + \phi^{ij},$$

where $\phi^{ij} = d\alpha^{ij} + h^{ij} \in L^\infty(I, L^2)$ satisfies $d\phi^{ij} = 0$ weakly and:

(4.5) $$||\phi^{ij}||_{L^\infty(I,L^2)} \leq c(||\delta\beta^{ij}||_{L^\infty(I,L^2)} + ||a^{ij}||_{L^\infty(I,L^2)}) \leq cE_0^{\frac{1}{2}}.$$

From the harmonic map flow equation (1.1) we derive two relations. *First,* denoting by $\{e_a, a = 1,2\}$ a local orthonormal frame, we have (using $|u|^2 = 1 \quad a.e.$):

$$\begin{aligned} u_t^i - \Delta u^i &= \sum_{j,a} (u^i du^j.e_a - u^j du^i.e_a) du^j.e_a \\ &= \sum_{j,a} (a^{ij}.e_a)(du^j.e_a) = \sum_j \langle a^{ij}, du^j \rangle \\ &= \sum_j [\langle \delta\beta^{ij}, du^j \rangle + \langle \phi^{ij}, du^j \rangle], \end{aligned}$$

which we write in abbreviated form:

(4.6) $$u_t - \Delta u = \langle \delta\beta, du \rangle + \langle \phi, du \rangle.$$

Second, from:

$$\delta a^{ij} = u^i \Delta u^j - u^j \Delta u^i = u^i u_t^j - u^j u_t^i \in L^2(M \times I)$$

and

(4.7) $$\delta a^{ij} = \Delta \alpha^{ij},$$

which follows from (4.2), we conclude via the Calderón-Zygmund inequality that $\alpha^{ij} \in L^2(I, H^2(M))$ and:

(4.8) $$||d\alpha^{ij}||_{H^1(M_t)} \leq c||\delta a^{ij}||_{L^2(M_t)} \leq c||u_t||_{L^2(M_t)}$$

for a.e. $t \in [0,T]$.

4.2. Rewriting the equation for w

We may write for v an equation analogous to (4.6) for u:

$$v_t - \Delta v = \langle \delta\eta, dv\rangle + \langle \psi, dv\rangle,$$

where $\eta \in \Lambda^2 L^\infty(I, H^1)$ with $||\eta||_{L^\infty(I,H^1)} \leq cE_0^{\frac{1}{2}}$ and $\psi = (\psi^{ij}) \in L^\infty(I, L^2)$ satisfies $||\psi||_{L^\infty(I,L^2)} \leq cE_0^{\frac{1}{2}}$. A simple calculation gives for $w = u - v$:

$$(4.9)\qquad \begin{cases} w_t - \Delta w = \langle \delta\beta, dw\rangle + f, \\ w(.,0) = 0 \\ w(x,t) = 0, t \in I, x \in \partial M, \end{cases}$$

where

$$f(x,t) = \langle \phi, dw\rangle + \langle \delta(\beta - \eta), dv\rangle + \langle \phi - \psi, dv\rangle\ .$$

Claim. *$f \in L^4(I, L^{\frac{4}{3}}(M))$ and*

$$||f||_{L^4(I,L^{\frac{4}{3}})} \leq cT^{\frac{1}{4}}E_0 + cE_0^{\frac{3}{4}}\left(||u_t||^{\frac{1}{2}}_{L^2(M\times I)} + ||v||^{\frac{1}{2}}_{L^2(I,H^2)}\right)$$

4.3. Proof of claim

f consists of three terms, which we estimate in turn.

4.3.1. Recall $\phi^{ij} = d\alpha^{ij} + h^{ij}$. The h^{ij} are smooth harmonic forms, so:

$$(4.10)\quad \sup_{M_t} |h^{ij}| \leq c||h^{ij}||_{L^2(M_t)} \leq c[||a^{ij}||_{L^2} + ||d\alpha^{ij}||_{L^2} + ||d\beta^{ij}||_{L^2}] \leq cE_0^{\frac{1}{2}}$$

for a.e. t. The second inequality follows from (4.2) and the third from (4.3) and the bounds in the Hodge theorem; the first inequality is clear, since the space of harmonic forms is finite-dimensional. Thus for a.e. t:

$$(4.11)\qquad ||\langle h^{ij}, dw^i\rangle||_{L^{4/3}(M_t)} \leq c(\sup_{M_t}|h^{ij}|)||dw^i||_{L^2(M_t)} \leq cE_0.$$

Now by Hölder's inequality:

$$(4.12)\qquad ||\langle d\alpha^{ij}, dw^i\rangle||_{L^{\frac{4}{3}}(M_t)} \leq c||d\alpha^{ij}||_{L^4(M_t)}||dw^i||_{L^2(M_t)}$$

(recall from (4.3) that $d\alpha^{ij} \in H^1(M_t)$, hence is in $L^p(M_t)$ for any $p < \infty$, a.e. (t)). We now apply the interpolation inequality (2.1) to $\nabla\alpha^{ij}$ and conclude:

$$\begin{aligned} \int_{M_t} |d\alpha^{ij}|^4 dx &\leq c\left(\int_{M_t} \nabla^2\alpha^{ij}|^2 dx\right)\left(\int_{M_t} |d\alpha^{ij}|^2 dx\right) \\ &\leq c||\delta a^{ij}||^2_{L^2(M_t)}E_0 \quad a.e.(t)\ , \end{aligned}$$

where the second inequality follows from (4.3), the Calderón-Zygmund inequality and (4.7). Combined with (4.8) this gives:

$$(4.13)\quad \int_0^T \left(\int_{M_t} |d\alpha^{ij}|^4 dx\right) dt \le cE_0 \int_0^T \left(\int_{M_t} |u_t|^2 dx\right) dt = cE_0 ||u_t||^2_{L^2(M\times I)} .$$

Finally from (4.12) and (4.13) we obtain:

$$(4.14)\qquad \int_0^T ||\langle d\alpha^{ij}, dw^i\rangle||^4_{L^{\frac{4}{3}}(M_t)} dt \le cE_0^3 ||u_t||^2_{L^2(M\times I)}.$$

and from (4.11):

$$\int_0^T ||\langle h^{ij}, dw^i\rangle||^4_{L^{\frac{4}{3}}(M_t)} dt \le cTE_0^4,$$

which combine to give the estimate for the first term in $f(x,t)$:

$$(4.15)\qquad ||\langle \phi, dw\rangle||_{L^4(I,L^{\frac{4}{3}})} \le cT^{\frac{1}{4}} E_0 + c||u_t||^{\frac{1}{2}}_{L^2(M\times I)} E_0^{3/4}.$$

4.3.2. For the second term, Hölder's inequality yields:

$$\begin{aligned} ||\langle \delta(\beta-\eta), dv\rangle||_{L^{4/3}(M_t)} &\le ||\delta(\beta-\eta)||_{L^2(M_t)} ||dv||_{L^4(M_t)} \quad a.e.(t) \\ &\le cE_0^{1/2} ||dv||_{L^4(M_t)}, \end{aligned}$$

using the estimate (4.3) for $\delta\beta$ and the analogous estimate for $\delta\eta$. We also used the fact that $v \in L^2(I, H^2(M))$ (recall we're assuming that v is smooth in $M\times(0,T)$). Applying once more the interpolation inequality (2.1) yields:

$$\begin{aligned} ||\langle \delta(\beta-\eta), dv\rangle||^4_{L^{4/3}(M_t)} &\le cE_0^2 \int_{M_t} |dv|^4 dx \\ &\le cE_0^2 \left(\int_{M_t} |dv|^2 dx\right)\left(\int_{M_t} |\nabla^2 v|^2 dx\right) \quad a.e.(t), \end{aligned}$$

so

$$\int_0^T ||\langle \delta(\beta-\eta), dv\rangle||^4_{L^{4/3}(M_t)} dt \le cE_0^3 \int_0^T ||v||^2_{H^2(M_t)} dt.$$

This gives the estimate for the second term in $f(x,t)$:

$$(4.16)\qquad ||\langle \delta(\beta-\eta), dv\rangle||_{L^4(I,L^{\frac{4}{3}})} \le cE_0^{3/4} ||v||^{1/2}_{L^2(I,H^2)}.$$

4.3.3. The estimate for the third term is similar. Notice that (4.5) for ϕ and the corresponding inequality for ψ imply $\phi-\psi \in L^\infty(I, L^2)$ and

$$||\phi-\psi||_{L^\infty(I,L^2)} \le cE_0^{1/2}.$$

This implies, exactly as in 4.3.2:

$$||\langle \phi-\psi, dv\rangle||^4_{L^{4/3}(M_t)} \le cE_0^3 ||v||^2_{H^2(M_t)} \quad a.e.(t)\ ,$$

so:

$$(4.17)\qquad ||\langle \phi-\psi, dv\rangle||_{L^4(I,L^{\frac{4}{3}})} \le cE_0^{3/4} ||v||^{1/2}_{L^2(I,H^2)}.$$

(4.15), (4.16) and (4.17) prove the claim.

4.4. Decomposition of the 'most singular' term in (4.9)

In 4.4 and 4.5 we set $I = [0, T']$. Given $\varepsilon > 0$ there exists $T' \in (0, T)$ such that:

$$\beta = \beta_\epsilon + \beta'_\epsilon$$

where $||\beta_\epsilon||_{L^\infty(I,H^1)} < \epsilon$ and $\beta'_\epsilon \in C^\infty(I, \Lambda^2(M))$ satisfies:

$$\sup_{M\times[0,T]} |\delta\beta'_\epsilon| \leq cE_0^{1/2} .$$

(Write the decomposition at time zero and note the initial data is achieved in H^1.) Thus:

$$(4.18) \qquad ||\langle\delta\beta'_\epsilon, dw\rangle||_{L^4(I,L^{\frac{4}{3}})} \leq cT^{1/4}E_0^{1/2}||dw||_{L^\infty(I,L^2)} \leq cT^{1/4}E_0.$$

We rewrite (4.9) as:

$$(4.19) \qquad \begin{cases} w_t - \Delta w = \langle\delta\beta_\epsilon, dw\rangle + f_\epsilon \\ w(x, .) = 0, \quad x \in \partial M \\ w(., 0) = 0 \quad \text{in } M, \end{cases}$$

where $f_\epsilon = f + \langle\delta\beta'_\epsilon, dw\rangle \in L^4(I, L^{\frac{4}{3}})$ (by the claim in 4.2 and (4.18)).

4.5. Conclusion of the proof

We now use Theorem 2.3.2 to show that, for small enough $\epsilon > 0$, (4.19) has a unique solution $\Phi \in L^4_0(I, W^{2,4/3})$ (where we regard f_ϵ as given). Observe that (4.19) may be written as (replacing w by Φ):

$$\Phi - \mathscr{L}(\langle\delta\beta_\epsilon, d\Phi\rangle) = \mathscr{L}(f_\epsilon), \quad \Phi \in L^4_0(I, W^{2,4/3}).$$

Showing that $\Phi \mapsto \Phi - \mathscr{L}(< \delta\beta_\epsilon, d\Phi >)$ is an isomorphism of $L^4_0(I; W^{2,4/3})$ will establish the existence and uniqueness claimed for (4.19). Let $\Phi \in L^4_0(I; W^{2,4/3})$. The Sobolev embedding $W^{1,4/3}(M) \hookrightarrow L^4(M)$ implies $d\Phi \in L^4(I; L^4)$. By Hölder's inequality we have, for almost every t:

$$||\langle\delta\beta_\epsilon, d\Phi\rangle||_{L^{4/3}(M_t)} \leq c||\delta\beta_\epsilon||_{L^2(M_t)}||d\Phi||_{L^4(M_t)} \leq c\epsilon||d\Phi||_{L^4(M_t)},$$

and hence:

$$||\langle\delta\beta_\epsilon, d\Phi\rangle||_{L^4(I;L^{4/3})} \leq c\epsilon||\Phi||_{L^4(I;W^{2,4/3})}.$$

Since $\mathscr{L} : L^4(I; L^{4/3}) \to L^4_0(I; W^{2,4/3})$ is bounded, choosing $\epsilon = \epsilon_0$ sufficiently small establishes the claim. (The bound on $\mathscr{L}$ depends on T but not on T'.)

As in [3] and [4] we must also consider problem (4.19) in the Hilbert spaces $L^2(I; H^s(M))$. In order to apply theorem 2.3.1, we must verify that $f_\epsilon \in L^2(I; H^{-1})$ and $< \delta\beta_\epsilon, d\Psi > \in L^2(I; H^{-1})$ if $\Psi \in W_0$. The first assertion follows from $L^{4/3}(M) \hookrightarrow H^{-1}(M)$ and $f_\epsilon \in L^4(I; L^{4/3})$. The second assertion follows from Wente's theorem 2.4. Indeed we have, for a.e. t:

$$\begin{aligned} ||\langle\delta\beta_\epsilon, d\Psi\rangle||_{H^{-1}(M_t)} &\leq c||\Psi||_{H^1(M_t)}||\delta\beta_\epsilon||_{H^1(M_t)} \\ &\leq c\epsilon||\Psi||_{H^1(M_t)}, \end{aligned}$$

so that:

$$||\langle\delta\beta_\epsilon, d\Psi\rangle||_{L^2(I;H^{-1})} \leq c\epsilon||\Psi||_{L^2(I;H^1)}.$$

As before, this implies that choosing $\epsilon = \epsilon_1 < \epsilon_0$ sufficiently small, the map $\Psi \mapsto \Psi - \mathcal{U}(< \delta\beta_\epsilon, d\Psi >)$ is an isomorphism of W_0. Equivalently, (4.19) has a unique solution in W_0. Since both the solution Φ obtained in the previous paragraph and the original w are solutions of (4.19) in W_0 (note $\Phi_t \in L^4(I; L^{4/3}) \hookrightarrow L^2(I; H^{-1})$), it follows that $\Phi = w$, hence $w \in L^4(I, W^{2,4/3}) \subset L^4(I; W^{1,4})$. This concludes the proof of lemma 3.1, and with it the proof of theorem 1.3.

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