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Completeness Results for Basic Narrowing*

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Abstract. In this paper we analyze completeness results for basic narrowing. We show that basic narrowing is not complete with respect to normalizable solutions for equational theories defined by confluent term rewriting systems, contrary to what has been conjectured. By imposing syntactic restrictions on the rewrite rules we recover completeness. We refute a result of Hölldobler which states the completeness of basic conditional narrowing for complete (i.e. confluent and terminating) conditional term rewriting systems without extra variables in the conditions of the rewrite rules. In the last part of the paper we extend the completeness results of Giovannetti and Moiso for level-confluent and terminating conditional systems with extra variables in the conditions to systems that may also have extra variables in the right-hand sides of the rules.

Keywords: Narrowing, Basic narrowing, Conditional narrowing, Completeness, Term rewriting systems

1. Introduction

The aim of this paper is to analyze the various completeness results for narrowing in a uniform setting. In order to avoid biting off more than we can chew, we restrict ourselves to ordinary narrowing, basic narrowing, conditional narrowing and basic conditional narrowing. In particular, we do not consider normal narrowing (Fay

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[14]), the combination of basic and normal narrowing (Réty [37], Nutt et al. [36]), narrowing modulo equational theories (Kirchner [29]), nor various narrowing strategies like innermost and lazy narrowing (Fribourg [15] and You [42] respectively, see also Echahed [12, 13]).

Recently there has been much interest in incorporating the logic and functional programming paradigms in a single language. The computational mechanism underlying many of these imalgamated languages is conditional narrowing. Examples include ALF (Hanus [19]), BABEL (Moreno-Navarro and Rodríguez-Artalejo [35]), EQLOG (Goguen and Meseguer [18]), K-LEAF (Giovannetti et al. [16]) and SLOG (Fribourg [15]).

Narrowing was first studied in the context of semantic or E-unification. Fay [14] and Hullot [23] showed that narrowing is a complete method for solving equations in the theory defined by a confluent and terminating term rewriting system. Completeness means that for every solution to a given equation, a more general solution can be found by narrowing. It is well-known that the termination requirement can be dropped, provided we restrict ourselves to normalizable solutions. In other words, narrowing is complete for confluent term rewriting systems with respect to normalizable solutions. In order to reduce the search space of narrowing, Hullot [23] introduced the concept of basic narrowing. He showed that basic narrowing is complete for confluent and terminating term rewriting systems. In this paper we show that basic narrowing is not complete for confluent term rewriting systems with respect to normalizable solutions, thereby disproving a conjecture of Yamamoto [41].

Narrowing has been extended to conditional theories by Kaplan [28], Hußmann [25] and Dershowitz and Plaisted [10, 11], among others. Giovannetti and Moiso [17] observed that extra variables in the conditions of the rewrite rules may cause incompleteness (cf. Hußmann [26]). They showed that this incompleteness can be avoided by strengthening confluence to level-confluence. We extend their result to conditional term rewriting systems with extra variables in the right-hand side of the rules. Hölldobler [22] was one of the first to perform a systematic and extensive analysis of various versions of conditional narrowing for conditional term rewriting systems without extra variables. However, we will show that his completeness result for basic conditional narrowing with respect to confluent and terminating conditional term rewriting systems is incorrect. Our counterexample might influence the completeness of ALF (Hanus [19]) since its operational semantics is in essence basic conditional narrowing.

The paper is organized as follows. Section 2 contains a concise introduction to term rewriting and some elementary unification theory. In Sect. 3 we introduce narrowing and review its completeness. Section 4 is concerned with basic narrowing. We show that completeness is lost if we drop the termination requirement in exchange for the restriction to normalizable solutions, contrary to what is generally believed. In Sect. 5 we show that orthogonality and right-linearity are sufficient syntactic restrictions for recovering completeness. Conditional narrowing is introduced in Sect. 6. In Sect. 7 we show that basic conditional narrowing is not complete for confluent and terminating conditional term rewriting systems. We show that basic conditional narrowing is complete if we strengthen termination to decreasingness, a property of conditional term rewriting systems that implies the decidability of the rewrite relation. In Sect. 7 we also refute a conjecture of Giovannetti and Moiso [17] about the completeness of basic conditional narrowing for orthogonal conditional term rewriting systems. Section 8 contains a detailed account of the results of Giovannetti and Moiso [17] concerning the completeness of conditional narrowing for level-confluent systems. In Sect. 9 we show that conditional narrowing is complete for level-complete systems that have extra variables in the right-hand sides of the rewrite rules. Section 10 summarizes the results discussed in detail in previous sections. We mention some open problems and give suggestions for further research.

It is well-known that the correct use of variables and substitutions in completeness proofs requires great care. Several completeness proofs presented in the literature are incorrect due to incorrect assumptions about variables occurring in narrowing derivations and substitutions. Especially the so-called lifting lemma is notorious in this respect. This phenomenon is well-known in logic programming (cf. Shepherdson [40]). In the present paper it is our endeavour to give complete and rigorous proofs of the various lifting lemma's and other results. In particular, we take great efforts to motivate all assumptions about variables and substitutions. We are aware that easy readability is strained by a fully rigorous treatment of these matters. In order to enhance readability, the technical proofs of the propositions that relate certain rewrite sequences to basic narrowing derivations are deferred till the Appendix.

2. Preliminaries

In this section we review the basic notions of terms rewriting and unification. We refer to Dershowitz and Jouannaud [5] and Klop [31] for extensive surveys.

A *signature* is a set $\mathscr{F}$ of *function symbols*. Associated with every $f \in \mathscr{F}$ is a natural number denoting its arity. Function symbols of arity 0 are called *constants*. The set $\mathscr{T}(\mathscr{F}, \mathscr{V})$ of *terms* built from a signature $\mathscr{F}$ and a countably infinite set of *variables* $\mathscr{V}$ is the smallest set such that $\mathscr{V} \subset \mathscr{T}(\mathscr{F}, \mathscr{V})$ and if $f \in \mathscr{F}$ has arity n and $t_1, \ldots, t_n \in \mathscr{T}(\mathscr{F}, \mathscr{V})$ then $f(t_1, \ldots, t_n) \in \mathscr{T}(\mathscr{F}, \mathscr{V})$. We write c instead of $c(\,)$ whenever c is a constant. Identity of terms is denoted by $\equiv$. The set of variables occurring in a term t is denoted by $\mathscr{V}(t)$.

A precise formalism for describing subterm occurrences is obtained through the notion of *position*. The set $O(t)$ of *positions* is a term t is inductively defined as follows:

$$O(t) = \begin{cases} \{\varepsilon\} & \text{if } t \in \mathscr{V}, \\ \{\varepsilon\} \cup \{i \cdot p \mid 1 \leqq i \leqq n \text{ and } p \in O(t_i)\} & \text{if } t \equiv f(t_1, \ldots, t_n). \end{cases}$$

So positions are sequences of natural numbers denoting subterm occurrences. If $p \in O(t)$ then $t_{|p}$ denotes the *subterm* of t at position p, i.e.

$$t_{|p} = \begin{cases} t & \text{if } p = \varepsilon, \\ (t_i)_{|q} & \text{if } t \equiv f(t_1, \ldots, t_n) \text{ and } p = i \cdot q. \end{cases}$$

We write $s \trianglelefteq t$ to indicate that s is a subterm of t. If $s \trianglelefteq t$ and $s \not\equiv t$ then s is a *proper* subterm of t. The set $O(t)$ is partitioned into $\bar{O}(t)$ and $O_{\mathscr{V}}(t)$ as follows: $\bar{O}(t) = \{p \in O(t) \mid t_{|p} \notin \mathscr{V}\}$ and $O_{\mathscr{V}}(t) = \{p \in O(t) \mid t_{|p} \in \mathscr{V}\}$. If $p \in O(t)$ then $t[s]_p$ denotes the term that is obtained from t by replacing the subterm at position p by the term s. Formally:

$$t[s]_p = \begin{cases} s & \text{if } p = \varepsilon, \\ f(t_1, \ldots, t_i[s]_q, \ldots, t_n) & \text{if } t \equiv f(t_1, \ldots, t_n) \text{ and } p = i \cdot q. \end{cases}$$

Positions are partially ordered by the prefix ordering $\leqq$, i.e. $p \leqq q$ if there exists an r such that $p \cdot r = q$. We write $p < q$ if $p \leqq q$ and $p \neq q$. Positions p, q are *disjoint*, denoted by $p \perp q$, if neither $p \leqq q$ nor $q \leqq p$.

A *substitution* σ is a mapping from $\mathscr{V}$ to $\mathscr{T}(\mathscr{F}, \mathscr{V})$ such that $\{x \in \mathscr{V} \mid \sigma(x) \not\equiv x\}$ is finite. This set is called the *domain* of σ and denoted by $\mathscr{D}\sigma$. We frequently identify a substitution σ with the set $\{x \mapsto \sigma(x) \mid x \in \mathscr{D}\sigma\}$. The *empty* substitution will be denoted by ε. So $\varepsilon = \varnothing$ by abuse of notation. Substitutions are extended to morphisms from $\mathscr{T}(\mathscr{F}, \mathscr{V})$ to $\mathscr{T}(\mathscr{F}, \mathscr{V})$, i.e. $\sigma(f(t_1, \ldots, t_n)) \equiv f(\sigma(t_1), \ldots, \sigma(t_n))$ for every n-ary function symbol f and terms $t_1, \ldots, t_n$. In the following we write σt instead of $\sigma(t)$. The set of variables *introduced* by σ is denoted by $\mathscr{I}\sigma$, i.e. $\mathscr{I}\sigma = \bigcup_{x \in \mathscr{D}\sigma} \mathscr{V}(\sigma x)$. Composition of substitutions is denoted by '$\circ$', i.e. $(\sigma \circ \tau)x = \sigma(\tau x)$ for all $x \in \mathscr{V}$. We say that a substitution σ is *more general* than a substitution τ, denoted by $\sigma \leqq \tau$, if there exists a substitution ρ such that $\rho \circ \sigma = \tau$. Let $V \subseteq \mathscr{V}$. The *restriction* $\sigma \restriction_V$ of σ to V is defined as follows:

$$\sigma \restriction_V x = \begin{cases} \sigma x & \text{if } x \in V, \\ x & \text{otherwise.} \end{cases}$$

A *variable renaming* is a bijective substitution from $\mathscr{V}$ to $\mathscr{V}$. We write $\sigma = \tau[V]$ if $\sigma \restriction_V = \tau \restriction_V$ and $\sigma \leqq \tau[V]$ denotes the existence of a substitution ρ such that $\rho \circ \sigma = \tau[V]$. Two terms s that t are *unifiable* if there exists a substitution σ, a so-called *unifier* of s and t, such that $\sigma s \equiv \sigma t$. It is well-known that unifiable terms s, t posses a *most general unifier* σ, i.e., $\sigma \leqq \tau$ for every other unifier τ of s and t. Most general unifiers are unique up to variable renaming. A substitution σ is *idempotent* if $\sigma \circ \sigma = \sigma$. It is easy to show that a substitution σ is idempotent if and only if $\mathscr{D}\sigma \cap \mathscr{I}\sigma = \varnothing$. Given two unifiable terms s and t, the unification algorithms of Robinson [39] and Martelli and Montanari [33] produce an idempotent most general unifier σ that satisfies $\mathscr{D}\sigma \cup \mathscr{I}\sigma \subseteq \mathscr{V}(s) \cup \mathscr{V}(t)$. In the sequel we make use of the following proposition. Its routine proof is omitted.

Proposition 2.1. *If σ is an idempotent most general unifier of two terms s, t that have no variables in common then $\mathscr{D}\sigma \cup \mathscr{I}\sigma = \mathscr{V}(s) \cup \mathscr{V}(t)$.* □

Let $\sim$ be a binary relation on terms. We say that $\sim$ is *closed under contexts* if $s \sim t$ implies that $u[s]_p \sim u[t]_p$, for all terms u and positions $p \in O(u)$. The relation $\sim$ is *closed under substitutions* if $\sigma s \sim \sigma t$ whenever $s \sim t$, for all substitutions σ. A relation that is closed under contexts and substitutions is called a *rewrite relation.*

An *equation* is a pair (s, t) of terms, written as $s = t$. Let E be a set of equations. The smallest symmetric relation that contains E and is closed under contexts and substitutions is denoted by $\leftrightarrow_E$. So $s \leftrightarrow_E t$ if there exist an equation $l = r$ with $l = r \in E$ or $r = l \in E$, a position $p \in O(s)$, and a substitution σ such that $s_{|p} \equiv \sigma l$ and $t \equiv s[\sigma r]_p$. The transitive-reflexive closure of $\leftrightarrow_E$ is denoted by $=_E$. This relation is extended to substitutions as follows: $\sigma =_E \tau$ if $\sigma x =_E \tau x$ for all $x \in \mathscr{V}$. We write $\sigma \leqq_E \tau$ if there exists a substitution ρ such that $\rho \circ \sigma =_E \tau$. We define $\sigma =_E \tau[V]$ and $\sigma \leqq_E \tau[V]$ as above.

Two terms s and t are *E-unifiable* if there exists a substitution σ such that $\sigma s =_E \sigma t$. In the context of a set of equations E, the notion of most general unifier generalizes to *complete sets of E-unifiers.* A set of substitutions Σ is a complete set of E-unifiers of two terms s and t if the following three conditions are satisfied:

- $\mathscr{D}\sigma \subseteq \mathscr{V}(s) \cup \mathscr{V}(t)$ for all $\sigma \in \Sigma$,

- every $\sigma \in \Sigma$ is an E-unifier of s and t,
- if τ is an E-unifier of s and t then there exists a $\sigma \in \Sigma$ such that $\sigma \leq_E \tau[\mathscr{V}(s) \cup \mathscr{V}(t)]$.

Every set consisting of a most general unifier σ of terms s and t with $\mathscr{D}\sigma \subseteq \mathscr{V}(s) \cup \mathscr{V}(t)$ is a complete set of $\varnothing$-unifiers of s and t.

A *rewrite rule* is a directed equation $l \to r$ satisfying $l \notin \mathscr{V}$ and $\mathscr{V}(r) \subseteq \mathscr{V}(l)$. If $l \to r$ is a rewrite rule and σ a variable renaming then the rewrite rule $\sigma l \to \sigma r$ is called a *variant* of $l \to r$. A *term rewriting system* (TRS for short) is a set of rewrite rules. A rewrite rule $l \to r$ is *left-linear* (*right-linear*) if l (r) does not contain multiple occurrences of the same variable. A *left-linear* (*right-linear*) TRS only contains left-linear (right-linear) rewrite rules.

The rewrite relation $\to_{\mathscr{R}}$ associated with the TRS $\mathscr{R}$ is defined as follows: $s \to_{\mathscr{R}} t$ if there exist a variant[1] $l \to r$ of a rewrite rule in $\mathscr{R}$, a position $p \in O(s)$, and a substitution σ such that $s_{|p} \equiv \sigma l$ and $t \equiv s[\sigma r]_p$. The term σl is called a *redex* and we say that s rewrites to t by *contracting* redex σl. We call $s \to_{\mathscr{R}} t$ a *rewrite step*. Occasionally we write $s \to_{[p, l \to r, \sigma]} t$ or $s \to_{[p, l \to r]} t$. The transitive-reflexive closure of $\to_{\mathscr{R}}$ is denoted by $\twoheadrightarrow_{\mathscr{R}}$. If $s \twoheadrightarrow_{\mathscr{R}} t$ we say that s *reduces* to t. The transitive closure of $\to_{\mathscr{R}}$ is denoted by $\to_{\mathscr{R}}^{+}$. We write $s \leftarrow_{\mathscr{R}} t$ if $t \to_{\mathscr{R}} s$; likewise for $s \twoheadleftarrow_{\mathscr{R}} t$. The transitive-reflexive-symmetric closure of $\to_{\mathscr{R}}$ is called *conversion* and denoted by $=_{\mathscr{R}}$. If $s =_{\mathscr{R}} t$ then s and t are *convertible*. If E is the set of equations corresponding to $\mathscr{R}$, i.e. $E = \{l = r \mid l \to r \in \mathscr{R}\}$, then $=_{\mathscr{R}}$ and $=_E$ coincide. Via this correspondence the notion of $\mathscr{R}$-unification is implicitly defined. Two terms t_1, t_2 are *joinable*, denoted by $t_1 \downarrow_{\mathscr{R}} t_2$, if there exists a term t_3 such that $t_1 \twoheadrightarrow_{\mathscr{R}} t_3 \twoheadleftarrow_{\mathscr{R}} t_2$. Such a term t_3 is called a *common reduct* of t_1 and t_2. When no confusion can arise, we omit the subscript $\mathscr{R}$.

A term s is a *normal form* if there is no term t with $s \to t$. We also say that s is *normalized*. A term s *has* a normal form if there exists a reduction sequence $s \twoheadrightarrow t$ with t a normal form. A TRS is *weakly normalizing* if every term has a normal form. A TRS is *strongly normalizing* if there are no infinite reduction sequences $t_1 \to t_2 \to t_3 \to \cdots$. In other words, every reduction sequence eventually ends in a normal form. A TRS is *locally confluent* if for all terms s, t_1, t_2 with $t_1 \leftarrow s \to t_2$ we have $t_1 \downarrow t_2$. A TRS is *confluent* or has the *Church–Rosser* property if for all terms s, t_1, t_2 with $t_1 \twoheadleftarrow s \twoheadrightarrow t_2$ we have $t_1 \downarrow t_2$. A well-known equivalent formulation of confluence is that every pair of convertible terms is joinable ($t_1 = t_2 \Rightarrow t_1 \downarrow t_2$). The renowned Newman's Lemma states that every locally confluent and strongly normalizing TRS is confluent. A *complete* TRS is confluent and strongly normalizing. A *semi-complete* TRS is confluent and weakly normalizing. Each term in a (semi-) complete TRS has a unique normal form. The above properties of TRSs (weak normalization, strong normalization, local confluence, confluence, completeness, and semi-completeness) specialize to terms in the obvious way. For instance, a term s is confluent if $t_1 \downarrow t_2$ whenever $t_1 \twoheadleftarrow s \twoheadrightarrow t_2$. If a term t has a unique normal form then we denote this normal form by $t\downarrow$.

A substitution σ is *normalized* (with respect to a TRS $\mathscr{R}$) is σx is a normal form for every $x \in \mathscr{D}\sigma$. A substitution σ is *normalizable* if σx has a normal form for every $x \in \mathscr{D}\sigma$. Let σ be a normalizable substitution. A normalized substitution τ is called a *normal form* of σ if $\sigma x \twoheadrightarrow \tau x$ for all $x \in \mathscr{V}$.

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[1] The use of variants is not essential for defining the rewrite relation since rewriting is *variant independent*, meaning that if $s \to_{[p, l \to r, \sigma]} t$ and $l' \to r'$ is a variant of $l \to r$ then also $s \to_{[p, l' \to r', \sigma']} t$ for some substitution σ'. However, it states explicitly that we may rename variables when necessary, e.g. when we relate rewriting to narrowing, which is not variant independent in the above sense.

Let $l_1 \to r_1$ and $l_2 \to r_2$ be variants of rewrite rules of a TRS $\mathscr{R}$ without common variables. Suppose $p \in \bar{O}(l_1)$ such that $(l_1)_{|p}$ and l_2 are unifiable, so $\sigma(l_1)_{|p} \equiv \sigma l_2$ for a most general unifier σ. The term $\sigma l_1 \equiv \sigma l_1[\sigma l_2]_p$ is subject to the reduction steps $\sigma l_1 \to \sigma r_1$ and $\sigma l_1 \to \sigma l_1[\sigma r_2]_p \equiv \sigma(l_1[r_2]_p)$. The pair of reducts $\langle \sigma(l_1[r_2]_p), \sigma r_1 \rangle$ is a *critical pair of* $\mathscr{R}$. If $l_1 \to r_1$ and $l_2 \to r_2$ are variants, we do not consider the case $p \in \varepsilon$. A critical pair $\langle s, t \rangle$ is *convergent* if $s \downarrow t$. The well-known Critical Pair Lemma states that a TRS is locally confluent if and only if all its critical pairs are convergent.

A TRS is called *non-ambiguous* or *non-overlapping* if it has no critical pairs. An *orthogonal* TRS is both left-linear and non-ambiguous. For orthogonal TRSs a considerable amount of theory has been developed, see Klop [31] for a comprehensive survey. The most prominent fact is that every orthogonal TRS is confluent. In Sect. 4 we make use of the work of Huet and Lévy [21] on needed reductions in orthogonal TRSs.

We conclude this section with some information on *multiset orderings*. A *multiset* over a set A is an unordered collection of elements of A in which elements may have multiple occurrences. Every partial order (i.e. transitive and irreflexive relation) $\succ$ on A can be extended to a partial order $\succ\!\!\succ$ on the set of finite multisets over A as follows: $M \succ\!\!\succ N$ if there exists multisets X and Y such that

- $\varnothing \neq X \subseteq M$,
- $N = (M - X) \cup Y$,
- for every $y \in Y$ there exists an $x \in X$ such that $x \succ y$.

The partial order $\succ\!\!\succ$ is called the *multiset extension* of $\succ$. Dershowitz and Manna [6] showed that the multiset extension of a well-founded order is again well-founded.

3. Narrowing

In this section we introduce narrowing and review some completeness results. The narrowing relation defined below was introduced by Hullot [23].

Definition 3.1. We say that a term t is *narrowable* into a term t' if there exist a position $p \in \bar{O}(t)$, a variant[2] $l \to r$ of a rewrite rule in $\mathscr{R}$, and a substitution σ such that

- σ is a most general unifier of $t_{|p}$ and l,
- $t' \equiv \sigma(t[r]_p)$.

We write $t \rightsquigarrow_{[p, l \to r, \sigma]} t'$ or simply $t \rightsquigarrow_\sigma t'$. The relation $\rightsquigarrow$ is called *narrowing*.

Notation. We write $t \rightsquigarrow_\sigma^* t'$ if there exists a narrowing derivation

$$t \equiv t_1 \rightsquigarrow_{\sigma_1} t_2 \rightsquigarrow_{\sigma_2} \cdots \rightsquigarrow_{\sigma_{n-1}} t_n \equiv t'$$

such that $\sigma = \sigma_{n-1} \circ \cdots \circ \sigma_2 \circ \sigma_1$. If $n = 1$ then $\sigma = \varepsilon$.

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[2] Renaming of rewrite rules is mandatory for ensuring completeness. The idea is to use a single variant of a rewrite rule and a single most general unifier, in order to avoid unnecessary computations. We always require that the rewrite rule has no variables in common with the term to be narrowed, i.e. $\mathscr{V}(l) \cap \mathscr{V}(t) = \varnothing$, but in general this is not sufficient for completeness. From the proof of Lemma 3.4 below, the precise requirements of freshness can be deduced. That proof makes also clear that any idempotent most general unifier is adequate for completeness.

In a rewrite step $s \to_{[p, l \to r, \sigma]} t$ we may always assume that the applied rewrite rule has no variables in common with s and σ is restricted to variables occurring in l. Consequently, σ is a most general unifier of $s_{|p}$ and l, and $t \equiv s[\sigma r]_p \equiv \sigma(s[r]_p)$. Hence rewriting can be viewed as a special case of narrowing.

A nice explanation of the word "narrowing" can be found in Klop [30]. We now explain how narrowing can be used for equational unification. In order to facilitate the exposition, we extend the set of function symbols with a fresh binary function symbol $=^?$ and a fresh constant *true*. We furthermore assume that $\mathscr{R}$ contains the rewrite rule $x =^? x \to true$.[3] We consider only terms of the following form:

- terms that do not contain any occurrences of $=^?$ and *true*,
- terms $s =^? t$ with s and t satisfying the previous condition,
- the constant *true*.

Terms of the second form are called *goals*. It should be stressed that confluence, completeness, and semi-completeness are retained under the addition of the rule $x =^? x \to true$.[4]

Example 3.2. Consider the TRS

$$\mathscr{R} = \begin{cases} 0 + x & \to x \\ S(x) + y \to S(x+y). \end{cases}$$

Suppose we want to solve the goal $z + z =^? S(S(0))$. Figure 1 shows that narrowing is able to find the (unique) solution $\{z \mapsto S(0)\}$. This is not a coincidence: below we will see that narrowing is able to find all elements of a complete set of $\mathscr{R}$-solutions[5] of a given goal, provided $\mathscr{R}$ satisfies certain conditions.

The soundness of narrowing is expressed in the next lemma.

Lemma 3.3. *Let $\mathscr{R}$ be a TRS. If $s =^? t \rightsquigarrow_\sigma^* true$ then σ is an $\mathscr{R}$-unifier of s and t.*

Proof. Easy induction on the length of the narrowing derivation $s =^? t \rightsquigarrow_\sigma^* true$, using the observation that $\sigma' s' \to_{\mathscr{R}} t'$ whenever $s' \rightsquigarrow_{\sigma'} t'$. □

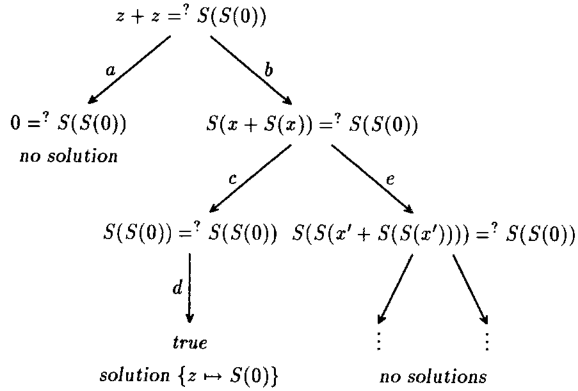
The following lemma of Hullot [23] is the key to completeness. It states that rewrite sequences can be 'lifted' to narrowing derivations.

Lemma 3.4. *Let $\mathscr{R}$ be a TRS. Suppose we have terms s and t, a normalized substitution θ and a set of variables V such that $\mathscr{V}(s) \cup \mathscr{D}\theta \subseteq V$ and $t \equiv \theta s$. If $t \twoheadrightarrow_{\mathscr{R}} t'$ then there exist a term s' and substitutions θ', σ such that*

- $s \rightsquigarrow_\sigma^* s'$,
- $\theta' s' \equiv t'$,
- $\theta' \circ \sigma = \theta[V]$,
- *θ' is normalized.*

Furthermore, we may assume that the narrowing derivation $s \rightsquigarrow_\sigma^* s'$ and the rewrite sequence $t \twoheadrightarrow_{\mathscr{R}} t'$ employ the same rewrite rules at the same positions. □

[3] This assumption will not be make when we consider orthogonal TRSs.
[4] This even holds if we would allow unrestricted term formation, due to modularity considerations; see Middeldorp [34]
[5] An $\mathscr{R}$-solution of a goal $s =^? t$ is an $\mathscr{R}$-unifier of the terms s and t.



step	rewrite rule			narrowing substitution
a	$0 + x$	$\rightarrow$	x	$\{x, z \mapsto 0\}$
b	$S(x) + y$	$\rightarrow$	$S(x + y)$	$\{y, z \mapsto S(x)\}$
c	$0 + x'$	$\rightarrow$	x'	$\{x \mapsto 0,\ x' \mapsto S(0)\}$
d	$x =^? x$	$\rightarrow$	*true*	$\{x \mapsto S(S(0))\}$
e	$S(x') + y'$	$\rightarrow$	$S(x' + y')$	$\{x \mapsto S(x'),\ y' \mapsto S(S(x'))\}$

Fig. 1.

The proof presented in [23] is incorrect with regard to the normalization of the resulting substitution θ'. Before giving a rigorous proof of this lemma we present three easy propositions that are heavily used in the proofs of all lifting lemma's in this paper.

Proposition 3.5. *If t is a term and σ a substitution then $\mathscr{V}(\sigma t) = (\mathscr{V}(t) - \mathscr{D}\sigma) \cup \mathscr{I}\sigma\restriction_{\mathscr{V}(t)}$.*

Proof. Obvious. □

Proposition 3.6. *Suppose we have substitutions σ, θ, θ' and sets A, B of variables such that $(B - \mathscr{D}\sigma) \cup \mathscr{I}\sigma \subseteq A$. If $\theta = \theta'$ $[A]$ then $\theta \circ \sigma = \theta' \circ \sigma$ $[B]$.*

Proof. We have $(\theta \circ \sigma)\restriction_B = (\theta\restriction_{\mathscr{I}\sigma} \circ \sigma)\restriction_{B \cap \mathscr{D}\sigma} \cup \theta\restriction_{B - \mathscr{D}\sigma} = (\theta'\restriction_{\mathscr{I}\sigma} \circ \sigma)\restriction_{B \cap \mathscr{D}\sigma} \cup \theta'\restriction_{B - \mathscr{D}\sigma} = (\theta' \circ \sigma)\restriction_B$. The assumptions are used in the second equality. □

Proposition 3.7. *Let $\mathscr{R}$ be a TRS and suppose we have sets A, B of variables and substitutions σ, θ, θ' such that the following conditions are satisfied:*
- *$\theta\restriction_A$ is $\mathscr{R}$-normalized,*
- *$\theta' \circ \sigma = \theta$ $[A]$,*
- *$B \subseteq (A - \mathscr{D}\sigma) \cup \mathscr{I}\sigma\restriction_A$.*

Then $\theta'\restriction_B$ is also $\mathscr{R}$-normalized.

Proof. Let $x \in B$. We have to show that $\theta' x$ is an $\mathscr{R}$-normal form. If $x \in A - \mathscr{D}\sigma$ then $\theta' x \equiv (\theta' \circ \sigma)x \equiv \theta x$ which is an $\mathscr{R}$-normal form by assumption. If $x \in \mathscr{I}\sigma\restriction_A$ then there exists a variable $y \in A$ such that $x \in \mathscr{V}(\sigma y)$. We have $\theta' x \subseteq \theta'(\sigma y) \equiv \theta y$. By assumption θy is an $\mathscr{R}$-normal form and hence its subterm $\theta' x$ is also an $\mathscr{R}$-normal form. □

Proof of Lemma 3.4. We use induction on the length of the reduction sequence from t to t'. The case of zero length is trivial. Suppose $t \rightarrow_{\mathscr{R}} t_1 \twoheadrightarrow_{\mathscr{R}} t'$ is a reduction sequence of length $n+1$. Let $t \rightarrow_{[p, l \rightarrow r]} t_1$. We may assume that $\mathscr{V}(l) \cap V = \varnothing$.[6] We have $(\theta s)_{|p} \equiv \tau l$ for some substitution τ with $\mathscr{D}\tau \subseteqq \mathscr{V}(l)$. Since θ is normalized we have $p \in \bar{O}(s)$ and hence $(\theta s)_{|p} = \theta(s_{|p})$. Let $\mu = \tau \cup \theta$. We have $\mu(s_{|p}) \equiv \theta(s_{|p}) \equiv \tau l \equiv \mu l$, so $s_{|p}$ and l are unifiable. Let σ_1 be an idempotent most general unifier of $s_{|p}$ and l. Proposition 2.1 yields $\mathscr{D}\sigma_1 \cup \mathscr{I}\sigma_1 = \mathscr{V}(s_{|p}) \cup \mathscr{V}(l)$. Let $s_1 \equiv \sigma_1(s[r]_p)$. By definition

$$s \rightsquigarrow_{[p, l \rightarrow r, \sigma_1]} s_1. \tag{1}$$

Since $\sigma_1 \leqq \mu$, there exists a substitution ρ such that $\rho \circ \sigma_1 = \mu$. Let $V_1 = (V - \mathscr{D}\sigma_1) \cup \mathscr{I}\sigma_1$. Define $\theta_1 = \rho \restriction_{V_1}$. Clearly $\mathscr{D}\theta_1 \subseteqq V_1$. We have $\mathscr{V}(s_1) = \mathscr{V}(\sigma_1(s[r]_p)) \subseteqq \mathscr{V}(\sigma_1(s[l]_p)) = \mathscr{V}(\sigma_1 s) \subseteqq V_1$. The last inclusion follows from Proposition 3.5. Therefore

$$\mathscr{V}(s_1) \cup \mathscr{D}\theta_1 \subseteqq V_1. \tag{2}$$

Using $\theta_1 = \rho[V_1]$ we obtain $\theta_1 s_1 \equiv \rho s_1 \equiv \rho\sigma_1(s[r]_p) \equiv \mu(s[r]_p) \equiv \mu s[\mu r]_p$. Since $V \cap \mathscr{D}\tau = \varnothing$ we have $\mu = \theta_1[V]$. Likewise $\mu = \tau[\mathscr{V}(r)]$. Hence the term $\mu s[\mu r]_p$ equals $\theta s[\tau r]_p \equiv t_1$. Thus

$$\theta_1 s_1 = t_1. \tag{3}$$

Next we show that $\theta_1 \circ \sigma_1 = \theta[V]$. Proposition 3.6 yields $\theta_1 \circ \sigma_1 = \rho \circ \sigma_1[V]$. We already noticed that $\mu = \theta[V]$. Linking these two equalities via the equation $\rho \circ \sigma_1 = \mu$ yields

$$\theta_1 \circ \sigma_1 = \theta[V]. \tag{4}$$

Before we can apply the induction hypothesis, we have to verify that θ_1 is normalized. Since $\mathscr{D}\theta_1 \subseteqq V_1$ it suffices to show that $\theta_1 \restriction_{V_1}$ is normalized. Let $B = (V - \mathscr{D}\sigma_1) \cup \mathscr{I}\sigma_1 \restriction_V$. Proposition 3.7 (with $A = V$) yields the normalization of $\theta_1 \restriction_B$. We claim that $\mathscr{I}\sigma_1 \subseteqq B$ and hence $B = V_1$. Recall that $\mathscr{I}\sigma_1 \subseteqq \mathscr{V}(s_{|p}) \cup \mathscr{V}(l)$. Let $x \in \mathscr{I}\sigma_1$. Idempotence of σ_1 yields $x \notin \mathscr{D}\sigma_1$. If $x \in \mathscr{V}(s_{|p}) \subseteqq V$ then $x \in V - \mathscr{D}\sigma_1$. If $x \in \mathscr{V}(l)$ then $x \in \mathscr{V}(\sigma_1 l) = \mathscr{V}(\sigma_1(s_{|p}))$ and thus $x \in \mathscr{I}\sigma_1 \restriction_V$. So $\mathscr{I}\sigma_1 \subseteqq B$. Hence

$$\theta_1 \quad \text{is normalized.} \tag{5}$$

The induction hypothesis yields a term s' and substitutions θ', σ' such that

$$s_1 \rightsquigarrow^*_{\sigma'} s', \tag{6}$$

$$\theta' s' \equiv t', \tag{7}$$

$$\theta' \circ \sigma' = \theta_1[V_1], \tag{8}$$

$$\theta' \quad \text{is normalized.} \tag{9}$$

Moreover, we may assume that $s_1 \rightsquigarrow^*_{\sigma'} s'$ and $t_1 \twoheadrightarrow_{\mathscr{R}} t'$ apply the same rewrite rules at the same positions. Let $\sigma = \sigma' \circ \sigma_1$. Concatenating (1) and (6) yields $s \rightsquigarrow^*_\sigma s'$. By construction this narrowing derivation employs the same rewrite rules at the same positions as the rewrite sequence $t \twoheadrightarrow_{\mathscr{R}} t'$. It remains to show that $\theta' \circ \sigma = \theta[V]$. Proposition 3.6 applied to (8) yields $\theta' \circ \sigma' \circ \sigma_1 = \theta_1 \circ \sigma_1[V]$ and hence $\theta' \circ \sigma = \theta_1 \circ \sigma_1 = \theta[V]$ by (4). □

Theorem 3.8 (Hullot [23]). *Let $\mathscr{R}$ be a complete TRS. If $\sigma s =_{\mathscr{R}} \sigma t$ then there exists a narrowing derivation $s =^? t \rightsquigarrow^*_\tau$ true such that $\tau \leqq_{\mathscr{R}} \sigma[\mathscr{V}(s) \cup \mathscr{V}(t)]$.*

[6] This is justified by the variant independence of rewriting, cf. footnote 1 on page 217.

Proof. Let σ' be the normal form of σ, i.e., $\sigma' = \{x \mapsto (\sigma x)\downarrow_{\mathscr{R}} \mid x \in \mathscr{D}\sigma\}$. Notice that $\sigma =_{\mathscr{R}} \sigma'$. Clearly $\sigma' s =_{\mathscr{R}} \sigma' t$. Confluence of $\mathscr{R}$ yields $\sigma' s \downarrow_{\mathscr{R}} \sigma' t$. Hence there exists a rewrite sequence $\sigma'(s =^? t) \twoheadrightarrow_{\mathscr{R}} true$. According to Lemma 3.4 there exists a narrowing derivation $s =^? t \rightsquigarrow_{\tau}^{*} true$ and a substitution σ'' such that $\sigma'' \circ \tau = \sigma'[\mathscr{V}(s) \cup \mathscr{V}(t)]$. Therefore $\tau \leqq \sigma'[\mathscr{V}(s) \cup \mathscr{V}(t)]$. Since $\sigma =_{\mathscr{R}} \sigma'$ we conclude that $\tau \leqq_{\mathscr{R}} \sigma[\mathscr{V}(s) \cup \mathscr{V}(t)]$. □

In the following, statements like Theorem 3.8 will be abbreviated by saying that (a kind of) narrowing is complete for (a class of) TRS (with respect to certain goals and substitutions). The reason for this terminology becomes apparent in the following equivalent formulation of Theorem 3.8.

Corollary 3.9. *Let $\mathscr{R}$ be a complete TRS. The set $\{\sigma \upharpoonright_{\mathscr{V}(s) \cup \mathscr{V}(t)} \mid s =^? t \rightsquigarrow_{\sigma}^{*} true\}$ is a complete set of $\mathscr{R}$-unifiers of s and t.* □

From the above proof it is clear that the subscript $\mathscr{R}$ in $\tau \leqq_{\mathscr{R}} \sigma$ can be dropped if we only consider normalized substitutions. Strong normalization of $\mathscr{R}$ is only used in the normalization of σ into σ', hence we can strengthen Theorem 3.8 by dropping the strong normalization requirement and restricting ourselves to normalizable substitutions.

Theorem 3.10. *Narrowing is complete for confluent TRSs with respect to normalizable substitutions.* □

Since in a weakly normalizing TRS every substitution is normalizable, we obtain the following result of Yamamoto [41].

Corollary 3.11. *Narrowing is complete for semi-complete TRSs.* □

4. Basic Narrowing

The search space of narrowing is quite large. As a matter of fact, the narrowing procedure seldom terminates. Hullot [23] introduced a restricted form of narrowing, the so-called *basic narrowing*, which still is complete for complete TRSs.

Definition 4.1.

(1) Let $t_1 \rightsquigarrow_{[p_1, l_1 \to r_1, \sigma_1]} t_2 \rightsquigarrow_{[p_2, l_2 \to r_2, \sigma_2]} \cdots \rightsquigarrow_{[p_{n-1}, l_{n-1} \to r_{n-1}, \sigma_{n-1}]} t_n$ be a narrowing derivation. We inductively define sets of positions $B_1, \ldots, B_n$ as follows:

$$B_1 = \bar{O}(t_1),$$

$$B_{i+1} = \mathscr{B}(B_i, p_i, r_i) \quad \text{for } 1 \leqq i < n.$$

Here $\mathscr{B}(B_i, p_i, r_i)$ abbreviates $(B_i - \{q \in B_i \mid p_i \leqq q\}) \cup \{p_i \cdot q \mid q \in \bar{O}(r_i)\}$. Positions in B_i are referred to as *basic positions* and positions in $\bar{O}(t_i) - B_i$ are called *non-basic* $(1 \leqq i \leqq n)$. We say that the above narrowing derivation is *basic* if $p_i \in B_i$ for $1 \leqq i < n$.

(2) A rewrite sequence $t_1 \to_{[p_1, l_1 \to r_1, \sigma_1]} t_2 \to_{[p_2, l_2 \to r_2, \sigma_2]} \cdots \to_{[p_{n-1}, l_{n-1} \to r_{n-1}, \sigma_{n-1}]} t_n$ is *based* on a set of positions $B_1 \subseteqq \bar{O}(t_1)$ if $p_i \in B_i$ for $1 \leqq i < n$ with $B_2, \ldots, B_{n-1}$ defined as above.

So in a basic derivation narrowing is never applied to a subterm introduced by a previous narrowing substitution. It should be noted that the concepts defined above do not depend on the used variants of rewrite rules. It is not difficult to show that the sets B_i defined above are closed under prefix[7] for every rewrite sequence that is based on a set which is closed under prefix. This observation will be used freely in the sequel.

Example 4.2. Consider the complete TRS $\mathscr{R} = \{f(f(x)) \to x\}$. The infinite sequence

$$f(x) =^? x \rightsquigarrow_{\{x \mapsto f(x')\}} x' =^? f(x') \rightsquigarrow_{\{x' \mapsto f(x'')\}} f(x'') =^? x'' \rightsquigarrow_{\{x'' \mapsto f(x''')\}} \cdots$$

is the only narrowing derivation issued from the goal $\underline{f}(x) =^? x$. It is not basic since the restriction $p_2 \in B_2$ is violated if we take $B_1 = \bar{O}(t_1) = \{\varepsilon, 1\}$ and $B_2 = \{\varepsilon\}$. In later examples, when we state that a given narrowing derivation is (non-)basic, the justification – i.e. the sets B_i – is given by underlining all non-basic positions.

Hullot showed that if all basic narrowing derivations starting at a right-hand side of a rewrite rule terminate, then the search space of basic narrowing is finite for any term. Recently, Chabin and Réty [4] argued that the termination behaviour of basic narrowing can be further improved by adopting a graph representation of the TRS and the goal to be solved.

Herold [20] showed that the sets B_i can be reduced by means of *left-to-right* basic narrowing, without losing completeness. The search space can also be reduced by means of the so-called *selection* narrowing of Bosco et al. [3]. In this paper we do not consider these optimizations, but we note that all our results concerning basic (conditional) narrowing – both positive and negative – extend to both left-to-right and selection narrowing. Krischer and Bockmayr [32] describe various criteria to detect redundant basic narrowing derivations.

A more elegant formulation of basic narrowing is obtained by partitioning goals into a *skeleton* and *environment part* as in Nutt et al. [36] and Hölldobler [22]. In such a formulation narrowing would be defined on pairs $\langle t, \theta \rangle$, consisting of a term t (the skeleton) and a substitution θ (the environment), as follows: $\langle t, \theta \rangle \rightsquigarrow_\sigma \langle t[r]_p, \sigma \circ \theta \rangle$ where $p \in \bar{O}(t)$ and σ is a most general unifier of $(\theta t)_{|p}$ and l for some rewrite rule $l \to r$. The main reason for adopting the "standard" definition is that we can still use Lemma 3.4 whereas the above formulation requires a more complicated lifting lemma (in order to ensure completeness of basic narrowing for complete TRSs).

Besides the lifting lemma, the completeness proof of basic narrowing employs Proposition 4.4. A proof of this proposition is given in the Appendix.

Definition 4.3. An *innermost* redex does not contain other redexes. In an innermost reduction sequence only innermost redexes are contracted.

Proposition 4.4 (Hullot [24], Yamamoto [41]). *Let $\mathscr{R}$ be a TRS and σ a normalized substitution. Every innermost reduction sequence starting from σt is based on $\bar{O}(t)$.* □

Theorem 4.5 (Hullot [23, 24]). *Basic narrowing is complete for complete TRSs.*

Proof. Let $\mathscr{R}$ be a complete TRS and suppose that $\sigma s =_{\mathscr{R}} \sigma t$. Let σ' be the normal form of σ. Just as in the proof of Theorem 3.8 we obtain $\sigma'(s =^? t) \twoheadrightarrow_{\mathscr{R}} true$. Because $\mathscr{R}$ is complete we may assume that this reduction sequence is innermost. According to the previous proposition the sequence is based on $\bar{O}(s =^? t)$. Since the narrowing

[7] That is, if $p < q$ and $q \in B_i$ then $p \in B_i$.

derivation constructed by Lemma 3.4 employs the same rewrite rules at the same positions, we know that it is basic. The remainder of the proof follows literally the proof of Theorem 3.8. □

Several authors (Yamamoto [41], Hölldobler [22]) reported a mistake in the proof of Hullot as given in [23]. Less well-known is the fact that Hullot himself was the first to repair the proof, see Hullot's thesis [24]. Yamamoto observed that strong normalization can be weakened to *weakly innermost normalization.* A TRS is called weakly innermost normalizing if every term has a normal form that can be reached by means of an innermost reduction sequence. More interesting is the following statement.

Conjecture 4.6 (Yamamoto [41]). *Basic narrowing is complete for semi-complete TRSs.* □

Counterexample 4.7. Consider the TRS

$$\mathcal{R} = \begin{cases} f(x) & \to g(x,x) \\ a & \to b \\ g(a,b) & \to c \\ g(b,b) & \to f(a). \end{cases}$$

Induction on the structure of terms and straightforward case analysis reveals that every term has a unique normal form. Hence $\mathcal{R}$ is semi-complete. However, the goal $f(a) =^? c$ cannot be solved by basic narrowing. Figure 2 shows all narrowing derivations starting from this goal. Recall that non-basic positions are marked by underlining. (Since the goal is variable-free, all narrowing steps in the figure are rewrite steps.) The steps marked with a star are non-basic because each of them rewrites an occurrence of the term a introduced by the substitution $\{x \mapsto a\}$ used in the step from $f(a) =^? c$ to $g(a,a) =^? c$. Since every successful derivation passes through a marked step, basic narrowing is not able to solve the goal $f(a) =^? c$. Basic narrowing is also unable to solve the *normalized* goal $g(x,x) =^? c$ since the only (basic) narrowing step starting from $g(x,x) =^? c$ produces the goal $f(a) =^? c$: $g(x,x) =^? c \rightsquigarrow_{\{x \mapsto b\}} f(a) =^? c$.

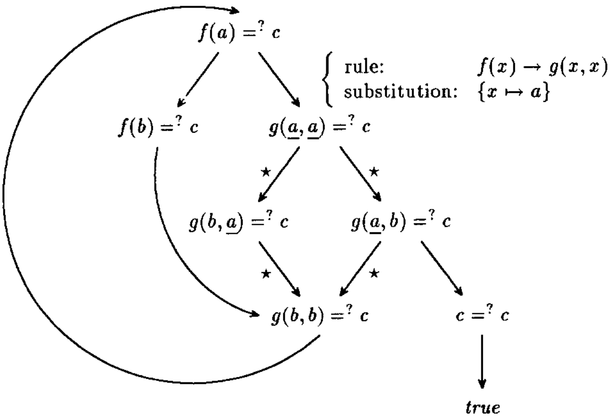


Fig. 2.

In particular basic narrowing is not complete for confluent TRS with respect to normalizable substitutions, contrary to what is generally believed. In the next section we recover the completeness of basic narrowing for semi-complete TRSs by imposing syntactic restrictions on the rewrite rules.

5. Restoring Completeness

Counterexample 4.7 suggests two sufficient conditions for the completeness of basic narrowing for semi-complete TRSs: *orthogonality* and *right-linearity*. We first show the sufficiency of orthogonality. The proof is based on the work of Huet and Lévy [21] on needed reductions. Before stating their main result, we introduce a few preliminary concepts.

Definition 5.1. Let $A: s \rightarrow_{[p, l \rightarrow r]} t$ be a reduction step in a TRS $\mathscr{R}$ and let $q \in O(s)$. The set $q \backslash A$ of *descendants* of q in t is defined as follows:

$$q \backslash A = \begin{cases} \{q\} & \text{if } q < p \text{ or } q \perp p, \\ \{p \cdot p_3 \cdot p_2 \mid r_{|p_3} \equiv l_{|p_1}\} & \text{if } q = p \cdot p_1 \cdot p_2 \text{ with } p_1 \in O_{\mathscr{V}}(l), \\ \varnothing & \text{otherwise.} \end{cases}$$

If $Q \subseteq O(s)$ then $Q \backslash A$ denotes the set $\bigcup_{q \in Q} q \backslash A$. The notion of descendant is extended to rewrite sequences in the obvious way. Orthogonal TRSs have the nice property that a descendant of a redex is again a redex (with respect to the same rewrite rule).

Definition 5.2. A redex s in a term t is *needed* if in every reduction sequence from t to normal form a descendant of s is contracted. A needed redex s in a term t is *innermost* if it does not contain other needed redexes. In an (innermost) needed reduction sequence only (innermost) needed redexes are contracted.

Theorem 5.3 (Huet and Lévy [21]). *Let t be a term in an orthogonal TRS.*

- *If t is not a normal form then t contains a needed redex.*
- *If t has a normal form, repeated contraction of needed redexes leads to that normal form.* □

Definition 5.4. Let $\mathscr{R}$ be a TRS. We write $s \mathrel{+\!\!\!\!\!\!\rightarrow} t$ if t can be obtained from s by contracting a set of pairwise disjoint redexes in s. The relation $\mathrel{+\!\!\!\!\!\!\rightarrow}$ is called *parallel reduction.*

Proposition 5.5. *Let $\mathscr{R}$ be an orthogonal TRS and σ a normalized substitution. Every innermost needed reduction sequence starting from σt is based on $\bar{O}(t)$.*

Proof. See the Appendix. □

The formulation of the completeness theorem for basic narrowing with respect to normalizable solutions in the context of orthogonal TRSs is slightly different than previous completeness results. The reason is that the rewrite rule $x =^? x \rightarrow true$ cannot be used since it disturbs left-linearity. This also explains why we have to require the normalizablity of σs and σt.

Theorem 5.6. *Let $\mathscr{R}$ be an orthogonal TRS. If $\sigma s =_{\mathscr{R}} \sigma t$ and σ, σs, and σt are normalizable then there exists a basic narrowing derivation $s =^? t \rightsquigarrow_{\tau}^* s' =^? t'$ and a most general unifier τ' of s' and t' such that $\tau' \circ \tau \leq_{\mathscr{R}} \sigma[\mathscr{V}(s) \cup \mathscr{V}(t)]$.*

Proof. Let σ' be the normal form of σ. By confluence, the terms σs, $\sigma' s$, σt, and $\sigma' t$ have the same normal form n. Thus there exists a sequence $\sigma'(s =^? t) \twoheadrightarrow_{\mathcal{R}} n =^? n$. Due to the absence of the rule $x =^? x \rightarrow true$, the term $n =^? n$ is a normal form. According to Theorem 5.3 we may assume that in the rewrite sequence from $\sigma'(s =^? t)$ to $n =^? n$ only innermost needed redexes are contracted. From Proposition 5.5 we learn that the sequence is based on $\bar{O}(s =^? t)$ and hence the narrowing derivation constructed by Lemma 3.4 is basic. The remainder of the proof is similar to the previous completeness proofs. □

The above completeness result has been independently obtained by Giovannetti and Moiso (Corrado Moiso, personal communication, August 1991).

Corollary 5.7. *Basic narrowing is complete for weakly normalizing orthogonal TRSs.* □

The following example shows that the normalizability requirement of σs and σt in Theorem 5.6 is essential.

Example 5.8. Consider the orthogonal TRS

$$\mathcal{R} = \begin{cases} f(x) \rightarrow h(x, x) \\ g(x) \rightarrow h(x, i(x)) \\ a \quad\ \rightarrow i(a). \end{cases}$$

The following narrowing derivation shows that the goal $f(a) =^? g(a)$ can be solved:

$$\begin{aligned} f(a) =^? g(a) &\rightsquigarrow h(\underline{a}, \underline{a}) =^? g(a) \\ &\rightsquigarrow h(\underline{a}, \underline{a}) =^? h(\underline{a}, i(\underline{a})) \\ &\rightsquigarrow h(\underline{a}, i(a)) =^? h(\underline{a}, i(\underline{a})). \end{aligned}$$

The third step in the above sequence is non-basic. One easily shows that $f(a)$ and $g(a)$ have no common reduct with respect to basic narrowing. Hence the goal $f(a) =^? g(a)$ cannot be solved by basic narrowing.

Furthermore, orthogonality cannot be weakened to non-ambiguity.

Example 5.9. Consider the TRS

$$\mathcal{R} = \begin{cases} f(x) \quad\ \rightarrow g(x, h(x)) \\ g(x, x)) \rightarrow a \\ b \qquad\quad \rightarrow h(b). \end{cases}$$

Since there are no critical pairs, $\mathcal{R}$ is non-ambiguous. With some effort we can show that $\mathcal{R}$ is confluent, notwithstanding the presence of the non-left-linear rule $g(x, x) \rightarrow a$. We have $f(b) \twoheadrightarrow_{\mathcal{R}} a$ but basic narrowing is not able to solve the goal $f(b) =^? a$. This goal is not normalized. If we add the rewrite rule $f'(b') \rightarrow f(b)$ to $\mathcal{R}$ then basic narrowing is unable to find the only solution $\{x \mapsto b'\}$ of the normalized goal $f'(x) =^? a$.

Notice that the TRS in Example 5.9 is not weakly normalizing. We conjecture that basic narrowing is complete for semi-complete non-ambiguous TRS. We now consider the sufficiency of right-linearity. The following useful notion is inspired by a similar notion introduced by Réty [37].

Definition 5.10. Let $A: s \rightarrow_{[p,l\rightarrow r]} t$ be a reduction step in a TRS $\mathcal{R}$. A position $q \in O(s)$ is called an *antecedent* of a position $q' \in O(t)$ if q' is a descendant of q. The set of antecedents of q' in s is denoted by A/q'. This notion is extended to sets of positions in the obvious way.

The next proposition is the key result for proving the sufficiency of right-linearity for the completeness of basic narrowing for confluent TRSs with respect to normalizable solutions. We will make a small concession with regard to our endeavour to rigorous proofs: statements that depend on the easy but tedious interplay between antecedents and basic positions are not proved in full detail. We feel that such detail would veil the structure of the proof. The transformation presented in the proof is illustrated in Example 5.12 below.

Proposition 5.11. *Let $\mathcal{R}$ be a right-linear TRS and σ a normalized substitution. Every reduction sequence starting from σt can be transformed into a reduction sequence that is based on $\bar{O}(t)$.*

Proof. We use induction on the length of the reduction sequence starting from σt. If the length equals zero then we have nothing to prove. Let

$$\sigma t \equiv t_1 \rightarrow_{[p_1,l_1\rightarrow r_1]} \cdots \rightarrow_{[p_{n-1},l_{n-1}\rightarrow r_{n-1}]} t_n \rightarrow_{[p_n,l_n\rightarrow r_n]} t_{n+1}$$

be a reduction sequence of n steps. Define $B_1, \ldots, B_n$ as usual. According to the induction hypothesis we may assume that $p_i \in B_i$ for $i = 1, \ldots, n-1$. If $p_n \in B_n$ then the whole sequence is based on $O(t)$. So assume that $p_n \notin B_n$. Define sets of positions $V_1, \ldots, V_n$, $W_1, \ldots, W_n$ as follows:

- $V_n = \{p_n\}$,
- $V_i = A_i/(V_{i+1} - B_{i+1})$ for $i = n-1, \ldots, 1$ (here A_i is the reduction step from t_i to t_{i+1}),
- $W_i = V_i \cap B_i$ for $i = 1, \ldots, n$.

Using the fact that $p_i \in B_i$, it is not difficult to show that $q \nleq p_i$ whenever $q \in V_{i+1} - B_{i+1}$, for $i = 1, \ldots, n-1$. From this we easily obtain that $(t_i)_{|q} \equiv (t_n)_{|p_n}$ for all $q \in V_i$. With some effort we can show that for every $q \in V_i$ either $q \perp p_i$ or q can be written as $q = p_i \cdot q' \cdot q''$ for some $q' \in O_{\mathcal{V}}(l_i)$. Moreover, if $q \in W_i$ then only the second case applies. Let m be the smallest index such that $V_m \neq \varnothing$. We now construct the diagram of Fig. 3. A few remarks are in order. First note that $V_m \subseteq B_m$: if $m > 1$ then this follows by definition; the normalization of σ yields $V_1 \subseteq B_1$. Therefore $V_m = W_m$.

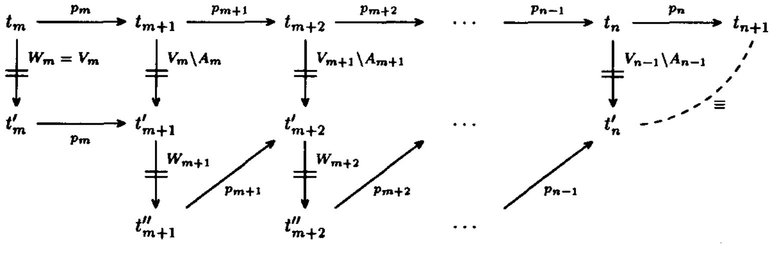


Fig. 3.

Right-linearity of $\mathcal{R}$ yields $V_i \backslash A_i \subseteq V_{i+1}$ and hence $V_i \backslash A_i \cup W_{i+1} = V_{i+1}$. Observe that p_i is a redex position in t_i' even if the rewrite rule $l_i \to r_i$ is non-left-linear. Since $V_{n-1} \backslash A_{n-1} = \{p_n\}$ we have $t_n' \equiv t_{n+1}$. Finally, it is straightforward to show that the reduction sequence

$$t_m \nrightarrow\!\!\!\!\!\rightarrow t_m' \to t_{m+1}' \nrightarrow\!\!\!\!\!\rightarrow t_{m+1}'' \to t_{m+2}' \nrightarrow\!\!\!\!\!\rightarrow t_{m+2}'' \to \cdots \to t_n'$$

is based on B_m and thus we succeeded in constructing a reduction sequence from σt to t_{n+1} that is based on $\bar{O}(t)$. $\square$

Example 5.12. Consider the right-linear TRS

$$\mathcal{R} = \begin{cases} f(x,x) \to g(i(b),x) \\ g(x,x) \to f(x,i(a)) \\ i(x) \quad\;\; \to j(x) \\ a \qquad\;\; \to b \end{cases}$$

and the non-basic reduction sequence

$$f(i(a),i(b)) \to f(i(b),i(b)) \to g(i(b),\underline{i}(\underline{b})) \to f(\underline{i}(\underline{b}),i(a)) \to f(j(\underline{b}),i(a)).$$

The information extracted from this sequence in the proof of Proposition 5.11 is summarized in the following table:

Table 1

i	t_i	p_i	B_i	V_i	W_i
1	$f(i(a),i(b))$	1.1	$O(t_1)$	$\varnothing$	$\varnothing$
2	$f(i(b),i(b)$	ε	$O(t_2)$	$\{1,2\}$	$\{1,2\}$
3	$g(i(b),i(b)$	ε	$\{\varepsilon,1,1.1\}$	$\{1,2\}$	$\{1\}$
4	$f(i(b),i(a))$	1	$\{\varepsilon,2,2.1\}$	$\{1\}$	$\varnothing$
5	$f(j(b),i(a))$				

This gives rise to the construction in Fig. 4, from which we obtain the basic reduction sequence

$$f(i(a),i(b)) \to f(i(b),i(b)) \nrightarrow\!\!\!\!\!\rightarrow f(j(\underline{b}),j(\underline{b})) \to g(i(b),\underline{j}(\underline{b})) \to g(j(\underline{b}),\underline{j}(\underline{b})) \to f(\underline{j}(\underline{b}),i(a)).$$

Notice that there are two further antecedents of $i(b)$, viz. the underlined subterms in $f(\underline{i(a)}, \underline{i(b)})$. These antecedents didn't make their presence into V_1, and with

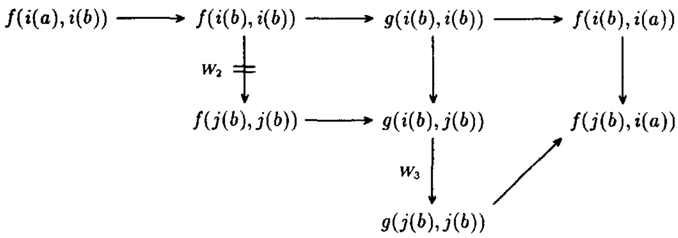


Fig. 4.

reason: if we start our detour at $f(i(a), i(b))$ instead of $f(i(b), i(b))$ we do not end up with a *basic* sequence.

Theorem 5.13. *Basic narrowing is complete for confluent right-linear TRSs with respect to normalizable substitutions.*

Proof. Similar to the proof of Theorem 4.5. The only difference is the replacement of Proposition 4.4 by Proposition 5.11. □

Corollary 5.14. *Basic narrowing is complete for semi-complete right-linear TRSs.* □

6. Conditional Narrowing

Before introducing conditional narrowing, we give a short review of conditional rewriting.

The rules of a *conditional term rewriting system* (CTRS for short) have the form $l \to r \Leftarrow c$. Here the conditional part c is a (possibly empty) sequence $s_1 = t_1, \ldots, s_n = t_n$ of equations. At present we only require that l is not a single variable. A rewrite rule without conditions will be written as $l \to r$. The rewrite relation $\to_{\mathcal{R}}$ associated with a CTRS $\mathcal{R}$ is obtained by interpreting the equality signs in the conditional part of a rewrite rule as joinability. Formally, $\to_{\mathcal{R}}$ is the smallest (w.r.t. inclusion) rewrite relation $\to$ with the property that $l\sigma \to r\sigma$ whenever there exist a variant $l \to r \Leftarrow c$ of a rewrite rule in $\mathcal{R}$ and a substitution σ such that $\sigma s' \downarrow \sigma t'$ for every equation $s' = t'$ in c. The existence of $\to_{\mathcal{R}}$ is easily proved (see e.g. Kaplan [27] or Giovannetti and Moiso [17]). An inductive definition of $\to_{\mathcal{R}}$ is given below (Definition 6.2). All notions that we defined in Sect. 2 for TRSs extend to CTRSs.

The various completeness results for conditional narrowing put different restrictions on the distribution of variables among rewrite rules. The next definition makes these restrictions explicit.

Definition 6.1. The set of variables occurring in a conditional rewrite rule R: $l \to r \Leftarrow c$ is denoted by $\mathcal{V}(R)$ and $\mathcal{E}(R)$ denotes the set of *extra* variables occurring in R, i.e., $\mathcal{E}(R) = \mathcal{V}(R) - \mathcal{V}(l)$. Every rewrite rule $l \to r \Leftarrow c$ is classified according to the distribution of variables among l, r, and c, as follows:

type	requirement
1	$\mathcal{V}(r) \cup \mathcal{V}(c) \subseteq \mathcal{V}(l)$
2	$\mathcal{V}(r) \subseteq \mathcal{V}(l)$
3	$\mathcal{V}(r) \subseteq \mathcal{V}(l) \cup \mathcal{V}(c)$
4	*no restrictions*

An n-CTRS contains only rules of type n. So a 1-CTRS contains no extra variables, a 2-CTRS may only contain extra variables in the conditions, and a 3-CTRS may even have extra variables in the right-hand sides provided these also occur in the corresponding conditional part. A 4-CTRS will simply be called CTRS.

Most of the literature on conditional term rewriting is concerned with 1 and 2-CTRSs. Just as in the unconditional case, we assume that our CTRSs contain the rule $x =^? x \to \mathit{true}$.

Notation. If c is the sequence of equations $s_1 = t_1, \ldots, s_n = t_n$ then $\tilde{c}$ denotes the multiset[8] $\{s_1 =^? t_1, \ldots, s_n =^? t_n\}$.

Definition 6.2. Let $\mathscr{R}$ be a CTRS. We inductively define TRSs $\mathscr{R}_n$ for $n \geqq 0$ as follows:[9]

$$\mathscr{R}_0 = \{x =^? x \rightarrow true\},$$
$$\mathscr{R}_{n+1} = \{\sigma l \rightarrow \sigma r \mid l \rightarrow r \Leftarrow c \in \mathscr{R} \text{ and } \sigma e \twoheadrightarrow_{\mathscr{R}_n} true \text{ for all } e \in \tilde{c}\}.$$

Observe that $\mathscr{R}_n \subseteq \mathscr{R}_{n+1}$ for all $n \geqq 0$. We have $s \rightarrow_{\mathscr{R}} t$ if and only if $s \rightarrow_{\mathscr{R}_n} t$ for some $n \geqq 0$ ([27], [17]). The minimum such n is called the *depth* of $s \rightarrow t$.

We are now ready to define conditional narrowing. In the literature several different formulations are given (e.g. Kaplan [28], Dershowitz and Plaisted [10], Hußmann [25], Giovannetti and Moiso [17], Bockmayr [2]). In this paper we follow the natural approach of Bockmayr [2]. In this approach narrowing is directly defined on finite multisets[10] of goals, the so-called *goal clauses.* In examples goal clauses are often presented as sequences of goals, i.e. we frequently omit the curly brackets. We also find it convenient to identify a goal e with the goal clauses $\{e\}$. The rewrite relation $\rightarrow_{\mathscr{R}}$ extends to goal clauses in the obvious way. The extended relation inherits all properties (confluence, strong normalization, ...) of the original $\rightarrow_{\mathscr{R}}$. The set of variables occurring in a goal clause T will be denoted by $\mathscr{V}(T)$.

Definition 6.3. Let $\mathscr{R}$ be a CTRS. A goal clause S *conditionally narrows* into a goal clause T if there exist a goal $e \in S$, a position $p \in \bar{O}(e)$, a variant $R: l \rightarrow r \Leftarrow c$ of a conditional rewrite rule in $\mathscr{R}$, and a substitution σ such that

- σ is a most general unifier of $e_{|p}$ and l,
- $T = \sigma((S - \{e\}) \cup \{e[r]_p\} \cup \tilde{c})$.

We write $S \leadsto_{[e,p,R,\sigma]} T$ or simply $S \leadsto_\sigma T$.

Example 6.4. Consider the CTRS

$$\mathscr{R} = \begin{cases} even(0) & \rightarrow t \\ even(S(x)) & \rightarrow odd(x) \\ odd(x) & \rightarrow t \quad \Leftarrow \; even(x) = f \\ odd(x) & \rightarrow f \quad \Leftarrow \; even(x) = t \end{cases}$$

and the goal $even(S(y)) =^? t$. The following derivation shows that the solution

[8] See footnote 11 on page 22.

[9] The usual definition

$$\mathscr{R}_0 = \varnothing,$$
$$\mathscr{R}_{n+1} = \{\sigma l \rightarrow \sigma r \mid l - r \Leftarrow c \in \mathscr{R} \text{ and } \sigma s \downarrow_{\mathscr{R}_n} \sigma t \text{ for all } s = t \text{ in } c\}$$

does not rely on the presence of the rule $x =^? x \rightarrow true$. For terms without occurrences of $=^?$ these relations coincide with the ones of Definition 6.2.

[10] Representing goal clauses by multisets facilitates the definition of basic narrowing in Sect. 7. As far as conditional narrowing is concerned, we might as well opt for a set representation.

$\{y \mapsto S(0)\}$ is found by conditional narrowing:

$$\begin{aligned} even(S(y)) =^? t &\rightsquigarrow odd(y) =^? t \\ &\rightsquigarrow t =^? t, even(y) =^? f \\ &\rightsquigarrow_{\sigma_1} t =^? t, odd(x) =^? f \\ &\rightsquigarrow t =^? t, f =^? f, even(x) =^? t \\ &\rightsquigarrow_{\sigma_2} t =^? t, f =^? f, t =^? t \\ &\rightsquigarrow^* true, true, true. \end{aligned}$$

Here $\sigma_1 = \{y \mapsto S(x))$ and $\sigma_2 = \{x \mapsto 0\}$.

Notation. We will use the symbol $\top$ as a generic notation for multisets consisting of a finite number of *true*'s.

Definition 6.5. Let $\mathcal{R}$ be a CTRS and T a goal clause. We write $\mathcal{R} \vdash T$ if $T \twoheadrightarrow_{\mathcal{R}}^{\bullet} \top$. The set of all such goal clauses is denoted by $\mathcal{G}_\top(\mathcal{R})$ or simply $\mathcal{G}_\top$. If $\mathcal{R}$ is confluent then $\mathcal{G}_\top$ is closed under $\rightarrow_{\mathcal{R}}$. The *level* of goal clause $T \in \mathcal{G}_\top$ is the least n such that $\mathcal{R}_n \vdash T$.

The soundness of conditional narrowing is expressed in the following lemma.

Lemma 6.6. *Let $\mathcal{R}$ be a CTRS and T a goal clause. If $T \rightsquigarrow_\sigma^* \top$ then $\mathcal{R} \vdash \sigma T$.*

Proof. Induction on the length of the narrowing derivation from T to $\top$. The case of zero length is trivial. Suppose

$$T \rightsquigarrow_{[e,p,l \to r \Leftarrow c,\sigma_1]} T_1 \rightsquigarrow_{\sigma_2}^* \top.$$

Let $\sigma = \sigma_2 \circ \sigma_1$. By definition $T_1 = \sigma_1((T - \{e\}) \cup \{e[r]_p\} \cup \tilde{c})$. The induction hypothesis yields $\mathcal{R} \vdash \sigma_2 T_1$. Hence we have both $\mathcal{R} \vdash \sigma((T - \{e\}) \cup \{e[r]_p\})$ and $\mathcal{R} \vdash \sigma\tilde{c}$. From the last observation we infer that $\sigma l \rightarrow_{\mathcal{R}} \sigma r$ and therefore

$$\sigma T \rightarrow_{\mathcal{R}} (\sigma T - \{\sigma e\}) \cup \{\sigma e[\sigma r]_p\}.$$

Since $(\sigma T - \{\sigma e\}) \cup \{\sigma e[\sigma r]_p\} = \sigma((T - \{e\}) \cup \{e[r]_p\})$ we obtain $\mathcal{R} \vdash \sigma T$. □

In order to compare conditional rewriting and conditional narrowing, Backmayr [2] introduced a further relation on goal clauses which he called *Reduktion ohne Answertung der Prämisse* (reduction without evaluating conditions). We will denote a slight invariant of this relation by $\rightarrowtail$.

Definition 6.7. Let $\mathcal{R}$ be a CTRS and suppose that S and T are goal clauses. We write $S \rightarrowtail_{\mathcal{R}} T$ if there exist a goal $e \in S$, a position $p \in O(e)$, a variant $l \rightarrow r \Leftarrow c$ of a rewrite rule in $\mathcal{R}$ and a substitution σ such that

- $e_{|p} \equiv \sigma l$,
- $T = (S - \{e\}) \cup \{e[\sigma r]_p\} \cup \sigma\tilde{c}$,
- $\mathcal{R} \vdash \sigma\tilde{c}$.

Occasionally we write $S \rightarrowtail_{[e,p,l \to r \Leftarrow c,\sigma]} T$ or $S \rightarrowtail_{[e,p,l \to r \Leftarrow c]} T$. The transitive-reflexive closure of $\rightarrowtail_{\mathcal{R}}$ is denoted by $\twoheadrightarrow_{\mathcal{R}}$. We define approximations $\rightarrowtail_{\mathcal{R}}^n$ $(n \geqq 0)$ of $\rightarrowtail_{\mathcal{R}}$ as in Definition 6.2. That is, $S \rightarrowtail_{\mathcal{R}}^0 T$ if $T = (S - \{e\}) \cup \{true\}$ with $e \equiv (s =^? s) \in S$ and $S \rightarrowtail_{\mathcal{R}}^{n+1} T$ if $S \rightarrowtail_{[e,p,l \to r \Leftarrow c,\sigma]} T$ with $\mathcal{R}_n \vdash \sigma\tilde{c}$. We have $\rightarrowtail_{\mathcal{R}}^n \subseteq \rightarrowtail_{\mathcal{R}}^{n+1}$ for all $n \geqq 0$ and $\rightarrowtail_{\mathcal{R}} = \bigcup_{n \geqq 0} \rightarrowtail_{\mathcal{R}}^n$.

The difference with the definition of Bockmayr is that we require $\mathscr{R} \vdash \sigma\tilde{c}$. For 1-CTRSs the relation $\rightarrowtail$ can be viewed as a special case of the conditional narrowing relation $\rightsquigarrow$, but in general $\rightarrowtail$ is not included in $\rightsquigarrow$ due to extra variables in conditional rewrite rules.

Proposition 6.8 (Bockmayr [2]). *Let $\mathscr{R}$ be a CTRS and T a goal clause. We have $\mathscr{R} \vdash T$ if and only if $T \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} \top$.*

Proof.

$\Rightarrow$ By induction on n we will show the existence of a sequence $T \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} \top$ whenever $\mathscr{R}_n \vdash T$. If $n = 0$ then there exists a rewrite sequence from T to $\top$ in which only the rule $x =^? x \to true$ is used. By definition, this sequence is also a $\rightarrowtail_{\mathscr{R}}$-sequence. Suppose $\mathscr{R}_{n+1} \vdash T$. So there exists and $\mathscr{R}_{n+1}$-sequence from T to $\top$. We use induction on the length of this sequence. The case of zero length is trivial. Let $T \to_{\mathscr{R}_{n+1}} T' \twoheadrightarrow_{\mathscr{R}_{n+1}} \top$. We obtain a sequence $T' \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} \top$ from the second induction hypothesis. There exist a goal $e \in T$, a position $p \in O(e)$, a variant $R: l \to r \Leftarrow c$ of a rewrite rule in $\mathscr{R}$, and a substitution σ such that $e_{|p} \equiv \sigma l$, $T' = (T - \{e\}) \cup \{e[\sigma r]_p\}$, and $\mathscr{R}_n \vdash \sigma\tilde{c}$. From the first induction hypothesis we obtain a sequence $\sigma\tilde{c} \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} \top$. We have $T \rightarrowtail_{\mathscr{R}} T' \cup \sigma\tilde{c}$. Combining this step with the sequences $T' \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} \top$ and $\sigma\tilde{c} \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} \top$ yields the desired result.

$\Leftarrow$ We use induction on the length of the sequence $T \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} \top$. The case of zero length is trivial. Suppose $T \rightarrowtail_{\mathscr{R}} T' \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} \top$. The induction hypothesis yields $\mathscr{R} \vdash T'$. There exist a goal $e \in T$, a position $p \in O(e)$, a variant $R: l \to r \Leftarrow c$ of a rewrite rule in $\mathscr{R}$, and a substitution σ such that $e_{|p} \equiv \sigma l$, $T' = T'' \cup \sigma\tilde{c}$, and $\mathscr{R} \vdash \sigma\tilde{c}$. Here $T'' = (T - \{e\}) \cup \{e[\sigma r]_p\}$. Clearly $T \to_{\mathscr{R}} T''$. Since $T'' \subseteq T'$ we infer $\mathscr{R} \vdash T''$ from $\mathscr{R} \vdash T'$. We conclude that $\mathscr{R} \vdash T$. □

In Sect. 7 we will see that $\rightarrowtail_{\mathscr{R}}$ does not inherit strong normalization of $\mathscr{R}$. Confluence is preserved, provided we are not particular about a few extra *true*'s.

Notation. We write $S \cong T$ if the goal clauses S and T are identical or they differ only in the number of *true*'s, i.e., $S - \top = T - \top$ by abuse of notation.

Proposition 6.9. *Let $\mathscr{R}$ be a CTRS and S a goal clause.*

(1) *If $S \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} T$ then T can be partitioned into T_1 and T_2 such that $S \twoheadrightarrow_{\mathscr{R}} T_1$ and $\mathscr{R} \vdash T_2$.*

(2) *If $S \twoheadrightarrow_{\mathscr{R}} T$ then there exists a goal clause T_1 such that $S \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} T \cup T_1$ and $\mathscr{R} \vdash T_1$.*

Proof. Straightforward. □

Lemma 6.10. *Let $\mathscr{R}$ be a confluent CTRS. If $S \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} T_1$ and $S \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} T_2$ then there exist goal clauses $T_3 \cong T_4$ such that $T_1 \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} T_3$ and $T_2 \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} T_4$.*

Proof. Proposition 6.9(1) yields goal clauses U_1, V_1, U_2 and V_2 such that $T_1 = U_1 \cup V_1$, $T_2 = U_2 \cup V_2$, $\mathscr{R} \vdash V_1$, $\mathscr{R} \vdash V_2$, $S \twoheadrightarrow_{\mathscr{R}} U_1$ and $S \twoheadrightarrow_{\mathscr{R}} U_2$. Since the relation $\to_{\mathscr{R}}$ is confluent on goal clauses, there exists a goal clause U_3 such that both $U_1 \twoheadrightarrow_{\mathscr{R}} U_3$ and $U_2 \twoheadrightarrow_{\mathscr{R}} U_3$. According to Proposition 6.9(2) there exist goal clauses W_1 and W_2 such that $\mathscr{R} \vdash W_1$, $\mathscr{R} \vdash W_2$, $U_1 \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} U_3 \cup W_1$ and $U_2 \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} U_3 \cup W_2$. Using Proposition 6.8, we obtain

$$T_1 = U_1 \cup V_1 \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} U_3 \cup W_1 \cup V_1 \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} U_3 \cup \top$$

and likewise

$$T_2 = U_2 \cup V_2 \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} U_3 \cup W_2 \cup V_2 \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} U_3 \cup \top. \quad \square$$

From Proposition 6.8 and Lemma 6.10 we immediately infer that $\mathscr{G}_{\mathrm{T}}$ is closed under $\rightarrowtail_{\mathscr{R}}$ for confluent CTRSs $\mathscr{R}$.

In the remainder of this section we show that conditional narrowing is complete for CTRSs without extra variables – the so-called 1-CTRSs.

Lemma 6.11. *Let $\mathscr{R}$ be a 1-CTRS. Suppose we have goal clauses S and T, a normalized substitution θ, and a set V of variables such that $\mathscr{V}(S) \cup \mathscr{D}\theta \subseteq V$ and $T = \theta S$. If $T \twoheadrightarrow_{\mathscr{R}} T'$ then there exist a goal clause S' and substitution θ', σ such that*

- $S \rightsquigarrow_{\sigma}^{*} S'$,
- $\theta' S' = T'$,
- $\theta' \circ \sigma = \theta[V]$,
- *θ' is normalized.*

Furthermore, we may assume that the narrowing derivation $S \rightsquigarrow_{\sigma}^{*} S'$ and the rewrite sequence $T \twoheadrightarrow_{\mathscr{R}} T'$ employ the same rewrite rules at the same positions in the corresponding goals.

Proof. Almost identical to the proof of the lifting lemma for TRSs (Lemma 3.4). The only difference is that we are dealing with goal clauses instead of terms. □

Bockmayr [2] presents an incorrect lifting lemma for 1-CTRSs with respect to a single $\rightarrowtail$-step. This "one step" lifting lemma is not powerful enough to lift rewrite sequence by an inductive proof. The proof of the lifting lemma for 1-CTRS presented in Kaplan [28] employs false assumptions about narrowing substitutions.

Definition 6.12. A substitution σ is called an *$\mathscr{R}$-solution* of a goal clause T if $\mathscr{R} \vdash \sigma T$.

Theorem 6.13. *Conditional narrowing is complete for complete 1-CTRSs.*

Proof. Let $\mathscr{R}$ be a complete 1-CTRSs and suppose σ is an $\mathscr{R}$-solution of a goal clause T. Let σ' be the normal form of σ. We obtain $\mathscr{R} \vdash \sigma' T$ from the confluence of $\mathscr{R}$. According to Proposition 6.8 there exists a sequence $\sigma' T \twoheadrightarrow_{\mathscr{R}} \mathrm{T}$. Lemma 6.11 yields a narrowing derivation $T \rightsquigarrow_{\tau}^{*} \mathrm{T}$ and a substitution σ'' such that $\sigma'' \circ \tau = \sigma'[\mathscr{V}(T)]$. Therefore $\tau \leq \sigma'[\mathscr{V}(T)]$ and hence $\tau \leq_{\mathscr{R}} \sigma[\mathscr{V}(T)]$. □

In the literature this completeness result is ascribed to different authors. It seems that Kaplan was the first who presented a detailed proof (in a different setting though). As was the case for TRSs, we may drop the requirement of strong normalization in exchange for the restriction to normalizable solutions.

Theorem 6.14. *Conditional narrowing is complete for confluent 1-CTRSs with respect to normalizable substitutions.* □

Corollary 6.15. *Conditional narrowing is complete for semi-complete 1-CTRSs.* □

7. Basic Conditional Narrowing

The formulation of basic narrowing for TRSs (Definition 4.1) does not immediately extend to the conditional case. The reason is that a goal clause consists of several goals, each to be equipped with its own constraint on "narrowable positions". In order to keep the administration of these constraints manageable we introduce the following concept.

Definition 7.1. Let T be a goal clause. A *position constraint* for T is a mapping B that assigns to every goal $e \in T$ a subset of $\bar{O}(e)$. The position constraint that assigns to every $e \in T$ the set $\bar{O}(e)$ will be denoted by $\bar{T}$.

Definition 7.2.

(1) A narrowing derivation $T_1 \rightsquigarrow_{[e_1,p_1,l_1 \to r_1 \Leftarrow c_1,\sigma_1]} \cdots \rightsquigarrow_{[e_{n-1},p_{n-1},l_{n-1} \to r_{n-1} \Leftarrow c_{n-1},\sigma_{n-1}]} T_n$ is *basic* if $p_i \in B_i(e_i)$ for $1 \leqq i \leqq n-1$ where the position constraints $B_1, \ldots, B_n$ are inductively defined by $B_1 = \bar{T}_1$ and

$$B_{i+1}(e) = \begin{cases} B_i(e') & \text{if } e' \in T_i - \{e_i\}, \\ \mathscr{B}(B_i(e_i), p_i, r_i) & \text{if } e' \equiv e_i[r_i]_{p_i}, \\ \bar{O}(e') & \text{if } e' \in \tilde{c}_i \end{cases}$$

for all $1 \leqq i < n$ and $e \equiv \sigma_i e' \in T_{i+1}$.[11]

(2) A rewrite sequence $T_1 \rightarrowtail_{[e_1,p_1,l_1 \to r_1 \Leftarrow c_1,\sigma_1]} \cdots \rightarrowtail_{[e_{n-1},p_{n-1},l_{n-1} \to r_{n-1} \Leftarrow c_{n-1},\sigma_{n-1}]} T_n$ is *based* on a position constraint B_1 for T_1 if $p_i \in B_i$ for $1 \leqq i \leqq n-1$ with $B_2, \ldots, B_n$ defined by

$$B_{i+1}(e) = \begin{cases} B_i(e) & \text{if } e \in T_i - \{e_i\}, \\ \mathscr{B}(B_i(e_i), p_i, r_i) & \text{if } e \equiv e_i[\sigma_i r_i]_{p_i}, \\ \bar{O}(e') & \text{if } e \equiv \sigma_i e' \text{ with } e' \in \tilde{c}_i \end{cases}$$

for all $1 \leqq i < n$ and $e \in T_{i+1}$.

Hölldobler [22] showed that basic conditional narrowing is complete for complete 1-CTRSs. This fact is also mentioned in the "summary of completeness results and open problems for conditional narrowing" in Giovannetti and Moiso [17]. However, the following example reveals that this result is incorrect.

Counterexample 7.3. Consider the 1-CTRS

$$\mathscr{R} = \begin{cases} f(x) \to a \Leftarrow x = b, x = c \\ d \quad \to b \\ d \quad \to c \\ b \quad \to c \Leftarrow f(d) = a. \end{cases}$$

Since the recursive path ordering is applicable (with precedence $f \succ a$ and $d \succ b \succ c$) to the unconditional part of $\mathscr{R}$, $\mathscr{R}$ certainly is strongly normalizing. We have $d \to b$ and $d \to c$, and hence $f(d) \to a$ and $b \to c$, which makes the only critical pair $\langle b, c \rangle$ convergent. Local confluence is obtained by some easy case analysis or an appeal to a result of Dershowitz et al. [8] which states that the Critical Pair Lemma holds for *overlay*[12] CTRSs. According to Newman's Lemma $\mathscr{R}$ is confluent. However, basic conditional narrowing is not able to solve the goal $f(d) =^? a$ as can be seen from Fig. 5 (in this figure trivial goals of the form $t =^? t$ are not shown), while the

[11] Recall that $T_{i+1} = \sigma_i((T_i - \{e_i\}) \cup \{e_i[r_i]_{p_i}\} \cup \tilde{c}_i)$. If we would have represented goal clauses by sets then the definition of B_{i+1} is ambiguous since $T_i - \{e_i\}$, $\{e_i[r_i]_{p_i}\}$, and $\tilde{c}_i$ do not have to be pairwise disjoint.

[12] An overlay CTRS is a CTRS with the property that if $l_1 \to r_1 \Leftarrow c_1$ and $l_2 \to r_2 \Leftarrow c_2$ are variants of rewrite rules and $p \in \bar{O}(l_1)$ such that $(l_1)_{|p}$ and l_2 are unifiable, then $p = \varepsilon$.

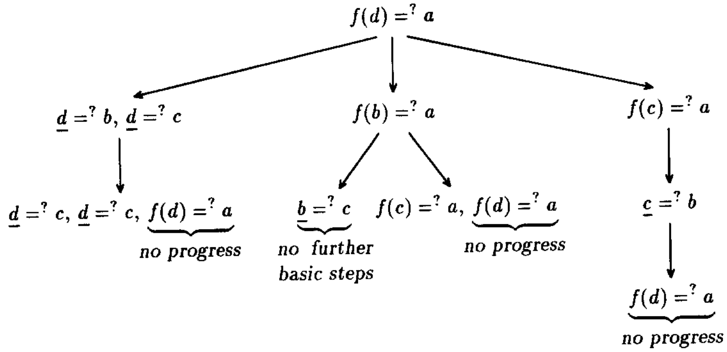


Fig. 5.

following non-basic narrowing derivation shows that the goal can be solved:

$$\begin{aligned} f(d) =^? a &\rightsquigarrow a =^? a, \underline{d} =^? b, \underline{d} =^? c \\ &\rightsquigarrow a =^? a, b =^? b, \underline{d} =^? c \\ &\rightsquigarrow a =^? a, b =^? b, c =^? c \\ &\rightsquigarrow^* \top. \end{aligned}$$

Basic conditional narrowing is also unable to solve the normalized goal $f(x) =^? a$.

The mistake in Hölldobler [22] is due to the incorrect assumption that the strong normalization of $\rightarrowtail_{\mathcal{R}}$ is implied by the strong normalization of $\mathcal{R}$. We now show that completeness of basic narrowing can be ensured by strengthening strong normalization. In the next section we show that completeness can also be recovered by strengthening confluence. The property defined below originates from Dershowitz et al. [9].

Definition 7.4. A 1-CTRS $\mathcal{R}$ is *decreasing* if there exists a well-founded extension $\succ$ of the rewrite relation $\rightarrow_{\mathcal{R}}$ with the following properties:

- $\succ$ has the *subterm property*, i.e. $t \succ t_{|p}$ for all positions $p \in O(t) - \{\varepsilon\}$,
- if $l \rightarrow r \Leftarrow c \in \mathcal{R}$ and σ is a substitution then $\sigma l \succ \sigma s, \sigma t$ for all $s = t$ in c.

Every decreasing 1-CTRS is strongly normalizing and moreover – when there are finitely many rewrite rules – its rewrite relation is decidable.

Example 7.5. The CTRS of Counterexample 7.3 is not decreasing: as $f(d) \rightarrow a \Leftarrow d = b, d = c$ is an instance of the first rewrite rule we must have $f(d) \succ b$, but the rule $b \rightarrow c \Leftarrow f(d) = a$ requires $b \succ f(d)$.

Lemma 7.6. *If $\mathcal{R}$ is a decreasing 1-CTRS then $\rightarrowtail_{\mathcal{R}}$ is strongly normalizing.*

Proof. With every goal clause S we associate a multiset $m(S)$ by replacing every goal $s =^? t$ is S by the terms s and t. The presence of *true* in S does not contribute to $m(S)$. Using the definition of $\succ$ it is easy to show that $m(S) \ggcurly m(T)$ whenever $S \rightarrowtail_{\mathcal{R}} T$. Here $\ggcurly$ is the multiset extension of $\succ$. Since the multiset extension of a well-founded ordering is well-founded, the relation $\rightarrowtail_{\mathcal{R}}$ is strongly normalizing. $\square$

The proof of Proposition 7.7 can be found in the Appendix.

Proposition 7.7. *Let $\mathcal{R}$ be a 1-CTRS, T a goal clause, and σ a normalized substitution. Every innermost $\rightarrowtail_{\mathcal{R}}$-sequence starting from σT is based on $\bar{T}$.* □

Theorem 7.8. *Basic conditional narrowing is complete for decreasing and confluent 1-CTRSs.*

Proof. Let $\mathcal{R}$ be a decreasing and confluent 1-CTRSs. Suppose σ is an $\mathcal{R}$-solution of a goal clause T and let σ' be its normal form. We obtain $\sigma' T \twoheadrightarrow_{\mathcal{R}} \top$ as in the proof of Theorem 6.13. Because $\rightarrowtail_{\mathcal{R}}$ is strongly normalizing (Lemma 7.6) there exists an innermost $\rightarrowtail_{\mathcal{R}}$-sequence from $\sigma' T$ to a normalized goal T'. From Lemma 6.10 we obtain $T' \cong \top$, i.e. $T' = \top$. According to Proposition 7.7 the innermost sequence $\sigma' T \twoheadrightarrow_{\mathcal{R}} \top$ is based on $\bar{T}$. It is not difficult to show that the narrowing derivation constructed by Lemma 6.11 is basic. The remainder of the proof follows literally the proof of Theorem 6.13. □

The CTRS of Counterexample 7.3 is not a so-called *normal* CTRS. In a normal CTRS $\mathcal{R}$ every right-hand side of an equation in the conditions of the rewrite rules is a ground normal form with respect to the unconditional TRS obtained from $\mathcal{R}$ by omitting the conditions. One might ask whether this is essential. The following example answers this question negatively.

Example 7.9. Consider the normal 1-CTRS

$$\mathcal{R} = \begin{cases} f(x) \to g(x,x) & \\ a \to b & \\ g(a,b) \to c & \\ g(b,b) \to c & \Leftarrow f(a) = c. \end{cases}$$

Completeness of $\mathcal{R}$ follows as in Counterexample 7.3. (The Critical Pair Lemma however is not applicable.) Notice that $g(b,b) \to c$ since $f(a) \to g(a,a) \to g(a,b) \to c$. One easily shows that the goal $f(a) =^? c$ cannot be solved by basic conditional narrowing. Moreover, basic conditional narrowing is not able to solve the normalized goal $g(x,x) =^? c$.

We conclude this section with a refutation of the following claim.

Conjecture 7.10 (Giovannetti and Moiso [17]). *Basic conditional narrowing is complete for semi-complete orthogonal 1-CTRSs.*[13] □

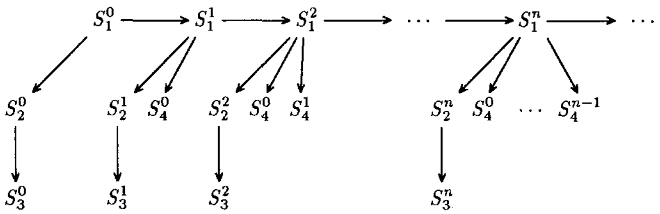
Since (weakly normalizing) orthogonal CTRSs are in general not confluent (Bergstra and Klop [1]), we cannot replace the phrase "semi-complete orthogonal" by "weakly normalizing orthogonal".

Counterexample 7.11. Consider the orthogonal 1-CTRS

$$\mathcal{R} = \begin{cases} f(x) \to a & \Leftarrow g(b) = c \\ g(x) \to c & \Leftarrow x = f(x) \\ b \to f(b). & \end{cases}$$

In the Appendix it is shown that $\mathcal{R}$ is semi-complete. The goal $g(b) =^? c$ can be solved

[13] Actually, Giovannetti and Moiso conjecture in [17] the completeness of basic conditional narrowing for orthogonal 1-CTRSs with respect to normalized solutions. By refuting the weaker statement, our counterexample becomes stronger.



Abbreviations ($i \geq 0$):

S_1^i : $g(f^i(b)) =^? c$ $\qquad$ S_2^i : $c =^? c,\ \underline{f}^i(\underline{b}) =^? f(\underline{f}^i(\underline{b}))$

S_3^i : $c =^? c,\ \underline{f}^i(\underline{b}) =^? a,\ g(b) =^? c$ $\qquad$ S_4^i : $g(f^i(a)) =^? c,\ g(b) =^? c$

$f^i(t)$: $\underbrace{f(\ldots f}_{i\ f\text{'s}}(t)\ldots)$

Fig. 6.

by conditional narrowing as follows:

$$\begin{aligned} g(b) =^? c &\rightsquigarrow c =^? c, \underline{b} =^? f(\underline{b}) \\ &\rightsquigarrow c =^? c, f(b) =^? f(\underline{b}) \\ &\rightsquigarrow^* \top. \end{aligned}$$

In this derivation the second step is not basic. Figure 6 reveals that all basic narrowing derivations issued from $g(b) =^? c$ come across a goal clause that contains the original goal $g(b) =^? c$. Hence basic conditional narrowing is not able to solve the goal $g(b) =^? c$. One easily shows that the normalized goal $g(x) =^? c$ also cannot be solved by basic conditional narrowing.

8. Level-Confluence

Hußmann claimed in [25] that conditional narrowing is also complete for complete CTRSs that have extra variables in the conditions of the rewrite rules, but the following example of Giovannetti and Moiso [17] shows that this is not the case.

Example 8.1. Consider the 2-CTRS

$$\mathscr{R} = \begin{cases} a \to b \\ a \to c \\ b \to c \Leftarrow x = b, x = c. \end{cases}$$

It is easy to show that $\mathscr{R}$ is complete. In particular we have $b \to_{\mathscr{R}} c$, but all narrowing derivations issued from the goal $b =^? c$ are infinite, e.g.

$$\begin{aligned} b =^? c &\rightsquigarrow c =^? c, x =^? b, x =^? c \\ &\rightsquigarrow c =^? c, x =^? c, x =^? c, x' =^? b, x' =^? c \\ &\rightsquigarrow c =^? c, x =^? c, x =^? c, x' =^? c,\ x' =^? c, x'' =^? b, x'' =^? c \\ &\rightsquigarrow \cdots. \end{aligned}$$

In order to cope with extra variables in the conditions of the rewrite rules, Giovannetti and Moiso proposed to strengthen confluence.

Definition 8.2. A CTRS $\mathscr{R}$ is called *level-confluent* if each $\mathscr{R}_n$ $(n \geqq 0)$ is confluent. We call $\mathscr{R}$ *level-complete* if each $\mathscr{R}_n$ $(n \geqq 0)$ is complete.

Example 8.3. The complete 1-CTRS of Counterexample 7.3 is not level-confluent: we have $d \to_{\mathscr{R}_1} b$ and $d \to_{\mathscr{R}_1} c$ but the depth of the joining step $b \to_{\mathscr{R}} c$ is 3. Likewise the 2-CTRS of Example 8.1 is not level-confluent.

Every strongly normalizing and level-confluent CTRS is level-complete, but the reverse does not hold.

Example 8.4. Consider the CTRS $\mathscr{R} = \{f(x) \to f(g(x)) \Leftarrow f(x) = f(a)\}$. We have $\mathscr{R}_0 = \{x =^? x \to true\}$ and $\mathscr{R}_{n+1} = \{f(g^n(a)) \to f(g^{n+1}(a))\} \cup \mathscr{R}_n$ for $n \geqq 0$. It is easy to see that every $\mathscr{R}_n$ is complete. Hence $\mathscr{R}$ is level-complete, but $\mathscr{R}$ is not strongly normalizing: $f(a) \to_{\mathscr{R}_1} f(g(a)) \to_{\mathscr{R}_2} f(g(g(a))) \to_{\mathscr{R}_3} \cdots$.

The next example shows that the lifting lemma as presented in Sect. 6 for 1-CTRSs does not carry over to (level-confluent) 2-CTRSs.

Example 8.5. Consider the strongly normalizing and level-confluent 2-CTRS

$$\mathscr{R} = \begin{cases} a \to b \\ b \to c \Leftarrow x = a, x = b. \end{cases}$$

Let $S = \{b =^? c\}$ and $\theta = \varepsilon$. The rewrite step $\theta S \rightarrowtail \{c =^? c, b =^? a, b =^? b\} = T'$ can be lifted to $S \rightsquigarrow_\varepsilon \{c =^? c, x =^? a, x =^? b\} = S'$. Any substitution θ' satisfying $\theta' S' = T'$ must have $\theta' x = b$ and thus θ' is not normalized as $b \to_{\mathscr{R}_2} c$. This is not really a problem since T' can be rewritten to $\top$ by using only $\rightarrowtail^1_{\mathscr{R}}$-steps and b is $\mathscr{R}_1$-normalized. That is, the rewrite sequence

$$\theta S \rightarrowtail^1_{\mathscr{R}} \{c =^? c, b =^? a, b =^? b\} \rightarrowtail^1_{\mathscr{R}} \{c =^? c, b =^? b, b =^? b\} \twoheadrightarrow^0_{\mathscr{R}} \top$$

can be lifted to

$$S \rightsquigarrow \{c =^? c, x =^? a, x =^? b\} \rightsquigarrow \{c =^? c, x =^? b, x =^? b\} \rightsquigarrow^* \top.$$

Now consider the rewrite sequence

$$\theta S \rightarrowtail^1_{\mathscr{R}} \{c =^? c, a =^? a, a =^? b\} \rightarrowtail^1_{\mathscr{R}} \{c =^? c, a =^? a, b =^? b\} \twoheadrightarrow^0_{\mathscr{R}} \top$$

in which the constant a is substituted for the extra variable of the conditional rewrite rule applied in the first step. This sequence cannot be lifted. The problem is that the introduced constant a is rewritten. This is possible because the subsequence from $\{c =^? c, a =^? a, a =^? b\}$ to $\top$ contains $\rightarrowtail^1_{\mathscr{R}}$-steps and a is normalized only with respect to $\mathscr{R}_0$.

In the next definition we restrict the relation $\rightarrowtail_{\mathscr{R}}$, based on the findings of the previous example.

Definition 8.6. Let $\mathscr{R}$ be an arbitrary CTRS. We define relations $\rightarrowtail\!\!\!\!\rightarrow^n_{\mathscr{R}}$ on $\mathscr{G}_\top$ for $n \geqq 0$ as follows: $\rightarrowtail\!\!\!\!\rightarrow^0_{\mathscr{R}}$ is the restriction of $\rightarrowtail^0_{\mathscr{R}}$ to $\mathscr{G}_\top$ and $S \rightarrowtail\!\!\!\!\rightarrow^{n+1}_{\mathscr{R}} T$ if there exist a goal $e \in S$, a position $p \in O(e)$, a variant $R: l \to r \Leftarrow c$ of a rewrite rule in $\mathscr{R}$, and a substitution σ such that

- $e_{|p} \equiv \sigma l$,
- $T = (S - \{e\}) \cup \{e[\sigma r]_p\} \cup \sigma \tilde{c}$,
- $\mathscr{R}_n \vdash \sigma \tilde{c}$,
- $\sigma \restriction_{\mathscr{E}(R)}$ is $\mathscr{R}_n$-normalized,
- the level of e is at least $n + 1$.

So $S \twoheadrightarrowtail^{n+1}_{\mathscr{R}} T$ if $S \rightarrowtail^{n+1}_{[e,p,R:l\to r\Leftarrow c,\sigma]} T$ with $\sigma\restriction_{\mathscr{E}(R)}$ $\mathscr{R}_n$-normalized and the level of e at least $n+1$. The union of all $\twoheadrightarrowtail^{n}_{\mathscr{R}}$ ($n \geqq 0$) is denoted by $\twoheadrightarrowtail_{\mathscr{R}}$.

It should be noted that in general the inclusions $\twoheadrightarrowtail^{n}_{\mathscr{R}} \subseteq \twoheadrightarrowtail^{n+1}_{\mathscr{R}}$ ($n \geqq 0$) do *not* hold. Moreover, $\twoheadrightarrowtail_{\mathscr{R}}$ is *properly* included in the restriction of $\rightarrowtail_{\mathscr{R}}$ to $\mathscr{G}_{\top}$. Lemma 8.11 below states that $\twoheadrightarrowtail_{\mathscr{R}}$ is powerful enough to rewrite all goals in $\mathscr{G}_{\top}$ to $\top$, provided $\mathscr{R}$ is a level-complete CTRS. In Lemma 8.13 we show that $\twoheadrightarrowtail_{\mathscr{R}}$-sequences can be lifted to narrowing derivations for any level-confluent 2-CTRSs $\mathscr{R}$. The completeness of conditional narrowing for level-complete 2-CTRS is an easy consequence of these two facts. We start with some easy propositions.

Proposition 8.7. *Let $\mathscr{R}$ be a level-confluent CTRS and e a goal with level n. If $e \to_{\mathscr{R}_m} e'$ with $m \leqq n$ then the level of e' is at most n.*

Proof. Since $\mathscr{R}_m \subseteq \mathscr{R}_n$ we have $e \to_{\mathscr{R}_n} e'$. By definition $e \twoheadrightarrow_{\mathscr{R}_n}$ *true*. Level-confluence of $\mathscr{R}$ yields $e' \twoheadrightarrow_{\mathscr{R}_n}$ *true*. Hence the level e' is at most n. □

Proposition 8.8. *Let $\mathscr{R}$ be a level-confluent CTRS and $S, T \in \mathscr{G}_{\top}$. If $S \twoheadrightarrowtail_{\mathscr{R}} T$ then the level of T does not exceed the level of S.*

Proof. If $S \twoheadrightarrowtail^{0}_{\mathscr{R}} T$ then S and T have the same level. So suppose that $S \rightarrowtail^{n+1}_{[e,p,R:l\to r\Leftarrow c,\sigma]} T$. Let m be the level of S. We have $m \geqq n+1$. We will show that the level of every goal $e' \in T = (S - \{e\}) \bigcup \{e[\sigma r]_p\} \cup \sigma\tilde{c}$ is bounded by m. For $e' \in S - \{e\}$ this is obvious. The level of every $e' \in \sigma\tilde{c}$ is at most $n < m$. In the only remaining case $e' \equiv e[\sigma r]_p$ we have $e \to_{\mathscr{R}_{n+1}} e'$ and hence the result follows from the previous proposition. □

Proposition 8.9. *Let $\mathscr{R}$ be a level-complete CTRS and $T \in \mathscr{G}_{\top}$. The following statements are equivalent:*
(1) *T is a $\to_{\mathscr{R}}$-normal form,*
(2) *T is a $\rightarrowtail_{\mathscr{R}}$-normal form,*
(3) *T is a $\twoheadrightarrowtail_{\mathscr{R}}$-normal form.*

Proof. The implications "(1) ⇒ (2)" and "(2) ⇒ (3)" are easy. Suppose there is a $\twoheadrightarrowtail_{\mathscr{R}}$-normal form $T \in \mathscr{G}_{\top}$ that is not a $\to_{\mathscr{R}}$-normal form. Take the smallest n such that $T \to_{\mathscr{R}_n} T'$ for some goal clause T'. If $n = 0$ then we obtain the impossible $T \twoheadrightarrowtail^{0}_{\mathscr{R}} T'$. So suppose that $n > 0$. By definition there exist a goal $e \in T$, a position $p \in O(e)$, a variant $R: l \to r \Leftarrow c$ of a rewrite rule in $\mathscr{R}$, and a substitution σ such that $e_{|p} \equiv \sigma l$, $T' = (T - \{e\}) \cup \{e[\sigma r]_p\}$, and $\mathscr{R}_{n-1} \vdash \sigma\tilde{c}$. Define a substitution τ as follows:

$$\tau x = \begin{cases} (\sigma x)\downarrow_{\mathscr{R}_{n-1}} & \text{if } x \in \mathscr{E}(R), \\ \sigma x & \text{otherwise.} \end{cases}$$

The well-definedness of τ follows from the completeness of $\mathscr{R}_{n-1}$. We have $\sigma\tilde{c} \twoheadrightarrow_{\mathscr{R}_{n-1}} \tau\tilde{c}$. Confluence of $\mathscr{R}_{n-1}$ yields $\mathscr{R}_{n-1} \vdash \tau\tilde{c}$. By construction $\tau\restriction_{\mathscr{E}(R)}$ is $\mathscr{R}_{n-1}$-normalized. If the level of e is at least n then $T \twoheadrightarrowtail^{n}_{[e,p,R,\tau]} (T - \{e\}) \cup \{e[\tau r]_p\} \cup \tau\tilde{c}$, contradicting the $\twoheadrightarrowtail_{\mathscr{R}}$-normalization of T. If the level of e is less than n then there exist an $m < n$ and a goal clause T'' such that $T \to_{\mathscr{R}_m} T''$, contradicting the minimality of n. We conclude that every $\twoheadrightarrowtail_{\mathscr{R}}$-normal form is also a $\to_{\mathscr{R}}$-normal form. □

Lemma 8.10. *If $\mathscr{R}$ is a level-complete CTRS then every $\twoheadrightarrowtail_{\mathscr{R}}$-sequence is finite.*

Proof. Let T be a goal clause. We will show that there are no infinite $\twoheadrightarrowtail_{\mathscr{R}}$-sequences starting from T. If $T \notin \mathscr{G}_{\top}$ then there are no $\twoheadrightarrowtail_{\mathscr{R}}$-sequences originating from T. So we may assume that T has some level n. We use induction on n. If $n = 0$ then only the rule $x =^? x \to true$ can be used, and the number of applications of this rule is clearly bounded by the cardinality of T. Suppose the level of T is $n+1$ and consider an infinite $\rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}}$-sequence starting from T. Since $\rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}}$-sequences issued from different goals in T do not interfere, we infer the pigeon-hole principle the existence of a goal $e \in T$ with an infinite $\rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}}$-sequence. Consider the first step

$$\{e\} \rightarrowtail\!\!\!\twoheadrightarrow_{[e,p,l \to r \Leftarrow c,\sigma]} \{e[\sigma r]_p\} \cup \sigma\tilde{c}$$

in this sequence. Since the level of e is $n+1$, we have $e \to_{\mathscr{R}}^{n+1} e[\sigma r]_p$. Proposition 8.8 shows that the level of $\{e[\sigma r]_p\} \cup \sigma\tilde{c}$ is at most $n+1$. Since the level of $\sigma\tilde{c}$ is less than $n+1$, we learn from the induction hypothesis that there are no infinite $\rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}}$-sequences starting from $\sigma\tilde{c}$. Hence there must be an infinite $\rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}}$-sequence starting from $e[\sigma r]_p$, and thus the level of $e[\sigma r]_p$ is $n+1$. We repeat the above process with $e[\sigma r]_p$. We end up with an infinite $\to_{\mathscr{R}}^{n+1}$-sequence, contradicting the strong normalization of $\mathscr{R}_{n+1}$. □

Lemma 8.1. *Let $\mathscr{R}$ be a level-complete CTRS and T a goal clause. We have $\mathscr{R} \vdash T$ if and only if $T \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} \top$.*

Proof. According to Proposition 6.8 it suffices to prove the equivalence of $T \rightarrowtail\!\!\twoheadrightarrow_{\mathscr{R}} \top$ and $T \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} \top$.

⇒ From Lemma 8.10 we infer that T has a normal form T' with respect to $\rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}}$. According to Proposition 8.9 T' is also a $\rightarrowtail_{\mathscr{R}}$-normal form. Clearly $T \rightarrowtail\!\!\twoheadrightarrow_{\mathscr{R}} T'$. Lemma 6.10 amounts to $T' \cong \top$, i.e. $T' = \top$. Therefore $T \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} \top$.

⇒ Trivial. □

Because every $\rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}}$-sequence is finite, we may assume that the normal form $T' = \top$ in the above proof is obtained by means of an innermost $\rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}}$-sequence. This observation will be used in the proof of the completeness of basic conditional narrowing for level-complete 2-CTRSs (Theorem 8.20).

Definition 8.12. A solution σ of a goal clause T is said to be *sufficiently normalized* if $\sigma\restriction_{\mathscr{V}(e)}$ is $\mathscr{R}_n$-normalized where n is the level of σe, for every goal $e \in T$.

The difficult part in the proof of the following lifting for level-confluent 2-CTRSs is the sufficient normalization of the resulting substitution θ'.

Lemma 8.13. *Let $\mathscr{R}$ be a level-confluent 2-CTRS. Suppose we have goal clauses S and T, a sufficiently normalized solution θ of S, and a set V of variables such that $\mathscr{V}(S) \cup \mathscr{D}\theta \subseteq V$ and $T = \theta S$. If $T \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} T'$ then there exist a goal clause S' and substitutions θ', σ such that*

- $S \rightsquigarrow_\sigma^* S'$,
- $\theta' S' = T'$,
- $\theta' \circ \sigma = \theta[V]$,
- *θ' is a sufficiently normalized solution of S'.*

Furthermore, we may assume that the narrowing derivation $S \rightsquigarrow_\sigma^* S'$ and the sequence $T \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} T'$ employ the same rewrite rules at the same positions in the corresponding goals.

Proof. We use induction on the length of the $\rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}}$-sequence from T to T'. The case of zero length is trivial. Suppose $T \rightarrowtail\!\!\!\twoheadrightarrow^n_{[\theta e,p,R:l \to r \Leftarrow c,\tau]} T_1 \rightarrowtail\!\!\!\twoheadrightarrow_{\mathscr{R}} T'$. We may assume that $\mathscr{V}(R) \cap V = \varnothing$ and $\mathscr{D}\tau \subseteq \mathscr{V}(R)$. We first show that $p \in \bar{O}(e)$. Let m be the level of θe.

By definition $n \leqq m$. We have $\tau l \to_{\mathscr{R}_n} \tau r$, $\mathscr{R}_{n-1} \vdash \tau\tilde{c}$, and $\tau\restriction_{\mathscr{E}(R)}$ is $\mathscr{R}_{n-1}$-normalized. Because θ is a sufficiently normalized solution of S, $\theta\restriction_{\mathscr{V}(e)}$ is $\mathscr{R}_m$-normalized and hence also $\mathscr{R}_n$-normalized. Thus $p \in \bar{O}(e)$ and so $(\theta e)_{|p} = \theta(e_{|p})$. Let $\mu \doteq \tau \cup \theta$. We have $\mu(e_{|p}) \equiv \theta(e_{|p}) \equiv \tau l \equiv \mu l$. Let σ_1 be an idempotent most general unifier of $e_{|p}$ and l. Proposition 2.1 yields $\mathscr{D}\sigma_1 \cup \mathscr{I}\sigma_1 = \mathscr{V}(e_{|p}) \cup \mathscr{V}(l)$. Let $S_1 = \sigma_1((S - \{e\}) \cup \{e[r]_p\} \cup \tilde{c})$. By definition $S \rightsquigarrow_{[e,p,R,\sigma_1]} S_1$. Let $V_1 = (V - \mathscr{D}\sigma_1) \cup \mathscr{I}\sigma_1 \cup \mathscr{E}(R)$. We now show that $\mathscr{V}(S_1) \subseteqq V_1$. Proposition 3.5 yields

$$\mathscr{V}(\sigma_1 S) \subseteqq (V - \mathscr{D}\sigma_1) \cup \mathscr{I}\sigma_1 \subseteqq V_1. \tag{1}$$

It is easy to show that

$$\mathscr{E}(R) \cap \mathscr{D}\sigma_1 = \varnothing. \tag{2}$$

Together with (1) and the inclusion $\mathscr{V}(r) \subseteqq \mathscr{V}(l) \cup \mathscr{E}(R)$,[14] this yields

$$\mathscr{V}(\sigma_1(e[r]_p)) \subseteqq \mathscr{V}(\sigma_1(e[l]_p)) \cup \mathscr{E}(R) = \mathscr{V}(\sigma_1 e) \cup \mathscr{E}(R) \subseteqq \mathscr{V}(\sigma_1 S) \cup \mathscr{E}(R) \subseteqq V_1. \tag{3}$$

The last inclusion follows from (1). From (2) and the inclusion $\mathscr{V}(\tilde{c}) \subseteqq \mathscr{V}(l) \cup \mathscr{E}(R)$ we obtain $\mathscr{V}(\sigma_1\tilde{c}) \subseteqq \mathscr{V}(\sigma_1 l) \cup \mathscr{E}(R)$. From formula (3) we learn that $\mathscr{V}(\sigma_1 l) \subseteqq V_1$ and thus

$$V(\sigma_1\tilde{c}) \subseteqq V_1. \tag{4}$$

Combining (1), (3), and (4) yields $\mathscr{V}(S_1) \subseteqq V_1$ Since $\sigma_1 \leqq \mu$, there exists a substitution ρ such that $\rho \circ \sigma_1 = \mu$. Define $\theta_1 = \rho\restriction_{V_1}$. By definition $\mathscr{D}\theta_1 \subseteqq V_1$ and $\theta_1 = \rho[V_1]$. From (2) and $\mathscr{V}(l) - \mathscr{D}\sigma_1 \subseteqq \mathscr{I}\sigma_1$ we infer that $\mathscr{V}(R) - \mathscr{D}\sigma_1 \subseteqq V_1$ and hence $((V \cup \mathscr{V}(R)) - \mathscr{D}\sigma_1) \cup \mathscr{I}\sigma_1 = V_1$. An application of Proposition 3.6 yields $\theta_1 \circ \sigma_1 = \rho \circ \sigma_1 = \mu[V \cup \mathscr{V}(R)]$. From $\mu\restriction_V = \theta$ and $\mu\restriction_{\mathscr{V}(R)} = \tau$ we infer that

$$\theta_1 \circ \sigma_1 = \theta[V] \tag{5}$$

and

$$\theta_1 \circ \sigma_1 = \tau[\mathscr{V}(R)]. \tag{6}$$

From these two equalities we obtain

$$\theta_1 S_1 = \theta_1\sigma_1(S - \{e\}) \cup \{\theta_1\sigma_1 e[\theta_1\sigma_1 r]_p\} \cup \theta_1\sigma_1\tilde{c} = \theta(S - \{e\}) \cup \{\theta e[\tau r]_p\} \cup \tau\tilde{c} = T_1. \tag{7}$$

Before we can apply the induction hypothesis, we have to show that θ_1 is a sufficiently normalized solution of S_1. Let $e' \in S_1$. By definition there exists an $e'' \in (S - \{e\}) \cup \{e[r]_p\} \cup \tilde{c}$ such that $e' \equiv \sigma_1 e''$. We distinguish three cases: (a) $e'' \in S - \{e\}$, (b) $e'' \equiv e[r]_p$, and (c) $e'' \in \tilde{c}$.

(a) Since $\mathscr{V}(e'') \subseteqq V$ we obtain $\theta_1 e' \equiv \theta e''$ from (5) and hence $\theta_1 e'$ has the same level as $\theta e''$, say k. By assumption $\theta\restriction_{\mathscr{V}(e'')}$ is $\mathscr{R}_k$-normalized. We have to show that $\theta_1\restriction_{\mathscr{V}(e')}$ is also $\mathscr{R}_k$-normalized. Since $\mathscr{V}(e') \subseteqq (\mathscr{V}(e'') - \mathscr{D}\sigma_1) \cup \mathscr{I}\sigma_1\restriction_{\mathscr{V}(e'')}$ (Proposition 3.5) this follows from Proposition 3.7.

(b) Let $e'' \equiv e[r]_p$. Formula (7) shows that $\theta_1 e' \equiv \theta e[\tau r]_p$. Therefore $\theta e \to_{\mathscr{R}_n} \theta_1 e'$. Proposition 8.7 shows that the level of $\theta_1 e'$ is at most m. So it suffices to show

[14] Since $\mathscr{R}$ is a 2-CTRS we have of course $\mathscr{V}(r) \subseteqq \mathscr{V}(l)$. However, in the next section we will reuse most parts of this proof in the context of 3-CTRS. Hence we avoid using the stronger inclusion $\mathscr{V}(r) \subseteqq \mathscr{V}(l)$ here.

that $\theta_1 \upharpoonright_{\mathscr{V}(e')}$ is $\mathscr{R}_m$-normalized. Since $\mathscr{R}$ is a 2-CTRS[15] we have $\mathscr{V}(r) \subseteq \mathscr{V}(l)$ and hence $\mathscr{V}(e') = \mathscr{V}(\sigma_1(e[r]_p)) \subseteq \mathscr{V}(\sigma_1(e[l]_p)) = \mathscr{V}(e')$. Using the fact that $\theta \upharpoonright_{\mathscr{V}(e)}$ is $\mathscr{R}_m$-normalized, we obtain the $\mathscr{R}_m$-normalization of $\theta_1 \upharpoonright_{\mathscr{V}(e')}$ from Propositions 3.5 and 3.7.

(c) Let $e'' \in \tilde{c}$. Since $\mathscr{V}(e'') \subseteq \mathscr{V}(R)$ we obtain $\theta_1 e' \equiv \tau e''$ from (6). By definition $\mathscr{R}_{n-1} \vdash \tau\tilde{c}$ and $\tau \upharpoonright_{\mathscr{E}(R)}$ is $\mathscr{R}_{n-1}$-normalized. So the level of $\theta_1 e'$ does not exceed $n-1$ and hence it suffices to show that $\theta_1 \upharpoonright_{\mathscr{V}(e')}$ is $\mathscr{R}_{n-1}$-normalized. From (4) we infer that $\mathscr{V}(e') \subseteq \mathscr{I}\sigma_1 \cup \mathscr{E}(R)$. In case (b) we noticed that $\theta_1 \upharpoonright_{\mathscr{I}\sigma_1}$ is $\mathscr{R}_m$-normalized and thus also $\mathscr{R}_{n-1}$-normalized. From $\mathscr{E}(R) \cap \mathscr{D}\sigma_1 = \varnothing$ and (6) we infer that $\theta_1 \upharpoonright_{\mathscr{E}(R)}$ equals $\tau \upharpoonright_{\mathscr{E}(R)}$, which by definition is $\mathscr{R}_{n-1}$-normalized.

The induction hypothesis yields a goal clause S' and substitutions θ', σ' such that $S_1 \rightsquigarrow^*_{\sigma'} S'$, $\theta' S' = T'$, $\theta' \circ \sigma' = \theta_1 [V_1]$, and θ' is a sufficiently normalized solution of S'. Moreover, we may assume that $S_1 \rightsquigarrow^*_{\sigma'} S'$ and $T_1 \ggg_{\mathscr{R}} T'$ employ the same rewrite rules at the same positions in the corresponding goals. Let $\sigma = \sigma' \circ \sigma_1$. Clearly $S \rightsquigarrow^*_{\sigma} S'$. By construction this narrowing derivation and the rewrite sequence $T \ggg_{\mathscr{R}} T'$ employ the same rewrite rules at the same positions in the corresponding goals. It remains to show that $\theta' \circ \sigma = \theta [V]$. This follows from $\theta' \circ \sigma' = \theta_1 [V_1]$ and (5), using Proposition 3.6. □

Giovannetti and Moiso [17] present a lifting lemma for level-confluent 2-CTRSs without proof. Dershowitz and Okada give a rather informal treatment of a lifting lemma for 1-CTRSs and level-confluent 2-CTRSs (Lemma 5.2 in [7]). This lemma is not suitable for proving the completeness of conditional narrowing for level-complete 2-CTRSs.[16]

Theorem 8.14 (Giovannetti and Moiso [17]). *Conditional narrowing is complete for level-complete 2-CTRSs.*

Proof. Let $\mathscr{R}$ be a level-complete 2-CTRS and suppose that σ is a solution of a goal clause T. Let n be the level of σT. Let σ' be the $\mathscr{R}_n$-normal form of σ. Confluence of $\mathscr{R}_n$ yields $\mathscr{R}_n \vdash \sigma' T$. Hence the level of $\sigma' T$ is at most n and therefore σ' is a sufficiently normalized solution of T. According to Lemma 8.11 there exists a sequence $\sigma' T \ggg_{\mathscr{R}} \top$. Lemma 8.13 yields a narrowing derivation $T \rightsquigarrow^*_{\tau} \top$ and a substitution σ'' such that $\sigma'' \circ \tau = \sigma' [\mathscr{V}(T)]$. Therefore $\tau \leq \sigma' [\mathscr{V}(T)]$ and hence $\tau \leq_{\mathscr{R}} \sigma [\mathscr{V}(T)]$. □

We have seen that in case of complete TRSs and 1-CTRSs strong normalization can be dropped, provided we restrict ourselves to normalizable solutions. This does not hold for level-complete 2-CTRSs as the following example of Giovannetti and Moiso [17] shows.

Example 8.15. Consider the level-confluent 2-CTRS

$$\mathscr{R} = \begin{cases} a \to b & \Leftarrow x = f(x) \\ c \to f(c). \end{cases}$$

We have $a \to_{\mathscr{R}} b$ because $c \to_{\mathscr{R}} f(c)$, but conditional narrowing is not able to solve the goal $a =^? b$, whose trivial solution ε is clearly normalizable.

[15] This is the only place in the proof where we use the fact that $\mathscr{R}$ is a 2-CTRS.
[16] As exemplified by the level-complete 2-CTRS $\{a \to b \Leftarrow x = a\}$.

However, it is not difficult to prove the equivalence of $\mathscr{R} \vdash T$ and $T \twoheadrightarrow_{\mathscr{R}} \top$ (cf. Lemma 8.11) for 2-CTRSs $\mathscr{R}$ with the property that that every $\mathscr{R}_n$ is semi-complete. The proof, which cannot be based on Lemma 8.10, has more or less the same structure as the proof of Proposition 6.8. Hence we can strengthen Theorem 8.14.

Definition 8.16. A CTRS $\mathscr{R}$ is called *level-semi-complete* if each $\mathscr{R}_n$ $(n \geqq 0)$ is semi-complete. Example 8.18 shows that level-semi-completeness is not the same as the combination of level-confluence and weak normalization.

Theorem 8.17. *Conditional narrowing is complete for level-semi-complete 2-CTRSs.* □

Example 8.18. Extend the CTRS of the previous example with the rule

$$f(x) \to d \Leftarrow y = f(y).$$

The new CTRS is level-confluent and weakly normalizing but not level-semi-complete as $\mathscr{R}_1$ is not weakly normalizing. Again the goal $a =^? b$ cannot be solved by conditional narrowing.

We conclude this section by proving that basic conditional narrowing is complete for level-complete 2-CTRS. This result is due to Giovannetti and Moiso. The Appendix contains a proof of the following proposition.

Proposition 8.19. *Let R be a level-confluent 2-CTRS and σ sufficiently normalized solution of a goal clause T. Every innermost $\twoheadrightarrow_{\mathscr{R}}$-sequence starting from σT is based on $\bar{T}$.* □

Theorem 8.20. *Basic conditional narrowing is complete for level-complete 2-CTRSs.*

Proof. Similar to the proof of Theorem 7.8. Let $\mathscr{R}$ be a level-complete 2-CTRS. Suppose σ is an $\mathscr{R}$-solution of a goal clause T and let σ' be its $\mathscr{R}_n$-normal form where n is the level of σT. We obtain $\sigma' T \twoheadrightarrow_{\mathscr{R}} \top$ as the proof of Theorem 6.13. We may assume that this sequence is innermost (cf. the remark after Lemma 8.11). According to Proposition 8.19 it is based on $\bar{T}$. It is not difficult to show that the narrowing derivation constructed by Lemma 8.13 is basic. The proof is completed as usual. □

9. Extra Variables in Right-Hand Sides

In this section we extend the main result of the previous section – the completeness of conditional narrowing for level-complete 2-CTRSs – to CTRSs that contain extra variables in the right-hand sides of the rewrite rules. An example of such a CTRS is the following system (inspired by [8]) which specifies the computation of Fibonacci numbers:

$$\begin{cases} 0 + x & \to x \\ S(x) + y & \to S(x+y) \\ f(0) & \to \langle 0, S(0) \rangle \\ f(S(x)) & \to \langle z, y+z \rangle \Leftarrow f(x) = \langle y, z \rangle \\ first(\langle x, y \rangle) & \to x \\ fib(x) & \to first(f(x)). \end{cases}$$

We require that extra variables in the right-hand side of a rule occur in its conditional part, i.e. we restrict ourselves to 3-CTRSs. This is not a real restriction as we consider only strongly normalizing CTRSs.

Unfortunately, the lifting lemma of the previous section does not extend to level-complete 3-CTRSs.

Example 9.1. Consider the level-complete 3-CTRS

$$\mathscr{R} = \begin{cases} a \to f(x) \Leftarrow x = b \\ b \to c. \end{cases}$$

Let $S = \{a =^? f(c)\}$ and $\theta = \varepsilon$. Clearly θ is a sufficiently normalized solution of S. We have the rewrite step

$$\theta S \rightarrowtail_{\mathscr{R}}^{1} \{f(b) =^? f(c),\, b =^? b\} = T'.$$

There is only one narrowing step originating from S:

$$S \rightsquigarrow_{\varepsilon} \{f(x) =^? f(c),\, x =^? b\} = S'.$$

Every substitution θ' satisfying $\theta' S' = T'$ must have $\theta' x = b$. But this conflicts with the sufficient normalization of θ' since the level of $\theta'(f(x) =^? f(c))$ is 1 and $\theta' x$ is $\mathscr{R}_1$-reducible. Suppose that we extend $\theta S \rightarrowtail_{\mathscr{R}}^{1} T'$ with the step $T' \rightarrowtail_{\mathscr{R}}^{0} \{f(b) =^? f(c),$ $\mathit{true}\} = T''$, i.e. we solve the condition of the rule applied in the previous step. The corresponding narrowing step is

$$S' \rightsquigarrow_{\{x, y \mapsto b\}} \{f(b) =^? f(c),\, \mathit{true}\} = S'',$$

where we used the rule $y =^? y \to \mathit{true}$. Now the problem has disappeared: Every substitution θ'' is sufficiently normalized with respect to S''. By solving the condition $x =^? b$ the problematic term b was transferred from the substitution θ' to the goal S''.

Example 9.1 suggests that Lemma 8.13 may hold if we restrict ourselves to $\rightarrowtail_{\mathscr{R}}$-sequences that first solve the introduced conditions after every application of a conditional rewrite rule. This indeed turns out to be the case. Observe that $\rightarrowtail_{\mathscr{R}}$-sequences complying with the obligation to solve conditions immediately after their introduction correspond to ordinary rewrite sequences, the only difference being the introduction of a few harmless constants *true* after a condition has been solved in a $\rightarrowtail_{\mathscr{R}}$-sequence.

Definition 9.2. Let $\mathscr{R}$ be a CTRS. A $\rightarrowtail_{\mathscr{R}}$-rewrite sequence is said to be *well-behaved* if it can be constructed according to the following two principles.

- Let $T \rightarrowtail_{[e,p,l \to r \Leftarrow c,\sigma]} (T - \{e\}) \cup \{e[\sigma r]_p\} \cup \sigma\tilde{c}$. If $\sigma\tilde{c} \twoheadrightarrow \top$ is well-behaved then
$$T \rightarrowtail_{[e,p,l \to r \Leftarrow c,\sigma]} (T - \{e\}) \cup \{e[\sigma r]_p\} \cup \sigma\tilde{c} \twoheadrightarrow (T - \{e\}) \cup \{e[\sigma r]_p\} \cup \top$$
is well-behaved. In particular every unconditional step $T \rightarrowtail_{[e,p,l \to r,\sigma]} (T - \{e\}) \cup$ $\{e[\sigma r]_p\}$ is well-behaved.
- If $T_1 \twoheadrightarrow_{\mathscr{R}} T_2$ and $T_2 \twoheadrightarrow_{\mathscr{R}} T_3$ are well-behaved then their concatenation $T_1 \twoheadrightarrow_{\mathscr{R}}$ $T_2 \twoheadrightarrow_{\mathscr{R}} T_3$ is well-behaved.

Proposition 9.3. *Let R be a level-complete CTRS and $T \in \mathscr{G}_\top$. There exists a well-behaved sequence $T \twoheadrightarrow_{\mathscr{R}} \top$.*

Proof. According to Lemma 8.11 there exists a sequence $T \twoheadrightarrow_{\mathcal{R}} \mathrm{T}$. This sequence can be transformed into a well-behaved sequence $T \twoheadrightarrow_{\mathcal{R}} \mathrm{T}$ by a straightforward reordering process. □

Lemma 9.4. *Let $\mathcal{R}$ be a level-complete 3-CTRS. Suppose we have goal clauses S and T, a sufficiently normalized solution θ of S, and a set V of variables such that $\mathcal{V}(S) \cup \mathcal{D}\theta \subseteq V$ and $T = \theta S$. If $T \twoheadrightarrow_{\mathcal{R}} T'$ is well-behaved then there exist a goal clause S' and substitutions θ', σ such that*

- $S \rightsquigarrow^*_\sigma S'$,
- $\theta' S' = T'$,
- $\theta' \circ \sigma = \theta[V]$,
- *θ' is a sufficiently normalized solution of S'.*

Furthermore, we may assume that the narrowing derivation $S \rightsquigarrow^*_\sigma S'$ and the sequence $T \twoheadrightarrow_{\mathcal{R}} T'$ employ the same rewrite rules at the same positions in the corresponding goals.

Proof. We use induction on the level of T. If the level of T equals 0 then only the rule $x =^? x \to true$ is used in the $\rightarrowtail_{\mathcal{R}}$-sequence from T to T'. Since $\{x =^? x \to true\}$ clearly constitutes a level-confluent 2-CTRS, the result follows from Lemma 8.13. Suppose the level of T equals $N+1$. We use induction on the length of the well-behaved $\rightarrowtail_{\mathcal{R}}$-sequence from T to T'. The case of zero length is trivial. Suppose $T \rightarrowtail^n_{[\theta e, p, R: l \to r \Leftarrow c, \tau]} T_1 \twoheadrightarrow_{\mathcal{R}} T'$ with $\mathcal{V}(R) \cap V = \varnothing$ and $\mathcal{D}\tau \subseteq \mathcal{V}(R)$. Because this sequence is well-behaved, the subsequence from $T_1 = (T - \{\theta e\}) \cup \{\theta e[\tau r]_p\} \cup \tau\tilde{c}$ to T' can be written as $T_1 \twoheadrightarrow_{\mathcal{R}} T_2 = (T - \{\theta e\}) \cup \{\theta e[\tau r]_p\} \cup \mathrm{T} \twoheadrightarrow_{\mathcal{R}} T'$. The structure of the proofs is illustrated in Fig. 7. Let m be the level of θe. Clearly $m \leqq N+1$. By definition $n \leqq m$. At this point we follow literally the proof of Lemma 8.13 until we reach formula (7). This takes care of diagram (1) in Fig. 7. Next we show the existence of a goal clause S_2 and substitutions θ_2, σ_2 such that $S_1 \rightsquigarrow^*_{\sigma_2} S_2$, $\mathcal{V}(S_2) \cup \mathcal{D}\theta_2 \subseteq V_2$, $\theta_2 S_2 = T_1$, $\theta_2 \circ \sigma_2 = \theta_1[V_1]$, and θ_2 is a sufficiently normalized solution of S_2. Here $V_2 = (V_1 - \mathcal{D}\sigma_2) \cup \mathcal{I}\sigma_2$. We distinguish two cases.

$\tilde{c} = \varnothing$ (This means that R is an unconditional rewrite rule.) We define $S_2 = S_1$, $\theta_2 = \theta_1$, and $\sigma_2 = \varepsilon$. Since in this case $\mathcal{V}(r) \subseteq \mathcal{V}(l)$ we can repeat cases (a) and (b) in the proof of Lemma 8.13 in order to conclude that θ_2 is sufficiently normalized solution of S_2. The other requirements are trivially satisfied.

$\tilde{c} \neq \varnothing$ In this case the substitution θ_1 is in general not a sufficiently normalized solution of S_1 since in case (b) of the proof of Lemma 8.13 the requirement $\mathcal{V}(r) \subseteq \mathcal{V}(l)$ is essential. Case (c), however, does not rely on the restriction to 2-CTRSs. Hence θ_1 is a sufficiently normalized solution of $\tau\tilde{c}$. From

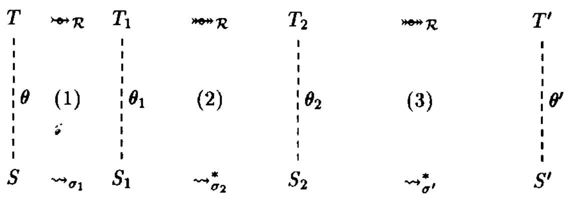


Fig. 7.

$\mathcal{R}_{n-1} \vdash \tau\tilde{c}$ we infer that the level of $\tau\tilde{c}$ is less than $N+1$. From $T_1 \twoheadrightarrow_{\mathcal{R}} T_2$ we extract a well-behaved sequence $\tau\tilde{c} \twoheadrightarrow_{\mathcal{R}} \top$. Applying the first induction hypothesis yields substitutions θ_2 and σ_2 such that $\sigma_1\tilde{c} \rightsquigarrow^*_{\sigma_2} \top$ and $\theta_2 \circ \sigma_2 = \theta_1[V_1]$. Let $S_2 = \sigma_2\sigma_1((S - \{e\}) \cup \{e[r]_p\}) \cup \top$. Clearly $S_1 \rightsquigarrow^*_{\sigma_2} S_2$. Using $\mathcal{V}(S_1) \subseteq V_1$ we easily obtain $\mathcal{V}(S_2) \subseteq V_2$. It is not difficult to show that $\theta_2 S_2 = T_1$. Since $\theta_2 \restriction_{V_2}$ also satisfies the above requirements (i.e. $\theta_2 \restriction_{V_2} \circ \sigma_2 = \theta_1[V_1]$ and $\theta_2 \restriction_{V_2} S_2 = T_1$), we may assume that $\mathcal{D}\theta_2 \subseteq V_2$. It remains to show that θ_2 is a sufficiently normalized solution of S_2. Let $e' \in S_2 - \top$. There exists an $e'' \in (S - \{e\}) \cup \{e[r]_p\}$ such that $e' \equiv \sigma_2\sigma_1 e''$. We distinguish two cases: (a) $e'' \in S - \{e\}$ and (b) $e'' \equiv e[r]_p$.

(a) Since $\mathcal{V}(e'') \subseteq V$ and $\mathcal{V}(\sigma_1 e'') \subseteq V_1$ we obtain $\theta_2 e' \equiv \theta e''$ from $\theta_1 \circ \sigma_1 = \theta[V]$ and $\theta_2 \circ \sigma_2 = \theta_1[V_1]$. Because θ is a sufficiently normalized solution of S, $\theta \restriction_{\mathcal{V}(e'')}$ is $\mathcal{S}_l$-normalized where l is the level of $\theta e''$. We have to show that $\theta_2 \restriction_{\mathcal{V}(e')}$ is also $\mathcal{S}_l$-normalized. This follows from $\theta_1 \circ \sigma_1 = \theta[V]$, $\theta_2 \circ \sigma_2 = \theta_1[V_1]$, and two applications of Propositions 3.5 and 3.7.

(b) Let $e'' \equiv e[r]_p$. We have $\theta e \rightarrow_{\mathcal{R}} \theta e[\tau r]_p \equiv \theta_2 e'$. Proposition 8.7 shows that the level of $\theta_2 e'$ is at most m and hence it suffices to show that $\theta_2 \restriction_{\mathcal{V}(e')}$ is $\mathcal{R}_m$-normalized. The crucial observation is that we have the following inclusion:

$$\mathcal{V}(e') = \mathcal{V}(\sigma_2\sigma_1(e[r]_p)) \subseteq \mathcal{V}(\sigma_2\sigma_1 e). \tag{1}$$

Suppose to the contrary that there exists a variable $x \in \mathcal{V}(\sigma_2\sigma_1(e[r]_p)) - \mathcal{V}(\sigma_2\sigma_1 e)$. This implies that $x \in \mathcal{V}(\sigma_2\sigma_1 r)$. Since $\mathcal{V}(\sigma_2\sigma_1 l) = \mathcal{V}(\sigma_2\sigma_1(e_{|p})) \subseteq \mathcal{V}(\sigma_2\sigma_1 e)$, we have $x \in \mathcal{V}(\sigma_2\sigma_1 r) - \mathcal{V}(\sigma_2\sigma_1 l)$. According to Lemma 6.6 we may infer $\mathcal{R} \vdash \sigma_2\sigma_1\tilde{c}$ from $\sigma_1\tilde{c} \rightsquigarrow^*_{\sigma_2} \top$. Hence $\sigma_2\sigma_1 l \rightarrow_{\mathcal{R}_i} \sigma_2\sigma_1 r$ from some $i \geq 0$. However, since $\rightarrow_{\mathcal{R}_i}$ is closed under substitutions (in particular under the substitution $\{x \mapsto \sigma_2\sigma_1 l\}$), we obtain an infinite $\rightarrow_{\mathcal{R}_i}$-sequence starting from $\sigma_2\sigma_1 l$, contradicting the level-completeness of $\mathcal{R}$. Therefore inclusion (1) is valid. Using $\theta_1 \circ \sigma_1 = \theta[V]$, $\mathcal{V}(e) \subseteq V$, and the $\mathcal{R}_m$-normalization of $\theta \restriction_{\mathcal{V}(e)}$, we obtain the $\mathcal{R}$-normalization of $\theta_1 \restriction_{\mathcal{V}(\sigma_1 e)}$ by means of Propositions 3.5 and 3.7. The $\mathcal{R}_m$-normalization of $\theta_2 \restriction_{\mathcal{V}(\sigma_2\sigma_1 e)}$ follows in the same way. From (1) we obtain the $\mathcal{R}_m$-normalization of $\theta_2 \restriction_{\mathcal{V}(e')}$.

This includes the construction of diagram (2) in Fig. 7. According to Proposition 8.8 the level of T_2 is at most $N+1$. If it is less than $N+1$ then we apply the first induction hypothesis. Otherwise we apply the second induction hypothesis. In both cases we obtain a goal clause S' and substitution θ', σ' such that $S_2 \rightsquigarrow^*_{\sigma'} S'$, $\theta' S' = T'$, $\theta' \circ \sigma' = \theta_2[V_2]$, and θ' is a sufficiently normalized solution of S'. Now that we have completed diagram (3) in Fig. 7, it is time to glue the three diagrams together. Let $\sigma = \sigma' \circ \sigma_2 \circ \sigma_1$. We clearly have $S \rightsquigarrow^*_{\sigma} S'$. From $\theta_1 \circ \sigma_1 = \theta[V]$, $\theta_2 \circ \sigma_2 = \theta_1[V_1]$, $\theta' \circ \sigma' = \theta_2[V_2]$, and the definitions of V_1 and V_2, we obtain $\theta' \circ \sigma = \theta[V]$ by two applications of Proposition 3.6. □

Combining Lemma 8.11, Proposition 9.3, and Lemma 9.4 yields the final result of this paper.

Theorem 9.5. *Conditional narrowing is complete for level-complete 3-CTRSs.* □

It is unclear whether basic conditional narrowing is complete for level-complete 3-CTRSs. The problem is that Proposition 8.19 does not extend to (level-complete) 3-CTRSs.

Example 9.6. Consider again the CTRS of Example 9.1 and the goal $S = \{a =^? f(c)\}$. The sequence

$$\begin{aligned} S &\rightarrowtail\!\!\!\!\rightarrow^1_{\mathscr{R}} f(\underline{b}) =^? f(c), \underline{b} =^? b \\ &\rightarrowtail\!\!\!\!\rightarrow^0_{\mathscr{R}} f(\underline{b}) =^? f(c), true \\ &\rightarrowtail\!\!\!\!\rightarrow^1_{\mathscr{R}} f(c) =^? f(c), true \\ &\rightarrowtail\!\!\!\!\rightarrow^0_{\mathscr{R}} \top \end{aligned}$$

is innermost and well-behaved, but not based on $\bar{S}$. Nevertheless, basic conditional narrowing is able to solve the goal $a =^? f(c)$:

$$\begin{aligned} S &\rightsquigarrow \quad f(x) =^? f(c), x =^? b \\ &\rightsquigarrow_{\{x \mapsto c\}} true, \underline{c} =^? b \\ &\rightsquigarrow \quad true, \underline{c} =^? c \\ &\rightsquigarrow \quad \top, \end{aligned}$$

but the corresponding $\rightarrowtail_{\mathscr{R}}$-sequence

$$\begin{aligned} S &\rightarrowtail^2_{\mathscr{R}} f(\underline{c}) =^? f(c), \underline{c} =^? b \\ &\rightarrowtail^0_{\mathscr{R}} true, \underline{c} =^? b \\ &\rightarrowtail^1_{\mathscr{R}} true, \underline{c} =^? c \\ &\rightarrowtail^0_{\mathscr{R}} \top \end{aligned}$$

is not a $\rightarrowtail\!\!\!\!\rightarrow_{\mathscr{R}}$-sequence since the level of S equals 1.

10. Conclusion

In this paper we have tried to perform a thorough study of the completeness of narrowing and basic narrowing for TRSs and CTRSs. The main results are summarized below. Results preceded with "∘" are new.

Norrowing is complete for
- complete TRSs (Theorem 3.8),
- semi-complete TRSs (Corollary 3.11),
- confluent TRSs with respect to normalizable solutions (Theorem 3.10).

Basic narrowing is complete for
- complete TRSs (Theorem 4.5),
- ∘ orthogonal TRSs with respect to normalizable solutions and goals (Theorem 5.6),
- ∘ confluent right-linear TRSs with respect to normalizable solutions (Theorem 5.13),
- ∘ weakly normalizing orthogonal TRSs (Corollary 5.7),
- ∘ semi-complete right-linear TRSs (Corollary 5.14).

Basic narrowing is not complete for
- ∘ semi-complete TRSs (Counterexample 4.7).

Conditional narrowing is complete for
- complete 1-CTRSs (Theorem 6.13),
- semi-complete 1-CTRSs (Corollary 6.15),

- confluent 1-CTRSs with respect to normalizable solutions (Theorem 6.14),
- level-complete 2-CTRSs (Theorem 8.14),
- ○ level-semi-complete 2-CTRSs (Theorem 8.17),
- ○ level-complete 3-CTRSs (Theorem 9.5).

Conditional narrowing is not complete for

- complete 2-CTRSs (Example 8.1),
- level-confluent 2-CTRSs with respect to normalizable solutions (Example 8.15).

Basic conditional narrowing is complete for

- ○ decreasing and confluent 1-CTRSs (Theorem 7.8),
- level-complete 2-CTRSs (Theorem 8.20).

Basic conditional narrowing is not complete for

- ○ complete 1-CTRSs (Counterexample 7.3),
- ○ semi-complete orthogonal 1-CTRSs (Counterexample 7.11).

We except that the completeness of basic narrowing for level-complete 2-CTRSs carries over to 3-CTRSs. It is less clear whether level-semi-completeness is sufficient for the completeness of conditional narrowing for 3-CTRSs (cf. Theorem 8.17). As a matter of fact, it seems reasonable to conjecture that Theorem 8.17 does not extend to 3-CTRSs since strong normalization of every $\mathscr{R}_n$ is crucial in the proof of the lifting lemma for 3-CTRSs (Lemma 9.4).

Giovannetti and Moiso [17] observed that the confluence proof of Bergstra and Klop [1] for orthogonal and normal 2-CTRSs (III_n systems in the terminology of [1]) actually shows level-confluence. Such a result (if at all true) makes less sense for 3-CTRSs since 3-CTRSs typically are not normal, see for example the 3-CTRS at the beginning of Section 9. Thus it is important to develop other criteria that are easy to check and which ensure the level-confluence of 3-CTRSs. Useful techniques which ensure the strong normalization of 2 and 3-CTRSs need also to be developed.

If we extend the set of basic positions as in the combination of basic and normal narrowing (see Réty [37]), our counterexamples (4.7 and 7.3) no longer work. It is worthwhile to investigate whether such a relaxed form of basic narrowing suffices for completeness.

As explained in the introduction, we restricted ourselves in this paper to narrowing and basic narrowing for TRSs and CTRSs. It would be interesting to treat the other variants of narrowing in the same systematic way.

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Appendix

In this Appendix we present proofs of Propositions 4.4, 5.5, 7.7 and 8.19. The proofs are very much alike, but it is difficult to capture the similarities in a separate, general proposition.

Proposition 4.4. *Let $\mathscr{R}$ be a TRS and σ a normalized substitution. Every innermost reduction sequence starting from σt is based on $\bar{O}(t)$.*

Proof. Suppose

$$\sigma t \equiv t_1 \to_{[p_1, l_1 \to r_1, \sigma_1]} \cdots \to_{[p_{n-1}, l_{n-1} \to r_{n-1}, \sigma_{n-1}]} t_n$$

is an innermost reduction sequence. Let $B_1 = \bar{O}(t)$ and define $B_2, \ldots, B_{n-1}$ as in Definition 4.1. By induction on i we will show that $(t_i)_{|p}$ is a normal form whenever $p \in O(t_i) - B_i$ for $1 \leqq i < n$. The case $i = 1$ follows from the normalization of σ. Suppose the statement holds for $i = 1, \ldots, m$ and let $p \in O(t_{m+1}) - B_{m+1}$. We distinguish two cases: $p \perp p_m$ and $p \geqq p_m$. (The case $p < p_m$ is impossible since this

would imply $p \in B_{m+1}$ as we already know that B_m is closed under prefix and $p_m \in B_m$.)
(1) If $p \perp p_m$ then clearly $p \in O(t_m) - B_m$ and $(t_{m+1})_{|p} \equiv (t_m)_{|p}$. The induction hypothesis yields the desired result.
(2) If $p \geqq p_m$ then there exist positions $q \in O_{\mathscr{V}}(r_m)$ and q' such that $p = p_m \cdot q \cdot q'$ (otherwise $p \in B_{m+1}$). Hence $(t_{m+1})_{|p} \equiv (\sigma_m r_m)_{|q \cdot q'} \equiv (\sigma_m x)_{|q'}$ where x is the variable in r_m at position q. So $(t_{m+1})_{|p}$ is a proper subterm of $\sigma_m l_m$ and because $t_m \rightarrow_{[p_m, l_m \rightarrow r_m, \sigma_m]} t_{m+1}$ is an innermost reduction step, $(t_{m+1})_{|p}$ is a normal form. □

Before proving Proposition 5.5 we give a few elementary properties of orthogonal TRSs. The following lemma expresses a famous result in the theory of orthogonal TRSs (see e.g. Huet and Lévy [21]). Confluence of orthogonal TRSs is an easy consequence of this lemma.

Parallel Moves Lemma. *Let $\mathscr{R}$ be an orthogonal TRS. If $t \mathrel{-\!\!\Vert\!\!\rightarrow} t_1$ and $t \mathrel{-\!\!\Vert\!\!\rightarrow} t_2$ then there exists a term t_3 such that $t_1 \mathrel{-\!\!\Vert\!\!\rightarrow} t_3$ and $t_2 \mathrel{-\!\!\Vert\!\!\rightarrow} t_3$. Moreover, the redexes contracted in $t_1 \mathrel{-\!\!\Vert\!\!\rightarrow} t_3$ ($t_2 \mathrel{-\!\!\Vert\!\!\rightarrow} t_3$) are the descendants in t_1 (t_2) of the redexes contracted in $t \mathrel{-\!\!\Vert\!\!\rightarrow} t_2$ ($t \mathrel{-\!\!\Vert\!\!\rightarrow} t_1$).* □

The following consequence of the Parallel Moves Lemma is used in the proof of Proposition 5.5 below.

Proposition A.1. *Let $\mathscr{R}$ be an orthogonal TRS. Suppose s contains a redex r which is not needed. If $s \rightarrow t$ then the descendants of r in t are not needed.*

Proof. Because r is not needed there exists a normalizing reduction sequence

$$s \equiv s_1 \rightarrow s_2 \rightarrow \cdots \rightarrow s_n$$

in which no descendant of r is contracted. Using the Parallel Moves Lemma we construct the diagram of Fig. 8. The contracted redexes in $t_i \mathrel{-\!\!\Vert\!\!\rightarrow} t_{i+1}$ are the descendants of the redex contracted in the step $s_i \rightarrow s_{i+1}$. Hence no descendant of r is contradicted in the sequence $t_1 \twoheadrightarrow t_n$ and because $s_n \equiv t_n$ no descendant of r in t is needed. □

Proposition 5.5. *Let $\mathscr{R}$ be an orthogonal TRS and σ a normalized substitution. Every innermost needed reduction sequence starting from σt is based on $\bar{O}(t)$.*

Proof. The proof has the same structure as the proof of Proposition 4.4. Suppose

$$\sigma t \equiv t_1 \rightarrow_{[p_1, l_1 \rightarrow r_1, \sigma_1]} \cdots \rightarrow_{[p_{n-1}, l_{n-1} \rightarrow r_{n-1}, \sigma_{n-1}]} t_n$$

is an innermost needed reduction sequence and define $B_1, \ldots, B_{n-1}$ as usual. By induction on i we will show that $(t_i)_{|p}$ contains no needed redexes whenever $p \in O(t_i) - B_i$ for $1 \leqq i < n$. The case $i = 1$ is trivial. Suppose the statement holds for $i = 1, \ldots, m$ and let $p \in O(t_{m+1}) - B_{m+1}$. The case $p \perp p_m$ easily follows from the induction

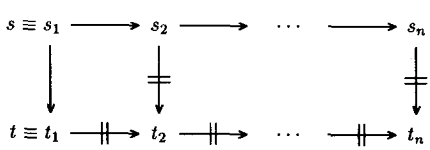

Fig. 8.

hypothesis. If $p \geqq p_m$ then $(t_{m+1})_{|p}$ is a proper subterm of $\sigma_m l_m$, just as in the proof of Proposition 4.4. Suppose $(t_{m+1})_{|p}$ contains a redex r. Since $t_m \to_{[p_m, l_m \to r_m, \sigma_m]} t_{m+1}$ is an innermost needed reduction step, r is not needed in t_m. Proposition A.1 shows that r is not needed in t_{m+1}. □

Proposition 7.7. *Let $\mathscr{R}$ be a 1-CTRS, T a goal clause and σ a normalized substitution. Every innermost $\rightarrowtail_{\mathscr{R}}$-sequence starting from σT is based on $\bar{T}$.*

Proof. Suppose

$$\sigma T = T_1 \rightarrowtail_{[e_1, p_1, R_1, \sigma_1]} \cdots \rightarrow_{[e_{n-1}, p_{n-1}, R_{n-1}, \sigma_{n-1}]} T_n$$

is an innermost $\rightarrowtail_{\mathscr{R}}$-sequence. Let $B_1 = \bar{T}$ and define the position constraints $B_2, \ldots, B_{n-1}$ as in Definition 7.2(2). By induction on i we will show that $e_{|p}$ is a normal form whenever $e \in T_i$ and $p \in O(e) - B_i(e)$ for $1 \leqq i < n$. For $i = 1$ this is a consequence of the normalization of σ. Suppose the statement holds for $i = 1, \ldots, m$. Let R_m be the rule $l_m \to r_m \Leftarrow c_m$ and take $e \in T_{m+1}$. We distinguish three cases: $e \in T_m - \{e_m\}$, $e \equiv e_m[\sigma_m r_m]_{p_m}$, and $e \in \sigma_m \tilde{c}_m$.

(1) If $e \in T_m - \{e_m\}$ then $B_{m+1}(e) = B_m(e)$ and hence the result follows from the induction hypothesis.
(2) The case $e \equiv e_m[\sigma_m r_m]_{p_m}$ follows as in the proof of Proposition 4.4.
(3) If $e \in \sigma_m \tilde{c}_m$ then $B_{m+1}(e) = \bar{O}(e')$ where $e = \sigma_m e'$. Hence it suffices to show that $\sigma_m \restriction_{\mathscr{V}(e')}$ is normalized. Since $\mathscr{R}$ is a 1-CTRS, we have $\mathscr{V}(e') \subseteq \mathscr{V}(l_m)$ and because $\sigma_m l_m$ is an innermost redex in e_m we know that $\sigma_m \restriction_{\mathscr{V}(l_m)}$ is normalized. □

Proposition 8.19. *Let $\mathscr{R}$ be a level-confluent 2-CTRS and σ a sufficiently normalized solution of a goal clause T. Every innermost $\rightarrowtail\!\!\!\!\leftrightarrow_{\mathscr{R}}$-sequence starting from σT is based on $\bar{T}$.*

Proof. Suppose

$$\sigma T = T_1 \rightarrowtail\!\!\!\leftrightarrow_{[e_1, p_1, R_1 : l_1 \to r_1 \Leftarrow c_1, \sigma_1]} \cdots \rightarrowtail\!\!\!\leftrightarrow_{[e_{n-1}, p_{n-1}, R_{n-1} : l_{n-1} \to r_{n-1} \Leftarrow c_{n-1}, \sigma_{n-1}]} T_n$$

is an innermost $\rightarrowtail\!\!\!\leftrightarrow_{\mathscr{R}}$-sequence. Let $B_1 = \bar{T}$ and define the position constraints $B_2, \ldots, B_{n-1}$ as in Definition 7.2(2). By induction on i we will show that $e_{|p}$ is $\mathscr{R}_j$-normalized whenever $e \in T_i$ and $p \in O(e) - B_i(e)$ for $1 \leqq i < n$. Here j is the level of e. For $i = 1$ this is a consequence of the sufficient normalization of σ. Suppose the statement holds for $i = 1, \ldots, m$ and let k be the level of e_m. We have

$$T_m \rightarrowtail\!\!\!\leftrightarrow^{l}_{[e_m, p_m, R_m, \sigma_m]} T_{m+1} = (T_m - \{e_m\}) \cup \{e_m[\sigma_m r_m]_{p_m}\} \cup \sigma_m \tilde{c}_m$$

for some $l \leqq k$. Hence $\mathscr{R}_{l-1} \vdash \sigma_m \tilde{c}_m$ and $\sigma_m \restriction_{\mathscr{E}(R_m)}$ is $\mathscr{R}_{l-1}$-normalized. Take $e \in T_{m+1}$. We distinguish three cases.

(1) If $e \in T_m - \{e_m\}$ then $B_{m+1}(e) = B_m(e)$ and hence the result follows from the induction hypothesis.
(2) Let $e \equiv e_m[\sigma_m r_m]_{p_m}$. We have $e_m \to_{\mathscr{R}_l} e$. Proposition 8.7 shows that the level of e is at most k and hence it suffices to show that $e_{|p}$ is $\mathscr{R}_k$-normalized. As in the proof of Proposition 4.4 we distinguish the two cases $p \perp p_m$ and $p \geqq p_m$.
    (a) If $p \perp p_m$ then $e_{|p} \equiv (e_m)_{|p}$ and $p \notin B_m(e_m)$. Hence the result follows from the induction hypothesis.
    (b) Let $p \geqq p_m$. Since $\mathscr{R}$ is a 2-CTRS we have $\mathscr{V}(r_m) \subseteq \mathscr{V}(l_m)$ and hence we infer that $e_{|p}$ is a proper subterm of $\sigma_m l_m$. Because $\sigma_m l_m$ is an innermost $\mathscr{R}_k$-redex in e_m, $\sigma_m \restriction_{\mathscr{V}(l_m)}$ is $\mathscr{R}_k$-normalized.

(3) If $e \in \sigma_m \tilde{c}_m$ then the level e does not exceed $l-1$ and $B_{m+1}(e) = \bar{O}(e')$ where $e = \sigma_m e'$. So it is sufficient to show that $\sigma_m \restriction_{\mathscr{V}(e')}$ is $\mathscr{R}_{l-1}$-normalized. Clearly $\mathscr{V}(e') \subseteq \mathscr{V}(l_m) \cup \mathscr{E}(R_m)$. We already observed that $\sigma_m \restriction_{\mathscr{E}(R_m)}$ is $\mathscr{R}_{l-1}$-normalized and $\sigma_m \restriction_{\mathscr{V}(l_m)}$ is $\mathscr{R}_k$-normalized. Since $k > l-1$, $\sigma_m \restriction_{\mathscr{V}(l_m)}$ is certainly $\mathscr{R}_{l-1}$-normalized. □

We conclude the Appendix by showing that the orthogonal 1-CTRS $\mathscr{R}$ of Counterexample 7.11 is semi-complete.

Proof. We transform

$$\mathscr{R} = \begin{cases} f(x) \to a & \Leftarrow\ g(b) = c \\ g(x) \to c & \Leftarrow\ x = f(x) \\ b \quad \to f(b) \end{cases}$$

into a semi-complete TRS $\mathscr{R}'$ such that the relations $\to_{\mathscr{R}}^{+}$ and $\to_{\mathscr{R}'}^{+}$ coincide. First observe the $g(b) \downarrow_{\mathscr{R}} c$. Hence $\mathscr{R}$ generates the same rewrite relation as the CTRS

$$\mathscr{R}_1 = \begin{cases} f(x) \to a \\ g(x) \to c & \Leftarrow\ x = f(x) \\ b \quad \to f(b). \end{cases}$$

It is not difficult to show that $\{a, b\} \cup \{f(s) \mid s \text{ is an arbitrary term}\}$ is the set of all terms t that satisfy $t \downarrow_{\mathscr{R}_1} f(t)$. As a consequence, the rewrite relations of $\mathscr{R}_1$ and the TRS

$$\mathscr{R}_2 = \begin{cases} f(x) & \to a \\ g(a) & \to c \\ g(b) & \to c \\ g(f(x)) & \to c \\ b & \to f(b) \end{cases}$$

coincide. Define

$$\mathscr{R}' = \begin{cases} f(x) \to a \\ g(a) \to c \\ b \quad \to f(b). \end{cases}$$

Clearly $\to_{\mathscr{R}'} \subseteq \to_{\mathscr{R}_2}$. One easily shows that $\to_{\mathscr{R}_2} \subseteq \to_{\mathscr{R}'}^{+}$. We conclude that $\to_{\mathscr{R}}^{+} = \to_{\mathscr{R}'}^{+}$. Confluence of $\mathscr{R}'$ is an immediate consequence of orthogonality. Weak normalization of $\mathscr{R}'$ easily follows by induction on the structure of terms. □