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On complements in lattices with covering properties

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Dedicated to the memory of Herbert Gross

1. Introduction

Stenström [4] investigated joins of finitely many atoms in a complete modular lattice. Among others, he proved a necessary and sufficient condition for such a join to have a complement. We show that Stenström's criterion also holds under assumptions weaker than modularity. These weaker assumptions are the so-called covering properties and neighborhood conditions.

2. Preliminaries

All lattices are assumed to be complete. The least and greatest elements will be denoted by 0 and 1, respectively. We say that x is covered by y and write $x \prec y$ if $x < y$ and if $x \leqq z < y$ implies $z = x$ for all z. An element x is called an atom if $0 \prec x$ and a dual atom if $x \prec 1$.

A lattice L is called atomic if, for each $x \neq 0$, there exists an atom p such that $p \leqq x$. A lattice is called atomistic if each of its elements is a join of atoms. Similarly a lattice is called dually atomistic if each of its elements is a meet of dual atoms.

In investigations on nonmodular lattices so-called modular pairs play an important role. For a detailed background concerning this notion (which is due to L. R. Wilcox) we refer to Maeda–Maeda [3].

DEFINITION. Let L be a lattice and $a, b \in L$. We say that a, b is a modular pair and write $(a, b)M$ if

$c \leqq b$ implies $(c \vee a) \wedge b = c \vee (a \wedge b)$.

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We say that a, b is a dual-modular pair and write $(a, b)M^*$ if

$$c \geqq b \quad \text{implies} \quad (c \wedge a) \vee b = c \wedge (a \vee b).$$

A lattice is obviously modular if and only if $(a, b)M$ holds for each $a, b \in L$. From Dedekind's isomorphism principle it follows that each modular lattice satisfies the neighborhood condition

$$a \wedge b \prec a \quad \text{implies} \quad b \prec a \vee b \tag{N}$$

and the dual neighborhood condition

$$b \prec a \vee b \quad \text{implies} \quad a \wedge b \prec a. \tag{N*}$$

In lattices of finite length, (N) and (N*) together imply modularity; for lattices in general, this is no longer true (cf. Crawley and Dilworth [2]).

For lattices with continuous chains the implications (N) and (N*) are trivially fulfilled. On the other hand, for lattices of infinite length which have discrete chains (in particular, for atomistic lattices) it makes sense to single out those lattices which satisfy (N) or (N*) or both in a nontrivial manner.

Examples of such lattices are atomistic modular lattices (as, for instance, the subspace lattices of projective geometries). There are also atomistic lattices satisfying both (N) and (N*) but not being modular: for instance, the lattice of all closed subspaces of an infinite dimensional Hausdorff topological vector space (cf. Maeda–Maeda [3]).

For a lattice L, the interval $[a, b]$ $(a, b \in L, a \leqq b)$ is the set of all $x \in L$ for which $a \leqq x \leqq b$.

An element b is said to be of finite height, if the length of the interval $[0, b]$ is finite. Any atom is an element of height 1. In general, an element of finite height need not be a join of atoms.

In lattices with (N) each join of finitely many atoms is of finite height. In atomistic lattices with (N) each element of finite height can be represented as a join of finitely many atoms. By a finite element we mean either 0 or an element which is a join of finitely many atoms.

An element b of a lattice L is called modular if $(x, b)M$ holds for every $x \in L$.

Atomistic lattices with (N) and (N*) have the property that each finite element is a modular element. More precisely, if L is an atomistic lattice with (N) and (N*), and if b is a finite element, then $(x, b)M$ and $(b, x)M^*$ hold for each $x \in L$ (cf. Maeda–Maeda [3], Corollary 9.4). Because of this property, an atomistic lattice with (N) and (N*) is said to be finite-modular. Note that, in general, such a lattice is neither finite nor modular. As a more general version of Dedekind's isomorphism principle we have

PROPOSITION 1 (cf. Maeda–Maeda [3], Lemma 1.3). *Let a, b be elements of a lattice L. If both $(a, b)M$ and $(b, a)M^*$ hold, then the transposed intervals $[a, a \vee b]$ and $[a \wedge b, b]$ are isomorphic (by the mutually inverse mappings $x \to x \wedge b$ and $y \to y \vee a$).*

Proposition 1 and the foregoing observations imply that if, in an atomistic lattice with (N) and (N*), b is a join of finitely many atoms, then the transposed intervals $[b \wedge x, b]$ and $[x, b \vee x]$ are isomorphic. We shall use this fact in the next section.

3. Elements of finite height with complements

We say that an atom p of a lattice L has the covering property (C) if, for any $x \in L$, $x \wedge p = 0$ implies $x \prec x \vee p$. A lattice is said to have the covering property (C) if each of its atoms has the covering property (C). A lattice with neighborhood condition (N) has clearly the covering property (C) whereas the converse is not true. However, if an atomistic lattice has the covering property (C), then it also satisfies the neighborhood condition (N) (cf. Maeda–Maeda [3], Theorem 7.10). Atomistic lattices with (N) are therefore also called AC-lattices.

Similarly, a dual atom m of a lattice L is said to possess the dual covering property (C*) if, for any $x \in L$, $x \vee m = 1$ implies $x \wedge m \prec x$. A lattice is said to have the dual covering property (C*) if each of its dual atoms has the dual covering property (C*). Again, it is clear that a lattice with (N*) also satisfies (C*), but not conversely.

In what follows we denote by x_+ the meet of all lower covers of the element x.

PROPOSITION 2. *Let L be a complete lattice with dual covering property (C*). Then $b_+ \leqq 1_+$ holds for all $b \in L$.*

Proof. If L has no dual atom, then $1_+ = 1$ and the assertion trivially holds. Next we observe that for an arbitrary dual atom m_i and for an arbitrary $b \in L$ we have either $b \leqq m_i$ or $b \nleqq m_i$ implying by (C*) that $b \wedge m_i \prec b$. Hence $b_+ \leqq \bigwedge (b \wedge m_i) \leqq \bigwedge m_i = 1_+$.

PROPOSITION 3. *Let L be a complete lattice satisfying the neighborhood condition (N) and its dual (N*), and let $(0 \neq)\ b \in L$ be an element of finite height. If $b \wedge 1_+ = 0$, then b is a join of finitely many atoms.*

Proof. From $b \wedge 1_+ = 0$ it follows by Proposition 2 that

$$b \wedge b_+ = 0. \tag{1}$$

Since $b_+ < b$ for any element b $(\neq 0)$ of finite height we conclude from (1) that $b_+ = 0$. This means that $[0, b]$ is a sublattice of finite length with (N) and (N*) in which 0 is the meet of all dual atoms $[0, b]$. It follows that $[0, b]$ is a modular and atomistic sublattice (cf. Birkhoff [1], Chapter VII). In particular, b is a join of finitely many atoms.

Next we show that if, in a complete lattice with (N) and (N*), an element b of finite height satisfies the condition $b \wedge 1_+ = 0$, then b has a complement. As a preparation, we prove

LEMMA 4. *In a complete lattice L with* (N) *let m be a dual atom satisfying* (C*) *and b an element such that $b \nleq m$. If $b \wedge m$ has a complement c, then $c \wedge m$ is a complement of b.*

Proof. The main steps are visualized in Figure 1. From $b \nleq m$ we obtain

$$b \vee m = 1 \succ m \tag{2}$$

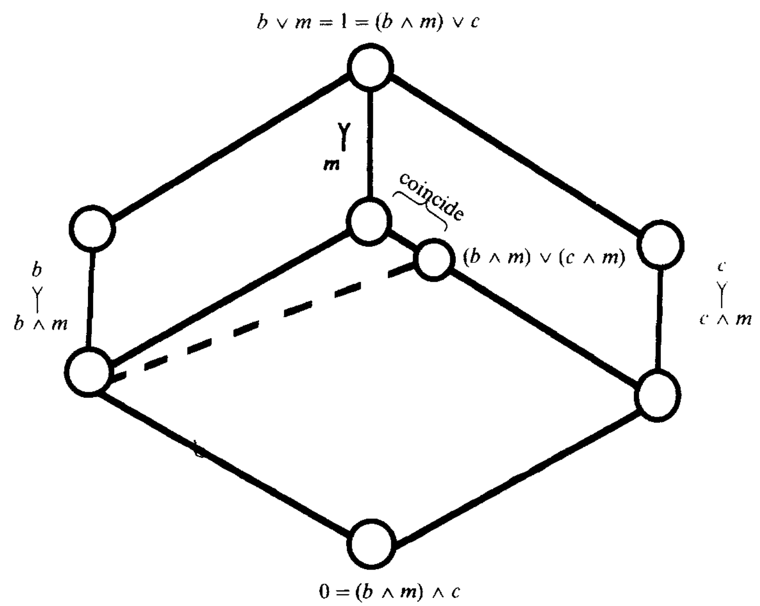

Figure 1

which implies by (C*) (which was assumed to hold for m) that $b \wedge m \prec b$. Since c is a complement of $b \wedge m$ we have

$$(b \wedge m) \vee c = 1 \tag{3}$$

and

$$(b \wedge m) \wedge c = 0. \tag{4}$$

It follows that $c \nleqq m$ since otherwise $c \vee (b \wedge m) \leqq m \prec 1$ contradicting our assumption that c is a complement of $b \wedge m$. Thus $c \vee m = 1 \succ m$ which yields that

$$c \wedge m \prec c \tag{5}$$

since m satisfies (C*). Next we observe that

$$c \wedge m < (c \wedge m) \vee (b \wedge m) \leqq m. \tag{6}$$

We show that here the right inequality is in fact an equality. To see this, note first that (5) and (6) yield $c \wedge [(c \wedge m) \vee (b \wedge m)] = c \wedge m \prec c$. Hence we obtain by (N) and by (3) that $(c \wedge m) \vee (b \wedge m) \prec c \vee [(c \wedge m) \vee (b \wedge m)] = c \vee (b \wedge m) = 1$. Since $(c \wedge m) \vee (b \wedge m) \leqq m \prec 1$ we conclude that

$$(c \wedge m) \vee (b \wedge m) = m. \tag{7}$$

Using (7) it is now easy to see that $c \wedge m$ is a complement of b. Namely, from (2) and (7) it follows that $1 = b \vee m = b \vee [(c \wedge m) \vee (b \wedge m)] = [b \vee (b \wedge m)] \vee (c \wedge m) = b \vee (c \wedge m)$. On the other hand, we get from (4) that $b \wedge (c \wedge m) = (b \wedge m) \wedge c = 0$. This proves the lemma.

For complete modular lattices the assertion of the preceding lemma was proved in Stenström [4]. The nonmodular (but semimodular) lattice of Figure 1 indicates that in fact we need only (N) and a weaker requirement than (N*). Of course, our following applications concern lattices satisfying both (N) and (N*) which means, in particular, that (C*) holds for all dual atoms. But even then we do not have to assume modularity which shows that our approach comprises a more general situation than Stenström [4].

COROLLARY 5. *Let L be a complete lattice with neighborhood condition* (N) *and its dual* (N*), *and let $b \in L$ be an element of finite height. If $b \wedge 1_+ = 0$, then b (is a join of finitely many atoms and) has a complement in L.*

Proof. Proposition 3 implies that b is a join of finitely many atoms. If the other statement were false, one could find an element b of minimal height having no complement. From $b \wedge 1_+ = 0$ we obtain the existence of a dual atom m such that $b \nleq m$. By Lemma 4, $b \wedge m$ cannot possess a complement, a contradiction.

The preceding corollary applies, in particular, to finite-modular AC-lattices and to modular lattices (the latter case was shown in Stenström [4]). We proceed now to prove a converse to Corollary 5.

LEMMA 6. *Let L be a complete lattice with neighborhood condition* (N) *and let b be a join of finitely many atoms. Assume moreover that $(b, x)M$ and $(x, b)M^*$ hold for all $x \in L$. If b has a complement, then $b \wedge 1_+ = 0$.*

Proof. Let b' be a complement of b. It is sufficient to show that b' is a meet of dual atoms. To see this, we first observe that $[0, b]$ is a modular sublattice of finite length whose greatest element is a join of atoms. Thus $[0, b]$ is complemented and hence relatively complemented. It follows that 0 can be represented as the meet of (all) dual atoms in the interval $[0, b]$. Next we observe that, by assumption, we have $(b, b')M$ and $(b', b)M^*$. This implies by Proposition 1 that the interval $[b', b \vee b'] = [b', 1]$ is isomorphic to the interval $[0, b]$ with the canonical mappings establishing an isomorphism. We conclude that the complement b' of b must be the meet of (all) dual atoms of the interval $[b', 1]$. Thus we obtain $1_+ \leqq b'$. In view of $b \wedge b' = 0$ this implies $b \wedge 1_+ = 0$ which was to be shown.

In particular, Lemma 6 applies to finite-modular AC-lattices as well as atomic modular lattices. The latter statement is proved in Stenström [4]. We note that the assumption "b is a join of finitely many atoms" cannot be replaced in the preceding lemma by the weaker requirement "b is of finite height" as the lattice of Figure 2 shows.

From Corollary 5 and Lemma 6 we obtain

THEOREM 7. *Let L be a complete finite-modular AC-lattice and let b be a finite element of L. Then b has a complement if and only if $b \wedge 1_+ = 0$.*

The preceding theorem generalizes Maeda–Maeda [3], Theorem 27.10, where it was shown that in a DAC-lattice (i.e. an AC-lattice whose dual is also an AC-lattice) each finite element has a complement. This is an easy consequence of

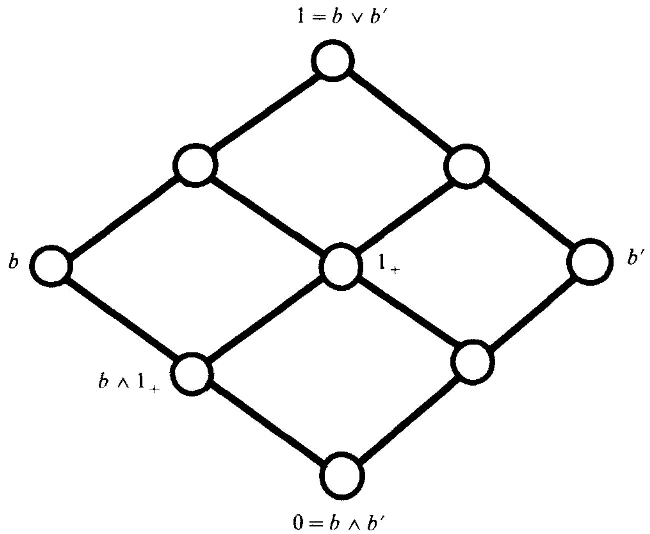

Figure 2

Theorem 7 since a DAC-lattice is dually atomistic and therefore has the property $1_+ = 0$.

As a by-product we get

THEOREM 8. *Let L be a complete atomic lattice with* (N) *and* (N*). *Each atom of L has a complement if and only if each finite join of atoms has a complement.*

Proof. Assume that each atom has a complement. Observing that a complement of an atom must be a dual atom, it follows that $1_+ = 0$. Hence $b \wedge 1_+ = 0$ holds for all b which are joins of finitely many atoms. By Corollary 5, each such b has a complement. The converse is trivial.

Again, the preceding result applies, in particular, to the finite-modular and the modular case. As already remarked, in a DAC-lattice each finite element has a complement. It seems therefore natural to ask whether a complete finite-modular AC-lattice in which each finite element has a complement is already a DAC-lattice. The following example shows that this is not the case:

EXAMPLE. Let Λ be the lattice of all subspaces of an infinite dimensional Hilbert space H. Let L be the set formed by removing from Λ all non-closed subspaces of finite codimension, that is, all subspaces M such that M has a finite complement $M^\perp$ and $M < (M^\perp)^\perp$. By Maeda–Maeda [3], Theorem 15.15, L (when partially ordered by set inclusion) is a finite-modular AC-lattice in which every atom and hence every finite element (cf. Theorem 8) have a complement. It is well-known that H has a nonclosed subspace M having the property that both M and any complement of M are infinite dimensional, whence $M \in L$. Letting N denote the closure of M, we have $M \subseteq N$ with $M^\perp = N^\perp$. Every dual atom of L that lies above N also lies above M. Since the dual atoms of L are all closed subspaces, the converse is also true. It follows that L is not a DAC-lattice.

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