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The maximal ring of quotient f-ring

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Abstract. In this article it is shown that the maximal quotient ring QA of a commutative semiprime f-ring A can be obtained by the formation of the orthocompletion of A, followed by that of the classical quotient ring; for archimedean f-rings the order of these can be inverted. It is shown that if $C = C(X, Z)$, where X is a zero-dimensional Hausdorff space, then the integral closure is the Dedekind–McNeille completion of C. The paper closes with a number of observations and examples.

1. Introduction

In this article all rings will be commutative and be endowed with an identity. We will rely upon Banaschewski's [Ba] construction of the maximal quotient ring for semiprime ring – rings in which the intersection of the prime ideals is trivial – and chiefly for this reason will restrict our attention to semiprime f-rings.

It is not our intention to introduce the notation of the maximal quotient ring as an f-ring extension. This has already been done, and in greater generality by F. W. Anderson in [An], an article which is interesting and not sufficiently well known, it appears.

An f-ring A is a lattice-ordered ring in which $a \wedge b = 0$ implies that $a \wedge bc = 0$, for all $c \geq 0$. In the context of ZFC this requirement is equivalent to the condition that A be embeddable as a subdirect product of totally ordered rings. For basic information on lattice-ordered groups and f-rings, the reader is encouraged to consult [BKW] and [AF]. In an f-ring every minimal prime l-ideal is a (ring) ideal. If A is semiprime then every minimal prime ideal is an l-ideal, and therefore a minimal prime l-ideal. In particular, if A is a semiprime f-ring then $ab = 0$ precisely when $|a| \wedge |b| = 0$. Thus, an element is a weak order unit exactly when it is regular in the ring. Also, 'polar' and 'annihilator' signify the same thing.

To recall, if $X \subseteq A$, then $X^{\perp}$ denotes the *polar* of X; that is to say, $X^{\perp} = \{a \in A : |a| \wedge |x| = 0, \text{ for all } x \in X\}$.

Recall the following useful condition, due to Mel Henriksen: suppose that A is an f-ring; A is said to satisfy the *bounded inversion* property if $a > 1$ implies that it

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is a multiplicative unit. The following convenient characterization first appeared in [HIJ].

LEMMA 1.0. *Suppose that A is a semiprime f-ring. Then A satisfies the bounded inversion property if and only if every maximal ideal of A is an l-ideal.*

We denote by Max(A) the topological space of maximal ideals under the hull-kernel topology. This topology has as its base the sets

$$m(a) = \{M \in \mathrm{Max}(A) : a \notin M\}.$$

Min(A) stands for the space of minimal prime ideals (which if the f-ring is semiprime, is also the space of minimal prime l-ideals.) Min(A) is also endowed with the hull-kernel topology, in which it is a (Hausdorff) zero-dimensional space; that is, one having a base of clopen sets.

A consequence of the lemma quoted above is that for a semiprime f-ring A with bounded inversion Max(A) is a Hausdorff space; it is compact regardless. We recall – see [CM1] and [HJ] for different versions – that, for a semiprime f-ring A, Min(A) is compact if and only if A is *complemented*: that is, for each $a \in A$ there is a $b \in A$ so that $ab = 0$ and $a + b$ is regular.

If A is any ring, we denote by qA the classical ring of quotients. If A is an f-ring then qA has a natural f-ring structure extending that of A, so that qA is a semiprime f-ring with bounded inversion, and A is an f-subring. Let us sketch how this is done: first any fraction a/b can be written with $b > 0$, because $a/b = ab/b^2$. Then define $a/b \vee 0 = (a \vee 0)/b$; this is well-defined, and endows qA with the lattice-order we want. It has the bounded inversion because $a/b > 1$ implies that $a > b$, and since b is regular, so is a, proving that the fraction a/b is a multiplicative unit of qA.

Recall that the prime ideals of qA are in one-to-one correspondence with the prime ideals of A which miss all the regular elements; see [AMc]. Therefore Max(qA) consists of the extensions of the prime ideals of A which are maximal with respect to excluding all regular elements. The maximal ideals of qA are l-ideals, since qA has bounded inversion. It can also be shown directly that the ideals of A which are maximal with respect to excluding all the regular elements are l-ideals.

For the remainder of this article we assume that, unless the contrary is stipulated, every f-ring is semiprime.

From the way the order on A was extended to qA, the respective spaces of minimal prime ideals are homeomorphic. This can easily be seen by recalling the notion of rigid containment in a lattice-ordered group.

Suppose that G is an l-subgroup of the lattice-ordered group H. We say that G is *rigid* in H if for each $h \in H$ there exists a $g \in G$ such that $h^{\perp\perp} = a^{\perp\perp}$. (The symbol $\perp$, refers to polars in the larger of the two lattice-ordered groups.) Now, returning to the f-ring A, note that $(a/b)^{\perp\perp} = a^{\perp\perp}$, which shows that A is rigid in qA. Recalling from [CM2] that, if G is rigid in H, then Min(G) is homeomorphic to Min(H), and via the contraction map $P \to P \cap G$, we conclude:

PROPOSITION 1.1. *For any semiprime* f-ring A, $\mathrm{Min}(A) = \mathrm{Min}(qA)$, *via the contraction map.*

The following proposition helps clarify the force of certain related conditions on qA. First, let us recall some definitions. A is said to be *von Neumann* (*regular*) if for each $a \in A$ there is an $x \in A$ such that $a^2x = a$; it is well known that this is equivalent to the condition that every principal ideal of A be generated by an idempotent.

A is projectable if for each $a \in A$, $A = a^\perp + a^{\perp\perp}$. If A is von Neumann and $a \in A$, then there is an idempotent e such that $Aa = Ae$. Then e and $1 - e$ are disjoint idempotents and $a^{\perp\perp} = Aa$, while $a^\perp = A(1 - e)$, which shows that A is projectable.

PROPOSITION 1.2. *Suppose that* $A = qA$ *is a semiprime* f-ring. *The following are equivalent*:

(1) *A is von Neumann.*
(2) *A is projectable.*
(3) *A is complemented.*

Proof. We have already seen that (1) implies (2). That (2) implies (3) is trivial (and well known). Finally, if A is complemented and $a \in A$, then suppose $b \in A$ satisfies $ab = 0$ and $a + b$ regular; since $A = qA$, $(a + b)c = 1$, for a suitably chosen $c \in A$. It is easy to show that ac is idempotent and generates Aa. □

2. The maximal quotient ring

We sketch here the construction of QA, the maximal quotient ring of a semiprime ring A, given by Banaschewski in [Ba], which we shall refer to in this development as the *Gel'fand–Banaschewski representation*. We shall recall as well, because it is very closely related to Banaschewski's construction, Bleier's development of the orthocompletion of a (representable) lattice-ordered group in [Bl].

First, let us give a definition of the general notion of quotient ring. For a comprehensive account the reader may refer to Lambek's book [L], where the subject is treated for the non-commutative case as well. Once more, let us mention [An], for a discussion of the subject in the context of f-rings.

If A is a subring of B, we call B a *quotient ring* of A if for each pair b_1 and b_2 in B, with $b_2 \neq 0$, there is an $a \in A$ such that ab_1 and ab_2 belong to A and $ab_2 \neq 0$. It is in this sense then, that every semiprime ring A has a unique *maximal quotient ring* QA.

Since A is semiprime, one can regard it as a subcartesian product of fields: $A \subseteq \pi\{F_x : x \in X\}$. We consider the Zariski topology on X, for which the basic open sets are the

$$S(f) = \{x \in x : f(x) \neq 0\}, \qquad \text{for all } f \in A.$$

Let D stand for the family of dense open subsets of X, and for each $W \in D$ set A_W defined as follows: $f \in A_W$ if and only if $f \in \pi\{F_x : x \in W\}$, and for each $y \in W$ thee is a neighborhood U of y, $U \subseteq W$, and $a, b \in A$ such that $f(z) = a(z)/b(z)$, all $z \in U$. Then A_W is a subring of the direct product $\pi\{F_x : x \in W\}$, and if $W \supseteq V$, both dense open subsets of X, there is an obvious induced homomorphism $\pi_{W,V} : A_W \to A_V$ by restriction. The rings A_W together with the connecting maps $\pi_{W,V}$ form a direct system, and Banaschewski showed in [Ba] that the direct limit is QA.

It is well worthwhile to observe that part of Banaschewski's achievement, is to demonstrate that the construction of *QA does not depend* on the (particular) Gel'fand–Banaschewski representation.

In particular, and since A is semiprime, one can use the space Min(A) to obtain the above representation; (the Zariski topology coincides with the hull-kernel topology.) A is a subdirect product of the residues A/P, and hence a subcartesian product of their fields of fractions.

For our purposes, and beginning with a (semiprime) f-ring A, let us see what the ordering adds to the picture. Each minimal prime ideal is an l-ideal, and so each A/P is a totally ordered integral domain, whence $q(A/P)$ is a totally ordered field.

The direct product $\pi\{q(A/P) : P \in \text{Min}(A)\}$ is a semiprime f-ring with coordinatewise operations, and so is the analogous direct product over any dense open subset of Min(A). It is easy to verify that the A_W's constructed above are f-subrings of their respective direct products, and that each $\pi_{W,V}$ (with $W \supseteq V$) is an l-homomorphism. All of this establishes most of:

THEOREM 2.1. *For each semiprime f-ring A, the maximal ring of quotients QA admits a lattice-ordering making it a semiprime f-ring and containing A as an f-subring. $f \in QA$ is positive iff for each dense open subset W of* Min(A), *and each $P \in W$ there exists an open set U and positive elements a and b in A, so that*

$P \in U \subseteq W$ and for each $Q \in U$ $f(Q) = (a+Q)/(b+Q)$. This is the unique lattice-ordering on QA making it an f-ring and A an f-subring of it.

Finally, QA is a von Neumann ring.

(Note: It should be obvious from the definition of the ordering that it is unique. The fact that QA is von Neumann is quite easy to prove, and may, in any event, be found in [L].)

Now let's review Bleier's construction of the orthocompletion, translating it to the topological language introduced here: for each dense open subset W of Min(A) define B_W to be the subring of A_W of all $f \in \pi\{q(A/P) : P \in W\}$ such that for each $P \in W$ there exists an open set U and $a \in A$ such that $P \in U \subseteq W$, and for each $Q \in U$, $f(Q) = a + Q$.

B_W is, indeed, an f-subring of A_W, and if V is a dense open subset of W, then the restriction $\pi_{W,V'}$ of $\pi_{W,V}$ to B_W is an l-homomorphism. The direct limit of the B_W, according to the account in [Bl], is the *orthocompletion* oA of A. It should be noted that this construction is independent of the dense open set W.

To review, for a representable lattice-ordered group G (one which is representable as a subdirect product of totally ordered groups) the *orthocompletion* is a lattice-ordered group H, for which the following conditions are satisfied:

(orth-1) H is *laterally complete*; that is to say, every set of pairwise disjoint elements has a supremum;
(orth-2) H is projectable, and
(orth-3) no proper lattice subgroup of H containing G is both laterally complete and projectable.

By a routine argument with direct limits, there is a natural inclusion of oA in QA, which is as an f-subring. Thus:

PROPOSITION 2.2. *The maximal ring of quotients QA of A contains the orthocompletion of A.*

We mention here the excellent work in [FGL], where the authors consider various rings of quotients, but, in particular, the maximal ring of quotients of $C(X)$, the ring of all continuous real-valued functions defined on the *Tychonoff* space X (and recall that a space is Tychonoff if it has a base of cozero-sets). Denoting $Q(C(X)) \equiv Q(X)$, it is shown in [FGL] that $Q(X)$ is the ring of continuous functions defined on dense open subsets of X, subject to identification on the intersection of common domains; formally,

$$Q(X) = \varinjlim\{C(U) : U \text{ a dense open subset of } X\}.$$

By contrast,

$$q(X) \equiv q(C(X)) = \varinjlim\{C(V) : V \text{ is a dense cozero-set in } X\}.$$

But more than what is asserted in Proposition 2.2 is true: the same argument that Bleier uses to prove that oA is orthocomplete applies to show that QA is orthocomplete. In fact:

PROPOSITION 2.3. *$A = QA$ if an only if A is orthocomplete and every regular element of A is invertible; that is, $A = qA$.*

Proof (of the sufficiency). Suppose $A = qA$ and A is orthocomplete. It suffices to prove that for each dense open subset W of $\mathrm{Min}(A)$, $A_W = B_W$. If $f \in A_W$ and $P \in W$, there is a neighborhood U of P and elements a and b in A, so that $U \subseteq W$ and for each $Q \in U$ $f(Q) = (a + Q)/(b + Q)$.

Since A is projectable, we may without loss of generality take b to be regular: $b + Q = b + (1 - e) + Q$, where $e^2 = e$ and $b^{\perp\perp} = e^{\perp\perp}$. Since $A = qA$, b is invertible, and, in particular, invertible mod Q. Thus $f(Q) = ab^{-1} + Q$, for each $Q \in U$, proving that $f \in B_W$. □

By a slight modification of the above proof, we obtain the first main theorem of this section.

THEOREM 2.4. *If A is any projectable f-ring, then $QA = o(qA)$.*

Proof. The only part that requires checking is that qA is projectable. If a/b and c/d are in qA we may, without loss of generality, suppose that $b = d > 0$. Write $a = a_1 + a_2$, so that $a_1 \in c^{\perp\perp}$ and $a_2 \in c^{\perp}$. It follows that $a/b = (a_1/b) + (a_2)/b$, and $a_1/b \in (c/b)^{\perp\perp}$ while $a_2/b \in (c/b)^{\perp}$. □

It turns out that the order of the operators q and o can be reversed without peril; this is the subject of the second main theorem of the section. (It is Theorem 2.4 which is difficult to extend.)

THEOREM 2.5. *If A is orthocomplete then so is qA, so that $qA = QA$. In general, $QA = q(oA)$.*

Proof. Consider $\{a_\sigma/b_\sigma : b_\sigma > 0, a_\sigma \geq 0, \sigma \in \Sigma\}$, pairwise disjoint in qA. In view of the projectability of A, we can assume that for each $\sigma \in \Sigma$, the projection of b_σ on $a_\sigma{}^\perp$ is an idempotent (because if $b_\sigma = x_\sigma + y_\sigma$, so that $x_\sigma \in a_\sigma{}^{\perp\perp}$ and $y_\sigma \in a_\sigma{}^\perp$, and e_σ is the idempotent generator of $a_\sigma{}^{\perp\perp}$, then $a_\sigma/b_\sigma = a_\sigma/(x_\sigma + (1 - e_\sigma))$).

The x_σ, projections of b_σ on $a_\sigma{}^{\perp\perp}$, are then also pairwise disjoint. Now form $a = \bigvee a_\sigma$, $x = \bigvee x_\sigma$ and $e = \bigvee e_\sigma$; then $e^{\perp\perp} = a^{\perp\perp}$, and it is easy to verify that $b = x + (1 - e)$ is regular. We leave it to the reader to verify that $a/b = \bigvee a_\sigma/b_\sigma$. This proves that q_A is laterally complete; this means that it is complemented, and, by 1.2, projectable. Then on account of Theorem 2.4, $qA = QA$.

In general, and since $q(oA)$ is orthocomplete, Proposition 2.3 implies that $q(oA) = Q(q(oA)) = Q(oA) \supseteq QA$. On the other hand, since $oA \subseteq QA$, it follows that $q(oA) \subseteq QA$, proving that $QA = q(oA)$. □

In attempting to decide whether the order of the application of the operators q and o matters in general, it will be useful to recall the notion of a locally inversion closed ring, as introduced in [Ba] by Banaschewski.

A (semiprime) ring A is *locally inversion closed* if for each $a \in A$, $a \neq 0$, and $P \in \mathrm{Min}(A)$ so that $a \notin P$ there is a neighborhood U of P, $U \subseteq U_a = \{Q \in \mathrm{Min}(A) : a \notin Q\}$, and an element $b \in A$, such that $ab + Q = 1 + Q$, for all $Q \in U$. Banaschewski points out that if A is locally inversion closed then $QA = oA$. It is shown in [AC], Theorem 3.4, $C(X)$ is locally inversion closed, for any Tychonoff space X.

Consider the following example: Let A be the subalgebra of $C(R)$ consisting of the functions which are piecewise polynomials (with finitely many pieces). The functions of oA are the ones which are defined on a dense open subset of the reals, and are local polynomials. This excludes the function $1/x$, which is in qA and hence in QA. So $oA \neq QA$. Observe that A is not locally inversion closed, nor does it satisfy the bounded inversion property.

However, A is complemented, which insures that qA is locally inversion closed, as we are about to see, and that in turn will be enough to make $QA = Q(qA) = o(qA)$.

It is easy to verify that A is complemented if and only if qA is projectable. Owing to Proposition 1.2, this occurs precisely when qA is von Neumann regular. Now, it should be clear that a semiprime f-ring which is von Neumann regular is locally inversion closed. This explains why the preceding example has a locally inversion closed classical quotient ring.

The next theorem, a converse to Banaschewski's observation concerning local inversion closure, is helpful.

THEOREM 2.6. *$QA = oA$ if and only if A is locally inversion closed. In general, $QA = o(qA)$ precisely when qA is locally inversion closed.*

Proof. Only the necessity has to be proved. Let us suppose that $a \in A$, and without loss of generality, suppose that $a > 0$. Complete $\{a\}$ to a maximal pairwise disjoint set $\{a(i) : i \in I\}$, with $a = a(j)$. Let $W = \bigcup\{u_{a(i)} : i \in I\}$; this is a dense open subset of Min(A). Let $b = \bigvee a(i)$ in oA, and observe that b is regular and therefore a multiplicative unit; say $bc = 1$, with $c \in QA = oA$.

Now for each $P \in \text{Min}(A)$ with $a \notin P$, there is an open set $U \subseteq W$ such that $P \in U$ and a $d \in A$ such that $c(Q) = d + Q$, for each $Q \in U$. Also, $b = a \vee x$, where x is the supremum of the $a(i)$, $i \neq j$, and $x \in Q$, for each $Q \in u_a$, since $a \wedge x = 0$; this means that $b(Q) = a + Q$ for all such Q. Therefore, if $Q \in U \cap u_a$, we have that

$$ad + Q = (a + Q)(d + Q) = b(Q)c(Q) = 1 + Q,$$

proving that $ad \equiv 1 \bmod Q$, for all Q in $U \cap u_a$, and hence that A is locally inversion closed.

As to the second assertion, if qA is locally inversion closed, then $QA = Q(qA) = o(qA)$, by Banaschewski's remark. Conversely, if $Q(qA) = QA = o(qA)$, then by the first part of this proof, qA is locally inversion closed. □

Now it is left to decide whether qA is locally inversion closed, for every semiprime f-ring.

As a first step, Theorem 2.6 guarantees that closure under local inversion is independent of the Gel'fand–Banaschewski representation; that is, independent of which family of prime ideals with trivial intersection one employs. As we have already observed, Bleier's construction of oA is also independent of which family of primes (with trivial intersection) one uses in the representation.

Next, let us recall a definition: if A is an f-ring then $A(1)$ denotes the *bounded subring* of A; meaning, the convex f-subring generated by 1. Let us further agree to call A *bounded* if $a = A(1)$. Recall that in any f-ring A, $a = (a \wedge 1)(a \vee 1)$, for each $a \in A$, Then observe that if A satisfies the bounded inversion property, then by this identity, $A \subseteq q(A(1))$, which means that A and its bounded subring have the same classical quotient ring.

We now settle, for archimedean, semiprime f-rings, the matter of the application of the operators o and q, in two stages.

THEOREM 2.7. *Suppose that A is an archimedean f-ring with the bounded inversion property. Then A is locally inversion closed, and therefore $QA = oA$.*

Proof. The final assertion follows from Theorem 2.6. By the remarks preceding this theorem, it suffices to assume that A is bounded. Then the Jacobson radical of A is trivial; in addition, for each maximal ideal M of A, A/M is an archimedean totally-ordered field. Armed with this, we will use the Gel'fand–Banaschewski representation on its maximal ideal space, to show that A is locally inversion closed.

Suppose that $0 < a \in A$ and $M \in \mathrm{Max}(A)$ with $a \notin M$. Then, since A/M is archimedean, there exist natural numbers m and n such that $a > m/n \bmod M$. Now let $c = a \vee m/n$; since A satisfies the bounded inversion property, c is a multiplicative unit. Next, observe that $(c - a)(a - m/n)^+ = 0$; dividing by c this reads as follows: $(1 - a/c)(a - m/n)^+ = 0$. Furthermore, with $d = (a - m/n)^+$, $a/c \equiv 1 \bmod N$, for each $N \in u_d$, proving that A is locally inversion closed. □

Since qA already has the bounded inversion property and it is archimedean whenever A is, we obtain the following corollary.

COROLLARY 2.7.1. *For any archimedean, semiprime f-ring A, qA is locally inversion closed; hence $QA = o(qA) = q(oA)$.*

In the non-archimedean case the matter of $o(qA)$ vs. $q(oA)$ remains unsettled; it is probable that the operators cannot be reversed, in general, but we know no counter-examples.

To conclude this section we comment on the injectivity of the maximal ring of quotients. It is well-known that QA is self-injective; mention of this occurs already in [FGL]. It can be proved directly, employing the so-called "Injective test" Lemma. However, from [L], p. 95, 4.3, we obtain that QA is, in fact, injective as a module over A. Since it is clear that QA is an A-essential extension of A, we can conclude the following:

PROPOSITION 2.8. *For any semiprime ring A, QA is the A-injective hull of A.*

3. Integral closure in QA

The broad goal of this section is to demonstrate that, under very reasonable hypotheses, the integral closure of a semiprime f-ring is determined by its additive and lattice-theoretic structure.

Let us begin by recalling some basic definitions; suppose that A is a subring of B (not necessarily semiprime). We say that $x \in B$ is *integral over A* if there is a monic polynomial $f(T) \in A[T]$ such that $f(x) = 0$. The collection of all elements of B which are integral over A form a subring of B, called the *integral closure* of A (in B).

Now, let us return to semiprime f-rings.

If A is such a ring, then let sA denote the *saturated hull* of A in QA; that is to say, sA is the l-subgroup generated by all the components of elements of A lying in

QA. (If $a = x + y$, and $x \wedge y = 0$, we say that x and y are components of a.) Notice that if x and y are components of a and b respectively, then xy is a component of ab; therefore, sA is, in fact, an f-subring of QA. Recall that QA is a von Neumann ring; since it is also orthocomplete, the boolean algebra $E(QA)$ of idempotents is complete. (Note: the Stone dual of $E(QA)$, namely Max(QA) is extremally disconnected.) Thus, sA is rigid in QA. Moreover, each component of an element of A is of the form ae, where $a \in A$ and $e \in E(QA)$. This means that sA is the A-submodule of QA generated by the idempotents of QA.

Note also that sA is, in fact, contained in oA.

Let us summarize the preceding paragraph as follows:

PROPOSITION 3.1. *For any semiprime f-ring A, sA, the saturated hull of A in QA is the A-submodule generated by $E(QA)$, the complete boolean algebra of idempotents of QA. Moreover, sA is rigid in QA, and a subring of oA.*

Let us denote by iA the integral closure of A in QA. If $0 < a \in A$ and b is a component of a in QA, then b satisfies the polynomial $T^2 - aT$, which implies that $sA \subseteq iA$. Furthermore, if $x \in iA$, then x^+ satisfies the polynomial $T^2 - xT$, whence x^+ is integral over iA and, therefore, over A. This shows that iA is an f-subring of QA.

Recall now the notion of a *Specker ring*, one which is generated as an abelian group by its idempotents (for references, see [C]). If A is a Specker ring then each $a \in A$ can be expressed uniquely as a sum $m_1e_1 + m_2e_2 + \cdots + m_ke_k$, where the $m_i \in Z$ and the e_i are pairwise disjoint idempotents. A Specker ring A is archimedean, projectable, semiprime and $qA = A$; moreover, QA is its lateral completion. In particular – see [BKW] – each idempotent in QA is a disjoint supremum of idempotents in A.

Topologically speaking, Specker rings can be viewed in the following way. Suppose that X is any zero-dimensional Hausdorff space (recall; a *zero-dimensional* space is one possessing a base of clopen sets). Let $C(X, \mathbf{Z})$ denote the f-ring of all integer-valued continuous functions on X. Then an f-ring A is a Specker ring if and only if $A = C(X, \mathbf{Z})$, for some compact zero-dimensional space X.

For any zero-dimensional space, $C = C(X, \mathbf{Z})$ is a projectable, semiprime f-ring.

Before stating the next result, recall the definition of the *Dedekind–McNeille completion*: if G is any archimedean lattice-ordered group, let dG denote this completion; dG is (conditionally) complete, and each positive element of dG is the supremum of elements of G. Also, recall from Lemma 2.3 in [CMc], that if an archimedean lattice-ordered group G is order-densely embedded in a complete l-group H, then dG is the convex hull of G in H.

PROPOSITION 3.2. *Suppose that A is a Specker ring. Then $sA = iA$ is the Dedekind-McNeille completion of A.*

This proposition will be a corollary of Theorem 3.3. However, we need a few preliminaries before stating it.

For each minimal prime ideal P of C, either C/P is the ring $\mathbf{Z}$ of integers, or else, by work of Norman Alling [Al], C/P is a non-standard model of the integers, which means that it satisfies all the first-order properties held by $\mathbf{Z}$, in the first-order theory of totally ordered rings. Being "integrally closed" is such a property (as opposed to being "integrally over", which is not). Thus, C/P is integrally closed in either event.

The fact that $\mathbf{Z}$ has no non-zero, proper convex ideals is also such a first-order property (as opposed to being archimedean, which is not). Thus, by Alling's work, if C/P is a non-standard model of $\mathbf{Z}$ it has no non-zero, proper convex ideals. This means that in C every minimal prime ideal is also maximal among l-ideals which are ring ideals.

If X is a Tychonoff space then EX stands for its *absolute*; this is the projective cover in the category of Tychonoff spaces; see [PW] for details. $C(EX, \mathbf{Z})$ is complete. Moreover, $oC = D(EX, \mathbf{Z})$; this is the algebra of all continuous functions on X with values in $\mathbf{Z} \cup \{\pm\infty\}$, which are finite on a dense set.

As we have seen, sC lies in oC; it is actually easy to see that $sC \subseteq C(EX, \mathbf{Z})$. Thus, by Lemma 2.3 in [CMc], C^c, the convex hull of C in $C(EX, \mathbf{Z})$, is the Dedekind–McNeille completion of C.

THEOREM 3.3. *For each zero-dimensional Haudorff space X, $iC(X, \mathbf{Z}) = sC(X, \mathbf{Z}) = dC(X, \mathbf{Z})$.*

Proof. Let $C = C(X, \mathbf{Z})$; to show that $iC = sC$, it obviously suffices to show that if $x \in QC$ is integral over C, then it lies in the saturated hull of C. In fact, let us make the following observation. If $Q \in \operatorname{Min}(sC)$ then, clearly, $Q \nsubseteq C$, and $Q \cap C$ is a prime ideal of C which is also an l-ideal. By the remark immediately preceding this theorem, it follows that $Q \cap C$ is a minimal prime ideal of C.

On the other hand, if b is any component of $a \in C$, then either b or $a - b$ belongs to Q, which proves that $sC = Q + C$. Now, by the comments preceding the theorem, $C/(Q \cap C)$ is integrally closed; furthermore, $C/(Q \cap C) = (C + Q)/Q = sC/Q$. This shows that for each $Q \in \operatorname{Min}(sC)$, sC/Q is a totally ordered integral domain, which is integrally closed and has no non-zero, proper convex ideals.

Suppose then that x is integral over C, and let $P \in \operatorname{Min}(QC)$; put $P' = P \cap sC$. Since x is integral over C, $x + P$ is integral over $C/(P \cap C) = sC/P' = sC/(sC \cap P) = (sC + P)/P$. Moreover, as each element of QC is locally a fraction of elements from C, we see that $q(QC/P) = q(sC/P')$, and so $x \in sC + P$. This means that $x + P = a(P) + P$, for some $a(P) \in sC$. We may, in fact, write $x = a(P) + y(P)$, with $y(P) \in P$, and so that $|a(P)| \wedge |y(P)| = 0$, since sC is projectable.

Consider now the basic open set u_x in the $\mathrm{Min}(QC)$; it is compact-open, and is covered by the sets $\{u_x(P) : P \in U_x\}$. Thus, we have finitely many indices $i = 1, 2, \ldots, k$, such that (with $a(i) = a(P_i)$) the $u_{a(i)}$ cover u_x. But this means that $x = a(1) \vee \cdots \vee a(k)$, and we conclude that $x \in sC$. This shows that $iC \subseteq sC$; since we had already observed the reverse containment, it follows that $iC = sC$.

Let us now prove that sC is, in fact, the Dedekind–McNeille competition of C. We already know that $dC = C^c$. We will show that $C^c = sC$; it is evident that $sC \subseteq C^c$.

Following Pierce [P], we observe that each function $f \in C$ can formally be expressed as $f = \sum_{n \in W} ne_n$, where e_n is the characteristic function of $\{x \in X : f(x) = n\}$.

Suppose then that $0 \leq y = \sum_{n \in N} nf_n \leq \sum_{n \in N} ne_n$. The e_n are pairwise disjoint idempotents in C, whereas the f_n are pairwise disjoint idempotents in $C(EX, \mathbf{Z})$. Then, for each $n \in N$, we have an increasing sequence of natural numbers $n(1), n(2), \ldots,$ such that, for a suitable choice of a component $f_k^{\sim}$ of $f_{n(k)}$, we have $y(n) \equiv \sum_k n(k) f_k^{\sim} \leq ne_n$. Thus the $n(k)$ are bounded, and consequently $y(n) = \sum_i s_{i(n)} g_{i(n)}$, where each $s_{i(n)}$ is a natural number, and each $g_{i(n)}$ is an idempotent formed by summing from among the $f_k^{\sim}$; thus, the $g_{i(n)}$ remain pairwise disjoint. In addition, for each $n \in N$, all but finitely many of the $s_{i(n)} = 0$.

Now, form (in $C(EX, \mathbf{Z})$) $g = \bigvee_n v_i g_{i(n)}$; this makes sense since $C(EX, \mathbf{Z})$ is Dedekind complete. Also form $a = \sum_{n \in N} (\sum_i s_{i(n)}) e_n$, which is an element of C. Finally, note that $y = \sum_{n \in N} y(n)$, and

$$ag = \sum_{n \in W} \left(\sum_i s_{i(n)} \right) (v_j g_{j(n)}) \in sC,$$

and that y is a component of ag, whence it follows that $y \in sC$.

This proves that $C^c = sC$, and we have finished the proof of the theorem. □

We should point out that, in general, the Dedekind–McNeille completion of $C(X, Z)$ is not $C(EX, Z)$. It is precisely when X is a so-called *weak c.b. space*; these are the spaces given by the following condition: whenever E_n is a decreasing sequence of regular closed sets for which $\bigcap E_n = \varnothing$, there is a decreasing sequence of zero-sets $Z_n \supseteq E_n$, such that $\bigcap Z_n = \varnothing$. (See [PW], Section 8.5, for a discussion of the $C(X)$ version of this.) The weak c.b. spaces include all the pseudo-compact ones.

4. Concluding remarks

Recall (Proposition 1.2) that if $A = qA$, then A is complemented precisely when it is projectable. In the proof of Theorem 2.4 it was verified that if A is projectable then so is qA; the same implication holds for complemented f-rings; indeed, since A is rigid in qA, it follows that A is complemented if and only if qA is. Now:

4.1. *An example of a semiprime f-ring A with the bounded inversion property which is not projectable, yet such that qA is.*

Let $A = C(X)$, where X is any metric space. It is not hard to see that A is complemented, whence qA is projectable. However, as long as X is not an F-space, then A is not projectable, because X is not basically disconnected. (Note: for any Tychonoff space, $C(X)$ is projectable precisely when X is basically disconnected; this is well-known, and, in any event, easy to derive.)

Recall that a lattice-ordered group G is said to have *stranded primes* is every prime l-ideal of G exceeds a unique minimal prime l-ideal. It is well known that if G is projectable then it has stranded primes, although the converse is false (see [AF] or [BKW]).

Then this same example shows that:

4.2. *If qA is projectable A need not have stranded primes.*

Since X was stipulated not to be an F-space, it follows from Theorem 14.25 in [GJ] that A does not have stranded primes; not even for its prime (ring) ideals.

4.3. *Even if $A = qA$, A may have stranded primes and fail to be projectable.*

Let $A = C(\beta N \backslash N)$; since $\beta N \backslash N$ is an F-space, A has stranded primes. However, $\beta N / N$ is not basically disconnected – see [GJ] – so that A is not projectable. Note that $A = qA$, as the space in question has no proper dense cozero sets.

4.4. *qA need not have stranded primes.*

Let D be an uncountable set with the discrete topology, and $A = C(\alpha D)$, where αX stands for the one-point compactification of X. Since D is uncountable, αD has no proper dense cozero sets, which means that $A = qA$. However, the root system of primes of A is not stranded: $\mathrm{Max}(A) = \alpha D$, but beneath the maximal ideal at infinity there are all the non-isolated points of βD.

We mention the following item without proof, although we do illustrate the converse.

4.5. *If A is a semiprime f-ring with bounded inversion, and A has stranded primes then so does qA, but the converse is false.*

Converse: let $A = C(\alpha N)$, where αN is the one-point compactification of N; $qA = QA = C(N)$, the lateral completion of A. A does not have stranded primes.

Before concluding we ought to mention a very recent contribution of Wickstead (see [Wi]), which fits very nicely in the context of this paper. We shall only apply it to semiprime f-rings, although it is valid in a more general – and non-order-theoretic – context.

Wickstead calls a semiprime, commutative ring A *fully regular* if for each subset D of mutually annihilating elements, and each partition $D = D_1 \cup D_2$ of D, there is an element $s \in A$ such that $d^2 s = d$, for each $d \in D_1$, and $ds = 0$, for each $d \in D_2$. The main theorem in [Wi] shows that a semiprime ring A is self-injective – that is, injective over itself – precisely when it is fully regular.

By way of summary, and for semiprime f-rings, let us tie in his result with the maximal ring of quotients and the material in the first section of this article.

THEOREM 4.6. *For a semiprime f-ring A the following are equivalent.*

(1) $A = QA$.

(2) *A is self-injective.*

(3) *A is fully regular.*

(4) *A is orthocomplete and every regular element of A is invertible.*

(5) *A is laterally complete and von Neumann regular.*

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