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Congruence distributive quasivarieties whose finitely subdirectly irreducible members form a universal class

JANUSZ CZELAKOWSKI AND WIESŁAW DZIOBIAK

To the memory of Basia Czelakowska

Abstract. By a congruence distributive quasivariety we mean any quasivariety **K** of algebras having the property that the lattices of those congruences of members of **K** which determine quotient algebras belonging to **K** are distributive. This paper is an attempt to study congruence distributive quasivarieties with the additional property that their classes of relatively finitely subdirectly irreducible members are axiomatized by sets of universal sentences. We deal with the problem of characterizing such quasivarieties and the problem of their finite axiomatizability.

For a quasivariety **K** of algebras and its member A denote by $\mathrm{Con}_{\mathbf{K}} A$ the set of all congruence relations Θ on A such that the quotient algebra A/Θ belongs to **K**. As the set $\mathrm{Con}_{\mathbf{K}} A$ is closed under arbitrary intersections, it forms a complete lattice. Hence for any pair of elements of A, say, a and b, we can form a least congruence relation on A, denoted $\Theta_{\mathbf{K}}(a, b)$, that contains (a, b) and belongs to $\mathrm{Con}_{\mathbf{K}} A$. By Lemma 2.2 of section 2, every such congruence relation is a compact element of $\mathrm{Con}_{\mathbf{K}} A$. So, as each element Θ of $\mathrm{Con}_{\mathbf{K}} A$ coincides with the lattice join (formed in $\mathrm{Con}_{\mathbf{K}} A$) of all $\Theta_{\mathbf{K}}(a, b)$ where $a \equiv b(\Theta)$, the lattice $\mathrm{Con}_{\mathbf{K}} A$ is algebraic.

We say that a quasivariety **K** of algebras is congruence distributive if, for every member A of **K**, the lattice $\mathrm{Con}_{\mathbf{K}} A$ is distributive. Due to Baker [1], [2], Jónsson [19], [20] and others we know that congruence distributive varieties possess very strong and nice properties. Our purpose for a long time before writing this paper was an intention of extending at least some of them into quasivarieties. This was partially realized by the first author in [7] (see also [8] and [9]) but in the area of propositional logics having a well behaved connective called disjunction. Another point was achieved when the second author had

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observed in [12] that within a congruence distributive quasivariety every finitely subdirectly irreducible algebra is finitely subdirectly irreducible in the absolute sense. The crucial point for writing this paper was overcome when we had realized due to the paper of Blok and Pigozzi [5] that the notion of disjunction connective from metalogical investigations can be put with a success into investigations of quasivarieties.

A quasivariety $\mathbf{K}$ of algebras is said to have equationally definable principal (congruence) meets (EDPM for short) if there exists a finite system $\Delta = \langle p_i(x, y, z, w), q_i(x, y, z, w)\rangle$, $i = 0, \ldots, n-1$, of pairs of 4-ary terms such that, for all $A \in \mathbf{K}$ and $a, b, c, d \in A$,

$$\Theta_{\mathbf{K}}(a, b) \cap \Theta_{\mathbf{K}}(c, d) = \bigvee_{i<n} \Theta_{\mathbf{K}}(p_i^A(a, b, c, d), q_i^A(a, b, c, d))$$

where the join $\bigvee$ is formed in $\mathrm{Con}_{\mathbf{K}} A$. In this event Δ is called a system of principal (congruence) intersection terms (in four variables). This notion for varieties was introduced and discussed in Blok and Pigozzi [5]. With the restriction $|\Delta| = 1$, it was previously considered in Baker [1] under the name of principal intersection property.

The aim of the paper is twofold. First, we deal with the problem of characterization of quasivarieties with EDPM. We prove two characterization theorems (Theorem 2.3 and Theorem 4.2). The first theorem reflects a result proved previously for varieties in Blok and Pigozzi [5] while the second one is in the style of Jónsson's Theorem characterizing congruence distributive varieties. From the first theorem it follows that for a given finite set M of finite similar algebras of finite type the problem whether or not the quasivariety generated by M has EDPM is decidable. The first characterization theorem also says that the quasivarieties with EDPM are exactly those which are congruence distributive and whose finitely subdirectly irreducible members form a universal class. Thus our paper can be viewed as an attempt to study congruence distributive quasivarieties with the additional property that their classes of finitely subdirectly irreducible members are axiomatized by sets of universal sentences. Many examples of these quasivarieties arise from the process of algebraization of deductive systems for propositional logics having a disjunction connective. For instance, quasivarieties generated by any set of finitely subdirectly irreducible Heyting or interior algebras are among them.

The second aim is subordinated to the axiomatization problem of quasivarieties with EDPM. We show (Theorem 3.4) that every quasivariety with EDPM and of finite type is finitely based provided that the class of its finitely subdirectly irreducible members is strictly elementary. The idea used in the proof of this result refers to Czelakowski [7, Theorem 3.2] where a related result but for propositional logics with disjunction connective was established (compare also Wojtylak [26, Theorem 3.4]). Among consequences of this result there is one saying that if **K** is a variety of finite type with EDPM and **M** is a subclass of $\mathbf{K}_{\mathrm{FSI}}$ then the least quasivariety $Q(\mathbf{M})$ containing **M** is finitely based iff $ISP_U(\mathbf{M})$ is a strictly elementary class. A similar result but under the additional assumption that **M** is a finite set of finite algebras was stated in Blok and Pigozzi [5, Corollary 2.7]. Moreover, having **K** and **M** as above we extend (see Corollary 3.8) Baker's idea of *UDE*-sentences to form a basis for $Q(\mathbf{M})$ provided that there is known a set of universal basic sentences axiomatizing $ISP_U(\mathbf{M})$. Applying arguments from Belkin [4] we also provide an answer to a question suggested in Blok and Pigozzi [5] by presenting a finite subdirectly irreducible lattice whose quasivariety is not finitely based.

1. Distributivity

Given a quasivariety **K** of algebras and $A \in \mathbf{K}$. An element Θ of $\mathrm{Con}_{\mathbf{K}} A$ is said to be finitely meet irreducible in $\mathrm{Con}_{\mathbf{K}} A$ if, for all $\Theta_0, \Theta_1 \in \mathrm{Con}_{\mathbf{K}} A$, $\Theta = \Theta_0 \wedge \Theta_1$ implies $\Theta = \Theta_0$ or $\Theta = \Theta_1$. If the identity relation on A, denoted ω_A, is finitely meet irreducible in $\mathrm{Con}_{\mathbf{K}} A$ then the algebra A is said to be finitely subdirectly irreducible in **K**. By $\mathbf{K}_{\mathrm{FSI}}$ we denote the class of all finitely subdirectly irreducible members of **K** and we assume the convention that trivial algebras belong to $\mathbf{K}_{\mathrm{FSI}}$. The lattice of all congruence relations on A will be denoted by $\mathrm{Con}\, A$. The lattice meet of $\mathrm{Con}_{\mathbf{K}} A$ as well as of $\mathrm{Con}\, \mathbf{A}$ coincides with the set-theoretical intersection and it will be denoted by $\wedge$ while to denote the lattice join of $\mathrm{Con}_{\mathbf{K}} A$ we shall use the symbols $+_{\mathbf{K}}$. For other notions occurring in this paper we refer to [17] and [22].

LEMMA 1.1 (cf. [10, Theorem 1] and [12, Lemma 2.1]). *For a quasivariety* **K** *of algebras the following conditions are equivalent*:

(i) **K** *is congruence distributive.*

(ii) *For every* $A \in \mathbf{K}$ *and* $\Theta_0, \Theta_1, \psi \in \mathrm{Con}_{\mathbf{K}} A$: *if* ψ *is finitely meet irreducible in* $\mathrm{Con}_{\mathbf{K}} A$ *and* $\Theta_0 \wedge \Theta_1 \leqslant \psi$ *then* $\Theta_0 \leqslant \psi$ *or* $\Theta_1 \leqslant \psi$.

(iii) *For every* $A \in \mathbf{K}$, $\Theta_0, \Theta_1 \in \mathrm{Con}\, A$ *and* $\psi \in \mathrm{Con}_{\mathbf{K}} A$: *if* ψ *is finitely meet irreducible in* $\mathrm{Con}_{\mathbf{K}} A$ *and* $\Theta_1 \wedge \Theta_1 \leqslant \psi$ *then* $\Theta_0 \leqslant \psi$ *or* $\Theta_1 \leqslant \psi$.

(iv) *For every* $A \in \mathbf{K}$, $a, b, c, d \in A$ *and* $\psi \in \mathrm{Con}_{\mathbf{K}} A$: *if* ψ *is finitely meet irreducible in* $\mathrm{Con}_{\mathbf{K}} A$ *and* $\Theta_{\mathbf{K}}(a, b) \wedge \Theta_{\mathbf{K}}(c, d) \leqslant \psi$ *then* $(a, b) \in \psi$ *or* $(c, d) \in \psi$.

Proof. The parts (i) implies (ii) and (iii) implies (iv) are obvious.

(ii) $\Rightarrow$ (iii): First, we show that (ii) implies

*) For every $A \in \mathbf{K}$, $\Theta_0, \Theta_1 \in \mathrm{Con}\, A$ and $\psi \in \mathrm{Con}_{\mathbf{K}} A$: if ψ is finitely meet irreducible in $\mathrm{Con}_{\mathbf{K}} A$, $\{\Theta_0, \Theta_1\} \cap \mathrm{Con}_{\mathbf{K}} A \neq \varnothing$ and $\Theta_0 \wedge \Theta_1 \leq \psi$ then $\Theta_0 \leq \psi$ or $\Theta_1 \leq \psi$.

Suppose that on a certain algebra $A \in \mathbf{K}$ we have congruence relations Θ_0, Θ_1 and ψ satisfying:

1. ψ is a finitely meet irreducible element of $\mathrm{Con}_{\mathbf{K}} A$.
2. $\{\Theta_0, \Theta_1\} \cap \mathrm{Con}_{\mathbf{K}} A \neq \varnothing$.
3. $\Theta_0 \wedge \Theta_1 \leq \psi$.
4. Neither $\Theta_0 \leq \psi$ nor $\Theta_1 \leq \psi$.

Assume $\Theta_0 \in \mathrm{Con}_{\mathbf{K}} A$; in the case $\Theta_1 \in \mathrm{Con}_{\mathbf{K}} A$ we proceed similarly. Let $B = \{(a, b) \in A \times A : (a, b) \in \Theta_1\}$. As B is a subalgebra of $A \times A$ whose projections $\pi_1, \pi_2 : B \to A$ fulfill $\pi_1(B) = \pi_2(B) = A$, we have $\pi_1^{-1}(\Theta_1) = \pi_2^{-1}(\Theta_1)$. From this it follows

5. $\pi_1^{-1}(\Theta_0) \wedge \pi_2^{-1}(\omega_A) \leq \pi_1^{-1}(\psi)$.

Indeed, $\pi_1^{-1}(\Theta_0) \wedge \pi_2^{-1}(\omega_A) \leq \pi_1^{-1}(\Theta_0) \wedge \pi_2^{-1}(\Theta_1) = \pi_1^{-1}(\Theta_0) \wedge \pi_1^{-1}(\Theta_1) = \pi_1^{-1}(\Theta_0 \wedge \Theta_1) \leq$ (by (3)) $\pi_1^{-1}(\psi)$. As $\Theta_0 \nleq \psi$ and $\pi_1(B) = A$, we have

6. $\pi_1^{-1}(\Theta_0) \nleq \pi_1^{-1}(\psi)$.

Take $(a, b) \in \Theta_1 \backslash \psi$; by (4) such a pair exists. As $(a, b), (b, b) \in B$, we obtain $(a, b) \equiv (b, b)(\pi_2^{-1}(\omega_A))$ and $(a, b) \not\equiv (b, b)(\pi_1^{-1}(\psi))$. Thus we also have

7. $\pi_2^{-1}(\omega_A) \nleq \pi_1^{-1}(\psi)$.

Since B, $A/\Theta_0 \in \mathbf{K}$ and $B/\pi_1^{-1}(\Theta_0) \cong A/\Theta_0$, we get $\pi_1^{-1}(\Theta_0) \in \mathrm{Con}_{\mathbf{K}} B$. Similarly, we have $\pi_2^{-1}(\omega_A), \pi_1^{-1}(\psi) \in \mathrm{Con}_{\mathbf{K}} B$. Moreover, $\pi_1^{-1}(\psi)$ is finitely meet irreducible in $\mathrm{Con}_{\mathbf{K}} B$ because ψ is finitely meet irreducible in $\mathrm{Con}_{\mathbf{K}} A$. Thus, by (5), (6) and (7), the condition (ii) is not satisfying, showing that (ii) implies *). Applying the same arguments it is an easy matter to show that *) yields (iii). Thus (ii) implies (iii).

(iv) $\Rightarrow$ (i): Assume (iv) and let $A \in \mathbf{K}$. As every element of $\mathrm{Con}_{\mathbf{K}} A$ is the meet of finitely meet irreducibles over it, we get

$$\Theta_{\mathbf{K}}(a, b) \wedge \bigvee_{(c,d) \in H} \Theta_{\mathbf{K}}(c, d) = \bigvee_{(c,d) \in H} \Theta_{\mathbf{K}}(a, b) \wedge \Theta_{\mathbf{K}}(c, d)$$

where $(a, b) \in A$, H is a finite subset of $A \times A$ and $\bigvee$ is the join formed in $\mathrm{Con}_{\mathbf{K}} A$. Therefore, as $\mathrm{Con}_{\mathbf{K}} A$ is algebraic, we obtain $\Theta_0 \wedge (\Theta_1 +_{\mathbf{K}} \Theta_2) = \Theta_0 \wedge \Theta_1 +_{\mathbf{K}} \Theta_0 \wedge \Theta_2$ for all $\Theta_0, \Theta_1, \Theta_2$ of $\mathrm{Con}_{\mathbf{K}} A$. Thus $\mathrm{Con}_{\mathbf{K}} A$ is distributive, showing that (iv) implies (i).

As a consequence of the above lemma we have

PROPOSITION 1.2. *Let* $\mathbf{K}$ *be a quasivariety with* EDPM *and system* $\Delta = \langle p_i(x, y, z, w), q_i(x, y, z, w)\rangle$, $i < n$, *of principal intersection terms. Then the following conditions are fulfilled*:

(i) $\mathbf{K}$ *is congruence distributive.*
(ii) *For every* $A \in \mathbf{K}$, $A \in \mathbf{K}_{\mathrm{FSI}}$ *iff* $A \vDash \forall xyzw[(\&_{i<n} p_i(x, y, z, w) = q_i(x, y, z, w)) \rightarrow (x = y \text{ or } z = w)]$.

Proof. (i) Let $a, b, c, d \in A \in \mathbf{K}$, and let Θ be a finitely meet irreducible element of $\mathrm{Con}_{\mathbf{K}} A$ such that $\Theta_{\mathbf{K}}(a, b) \wedge \Theta_{\mathbf{K}}(c, d) \leq \Theta$. Then $(p_i^A(a, b, c, d), q_i^A(a, b, c, d)) \in \Theta$ for all $i < n$. Hence $\Theta_{\mathbf{K}}([a]\Theta, [b]\Theta) \wedge \Theta_{\mathbf{K}}([c]\Theta, [d]\Theta) = \omega_{A/\Theta}$. Therefore $(a, b) \in \Theta$ or $(c, d) \in \Theta$ because $A/\Theta \in \mathbf{K}_{\mathrm{FSI}}$. Thus, by Lemma 1.1, $\mathbf{K}$ is congruence distributive.

(ii) Directly from the assumptions.

The next consequence of Lemma 1.1 provides a necessary condition for a quasivariety of algebras to be congruence distributive.

PROPOSITION 1.3 (see [12]). *If a quasivariety* $\mathbf{K}$ *of algebras is congruence distributive then* $\mathbf{K}_{\mathrm{FSI}} = V(\mathbf{K})_{\mathrm{FSI}} \cap \mathbf{K}$.

Proof. Let $A \in \mathbf{K}_{\mathrm{FSI}}$ and $\Theta_0 \wedge \Theta_1 = \omega_A$ where $\Theta_0, \Theta_1 \in \mathrm{Con}\, \mathbf{A}$. As ω_A is finitely meet irreducible in $\mathrm{Con}_{\mathbf{K}} A$, by Lemma 1.1, we get $\Theta_0 = \omega_A$ or $\Theta_1 = \omega_A$. Thus $A \in V(\mathbf{K})_{\mathrm{FSI}}$.

From the proposition it easily follows

COROLLARY 1.4. *Let* $\mathbf{K}$ *be a congruence distributive quasivariety of algebras. Then for a quasivariety* $\mathbf{L}$ *contained in* $\mathbf{K}$ *the following conditions are equivalent*:

(i) $\mathbf{L}$ *is congruence distributive.*
(ii) $\mathbf{L}_{\mathrm{FSI}} \subseteq \mathbf{K}_{\mathrm{FSI}}$.

Proof. (i) $\Rightarrow$ (ii). By Proposition 1.3.

(ii) $\Rightarrow$ (i): Let $\Theta_0 \wedge \Theta_1 \leq \psi$ where $A \in \mathbf{L}$, $\Theta_0, \Theta_1 \in \mathrm{Con}\, A$ and $\psi \in \mathrm{Con}_{\mathbf{L}} A$. As $\mathbf{L} \subseteq \mathbf{K}$, $\psi \in \mathrm{Con}_{\mathbf{K}} A$. Hence, by (ii) and Lemma 1.1, $\Theta_0 \leq \psi$ or $\Theta_1 \leq \psi$ whenever ψ is finitely meet irreducible in $\mathrm{Con}_{\mathbf{L}} A$. Thus, by Lemma 1.1, $\mathbf{L}$ is congruence distributive.

For a class $\mathbf{M}$ of similar algebras by $Q(\mathbf{M})$ we denote the least quasivariety containing $\mathbf{M}$. By a result of Grätzer and Lakser [18], $Q(\mathbf{M}) = \mathrm{ISPP}_U(\mathbf{M})$.

The origin of the following lemma stems from Jónsson [19].

LEMMA 1.5. *Let* $\mathbf{M}$ *be a class of similar algebras. Then every nontrivial member of* $Q(M)_{\mathrm{FSI}}$ *belongs to* $\mathrm{ISP}_U(\mathbf{M})$.

Proof. Let $A \in Q(\mathbf{M})_{\mathrm{FSI}}$ and $|A| > 1$. Then A is a subalgebra of $\Pi(C_i : i \in I)$, where $C_i \in \mathrm{ISP}_U(\mathbf{M})$, $i \in I$. For $S \subseteq I$ denote by Θ_S a congruence relation on $\Pi(C_i : i \in I)$ defined as follows: $a \equiv b(\Theta_S)$ iff $\{i \in I : a(i) = b(i)\} \supseteq S$. Obviously, $\Theta_S \upharpoonright A \in \mathrm{Con}_{Q(\mathbf{M})} A$ for all S. Let $\mathscr{F}$ denote the set of all filters F on I satisfying $\Theta_S \upharpoonright A = \omega_A$ for all $S \in F$. As $\{I\} \in \mathscr{F}$, $\mathscr{F}$ is non-empty. As the poset $(\mathscr{F}, \subseteq)$ is inductive, it has maximal elements. Choose one of them and denote it by U. We claim that U is an ultrafilter over I. By $|A| > 1$, $U \neq 2^I$. Suppose now that for a certain $S \subseteq I$, neither S nor $I \backslash S$ belong to U. Then for some $G \in U$, $\Theta_{G \cap S} \upharpoonright A \neq \omega_A$ and $\Theta_{G \cap (I \backslash S)} \upharpoonright A \neq \omega_A$. But $\Theta_{G \cap S} \upharpoonright A \wedge \Theta_{G \cap (I \backslash S)} \upharpoonright A = \Theta_G \upharpoonright A$. Therefore, by $\Theta_G \upharpoonright A = \omega_A$ and $A \in Q(\mathbf{M})_{\mathrm{FSI}}$, we have $\Theta_{G \cap S} \upharpoonright A = \omega_A$ or $\Theta_{G \cap (I \backslash S)} \upharpoonright A = \omega_A$, a contradiction. Thus for every $S \subseteq I$, $S \in U$ or $I \backslash S \in U$, showing the claim. Since $\bigvee (\Theta_S : S \in U) \upharpoonright A = \omega_A$, then by the claim, A is embeddable into the ultraproduct of C_i's modulo U. Hence $A \in \mathrm{ISP}_U(\mathbf{M})$ because $\mathrm{ISP}_U(\mathbf{M})$ is closed under I, S and P_U.

2. Characterization theorem

In this section we prove our first characterization theorem. It is in spirit of a corresponding result proved previously for varieties in Blok and Pigozzi [5, Theorem 1.5].

LEMMA 2.1. *For a quasivariety* $\mathbf{K}$ *of algebras,* $A, B \in \mathbf{K}$, $a, b \in A$, $\Theta_0, \Theta_1 \in \mathrm{Con}_{\mathbf{K}} A$ *and a surjective homomorphism* $h : A \to B$ *it holds*:

(i) $h(\Theta_{\mathbf{K}}(a, b) +_{\mathbf{K}} \mathrm{Ker}\, h) = \Theta_{\mathbf{K}}(h(a), h(b))$.

(ii) *If* $\mathrm{Con}_{\mathbf{K}} A$ *is distributive then* $h(\Theta_0 \wedge \Theta_1 +_{\mathbf{K}} \mathrm{Ker}\, h) = h(\Theta_0 +_{\mathbf{K}} \mathrm{Ker}\, h) \wedge h(\Theta_1 +_{\mathbf{K}} \mathrm{Ker}\, h)$.

Proof. (i) As $A/h^{-1}(\psi) \cong B/\psi$ and $h^{-1}h(\Theta) = \Theta$ for all $\Theta \geq \mathrm{Ker}\, h$ and all ψ, $h(\Theta_{\mathbf{K}}(a, b) +_{\mathbf{K}} \mathrm{Ker}\, h) \in \mathrm{Con}_{\mathbf{K}} B$. Hence $h(\Theta_{\mathbf{K}}(a, b) +_{\mathbf{K}} \mathrm{Ker}\, h) \geq \Theta_{\mathbf{K}}(h(a), h(b))$. Obviously, $\Theta_{\mathbf{K}}(a, b) +_{\mathbf{K}} \mathrm{Ker}\, h \leq h^{-1}(\Theta_{\mathbf{K}}(h(a), h(b)))$. Therefore, by $hh^{-1}(\psi) = \psi$, $h(\Theta_{\mathbf{K}}(a, b) +_{\mathbf{K}} \mathrm{Ker}\, h) \leq \Theta_{\mathbf{K}}(h(a), h(b))$.

(ii) It suffices to show that $h([\Theta_0 +_{\mathbf{K}} \mathrm{Ker}\, h] \cap [\Theta_1 +_{\mathbf{K}} \mathrm{Ker}\, h]) = h(\Theta_0 +_K \mathrm{Ker}\, h) \wedge h(\Theta_1 +_{\mathbf{K}} \mathrm{Ker}\, h)$. Let $(a, b) \in h(\Theta_0 +_{\mathbf{K}} \mathrm{Ker}\, h) \wedge h(\Theta_1 +_{\mathbf{K}} \mathrm{Ker}\, h)$. Then $(a, b) = (h(x), h(y))$ for some $(x, y) \in \Theta_0 +_{\mathbf{K}} \mathrm{Ker}\, h$. Hence $(x, y) \in h^{-1}h(\Theta_1 +_{\mathbf{K}} \mathrm{Ker}\, h)$, and, therefore, $(x, y) \in [(\Theta_0 +_{\mathbf{K}} \mathrm{Ker}\, h] \wedge [\Theta_1 +_{\mathbf{K}} \mathrm{Ker}\, h]$. Thus $(a, b) \in h([\Theta_0 +_{\mathbf{K}} \mathrm{Ker}\, h] \wedge [\Theta_1 +_{\mathbf{K}} \mathrm{Ker}\, h])$. The converse inclusion is immediate.

For a given type τ of algebras, let $Q(\tau)$ denote the set consisting of the following quasiidentities:

(1) $x = x$

(2) $x = y \rightarrow y = x$

(3) $x = y \;\&\; y = z \rightarrow x = z$

(4) $\&_{i<n} x_i = y_i \rightarrow f(x_0, \ldots, x_{n-1}) = f(y_0, \ldots, y_{n-1})$

where f is an n-ary function symbol from the first-order language of τ. Notice that in the above quasiidentities and others occurring in this paper we omit universal quantifiers.

In the following lemma Id (**K**) denotes the set of all identities valid in **K** and $\Theta_{\mathbf{K}}(H)$ denotes the least element of $\mathrm{Con}_{\mathbf{K}} A$ containing H.

LEMMA 2.2. *Let* **K** *be a quasivariety of algebras of type* τ, Γ *be a set of quasiidentities which are not identities, and let* $\mathbf{K} = \mathrm{Mod}\,\mathrm{Id}\,(\mathbf{K}) \cup \Gamma$. *Then for every* $A \in \mathbf{K}$, $H \subseteq A \times A$ *and* $a, b \in A$ *it holds*: $a \equiv b(\Theta_{\mathbf{K}}(H))$ *iff there exists a finite sequence* $(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})$ *of elements of* $A \times A$ *such that* $(a_{n-1}, b_{n-1}) = (a, b)$ *and, for every* $i < n$, $(a_i, b_i) \in H$ *or there exist a sequence* $j_0, \ldots, j_{k-1} < i$, *a quasiidentity* $r_0(\bar{x}) = s_0(\bar{x}) \;\&\; \cdots \;\&\; r_{k-1}(\bar{x}) = s_{k-1}(\bar{x}) \rightarrow r(\bar{x}) = s(\bar{x}) \in \Gamma \cup Q(\tau)$, *where* $\bar{x} = x_0, \ldots, x_{p-1}$, *and a sequence* $\bar{a} = a_0, \ldots, a_{p-1}$ *of elements of* A *such that* $\{(r^A_m(\bar{a}), s^A_m(\bar{a})) : m < k\} = \{(a_{j_m}, b_{j_m}) : m < k\}$ *and* $(r^A(\bar{a}), s^A(\bar{a})) = (a_i, b_i)$.

Proof. Define $\Theta \subseteq A \times A$ by $(c, d) \in \Theta$ iff the right hand side of the above pattern is satisfied by (c, d). Obviously, $H \subseteq \Theta$. Applying quasiidentities from the set $Q(\tau)$ we obtain that Θ is a congruence relation on A. Since $A/\Theta \models \mathrm{Id}\,(\mathbf{K}) \cup \Gamma$, $\Theta \in \mathrm{Con}_{\mathbf{K}} A$. Hence $\Theta_{\mathbf{K}}(H) \leqslant \Theta$. On the other hand, as $\Theta_{\mathbf{K}}(H) \in \mathrm{Con}_{\mathbf{K}} A$, $\Theta \leqslant \Theta_{\mathbf{K}}(H)$. Thus $\Theta_{\mathbf{K}}(H) = \Theta$, showing the lemma.

A lemma much more reflecting than the above one the spirit of Mal'cev lemma characterizing principal congruences the reader may find in Gorbunov [16, Lemma preceding Theorem 3].

By Proposition 1.2, we know that every quasivariety with EDPM is congruence distributive and its finitely subdirectly irreducible members form a universal class. It turns out that the converse implication is also true. This follows from the following theorem. The theorem also explains the connection between EDPM and congruence distributivity.

THEOREM 2.3. *For a quasivariety* $\mathbf{K}$ *of algebras the following conditions are equivalent*:

(i) $\mathbf{K}$ *has* EDPM.
(ii) *For every member* A *of* $\mathbf{K}$ *the lattice* $\mathrm{Con}_{\mathbf{K}} A$ *is distributive and the set of its compact elements forms a sublattice.*
(iii) *The lattice* $\mathrm{Con}_{\mathbf{K}} A_{\mathbf{K}}(\omega)$ *is distributive and the set of its compact elements forms a sublattice.*
(iv) $\mathbf{K}$ *is congruence distributive and* $\mathbf{K}_{\mathrm{FSI}}$ *forms a universal class.*
(v) *The lattice* $\mathrm{Con}_{\mathbf{K}} F_{\mathbf{K}}(\omega)$ *is distributive and* $\mathbf{K}_{\mathrm{FSI}}$ *forms a universal class.*
(vi) *The lattice* $\mathrm{Con}_{\mathbf{K}} F_{\mathbf{K}}(4)$ *is distributive and the set of its compact elements forms a sublattice and, moreover,* $S(\mathbf{K}_{\mathrm{FSI}}) \subseteq \mathbf{K}_{\mathrm{FSI}}$.
(vii) *There exists a finite system* $\langle p_i(x, y, z, w), q_i(x, y, z, w)\rangle$, $i<n$, *of pairs of* 4*-ary terms such that*

$$\mathbf{K}_{\mathrm{FSI}} \models \forall xyzw\left[\left(\underset{i<n}{\&} p_i(x, y, z, w) = q_i(x, y, z, w)\right) \leftrightarrow (x = y \text{ or } z = w)\right].$$

(viii) *There exists a finite system* $\langle p_i(x, y, z, w), q_i(x, y, z, w)\rangle$, $i<n$, *of pairs of* 4*-ary terms such that for every* $A \in \mathbf{K}$ *and all* $a, b, c, d \in A$ *it holds*:

$$\Theta_{\mathbf{K}}(a, b) \wedge \Theta_{\mathbf{K}}(c, d) = \omega_A \text{ iff } A \models \underset{i<n}{\&} p_i(x, y, z, w) = q_i(x, y, z, w)[a, b, c, d].$$

For $\mathbf{K}$ being a variety the equivalence of (i), (ii), (iii), (iv) was proved in Blok and Pigozzi [5]. Another theorem characterizing quasivarieties with EDPM will be given in the last section; see also Corollary 2.6 and 2.7 below.

Proof (*of Theorem* 2.3). By Proposition 1.2(i), (i) implies (ii). The part (ii) implies (iii) is trivial. Having Lemmas 2.1 and 2.2 we can proceed as in the proof of Theorem 1.5 in Blok and Pigozzi [5] to get that (ii) yields (i). Thus the conditions (i), (ii) and (iii) are equivalent. We shall show that the conditions (i), (iv), (v), (vi), (vii) and (vii) are equivalent as well. By Proposition 1.2, (i) implies (iv). That (iv) implies (v) is obvious.

(v)$\Rightarrow$(vi): Assume (v). Let x, y, z, w be free generators of $F_{\mathbf{K}}(4)$ and let $\langle p_\alpha, q_\alpha\rangle$, $\alpha<\beta$, be a system of generators $\Theta_{\mathbf{K}}(x, y) \wedge \Theta_{\mathbf{K}}(z, w)$. We claim $\mathbf{K}_{\mathrm{FSI}} \models \forall xyzw[(\&_{\alpha<\beta} p_\alpha(x, y, z, w) = q_\alpha(x, y, z, w)) \rightarrow (x = y \text{ or } z = w)]$. Let $A \in \mathbf{K}_{\mathrm{FSI}}$ and let $h: F_{\mathbf{K}}(4) \rightarrow A$ be a homomorphism such that $h(p_\alpha(x, y, z, w)) = h(q_\alpha(x, y, z, w))$ for all $\alpha<\beta$. Then $(p_\alpha, q_\alpha) \in \mathrm{Ker}\, h$ for all $\alpha<\beta$, and hence $\Theta_{\mathbf{K}}(x, y) \wedge \Theta_{\mathbf{K}}(z, w) \leqslant \mathrm{Ker}\, h$. But $\mathrm{Ker}\, h$ is finitely meet irreducible in $\mathrm{Con}_{\mathbf{K}} F_{\mathbf{K}}(4)$ because $F_{\mathbf{K}}(4)/\mathrm{Ker}\, h$ is embeddable into A and $S(\mathbf{K}_{\mathrm{FSI}}) \subseteq \mathbf{K}_{\mathrm{FSI}}$. Therefore, by distributivity of $\mathrm{Con}_{\mathbf{K}} F_{\mathbf{K}}(4)$, we get $\Theta_{\mathbf{K}}(a, b) \leqslant \mathrm{Ker}\, h$ or $\Theta_{\mathbf{K}}(c, d) \leqslant \mathrm{Ker}\, h$, that is, $h(x) = h(y)$ or $h(z) = h(w)$, proving the claim. By the claim and the fact that $\mathbf{K}_{FSI}$ is closed under ultraproducts we conclude the existence of a finite subsystem of $\langle p_\alpha, q_\alpha \rangle$, $\alpha < \beta$, say, $\langle p_i, q_i \rangle$, $i < n$, such that $\mathbf{K}_{FSI} \models \forall xyzw[(\&_{i<n} p_i(x, y, z, w) = q_i(x, y, z, w)) \rightarrow (x = y \text{ or } z = w)]$. As $\mathbf{K}_{FSI} \models \forall xyzw[(x = y \text{ or } z = w) \rightarrow p_\alpha(x, y, z, w) = q_\alpha(x, y, z, w)]$ for all $\alpha < \beta$, we get

$$*)\ \mathbf{K}_{FSI} \models \forall xyzw[(\&_{i<n} p_i(x, y, z, w) = q_i(x, y, z, w)) \leftrightarrow (x = y \text{ or } z = w)].$$

Let $a, b, c, d \in F_\mathbf{K}(4)$. We show $\Theta_\mathbf{K}(a, b) \wedge \Theta_\mathbf{K}(c, d) = \bigvee_{i<n} \Theta_\mathbf{K}(p_i(a, b, c, d), q_i(a, b, c, d))$ which, by distributivity of $\mathrm{Con}_\mathbf{K} F_\mathbf{K}(4)$, would imply that the set of compact elements of $\mathrm{Con}_\mathbf{K} F_\mathbf{K}(4)$ forms a sublattice; the rest of (vi) is immediate. Let $\bigvee_{i<n} \Theta_\mathbf{K}(p_i(a, b, c, d), q_i(a, b, c, d)) \leqslant \psi$, where ψ is a finitely meet irreducible element of $\mathrm{Con}_\mathbf{K} F_\mathbf{K}(4)$. Then $p_i([a]\psi, [b]\psi, [c]\psi, [d]\psi) = q_i([a]\psi, [b]\psi, [c]\psi, [d]\psi)$ for all $i < n$. Hence, by $F_\mathbf{K}(4)/\psi \in \mathbf{K}_{FSI}$ and $*$), we get $\Theta_\mathbf{K}(a, b) \leqslant \psi$ or $\Theta_\mathbf{K}(c, d) \leqslant \psi$ which in turn yields $\Theta_\mathbf{K}(a, b) \wedge \Theta_\mathbf{K}(c, d) \leqslant \psi$. Thus $\Theta_\mathbf{K}(a, b) \wedge \Theta_\mathbf{K}(c, d) \leqslant \bigvee_{i<n} \Theta_\mathbf{K}(p_i(a, b, c, d), q_i(a, b, c, d))$. Now, let $(e, f) \notin \Theta_\mathbf{K}(a, b) \wedge \Theta_\mathbf{K}(c, d)$. Then $(e, f) \notin \Theta_\mathbf{K}(a, b)$ or $(e, f) \notin \Theta_\mathbf{K}(c, d)$. Let $(e, f) \notin \Theta_\mathbf{K}(a, b)$; when $(e, f) \notin \Theta_\mathbf{K}(c, d)$ we proceed similarly. Then there exists a finitely meet irreducible element ψ of $\mathrm{Con}_\mathbf{K} F_\mathbf{K}(4)$ with $(e, f) \notin \psi$ and $\Theta_\mathbf{K}(a, b) \leqslant \psi$. As $[a]\psi = [b]\psi$ in $F_\mathbf{K}(4)/\psi$ and $F_\mathbf{K}(4)/\psi \in \mathbf{K}_{FSI}$, by $*$) we obtain $p_i([a]\psi, [b]\psi, [c]\psi, [d]\psi) = q_i([a]\psi, [b]\psi, [c]\psi, [d]\psi)$ in $F_\mathbf{K}(4)/\psi$ for all $i < n$. Hence $(e, f) \notin \bigvee_{i<n} \Theta_\mathbf{K}(p_i(a, b, c, d), q_i(a, b, c, d))$ and thus $\Theta_\mathbf{K}(a, b) \wedge \Theta_\mathbf{K}(c, d) \geqslant \bigvee_{i<n} \Theta_\mathbf{K}(p_i(a, b, c, d), q_i(a, b, c, d))$, completing the proof that (v) implies (vi).

(vi) $\Rightarrow$ (vii): Assuming (vi) and proceeding as in the part (v) implies (vi) we easily find a finite system of pairs of 4-ary terms satisfying (vii). Thus (vi) implies (vii).

(vii) $\Rightarrow$ (i): Assume (vii) and next proceed as in the proof of (v) implies (vi) to get that for all $a, b, c, d \in A \in \mathbf{K}$: $\Theta_\mathbf{K}(a, b) \wedge \Theta_\mathbf{K}(c, d) = \bigvee_{i<n} \Theta_\mathbf{K}(p_i(a, b, c, d), q_i(a, b, c, d))$. Thus the conditions (i)–(vii) are equivalent. But it is an easy matter to show that (i) implies (viii) and (viii) implies (vii). Thus all conditions of our theorem are equivalent.

Looking inside the proof of the part (vii) implies (i) we have

COROLLARY 2.4. *If* $\mathbf{K}_{FSI} \models \forall xyzw[(\&_{i<n} p_i(x, y, z, w) = q_i(x, y, z, w)) \leftrightarrow (x = y \text{ or } z = w)]$ *then* $\langle p_i(x, y, z, w), q_i(x, y, z, w) \rangle$, $i < n$, *is a system of principal intersection terms for* $\mathbf{K}$.

By the above theorem we obtain

COROLLARY 2.5. *For a given finite set* M *of finite algebras of finite type the problem whether or not* $Q(M)$ *has* EDPM *is decidable.*

Proof. As $Q(M)$ is locally finite of finite type and, by Lemma 1.5, every nontrivial member of $Q(M)_{\mathrm{FSI}}$ belongs to $\mathrm{IS}(M)$, we can decide in a finite number of steps whether or not there exists a finite system $\langle p_i(x, y, z, w), q_i(x, y, z, w)\rangle$, $i<n$, of pairs of 4-ary terms with $Q(M)_{\mathrm{FSI}} \models \forall xyzw[(\&_{i<n} p_i(x, y, z, w) = q_i(x, y, z, w)) \leftrightarrow (x = y \text{ or } z = w)]$. Hence, by Theorem 2.3, the corollary follows.

Although every quasivariety **K** with EDPM is always congruence distributive, the least variety $V(\mathbf{K})$ containing it need not be congruence distributive. We shall show that it even may happen that $V(\mathbf{K})$ satisfies no nontrivial congruence lattice identity.

Let $A = (\{0, 1, 2\}, f, 0)$ be an algebra of type $\langle 4, 0\rangle$ with f defined as follows: $f(a, b, c, d) = 0$ when $a = b$ or $c = d$, and $f(a, b, c, d) = 1$ otherwise. Take $p(x, y, z, w) := f(x, y, z, w)$ and $q(x, y, z, w) := 0$. We have $A \models \forall xyzw[p(x, y, z, w) = q(x, y, z, w) \leftrightarrow (x = y \text{ or } z = w)]$. Hence, by Lemma 1.5 and Theorem 2.3, $Q(A)$ has EDPM. Now, consider the algebra $A/\Theta(0, 1)$. As $\Theta(0, 1) = \{(0, 1), (1, 0)\} \cup \omega_A$, $A/\Theta(0, 1)$ is a 2-element algebra every operation of which takes the same fixed value for all its arguments. Hence every equivalence relation on any direct power of $A/\Theta(0, 1)$ is a congruence relation. Thus each finite partition lattice is embeddable into congruence lattice of some finite direct power of $A/\Theta(0, 1)$. Therefore, by the well-known result of Lattice Theory, it follows that congruence lattices of members of $V(\mathbf{K})$ cannot satisfy any nontrivial lattice identity.

In the case $V(\mathbf{K})$ is congruence distributive we have

COROLLARY 2.6. *Let* **K** *be a quasivariety of algebras such that* $V(\mathbf{K})$ *is congruence distributive. Then the following conditions are equivalent*:

(i) **K** *has* EDPM.

(ii) $\mathbf{K} = Q(\mathbf{M})$ *for some class* **M** *such that* $\mathrm{ISP}_U(\mathbf{M}) \subseteq V(\mathbf{K})_{\mathrm{FSI}}$.

Proof. (i) $\Rightarrow$ (ii): By Theorem 2.3 and Proposition 1.3, as **M** we can take $\mathbf{K}_{\mathrm{FSI}}$.

(ii) $\Rightarrow$ (i): By Lemma 1.5, $\mathbf{K}_{\mathrm{FSI}} \subseteq \mathrm{ISP}_U(\mathbf{M}^+)$ where $\mathbf{M}^+$ is obtained from **M** by adjoining a trivial algebra. As $\mathrm{ISP}_U(\mathbf{M}) \cap \mathbf{K} \subseteq V(\mathbf{K})_{\mathrm{FSI}}$, we have $\mathbf{K}_{\mathrm{FSI}} = \mathrm{ISP}_U(\mathbf{M}^+)$. Hence $\mathbf{K}_{\mathrm{FSI}}$ is a universal class. Moreover, by Corollary 1.4, **K** is congruence distributive. Thus, by Theorem 2.3, **K** has EDPM.

When **K** is contained in a variety with EDPM we can prove more (see also Proposition 4.4):

COROLLARY 2.7. *Let* **L** *be a variety with* EDPM, *and let* $\Delta = \langle p_i(x, y, z, w), q_i(x, y, z, w)\rangle$, $i<n$, *be a system of principal intersection terms for*

L. *Then for a quasivariety* **K** *contained in* **L** *the following conditions are equivalent*:

(i) **K** *has* EDPM *and* Δ *is a system of principal intersection terms for* **K**.
(ii) **K** *has* EDPM.
(iii) **K** *is congruence distributive.*
(iv) $\mathbf{K} = Q(\mathbf{M})$ *for some* $\mathbf{M} \subseteq \mathbf{L}_{\mathrm{FSI}}$.

Proof. (i)$\Rightarrow$(ii): Trivial. (ii)$\Rightarrow$(iii): By Theorem 2.3. (iii)$\Rightarrow$(iv): Use Proposition 1.3. (iv)$\Rightarrow$(i): By Theorem 2.3, $\mathbf{M} \models \forall xyzw[(\&_{i<n} p_i(x, y, z, w) = q_i(x, y, z, w)) \leftrightarrow (x = y \text{ or } z = w)]$. Hence, by Lemma 1.5 and Corollary 2.4, Δ is a system of principal intersection terms for **K** and hence **K** has EDPM.

An easy inspection of the above proof shows that Corollary 2.7 remains true when **L** is a quasivariety with EDPM.

3. Finite basis results

Let $\Delta = \langle p_i(x, y, z, w), q_i(x, y, z, w)\rangle$, $i < n$, be a fixed finite system of pairs of 4-ary terms of some type τ of algebras. With every quasiidentity $Q : \&_{j<m} r_j = s_j \rightarrow r = s$ expressed in the language of τ we associate the set $\Delta(Q)$ of n quasiidentities constructed from Q and the equations of Δ in the following way:

$$\mathop{\&}_{i<n} \mathop{\&}_{j<m} p_i(r_j, s_j, z, w) = q_i(r_j, s_j, z, w) \rightarrow p_k(r, s, z, w) = q_k(r, s, z, w)$$

where $k = 0, \ldots, n-1$ and the variables z, w are assumed to be distinct from the variables occurring in Q. Likewise as in Blok and Pigozzi [5] the sets of the form $\Delta(Q)$ will play a central role in proofs of our finite basis results. A related concept to $\Delta(Q)$ was also used in Czelakowski [7] and in the proof of a result announced in Wojtylak [26, Theorem 3.4] to obtain some finite basis results for propositional logics.

Let $\Sigma(\Delta)$ denote the set consisting of the following quasiidentities:

$$(1)_{ij} \quad [\&_{i<n} \&_{j<n} p_i(p_j(x, y, z, w), q_j(x, y, z, w), v, u) = q_i(p_j(x, y, z, w), q_j(x, y, z, w), v, u)] \rightarrow p_i(x, y, p_j(z, w, v, u), q_j(z, w, v, u)) = q_i(x, y, p_j(z, w, v, u), q_j(z, w, v, u))$$

$$(2)_{ij} \quad [\&_{i<n} \&_{j<n} p_i(x, y, p_j(z, w, v, u), q_j(z, w, v, u)) = q_i(x, y, p_j(z, w, v, u), q_j(z, w, v, u))] \rightarrow p_i(p_j(x, y, z, w), q_j(x, y, z, w), v, u) = q_i(p_j(x, y, z, w), q_j(x, y, z, w), v, u)$$

where $i, j = 0, \ldots, n-1$, and x, y, z, w, v, u are distinct individual variables.

LEMMA 3.1. *Suppose A is an algebra of type τ and $A \models \Sigma(\Delta)$. Then for every quasiidentity Q it holds: if $A \models \Delta(Q)$ then $A \models \Delta(R)$ for every $R \in \Delta(Q)$.*

Proof. Let Q be of the form $r_0 = s_0 \,\&\, \cdots \,\&\, r_{m-1} = s_{m-1} \rightarrow r = s$ and let $A \models \Delta(Q)$. Assume that z, w, v, u are distinct from the variables occurring in Q. Consider a quasiidentity

$$R: \Big[\mathop{\&}_{i<n} \mathop{\&}_{j<m} p_i(r_j, s_j, z, w) = q_i(r_j, s_j, z, w) \Big] \rightarrow p_k(r, s, z, w) = q_k(r, s, z, w)$$

where $k < n$. Since $A \models \Delta(Q)$, for every $i, k < n$ we have

$$A \models \Big[\mathop{\&}_{i<n} \mathop{\&}_{j<m} p_i(r_j, s_j, p_k(z, w, v, u), q_k(z, w, v, u))$$
$$= q_i(r_j, s_j\, p_k(z, w, v, u), q_k(z, w, v, u)) \Big]$$
$$\rightarrow p_i(r, s, p_k(z, w, v, u), q_k(z, w, v, u))$$
$$= q_i(r, s, p_k(z, w, v, u), q_k(z, w, v, u)).$$

Hence

$$A \models \Big[\mathop{\&}_{i<n} \mathop{\&}_{j<m} \mathop{\&}_{k<n} p_i(r_j, s_j, p_k(z, w, v, u), q_k(z, w, v, u))$$
$$= q_i(r_j, s_j, p_k(z, w, v, u), q_k(z, w, v, u)) \Big]$$
$$\rightarrow p_i(r, s, p_k(z, w, v, u), q_k(z, w, v, u))$$
$$= q_i(r, s, p_k(z, w, v, u), q_k(z, w, v, u))$$

for all $i, k < n$ which, by $(1)_{ik}$, yields

$$A \models \Big[\mathop{\&}_{i<n} \mathop{\&}_{j<m} \mathop{\&}_{k<n} p_i(p_k(r_j, s_j, z, w), q_k(r_j, s_j, z, w), v, u)$$
$$= q_i(p_k(r_j, s_j, z, w), q_k(r_j, s_j, z, w), v, u) \Big]$$
$$\rightarrow p_i(r, s, p_k(z, w, v, u), q_k(z, w, v, u))$$
$$= q_i(r, s, p_k(z, w, v, u), q_k(z, w, v, u))$$

for all $i, k < n$. Thus, by $(2)_{ik}$, $A \models \Delta(R)$, proving the lemma.

Now, let $\Gamma(\Delta)$ denote the set consisting of all elements of $\Sigma(\Delta)$ and the quasiidentities listed below, where $i = 0, \ldots, n-1$ and $\varphi(x, y, z, w)$ abbreviates the formula $\&_{i<n} p_i(x, y, z, w) = q_i(x, y, z, w)$.

$(1)_i$ $x = y \to p_i(x, y, z, w) = q_i(x, y, z, w)$.
$(2)_i$ $\varphi(x, y, z, w) \to p_i(y, x, z, w) = q_i(y, x, z, w)$.
$(3)_i$ $\varphi(x_0, x_1, z, w) \,\&\, \varphi(x_1, x_2, z, w) \to p_i(x_0, x_2, z, w) = q_i(x_0, x_2, z, w)$.
$(4_f)_i$ $\&_{j<m} \varphi(x_j, y_j, z, w) \to p_i(f(x_0, \ldots, x_{m-1}), f(y_0, \ldots, y_{m-1}), z, w)$ $= q_i(f(x_0, \ldots, x_{m-1}), f(y_0, \ldots, y_{m-1}), z, w)$ where f is a m-ary function symbol from the language of τ and $m \geq 1$.
$(5)_i$ $\varphi(x, y, z, w) \to p_i(z, w, x, y) = q_i(z, w, x, y)$.
(6) $\varphi(x, y, x, y) \to x = y$.

LEMMA 3.2. *Let* $\mathbf{K}$ *be a quasivariety with* Δ *as a system of principal intersection terms. Then* $\mathbf{K} \models \Gamma(\Delta)$, *and* $\mathbf{K} \models \Delta(Q)$ *for every quasiidentity* Q *valid in* $\mathbf{K}$.

Proof. Use Proposition 1.2(ii).

LEMMA 3.3. *Suppose* Σ *is a set of quasiidentities expressed in the language of* τ. *Then every finitely subdirectly irreducible member of the quasivariety* $\mathrm{Mod}\,\Sigma \cup \Gamma(\Delta) \cup \bigcup(\Delta(Q) : Q \in \Sigma \cup \Gamma(\Delta))$ *satisfies* $\forall xyzw[(\&_{i<n} p_i(x, y, z, w) = q_i(x, y, z, w)) \leftrightarrow (x = y \text{ or } z = w)]$.

Proof. By $(1)_i$ and $(5)_i$, it suffices to show that every member A of $\mathrm{Mod}\,\Sigma \cup \Gamma(\Delta) \cup \bigcup(\Delta(Q) : Q \in \Sigma \cup \Gamma(\Delta))_{\mathrm{FSI}}$ satisfies $\forall xyzw[(\&_{i<n} p_i(x, y, z, w) = q_i(x, y, z, w)) \to (x = y \text{ or } z = w)]$. Let $a, b, c, d \in A$ and $A \models \varphi(x, y, z, w)[a, b, c, d]$. Define

$$\Theta_0 = \{(e, f) \in A \times A : A \models \varphi(x, y, z, w)[e, f, c, d]\}$$

$$\Theta_1 = \{(e, f) \in A \times A : A \models \varphi(x, y, z, w)[g, h, e, f] \text{ for all } (g, h) \in \Theta_0\}.$$

We claim that Θ_0 and Θ_1 are congruence relations on A. By $(1)_i$–$(4_f)_i$, Θ_0 is a congruence on A. By $(1)_i$ and $(5)_i$, Θ_1 is reflexive. By $(2)_i$ and $(5)_i$, Θ_1 is symmetric, and, by $(3)_i$ and $(5)_i$, Θ_1 is transitive. That Θ_1 preserves operations follows from $(4_f)_i$ and $(5)_i$. Now, we show that A/Θ_0 and A/Θ_1 belong to $\mathrm{Mod}\,\Sigma \cup \Gamma(\Delta) \cup \bigcup(\Delta(Q) : Q \in \Sigma \cup \Gamma(\Delta))$. Let R be an element of $\Sigma \cup \Gamma(\Delta) \cup \bigcup(\Delta(Q) : Q \in \Sigma \cup \Gamma(\Delta))$ and assume that it is of the form $\&_{j<m} r_j(\bar{x}) = s_j(\bar{x}) \to r(\bar{x}) = s(\bar{x})$ where $\bar{x} = x_0, \ldots, x_{p-1}$. Let $A/\Theta_0 \models r_j(\bar{x}) = s_j(\bar{x})[[a_0]\Theta_0, \ldots, [a_{p-1}]\Theta_0]$ for all $j < m$. Then $(r_j(\bar{a}), s_j(\bar{a})) \in \Theta_0$

for all $j<m$ where $\bar{a}=a_0, \ldots, a_{p-1}$. Hence, by definition of Θ_0, $A \models \&_{j<m} \varphi(r_j(\bar{x}), s_j(\bar{x}), z, w)[\bar{a}, c, d]$. But $A \models \&_{j<m} \varphi(r_j(\bar{x}), s_j(\bar{x}), z, w) \rightarrow p_k(r(\bar{x}), s(\bar{x}), z, w) = q_k(r(\bar{x}), s(\bar{x}), z, w)[\bar{a}, c, d]$ for all $k<n$ since, by $\Sigma(\Delta) \subseteq \Gamma(\Delta)$ and Lemma 3.1, $A \models \Delta(R)$. Hence $A \models \varphi(r(\bar{x}), s(\bar{x}), z, w)[\bar{a}, c, d]$ and, therefore, $(r(\bar{a}), s(\bar{a})) \in \Theta_0$. Thus $A/\Theta_0 \models r(\bar{x}) = s(\bar{x})[[a_0]\Theta_0, \ldots, [a_{p-1}]\Theta_0]$, proving that $A/\Theta_0 \models R$. Similarly, applying $(5)_i$ one can show that $A/\Theta_1 \models R$. Thus A/Θ_0 and A/Θ_1 belong to $\operatorname{Mod} \Sigma \cup \Gamma(\Delta) \cup \bigcup (\Delta(Q): Q \in \Sigma \cup \Gamma(\Delta))$. From this, as $(a, b) \in \Theta_0$ and $(c, d) \in \Theta_1$, it follows that $\Theta_{\mathbf{K}}(a, b) \wedge \Theta_{\mathbf{K}}(c, d) \leq \Theta_0 \wedge \Theta_1$ where $\mathbf{K}$ abbreviates $\operatorname{Mod} \Sigma \cup \Gamma(\Delta) \cup \bigcup (\Delta(Q): Q \in \Sigma \cup \Gamma(\Delta))$. But, since $(6) \in \Gamma(\Delta)$, $\Theta_0 \wedge \Theta_1 = \omega_A$. Hence $\Theta_{\mathbf{K}}(a, b) \wedge \Theta_{\mathbf{K}}(c, d) = \omega_A$ which in conjunction with $A \in \mathbf{K}_{\mathrm{FSI}}$ yields $\Theta_{\mathbf{K}}(a, b) = \omega_A$ or $\Theta_{\mathbf{K}}(c, d) = \omega_A$. Thus $A \models x = y$ or $z = w[a, b, c, d]$, completing the proof.

Our original proof of Lemma 3.3 did not use congruences Θ_0 and Θ_1. It employed some idea from Czelakowski [7]. The idea of applying congruences Θ_0 and Θ_1 to the above proof is borrowed from Blok and Pigozzi [6].

We are ready now to prove the following

THEOREM 3.4. *Let* $\mathbf{K}$ *be a quasivariety with* EDPM *and let the type of* $\mathbf{K}$ *be finite. Then the following conditions are equivalent*:

(*i*) $\mathbf{K}$ *is finitely based.*

(*ii*) $\mathbf{K}_{\mathrm{FSI}}$ *is strictly elementary.*

Proof. (i) ⇒ (ii): By Proposition 1.2(ii).

(ii) ⇒ (i): Let the system $\Delta = \langle p_i(x, y, z, w), q_i(x, y, z, w) \rangle$, $i<n$, realizes EDPM for $\mathbf{K}$. Let φ be a first-order sentence axiomatizing $\mathbf{K}_{\mathrm{FSI}}$. By Proposition 1.2(ii) and compactness theorem, $\Sigma \cup \{\forall xyzw[(\&_{i<n} p_i(x, y, z, w) = q_i(x, y, z, w)) \leftrightarrow (x = y \text{ or } z = w)\} \models \varphi$ for some finite set Σ of quasiidentities valid in $\mathbf{K}$. Hence, by Lemma 3.3 and Proposition 1.2(ii), $\operatorname{Mod} \Sigma \cup \Gamma(\Delta) \cup \bigcup (\Delta(Q): Q \in \Sigma \cup \Gamma(\Delta)) \subseteq \mathbf{K}$. On the other hand, by Lemma 3.2, $\mathbf{K} \subseteq \operatorname{Mod} \Sigma \cup \Gamma(\Delta) \cup \bigcup (\Delta(Q): Q \in \Sigma \cup \Gamma(\Delta))$. Thus $\mathbf{K} = \operatorname{Mod} \Sigma \cup \Gamma(\Delta) \cup \bigcup (\Delta(Q): Q \in \Sigma \cup \Gamma(\Delta))$, showing that (ii) implies (i).

As corollary we obtain

COROLLARY 3.5. *Let* M *be a finite set of finite algebras of finite type, and let* $Q(M)$ *have* EDPM. *Then* $Q(M)$ *is finitely based.*

Proof. By Lemma 1.5, $Q(M)_{\mathrm{FSI}}$ is strictly elementary. Hence, by theorem 3.4, $Q(M)$ is finitely based.

We also have

COROLLARY 3.6. *Suppose M is a finite set of finite subdirectly irreducible algebras of finite type such that* $S(M) \subseteq V(M)_{\mathrm{FSI}}$, *and let* $V(M)$ *be congruence distributive. Then* $Q(M)$ *is finitely based.*

Proof. Use Corollary 2.6 and 3.5.

Due to Baker [2] we know that the variety generated by any finite algebra of a congruence distibutive variety of finite type is finitely based. The analogous result for the quasivariety generated by such an algebra is impossible to obtain because there are examples of finite lattices (Belkin [3], [4] and Tumanov [25]) as well as finite Heyting and interior algebras (Dziobiak [12] and Rybakov [24]) that individually generate non-finitely based quasivarieties. However, as the variety generated by a finite algebra A in a congruence distributive variety coincides with $Q(HS(A)_{\mathrm{FSI}})$, Blok and Pigozzi [5] asked whether the following natural generalization of Baker's finite basis result might still be possible: Is it true that every finite set of finite, subdirectly irreducible algebras in a congruence distributive variety of finite type generates a finitely based quasivariety? This question is equivalent to the following: Is Corollary 3.6 true without the assumption that $S(M) \subseteq V(M)_{\mathrm{FSI}}$? The lattice L of Figure 1 is subdirectly irreducible. Evidently, $S(L)$ is not contained in $V(L)_{\mathrm{FSI}}$. Applying arguments used in Belkin [4] we shall show that $Q(L)$ is not finitely based which would provide an answer to the above question.

L :

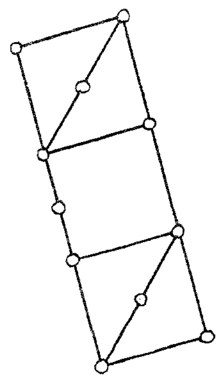

Figure 1

Let A_n, $n \geqslant 1$, denote the lattices depicted in Figure 2. We have

CLAIM (see Belkin [4]). For every $n \geqslant 1$ the following conditions are

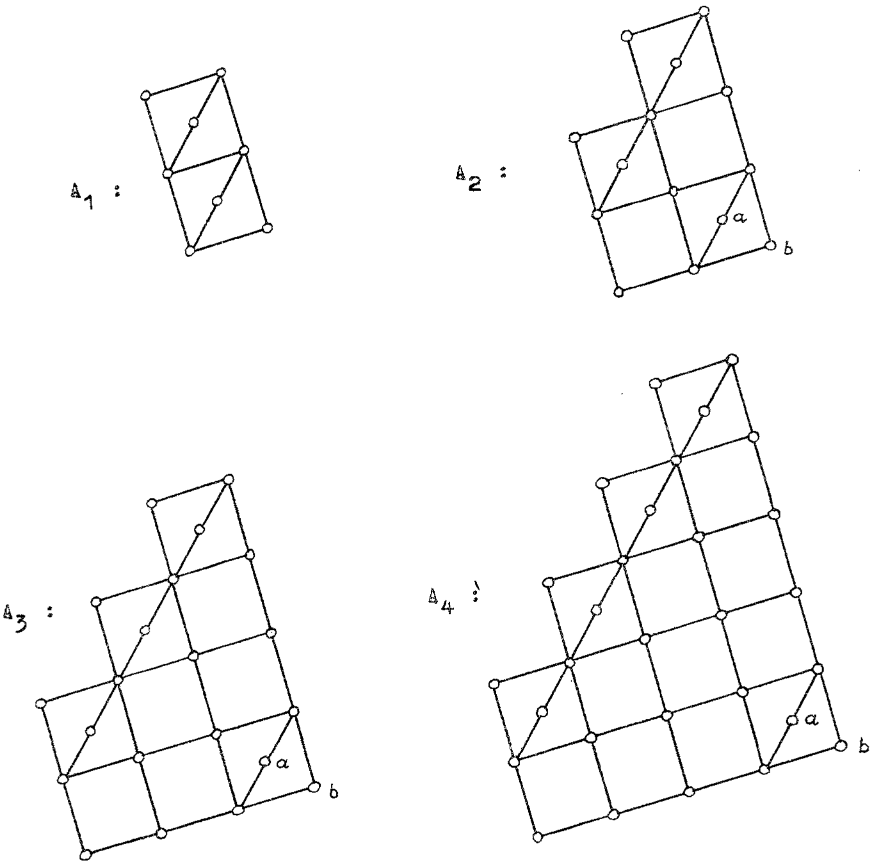

Figure 2

satisfied:

(1) Every proper sublattice of A_n belongs to $Q(M_{3-3})$.

(2) If Θ is an atom in $\operatorname{Con} A_n$ then either $a \equiv b(\Theta)$ or A_n/Θ and A_{n-1} are isomorphic via an isomorphism f such that $f(a) = a$ and $f(b) = b$.

As the lattice M_{3-3} of Figure 3 is a sublattice of L, it follows from claim (1) that every proper sublattice of every A_n belongs to $Q(L)$. Hence, as the lattices A_n, $n \geqslant 1$, belong to the locally finite variety $V(A_1)$, to show that $Q(L)$ is not finitely based it remains then to prove that $A_n \notin Q(L)$ for all $n \geqslant 1$. To this effect suppose otherwise. Then there exist congruence relations $\Theta_0, \ldots, \Theta_{k-1}$ on A_n such that $\Lambda(\Theta_j : j < k) = \omega_{A_n}$ and $A_n/\Theta_j \in IS(L)$ for all j. Since $|L| < |A_n|$ for all $n \geqslant 2$, then by claim (2), for every j, $A_n/\Theta_j \cong A_1$ or $a \equiv b(\Theta_j)$. But A_1 is not embeddable into L. So, for all $j < k$, $a \equiv b(\Theta_j)$, a contradiction. Thus $Q(L)$ cannot be finitely based.

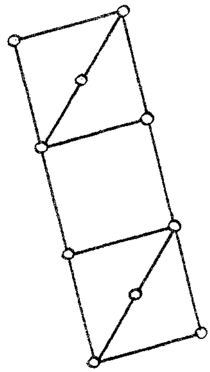

Figure 3

It is worth to mention that due to Corollary 3.6 the algebra $(L, a)_{a\in L}$ generates a finitely based quasivariety.

Blok and Pigozzi [5] proved that every finite set of finite subdirectly irreducible algebras of finite type belonging to a variety with EDPM generates a finitely based quasivariety. The following corollary strengthens this result.

COROLLARY 3.7. *Suppose* $\mathbf{K}$ *is a variety with* EDPM *and let the type of* $\mathbf{K}$ *be finite. Then for* $\mathbf{M} \subseteq \mathbf{K}_{\text{FSI}}$ *the following conditions are equivalent*:

(i) $Q(\mathbf{M})$ *is finitely based.*

(ii) $\text{ISP}_U(\mathbf{M})$ *is a strictly elementary class.*

Proof. By Corollary 2.7, $Q(\mathbf{M})$ has EDPM, and, by Lemma 1.5, $Q(M)_{\text{FSI}} = \text{ISP}_U(\mathbf{M}^+)$ where $\mathbf{M}^+$ is obtained from $\mathbf{M}$ by adjoining a trivial algebra. Hence, by Proposition 1.2(ii), (i) implies (ii). Assume now that $\text{ISP}_U(\mathbf{M}) = \text{Mod}\,\varphi$ for some first-order sentence φ. Then $\text{ISP}_U(\mathbf{M}^+) = \text{Mod}\,\varphi$ or $\forall xy[x = y]$. Hence $Q(\mathbf{M})_{\text{FSI}}$ is a strictly elementary class because, by Lemma 1.5, $Q(\mathbf{M})_{\text{FSI}} = \text{ISP}_U(\mathbf{M}^+)$. Thus, by Theorem 3.4, $Q(\mathbf{M})$ is finitely based, showing that (ii) implies (i).

The above corollary does not indicate to us how to construct a basis for $Q(\mathbf{M})$ even in the case we know a set axiomatizing $\text{ISP}_U(\mathbf{M})$. The discussion presented below removes partly this disadvantage.

A universal sentence whose matrix is of the form $\&_{i<k}\, r_i = s_i \rightarrow \bigvee_{j<m} t_j = w_j$ will be called a universal basic sentence or *UB*-sentence for short. The name is justified by the fact that every universal sentence is equivalent to the conjunction of a finite number of universal basic sentences. Hence, as the class $\text{ISP}_U(\mathbf{M})$ is universal, it can be axiomatized by a set of *UB*-sentences. Notice that if a trivial

algebra belongs to $\mathrm{ISP}_U(\mathbf{M})$ then every UB-sentence valid in $\mathrm{ISP}_U(\mathbf{M})$ must have at least one disjunct in its consequent. In the sequel by $\mathbf{M}^+$ we denote the class obtained from $\mathbf{M}$ by adjoining a trivial algebra.

For a universal basic sentence U whose matrix is of the above form with $m \geq 1$, we put $k(U) = m$ and define inductively the sets $\Sigma(U)^j$, $j < k(U)$,

$$\Sigma(U)^0 = \{t_0 = w_0\}$$

and, for $j < k(U) - 1$,

$$\Sigma(U)^{j+1} = \{p_i(r, s, t_{j+1}, w_{j+1}) = q_i(r, s, t_{j+1}, w_{j+1}) : i < n \text{ and } r = s \in \Sigma(U)^j\}.$$

Notice that $k(U)$ is the number of disjuncts occurring in the consequent of U. Denoting by $A(U)$ the formula $\&_{i<k}\, r_i = s_i$ we have the following corollary which extends Baker's idea of UDE-sentences (see Baker [1]) into quasivarieties.

COROLLARY 3.8. *Suppose* $\mathbf{K}$ *is a quasivariety with* EDPM *and a system of principal intersection terms* $\Delta = \langle p_i, q_i \rangle$, $i < n$. *Moreover, assume that the class* $\mathrm{ISP}_U(\mathbf{M}^+)$ *is axiomatized by a set* Σ *of UB-sentences, where* $\mathbf{M} \subseteq \mathbf{K}_{\mathrm{FSI}}$. *Then* $Q(\mathbf{M})$ *can be axiomatized relative to* $\mathbf{K}$ *by the set* $\Gamma \cup \bigcup(\Delta(Q) : Q \in \Gamma)$, *where* $\Gamma = \{A(U) \to r = s : U \in \Sigma \text{ and } r = s \in \Sigma(U)^{k(U)-1}\}$.

Proof. Since $\mathbf{M} \subseteq \mathbf{K}_{\mathrm{FSI}}$, then by Lemma 1.5 and Corollary 1.4 it follows that $Q(\mathbf{M})$ is congruence distributive. On the other hand, by Lemmas 3.2, 3.3 and Theorem 2.3, $\mathrm{Mod}_{\mathbf{K}}\, \Gamma \cup \bigcup(\Delta(Q) : Q \in \Gamma)$ is congruence distributive as well. Hence, by Corollary 1.4, $Q(\mathbf{M})_{\mathrm{FSI}}$ and $\mathrm{Mod}_{\mathbf{K}}\, \Gamma \cup \bigcup(\Delta(Q) : Q \in \Gamma)_{\mathrm{FSI}}$ are contained in $\mathbf{K}_{\mathrm{FSI}}$. So, to complete the proof it suffices to show that for every $A \in \mathbf{K}_{\mathrm{FSI}}$, $A \in A(\mathbf{M})$ iff $A \in \mathrm{Mod}_{\mathbf{K}}\, \Gamma \cup \bigcup(\Delta(Q) : Q \in \Gamma)$. Let $A \in \mathbf{K}_{\mathrm{FSI}}$ and $A \in Q(\mathbf{M})$. Then, by Lemma 1.5, $A \in \mathrm{ISP}_U(\mathbf{M}^+)$ which means $A \models \Sigma$, and, by $A \in \mathbf{K}_{\mathrm{FSI}}$, $A \models \Gamma \cup \bigcup(\Delta(Q) : Q \in \Gamma)$. Now, let $A \in \mathrm{Mod}_{\mathbf{K}}\, \Gamma \cup \bigcup(\Delta(Q) : Q \in \Gamma)$. Then $A \models \Gamma$ and, by $A \in \mathbf{K}_{\mathrm{FSI}}$, $A \models \Sigma$ which in turn, by $\mathrm{ISP}_U(\mathbf{M}^+) = \mathrm{Mod}\, \Sigma$, yields $A \in Q(\mathbf{M})$. Thus $Q(\mathbf{M}) = \mathrm{Mod}_{\mathbf{K}}\, \Gamma \cup \bigcup(\Delta(Q) : Q \in \Gamma)$, showing the corollary.

4. Another characterization

We want now to present an alternative characterization of quasivarieties with EDPM. First, notice

LEMMA 4.1. *For every* Δ, *the quasivariety* $\mathrm{Mod}\, \Gamma(\Delta) \cup \bigcup(\Delta(Q) : Q \in \Gamma(\Delta))$ *has* EDPM *and* Δ *is a system of principal intersection terms for it.*

Proof. Use Lemma 3.3, Theorem 2.3 and Corollary 2.4.

Now, we have:

THEOREM 4.2. *For a quasivariety* **K** *the following conditions are equivalent:*

(i) **K** *has* EDPM.
(ii) **K** *is congruence distributive and* $K \subseteq \mathrm{Mod}\, \Gamma(\Delta) \cup \bigcup (\Delta(Q) : Q \in \Gamma(\Delta))$ *for some finite system* Δ *of pairs of* 4*-ary terms.*
(iii) $\mathbf{K}_{\mathrm{FSI}} \subseteq V(\mathbf{K})_{\mathrm{FSI}}$ *and* $\mathbf{K} \subseteq \mathrm{Mod}\, \Gamma(\Delta) \cup \bigcup (\Delta(Q) : Q \in \Gamma(\Delta))$ *for some finite system* Δ *of pairs of* 4*-ary terms.*

Proof. (i)$\Rightarrow$(ii): By Theorem 2.3 and Lemma 3.2. (ii)$\Rightarrow$(iii): By Proposition 1.3. (iii)$\Rightarrow$(i): Assume (iii). Then $\mathbf{K}_{\mathrm{FSI}} \subseteq \mathrm{Mod}\, \Gamma(\Delta) \cup \bigcup (\Delta(Q) : Q \in \Gamma(\Delta))_{\mathrm{FSI}}$. Hence, by Lemma 4.1, $\mathbf{K}_{\mathrm{FSI}} \models \forall xyzw[(\&_{i<n} p_i(x, y, z, w) = q_i(x, y, z, w)) \leftrightarrow (x = y \text{ or } z = w)]$. Thus, by Theorem 2.3, **K** has EDPM.

We also want to mention that the quasivariety $\mathrm{Mod}\, \Gamma(\Delta) \cup \bigcup (\Delta(Q) : Q \in \Gamma(\Delta))$ is the largest among those which have EDPM with respect to Δ. This follows from Lemma 3.2. So, it is not a surprise that we also have

PROPOSITION 4.3. *Let* $\Delta = \langle p_i(x, y, z, w), q_i(x, y, z, w) \rangle$, $i < n$, *be a finite system of pairs of* 4*-ary terms of type* τ. *Then* $\mathrm{Mod}\, \Gamma(\Delta) \cup \bigcup (\Delta(Q) : Q \in \Gamma(\Delta)) = Q(\{A : A \text{ is of type } \tau \text{ and } A \models \forall xyzw[(\&_{i<n} p_i(x, y, z, w) = q_i(x, y, z, w)) \leftrightarrow (x = y \text{ or } z = w)]\})$.

Proof. $\subseteq$: The inclusion is obvious because, by Lemma 4.1, $\mathrm{Mod}\, \Gamma(\Delta) \cup \bigcup (\Delta(Q) : Q \in \Gamma(\Delta))_{\mathrm{FSI}} \models \forall xyzw[(\&_{i<n} p_i(x, y, z, w) = q_i(x, y, z, w)) \leftrightarrow (x = y \text{ or } z = w)]$.

$\supseteq$: By Lemma 1.5 and Corollary 2.4, $Q(\{A : A \text{ is of type } \tau \text{ and } A \models \forall xyzw[(\&_{i<n} p_i(x, y, z, w) = q_i(x, y, z, w)) \leftrightarrow (x = y \text{ or } z = w)]\})$ has EDPM with Δ as a system of principal intersection terms. So, by Lemma 3.2, the inclusion follows.

Concluding we notice the following

PROPOSITION 4.4. *Each of the conditions of Corollary* 2.7 *is equivalent to each of the following two:*

(v) *For every quasiidentity* Q, $\mathbf{K} \models Q$ *implies* $\mathbf{K} \models \Delta(Q)$.
(vi) $\mathbf{K} = \mathrm{Mod}_{\mathbf{L}}\, \Gamma \cup \bigcup (\Delta(Q) : Q \in \Gamma)$ *for some* Γ.

Proof. That (iv) implies (v) is obvious. Let $\mathbf{K} = \mathrm{Mod}_{\mathrm{L}}\, \Gamma$. Then, by (v), $\mathbf{K} = \mathrm{Mod}_{\mathrm{L}}\, \Gamma \cup \bigcup (\Delta(Q) : Q \in \Gamma)$. Thus (v) yields (vi). By Lemmas 3.2, 3.3 and Corollary 2.4, (vi) implies (i).

5. Questions

This section contains some questions that naturally arise from the context of the paper.

QUESTION 1. Assume A is a finite subdirectly irreducible algebra of finite type without nontrivial proper subalgebras. Is the quasivariety generated by A finitely based?

Actually, we do not even know whether $Q(A)$ is finitely based under the stronger assumption that A is simple. By a result of Pigozzi [23], a corresponding question for the variety generated by A has a negative answer. Notice also that if A is chosen from a congruence distributive variety then due to Corollaries 2.6 and 3.5 the quasivariety $Q(A)$ is finitely based. A related question whether there exists a finite hereditary simple algebra of finite type whose quasivariety is not finitely based was raised by Gorbunov [15].

Let $\mathbf{K}$ be a finitely generated quasivariety of algebras of finite type. By Theorem 2.3 and Corollary 3.5 we know that if $\mathbf{K}$ is congruence distributive and $\mathbf{K}_{\mathrm{FSI}}$ is a universal class then $\mathbf{K}$ is finitely based. This result is not true without the assumption that $\mathbf{K}$ is congruence distributive. Indeed, let $\mathbf{K} = Q(A)$ where $A = (\{0, 1, 2\}, f, g)$ is of type $\langle 1, 1\rangle$ with f and g defined by $f(0) = 1$, $g(0) = 2$ and $f(x) = g(x) = x$ for $x \neq 0$. Due to Gorbunov [14], $\mathbf{K}$ is not finitely based, and applying Lemma 1.5 one can verify that $\mathbf{K}_{\mathrm{FSI}}$ is a universal class. However, we do not know whether the second assumption in the above result is necessary. In other words, we have

QUESTION 2. Is every finitely generated and congruence distributive quasivariety $\mathbf{K}$ of algebras of finite type finitely based?

We also want to ask about a generalization of Theorem 3.4.

QUESTION 3. Suppose $\mathbf{K}$ is a congruence distributive quasivariety of algebras of finite type and let the class $\mathbf{K}_{\mathrm{FSI}}$ be strictly elementary. Is $\mathbf{K}$ finitely based?

Let M be a finite set of finite subdirectly irreducible algebras of finite type from a variety with EDPC in th sense of [13]. For each member A of M let A^+ denote the algebra resulting from A by adjoining two new constants representing in A distinct elements that are collapsed by every congruence relation on A different from ω_A. The variety $V(\{A^+ : A \in M\})$ is congruence distributive because, by a result of Köhler and Pigozzi [21], $V(M)$ is congruence distributive. Moreover, as $V(M)$ has the congruence extension property, every subalgebra of every A^+ is subdirectly irreducible. Hence, by Corollary 2.6, $Q(\{A^+ : A \in M\})$ has EDPM. Thus, by Corollary 3.5, $Q(\{A^+ : A \in M\})$ is finitely based. However, the following question remains still open.

QUESTION 4 (Blok and Pigozzi [5]). Does every finite set of finite subdirectly irreducible algebras of finite type from a variety with EDPC generate a finitely based quasivariety?

In [12], the second author proved that if in addition $\mathbf{K}$ is contained in a congruence distributive quasivariety Mod $\Sigma \cup \Gamma$, where Σ is a set of identities and Γ is a finite set of quasiidentities, then the answer to Question 2 is affirmative. Recently, in a letter to the first author Professor Don Pigozzi informed us that modifying the proof of Theorem 3.4 and applying some arguments from [6] he had answered Question 2 affirmatively without any additional assumptions.

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Section of Logic
Polish Academy of Sciences
Łódź, Poland

Institute of Mathematics
N. Copernicus University
Toruń, Poland