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New minimal geodesics in the group of symplectic diffeomorphisms

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Abstract. We compute the Hofer distance for a certain class of compactly supported symplectic diffeomorphisms of $\mathbb{R}^{2n}$. They are mainly characterized by the condition that they can be generated by a Hamiltonian flow φ_H^t which possesses only constant T-periodic solutions for $0 < T \leq 1$. In addition, we show that on this class Hofer's and Viterbo's distances coincide.

Subject Classifications: 58F

1 Hofer's metric

We consider the standard symplectic vector space $(\mathbb{R}^{2n}, \omega)$. By $\mathscr{D}$ we denote the group of time-1-maps φ_H^1 of a maybe time dependent Hamiltonian system

$$\dot{x}(t) = J\nabla H(t, x(t))$$

with $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$H \in \mathscr{C} := C_c^\infty([0,1] \times \mathbb{R}^{2n}, \mathbb{R}) .$$

In [Ho1] Hofer defined the energy of $\varphi \in \mathscr{D}$ as

$$E(\varphi) := \inf \left\{ \|H\| := \int_0^1 \sup H_t - \inf H_t \; dt \,|\varphi_H^1 = \varphi \right\}$$

where $H_t(x) := H(t, x)$. The crucial property of the energy is that

$$E(\varphi) = 0 \Rightarrow \varphi = \mathrm{id} .$$

It follows that

$$d(\varphi, \psi) := E(\varphi^{-1}\psi)$$

defines a bi-invariant metric on $\mathscr{D}$. $\|H\|$ can be seen as the length of the path

$$\varphi_H^{[0,1]} = \{\varphi_H^t | 0 \leq t \leq 1\}$$

in $\mathscr{D}$ connecting id and φ. In this language $E(\varphi) = d(\mathrm{id}, \varphi)$ gives the infimum over all lengths of paths $\psi^{[0,1]}$ in $\mathscr{D}$ with $\psi^0 = \mathrm{id}$ and $\psi^1 = \varphi$. The following definition is therefore natural.

Definition 1 *A path $\psi^{[0,1]}$ in $\mathscr{D}$ is called a minimal geodesic if*

$$\textit{length } \psi^{[0,1]} = d(\psi^0, \psi^1) .$$

Notice that in this case the infimum $d(\mathrm{id}, \varphi)$ becomes in fact a minimum. The actual calculation of $d(\mathrm{id}, \varphi)$ turns out to be a difficult task since it involves all Hamiltonians generating φ. In [Ho2] Hofer showed that autonomous Hamiltonians give rise to minimal geodesics provided that all T-periodic solutions of the corresponding flow with $0 < T \leq 1$ are constant. Prompted by the work of Bialy and Polterovich [B-P] the aim of our paper is to generalize Hofer's result to the non-autonomous case.

Definition 2 *A Hamiltonian $H \in \mathscr{C}$ is called admissible if $\varphi_H^T(x) = x$ with some $T \in (0, 1]$, $x \in \mathbb{R}^{2n}$ implies $\varphi_H^t(x) = x$ for every $t \in [0, T]$.*

$H \in \mathscr{C}$ is said to have fixed extremal points if there are two points $x_\pm \in \mathbb{R}^{2n}$ such that

$$\inf H_t = H_t(x_-) , \quad \sup H_t = H_t(x_+)$$

for all $t \in [0, 1]$.

In addition, we call $x_\pm$ isolated if there are open neighbourhoods $U_\pm \subset \mathbb{R}^{2n}$ of $x_\pm$ such that

$$U_\pm \cap \left(\bigcup_{T\in(0,1]} \bigcap_{t\in[0,T]} \textit{crit } H_t \right) = \{x_\pm\}$$

where crit H_t denotes the set of critical points of H_t.

We emphasize that the extremal values do not have to be fixed at all. The notion of fixed extremal points appeared in a preprint by Long [Lo]; Bialy and Polterovich [B-P] use the term "quasi-autonomous".

Theorem 1 *Suppose $H \in \mathscr{C}$ is admissible and has isolated fixed extremal points. Then H generates a minimal geodesic, i.e. $d(\mathrm{id}, \varphi_H^1) = \|H\|$.*

Loosely speaking, this result tells you that you should not move any point if you do not have to. By the bi-invariance of d, Theorem 1 translates to general (smooth) paths $\psi^{[0,1]}$ in $\mathscr{D}$. The following example indicates why the assumption on the existence of fixed extremal points is necessary for minimal geodesics; actually this was shown in full generality by Lalonde and McDuff [L-M2].

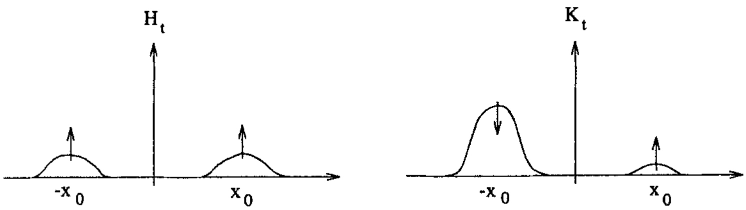
Example. We fix an $x_0 \in \mathbb{R}^{2n}$ with $|x_0| = 1$ and take as Hamiltonians

$$H(t,x) := [\rho(|x - x_0|^2) + \rho(|x + x_0|^2)]t$$

and

$$K(t,x) := \rho(|x - x_0|^2)t + \rho(|x + x_0|^2)(1 - t)$$

where $\rho : \mathbb{R} \to [0, \rho_0]$ is a bump function with $\rho(0) = \rho_0$ and $\rho(s) = 0$ if $|s| \geq \frac{1}{2}$. By choosing ρ_0 appropriately we may assume that H and K are admissible; note that H has fixed extremal points whereas K has not (cf. figure below). It is an elementary calculation to check that $\varphi_H^1 = \varphi_K^1$ and $\|H\| = \frac{\rho_0}{2} < \frac{3\rho_0}{4} = \|K\|$.



We remark that the set of admissible Hamiltonians has empty interior in the normed vector space $(\mathscr{E}, \|\cdot\|)$. Indeed, given any admissible H we can add a perturbation $K(t,x) = a(t)\rho(x)$ (where $\rho(x)$ is a bump function whose support is disjoint from that of each H_t) with arbitrarily small $\|K\|$ such that K generates the identity, i.e. $\varphi_{H+K}^1 = \varphi_H^1$. Then $H + K$ is not admissible. Therefore, also the class of sympletic diffeomorphisms generated by admissible Hamiltonians has empty interior in $(\mathscr{D}, d)$.

The same argument shows that a minimal geodesic between id and $\varphi \neq$ id is not unique.

2 The analytical setting

We are going to study the action functional

$$a_H(x) := a(x) + b_H(x) := \frac{1}{2}\int_0^1 \langle -J\dot{x}(t), x(t)\rangle dt - \int_0^1 H(t, x(t))dt$$

on the space $E := W^{1/2,2}(S^1, \mathbb{R}^{2n})$ as it is done in [H-Z]. The elements of E can be written in the form

$$x(t) = \sum_{k\in\mathbb{Z}} e^{2\pi kJt} x_k$$

where $x_k \in \mathbb{R}^{2n}$ such that $\sum_{k\in\mathbb{Z}} |k|\, |x_k|^2 < \infty$. Then E splits orthogonally into

$$E = E^- \oplus E^0 \oplus E^+$$

according to $k < 0$, $k = 0$, $k > 0$ with the norm

$$\|x\|_{1/2}^2 := |x_0|^2 + 2\pi \sum_{k\neq 0} |k|\ |x_k|^2 \ .$$

The action functional is expressed as

$$a_H(x) = \frac{1}{2}(\|x^+\|_{1/2}^2 - \|x^-\|_{1/2}^2) - \int_0^1 H(t, x(t))dt \ .$$

Now we consider for $H \in \mathscr{C}$ the fixed point set $\mathrm{Fix}(\varphi_H^1)$ and the set of corresponding actions

$$\{a_H(x)|x(0) \in \mathrm{Fix}(\varphi_H^1)\} \ .$$

We claim that these actions do not depend on H but only on the map $\varphi := \varphi_H^1$. Indeed, take $x(0) \in \mathrm{Fix}(\varphi)$ and x_∞ such that $H(t, x_\infty) = 0$ for all t. Then pick any path $g(s)$ in $\mathbb{R}^{2n}$ from $g(0) = x_\infty$ to $g(1) = x(0)$. If we define the surface

$$S := \{(t, \varphi_H^t(g(s)))|s, t \in [0, 1]\}$$

it is an easy calculation to show that

$$d(p\,dq - H\,dt)|_{TS} = 0$$

hence by Stokes' Theorem

$$\int_{\partial S} p\,dq - H\,dt = 0 \ .$$

This is equivalent to

$$a_H(x) = \int_{\varphi(g)} p\,dq - \int_g p\,dq$$

which depends only on φ and not on H. Actually also the choice of g is irrelevant.

Thus we may speak of the action $A(x, \varphi)$ of $x \in \mathrm{Fix}(\varphi)$ and define the action spectrum of a map $\varphi \in \mathscr{D}$ as

$$\sigma(\varphi) := \{A(x, \varphi)|x \in \mathrm{Fix}(\varphi)\} \ .$$

This is a compact, nowhere dense set in $\mathbb{R}$ and contains the critical values of a_H on E (cf. [H-Z]).

In [H-Z] Hofer and Zehnder singled out a special critical value by the minimax

$$\gamma(\varphi) := \sup_{F\in\mathscr{F}} \inf_{x\in F} a_H(x)$$

where the minimax set $\mathscr{F} := \{h(E^+)|h \in G\}$ does not depend on H; G stands for a certain group of homeomorphisms of E. This $\gamma(\varphi)$ has some remarkable properties.

Theorem 2 ([H-Z]) *The following relations hold true:*

1. $\gamma(\varphi) \in \sigma(\varphi)$, *and* $\gamma(\varphi) \geq 0$
2. $\gamma(\varphi_H^1) = 0$ *if* $H \geq 0$, *and* $\gamma(\varphi_H^1) > 0$ *if* $H \leq 0$, $H \neq 0$
3. $H \leq K \Rightarrow \gamma(\varphi_H^1) \geq \gamma(\varphi_K^1)$

4. $|\gamma(\varphi) - \gamma(\psi)| \leq d(\varphi, \psi)$
5. $d(id, \varphi) \geq \gamma(\varphi) + \gamma(\varphi^{-1})$

3 Proof of Theorem 1

We make use of an idea by Bialy and Polterovich [B-P] and consider the "time evolution" of the action spectrum, i.e. the set

$$\Sigma_H := \{(t, \sigma(\varphi_H^t)) | 0 < t \leq 1\} \subset \mathbb{R}^2 .$$

By assumption on H no φ_H^t (viewed as a map in $\mathscr{D}$) possesses a nontrivial 1-periodic solution therefore

$$\Sigma_H = \left\{ \left(t, -\int_0^t H(s,x)ds \right) | 0 < t \leq 1, \text{ and } x \in \bigcap_{s \in [0,t]} \operatorname{crit} H_s \right\}$$

where $\operatorname{crit} H_s$ is the set of critical points of H_s. In particular we have the two continuous curves

$$\left\{ \left(t, c_H^{\mp}(t) := -\int_0^t H(s, x_{\pm})ds \right) | 0 < t \leq 1 \right\} \subset \Sigma_H$$

with

$$c_H^+(1) - c_H^-(1) = \|H\| . \tag{1}$$

Lemma 1 *If $H \in \mathscr{C}$ has fixed extremal points then there exists a $t^* = t^*(H) > 0$ such that*

$$\gamma((\varphi_H^t)^{\pm 1}) \geq \pm c_H^{\pm}(t) \tag{2}$$

for $0 < t \leq t^*$. If, in addition, H is admissible equality holds.

Proof. In order to simplify the notation we set

$$H_\tau(t,x) := H(t, x - \tau x_-)$$
$$\theta_\tau(x) := x + \tau x_-$$

and observe that $\varphi_{H_\tau}^t = \theta_\tau \circ \varphi_H^t \circ \theta_\tau^{-1}$. Since the action spectrum is invariant under symplectic conjugation we obtain $\gamma(\varphi_{H_\tau}^t) \in \sigma(\varphi_{H_\tau}^t) = \sigma(\varphi_H^t)$ for all $\tau \in \mathbb{R}$. But $\tau \mapsto \gamma(\varphi_{H_\tau}^t)$ is continuous with image in a nowhere dense set. Whence it must be constant so that $\gamma(\varphi_{H_\tau}^t) = \gamma(\varphi_H^t)$ for all τ. This shows that we may assume that

$$x_- = 0 .$$

Let us define

$$h(t,x) := \tau H(\tau t, x) .$$

Then $\varphi_h^1 = \varphi_H^\tau$ and for small enough $\tau > 0$ we can estimate

$$h(t,x) \leq h(t,0) + \pi|x|^2$$

for all $t \in [0,1]$ and $x \in \mathbb{R}^{2n}$. The monotonicity of γ implies

$$\begin{aligned}\gamma(\varphi_H^\tau) &= \sup_{F\in\mathscr{F}} \inf_{x\in F} a_h(x)\\ &\geq -\int_0^1 h(t,0)dt + \inf_{x\in E^+}\left[\frac{1}{2}(\|x^+\|_{1/2}^2 - \|x^-\|_{1/2}^2) - \pi\int_0^1 |x|^2 dt\right]\\ &= -\int_0^\tau H(t,0)dt + 0\\ &= c_H^+(\tau)\,.\end{aligned}$$

In the case of an admissible H the reversed inequality $\max\sigma(\varphi_H^\tau) = c_H^+(\tau) \geq \gamma(\varphi_H^\tau) \in \sigma(\varphi_H^\tau)$ is trivial.

Analogously one proves the second relation. □

Notice that we would be done with the proof of Theorem 1 if we could show (2) for $t = 1$ because then by Theorem 2 and (1)

$$\|H\| \geq d(\mathrm{id}, \varphi_H^1) \geq \gamma(\varphi_H^1) + \gamma((\varphi_H^1)^{-1}) = \|H\|\,.$$

What are the possible obstructions? Since all the action spectra involved come from constant solutions the only bad phenomenon is the following. The curve $\{(t, c_H^+(t))\}$, or $\{(t, c_H^-(t))\}$, in $\sum_H$ is multiply covered (due to different critical points to the same critical value) and there is a $t_0 \in (0,1)$ at which one of these covering curves branches off. In this case $\gamma(\varphi_H^t)$ might follow this latter branch and end up at a level smaller than $c_H^+(1)$.

We now state a perturbation lemma which allows us to assume that (2) holds true even for all $t \in (0,1]$.

Lemma 2 *Let $H \in \mathscr{C}$ be admissible with isolated fixed extremal points $x_\pm$. Then given any $\epsilon > 0$ there is a modification $L \in \mathscr{C}$ of H satisfying*

1. $\|H - L\| < \epsilon$

2. L has the same fixed extremal points $x_\pm$ as H and

$$\gamma((\varphi_L^t)^{\pm 1}) \geq \pm c_L^\pm(t)$$

is valid for all $t \in (0,1]$.

From this we conclude that

$$d(\mathrm{id}, \varphi_L) = \|L\|$$

and

$$\|H\| \leq \|L\| + \|H - L\| \leq d(\mathrm{id}, \varphi_H^1) + d(\varphi_H^1, \varphi_L^1) + \epsilon \leq d(\mathrm{id}, \varphi_H^1) + 2\epsilon$$

with an arbitrary $\epsilon > 0$. Thus the proof of Theorem 1 is reduced to that of Lemma 2.

4 Proof of Lemma 2

The key idea will be to find a procedure which makes $x_\pm$ the unique fixed extremal points without creating new periodic orbits beyond control. In a first step we are going to remove all fixed minimal points except x_-. As in the proof of Lemma 1 we may restrict ourselves to the case where $x_- = 0$.

We know that

$$\bigcup_{0 \le t \le 1} \operatorname{supp} H_t \subset B_{\sqrt{R}}(0)$$

for some large $R > 0$. For a given $\epsilon > 0$ we pick a smooth function

$$f : [0, \infty) \to [0, \alpha]$$

satisfying

$$f(s) = \begin{cases} 0 & \text{if } s \in [0, \rho] \cup [R + \frac{\epsilon}{\pi}, \infty) \\ \alpha & \text{if } s \in [2\rho, R] \end{cases}$$

and

$$0 > f'(s) > -\pi$$

if $s \in (R, R + \frac{\epsilon}{\pi})$ as well as

$$\|f|_{[0,R]}\|_{C^2} < \frac{\epsilon}{2} \, .$$

Here $\alpha \in (0, \frac{\epsilon}{2})$ and $\rho \in (0, \frac{R}{2})$ are constants and we remark that for any given ρ there exists such an f with a suitable α.

We set

$$K(t,x) := H(t,x) + f(|(\varphi_H^t)^{-1}(x)|^2)$$

and state some properties of K. First of all, $K \in \mathscr{C}$ with $\|H - K\| < \frac{\epsilon}{2}$. Moreover, one can find $0 < r \le \rho < 2\rho \le r' < R$ such that $K(t,x) = H(t,x)$ if $|x|^2 < r$ and $K(t,x) = H(t,x) + \alpha$ whenever $r' < |x|^2 < R$, as well as $r' \to 0$ as $\rho \to 0$. This, together with the assumption that $x_- = 0$ is isolated, implies that K has 0 as its unique fixed minimal point provided we have taken ρ small enough. Using the fact that

$$K(t,x) \le H(t,x) + \frac{\epsilon}{2} \min\{1, |(\varphi_H^t)^{-1}(x)|^2\}$$

we then conclude as in the proof of Lemma 1 that there exists a $t^* = t^*(H, \epsilon) > 0$ such that

$$\gamma((\varphi_K^t)^{-1}) \ge -c_K^-(t) \tag{3}$$

for $0 < t \le t^*$, regardless what the perturbation f actually looks like.

Now we are going to prove that φ_K^T possesses only constant 1-periodic solutions whenever $\frac{t^*}{2} \le T \le 1$. Since 0 has become the only minimal point of K this will ensure that (3) holds true for all $t \in (0, 1]$ and finish the first part of the proof of Lemma 2.

By the transformation law of Hamiltonian vector fields we obtain

$$\varphi_K^t(x) = \varphi_H^t \circ \varphi_f^t(x) = \varphi_H^t(R(t, |x|^2)x)$$

with the "rotation matrix"

$$R(t,s) := e^{2f'(s)Jt} = \cos 2f'(s)t \cdot \mathbf{1} + \sin 2f'(s)t \cdot J \ .$$

Note that $\varphi_K^t(x) = \varphi_H^t(x)$ if $|x|^2 \notin (\rho, 2\rho) \cup (R, R + \frac{\epsilon}{\pi})$ and $t \in (0,1]$. Since H is admissible this means that φ_K^T with $0 < T \leq 1$ has no non-constant 1-periodic solutions starting in that region. Thus we only have to concentrate on the fixed point problem

$$(\varphi_H^T)^{-1}(x) = R(T, |x|^2)x \tag{4}$$

for $T \in [\frac{t^*}{2}, 1]$ and $|x|^2 \in (\rho, 2\rho) \cup (R, R + \frac{\epsilon}{2})$.

For sufficiently small $\rho > 0$ we have

$$\inf\left\{ |(\varphi_H^t)^{-1}(x) - x| \; |\rho \leq |x|^2 \leq 2\rho, \; \frac{t^*}{2} \leq t \leq 1 \right\} > 0$$

because 0 is an isolated fixed point of each φ_H^t. Hence (4) admits no solution with $|x|^2 \in (\rho, 2\rho)$ if we choose $\|f|_{[0,R]}\|_{C^2}$ small enough. If, on the other hand, $|x|^2 \in (R, R + \frac{\epsilon}{\pi})$ we know that $(\varphi_H^T)^{-1}(x) = x$ for every $T \in (0,1]$. In this case (4) reads

$$x = e^{2f'(|x|^2)JT}x$$

with $|x|^2 \in (R, R + \frac{\epsilon}{\pi})$ which has obviously no solution since $0 < 2|f'(|x|^2)|T < 2\pi$.

Just the same considerations applied to

$$L(t,x) := K(t,x) - f(|(\varphi_K^t)^{-1}(x)|^2)$$

lead to our final modification which satisfies all requirements, and Lemma 2 is completely proven.

5 Viterbo's metric

Another approach to defining a bi-invariant metric for the group $\mathscr{D}$ was found by Viterbo [Vi]. He considers the graph of a symplectic diffeomorphism $\varphi = \varphi_H^1 \in \mathscr{D}$ as an exact Lagrangian submanifold $L(\varphi)$ of the cotangent bundle $T^*\Delta^{2n}$ of the diagonal in $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \omega \oplus -\omega)$. A theorem of Sikorav [Si] guarantees the existence of a so-called generating function

$$S : \mathbb{R}^{2n} \times \mathbb{R}^N \to \mathbb{R}$$
$$(x,\xi) \mapsto S(x,\xi)$$

being a quadratic form in the ξ-variables outside a compact set, i.e.

$$L(\varphi) = \left\{ \left(x, \frac{\partial S}{\partial x}\right) \mid \frac{\partial S}{\partial \xi} = 0 \right\} .$$

Yet S is not uniquely determined by φ. Via a homological minimax principle Viterbo was able to pick out a critical value c of S ($-c_-$ in his original notation) which does not depend on the choice of S but only on φ. Moreover, there is an analogue to Theorem 2, namely

Theorem 3 ([Vi]) *The following relations hold true:*
1. $c(\varphi) \in \sigma(\varphi)$, *and* $c(\varphi) \geq 0$
2. $c(\varphi_H^1) = 0$ *if* $H \geq 0$, *and* $c(\varphi_H^1) > 0$ *if* $H \leq 0$, $H \neq 0$
3. $H \leq K \Rightarrow c(\varphi_H^1) \geq c(\varphi_K^1)$
4. $|c(\varphi) - c(\psi)| \leq d(\varphi, \psi)$
5. $d_V(\varphi, \psi) := c(\varphi^{-1}\psi) + c(\psi^{-1}\varphi)$ *defines a bi-invariant metric on* $\mathscr{D}$ *which satisfies* $d(\varphi, \psi) \geq d_V(\varphi, \psi)$.

For completeness we give the proofs for the inequalities in *5.* and *4.* since they are not explicitly contained in [Vi].

Proof. We first prove

$$d(\mathrm{id}, \varphi) \geq d_V(\mathrm{id}, \varphi)$$

by splitting it up into two parts. We claim that whenever $\varphi_H^1 = \varphi$ we have

$$-\int_0^1 \inf H_t \, dt \geq c(\varphi) \tag{5}$$

and

$$\int_0^1 \sup H_t \, dt \geq c(\varphi^{-1}) \, . \tag{6}$$

In order to show (5) we choose (a C^∞-approximation to) the Hamiltonian

$$K(t,x) := (\inf H_t)\rho(|x|^2) \leq 0$$

where $\rho : \mathbb{R} \to [0,1]$ has compact support and is identically 1 on a big ball containing all the $\operatorname{supp} H_t$; in addition, we take ρ such that K is admissible. Then $c(\varphi_K^1)$ has to be the action of a constant solution, and since $K \leq H$ we obtain by *2.* and *3.* that

$$c(\varphi_K^1) = -\int_0^1 \inf H_t \, dt \geq c(\varphi) \, .$$

For the proof of (6) we consider

$$L(t,x) := -(\sup H_t)\rho(|x|^2) \leq 0$$

and observe that $L(t,x) \leq -H(t, \varphi_H^t(x))$ whence

$$c(\varphi_L^1) = \int_0^1 \sup H_t \, dt \geq c(\varphi^{-1}) \, .$$

Finally we are going to show that

$$|c(\varphi) - c(\psi)| \leq d_V(\varphi, \psi)$$

which implies *4*. But this is an immediate consequence of the inequality $c(\varphi) = c(\psi^{-1}\varphi\psi) \leq c(\psi^{-1}\varphi) + c(\psi) \leq d_V(\varphi, \psi) + c(\psi)$. □

Suppose that $\varphi = \varphi_H^1$ for an admissible $H \in \mathscr{C}$ with isolated fixed extremal points $x_\pm$ as in Theorem 1. For small enough t the diffeomorphism φ_H^t is C^1-close to the identity hence $L(\varphi_H^t)$ is a graph over $T^*\Delta^{2n}$ and admits a classical generating function $S : \mathbb{R}^{2n} \to \mathbb{R}$. In this case Viterbo's minimax yields simply $\sup S$ thus we have for sufficiently small $t > 0$ that

$$c(\varphi_H^t) = -\int_0^t \inf H_s \, ds = -\int_0^t H(s, x_-) ds \; .$$

This argument replaces that given in the proof of Lemma 1 and shows that Lemma 1 holds for Viterbo's critical value c as well as for Hofer-Zehnder's γ. Now we can proceed exactly as in Sects. 3 and 4 and obtain

Theorem 4 *If $H \in \mathscr{C}$ is admissible and has isolated fixed extremal points then $d_V(id, \varphi_H^1) = \|H\|$.*

In particular, the two metrics d and d_V coincide on this class of symplectic diffeomorphisms in $\mathscr{D}$. It is still an open question whether they are generally the same or not.

6 Concluding remarks

There are obvious cases where one may relax the condition of having isolated fixed extremal points, e.g. if H has a fixed sign. Theorem 1 generalizes results by Hofer [Ho2] and Long [Lo] and gives a virtually ultimate answer to the minimal geodesic problem for Hofer's metric as long as no nontrivial fixed points occur.

Bialy and Polterovich [B-P] showed how to deduce Lemma 1 only from an axiomatic description of a critical value like that given in Theorem 2 or Theorem 3. Thus our results can be obtained without using the very definition of γ or c.

Hofer's metric was generalized to arbitrary symplectic manifolds by Lalonde and McDuff [L-M1] who also proved an analogue to Theorem 1 in this more general framework [La, L-M3].

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