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# Upward Drawings of Triconnected Digraphs[1]

P. Bertolazzi,[2] G. Di Battista,[3] G. Liotta,[3] and C. Mannino[2]

**Abstract.** A polynomial-time algorithm for testing if a triconnected directed graph has an upward drawing is presented. An upward drawing is a planar drawing such that all the edges flow in a common direction (e.g., from bottom to top). The problem arises in the fields of automatic graph drawing and ordered sets, and has been open for several years. The proposed algorithm is based on a new combinatorial characterization that maps the problem into a max-flow problem on a sparse network; the time complexity is $O(n + r^2)$, where $n$ is the number of vertices and $r$ is the number of sources and sinks of the directed graph. If the directed graph has an upward drawing, the algorithm allows us to construct one easily.

**Key Words.** Planarity, Automatic graph drawing, Hierarchical structures, Max-flow, *st*-Digraphs, Acyclic digraphs, Ordered sets.

**1. Introduction.** Planarity has been deeply investigated both in combinatorics and in graph-algorithms research [20]. Concerning undirected graphs, there are elegant characterizations of the graphs that have a planar representation and efficient algorithms for testing planarity (see, e.g., [17], [12], [2], [4], and [7]).

For acyclic directed graphs (in the following *digraphs*) the concept of planar drawing is naturally replaced by the concept of *upward drawing*, that is, a planar drawing with the additional constraint that all the edges are represented by curves increasing monotonically in the vertical direction. Figure 1 shows an example of upward drawing.

In practice (see, e.g., [25] and [10]) upward drawings are extensively used to display hierarchical structures such as PERT networks, ISA hierarchies in knowledge-representation diagrams, and subroutine call charts. Notice also that the construction of upward drawings can be viewed as computing "geometric realizations" of planar digraphs as monotone subdivisions.

Upward drawings have been investigated in the fields of ordered sets and automatic graph drawing (surveys on the automatic-graph-drawing problems can be found in [27] and [9]).

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[2] IASI–CNR, Viale Manzoni, 30-00185 Roma, Italy.

[3] Dipartimento di Informatica e Sistemistica, Università di Roma "La Sapienza," Via Salaria, 113-00198 Roma, Italy.

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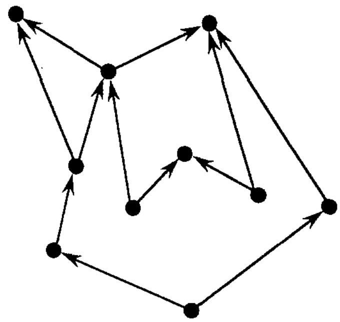

**Fig. 1.** An example of upward drawing.

Kelly and Rival [16] provided a characterization of *planar lattices*, i.e., lattices whose covering digraph admits an upward drawing, in terms of a family of forbidden subposets. Platt [21] showed that a lattice has an upward drawing if and only if the undirected graph obtained from the covering digraph by adding an edge between its source and its sink and by ignoring edge directions is planar. Other results on upward drawings in the field of ordered sets can be found in [30], [22], [14], [24], and [23].

Combinatorial characterizations of the digraphs that have an upward drawing are given in [15] and [6] (for general directed graphs) and in [29] (for single-source digraphs). In [15] and [6] a digraph is shown to have an upward drawing if and only if it is a subgraph of a planar *st-digraph*; in [29] a characterization is given in terms of forbidden circuits.

In [8] a lower bound on the area of upward drawings and an algorithm for constructing straight-line drawings that display the symmetries of a digraph are presented.

Recent results on the problem of testing if a given digraph has an upward drawing can be found in [5] and [13]. In [5] it is shown that a bipartite digraph has an upward drawing if and only if its underlying undirected graph is planar. In [13] an $O(n^2)$-time algorithm is given for upward-drawing testing in single-source digraphs with $n$ vertices.

In this paper we provide an $O(n + r^2)$-time algorithm for testing if a triconnected digraph $G$ has an upward drawing, where $n$ is the number of vertices of $G$ and $r$ is the number of its sources and sinks. The algorithm can decide within the same time bound if $G$ is a subgraph of a planar *st*-digraph and in this case can produce a planar *st*-digraph $G'$, with $n + 2$ vertices, that includes $G$. This has several implications; in fact $G'$ can be used for constructing:

1. An upward drawing of $G$ that maps each edge to a straight-line segment in $O(n \log n)$ time [6].
2. An upward drawing of $G$ that maps each edge to a polygonal line, such that the total number of bends in the drawing is at most $2n - 5$, in $O(n)$ time [8].
3. An upward drawing of $G$ that maps each edge to a straight-line segment in $O(n)$ time, if $G'$ has no transitive edges [6], [8].
4. A *directed visibility representation* of $G$ in $O(n)$ time [28], [6].

Our approach consists of two main phases.

First, we tackle the problem of testing if a planar digraph with a given planar embedding has an upward drawing that preserves the embedding. The problem is restated as a max-flow problem on a sparse network by using a new combinatorial characterization. This characterization is based on some geometric properties of the upward drawings of circuits.

Second, we exploit the fact that triconnected planar digraphs have a unique planar embedding.

The proposed approach allows us to represent implicitly all the possible *upward embeddings* of $G$, obtaining, for triconnected digraphs, a result that is similar to the one of Chiba *et al.* [3] on the planar embeddings of a planar biconnected graph.

The paper is organized as follows. In Section 2 we recall some basic definitions. In Section 3 we deal with the upward drawings of circuits. Section 4 contains the above-mentioned characterization. Section 5 contains the algorithm. In Section 6 we present related open problems.

**2. Preliminaries.** We assume familiarity with planar graphs, say from [11] and [20]. In the following some terminology and basic results are summarized.

We review some definitions on graph connectivity. A separating $k$-set of a graph $G$ is a set of $k$ vertices whose removal increases the number of connected components of $G$. Separating 1-sets and 2-sets are called *cutvertices* and *separation pairs*, respectively. A connected graph is said to be *biconnected* if it has no cutvertices. A graph is *triconnected* if it is biconnected and has no separation pairs. In the following, unless otherwise specified, we deal with biconnected graphs that do not have self-loops and multiple edges.

Let $\Gamma$ be a *drawing* of a graph $G$; $\Gamma$ maps each vertex of $G$ to a distinct point of the plane and each edge $(u, v)$ to a simple Jordan curve with endpoints $u$ and $v$. If each edge is mapped to a straight-line segment, $\Gamma$ is a *straight-line drawing*. Graph $G$ has a *planar drawing* if it has a drawing such that no two edges intersect, except at common endpoints. A graph is *planar* if it has a planar drawing.

Two planar drawings of a planar graph are *equivalent* when, for each vertex $v$, they have the same circular clockwise ordering of the edges incident on $v$. In this way the planar drawings of a planar graph are grouped into equivalence classes. In Figure 2 we show three planar straight-line drawings of a graph. The second

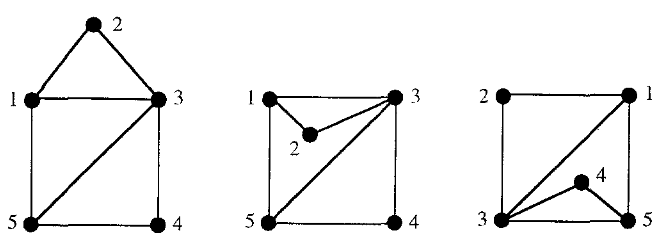


**Fig. 2.** Planar drawings of a graph.

and third drawings are equivalent. A triconnected planar graph has a unique equivalence class of planar drawings, up to a reflection [32].

Given a graph $G$, an *embedding* of $G$ associates to each vertex $v$ of $G$ a circular clockwise ordering of the incidence list of $v$. A graph with a given embedding is an *embedded graph*. In other words, an embedded graph is such that, for each vertex $v$, the edges that have $v$ as an endpoint are circularly clockwise sorted in the incidence list of $v$.

Given a planar graph $G$, we associate with each equivalence class $\Gamma$ of planar drawings of $G$ a *planarly embedded graph*, i.e., graph $G$ with the embedding (*planar embedding*) that is given by the circular clockwise ordering of the edges incident around each vertex of $\Gamma$. A triconnected planar graph has a unique planar embedding, up to a reflection.

A planar drawing $\Gamma$ divides the plane into connected regions called *facial components*. Each facial component is identified by the clockwise circular list of the vertices and the edges of its boundary. These lists are simple cycles of the planarly embedded graph that corresponds to $\Gamma$ and are called *faces*. All the drawings of a given planarly embedded graph have the same set of faces. The face that is the boundary of the unbounded facial component is usually called the *external face*.

Let $G$ be a digraph; the *underlying graph* of $G$ is the undirected graph obtained from $G$ by considering its edges as undirected. We define the planarity and connectivity properties of a digraph as the planarity and connectivity properties of its underlying graph.

An *upward drawing* of a digraph $G$ is a planar drawing of $G$, with the additional constraint that all the edges are represented by curves increasing monotonically in the vertical direction, according to which all the tangents to the curves representing edges make an angle $\theta$ with the horizontal satisfying $0° < \theta < 180°$. A trivial necessary condition for a digraph to have an upward drawing is to be acyclic.

An *st-digraph* is an acyclic digraph that has exactly one source (vertex without incoming edges) $s$, exactly one sink (vertex without outgoing edges) $t$, and contains the edge $(s, t)$. A characterization of the class of digraphs that have an upward drawing is given in [15] and [6] where the following two theorems are shown.

THEOREM 1. *Let $G$ be a digraph, the following statements are equivalent*:

*$G$ has an upward drawing*;
*$G$ has a straight-line upward drawing*;
*$G$ is a subgraph of a planar st-digraph.*

THEOREM 2. *Let $\Psi$ be a planarly embedded st-digraph. Let $\Gamma$ be the class of equivalent planar drawings associated with $\Psi$. Class $\Gamma$ contains upward drawings.*

According to Theorem 1, in the following we consider only straight-line upward drawings.

**3. Upward Drawings of Circuits.** We call an acyclic digraph whose underlying graph is a simple cycle a *circuit*. Let $C$ be a circuit and let $S$ ($T$) be the set of sources (sinks) of $C$. We have that $|S| = |T|$. We call *switches* the vertices of $S \cup T$; also, we call *source-switches* (*sink-switches*) the vertices of $S$ ($T$).

We observe that a circuit always has an upward drawing, in fact it can be trivially included into a planar *st*-digraph. Consider an upward drawing $\Gamma$ of $C$; $\Gamma$ is a *simple polygon* (i.e., a polygon such that no pair of nonconsecutive edges share a point) that divides the plane into two connected regions; we call the unbounded region *external*, the other one *internal*.

We focus on the drawing of a switch $v$ of $C$. Vertex $v$ is mapped into a vertex of the polygon. The two segments incident on $v$ define two angles, one in the internal region and one in the external region. Concerning the angle in the internal region there are two possibilities: if it is convex, we say that it is a *small* angle, else we say that it is a *big* angle. Figure 3 shows an upward drawing of a circuit. In the following we denote with $S_B$ ($S_S$) the set of source-switches that are drawn with a big (small) angle in the internal region. Analogously, we denote with $T_B$ ($T_S$) the set of sink-switches that are drawn with a big (small) angle in the internal region. We characterize $\Gamma$ by means of the following lemmas.

LEMMA 1. *$|S_S| \geq 1$ and $|T_S| \geq 1$.*

LEMMA 2. *If $|S| \geq 2$, then $|S_B| + |T_B| \geq 1$.*

PROOF. (By contradiction.) Suppose $|S_B| + |T_B| = 0$. Suppose that switches are labeled $s_1, t_1, \ldots, s_n, t_n$ following $\Gamma$ in clockwise order, where $s_i$ and $t_i$ ($i = 1, \ldots, n$) denote source-switches and sink-switches, respectively. Let $y(v)$ be the $y$-coordinate of vertex $v$ in $\Gamma$. We denote with $[a, b]$ a polygonal line connecting vertices $a$ and $b$. Suppose, without loss of generality, that $y(s_1) \leq y(s_2)$. See the example in Figure 4. Due to the fact that the angle in $s_2$ is a small angle, vertex $t_2$ has to be placed inside the region defined by the horizontal line through $s_2$, the polygonal line

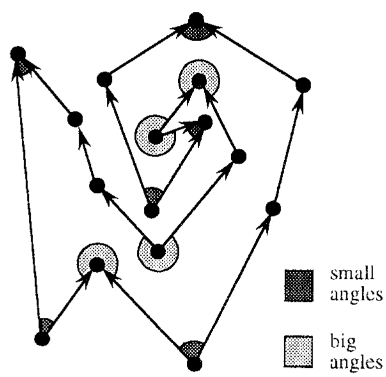


**Fig. 3.** Small and big angles in the internal region of an upward drawing of a circuit.

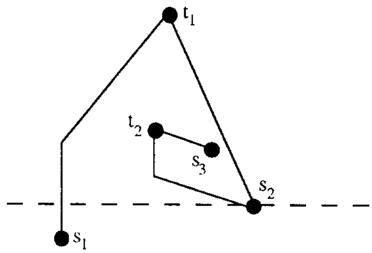


**Fig. 4.** An example for the proof of Lemma 2.

$[s_2, t_1]$, and the polygonal line $[s_1, t_1]$. It follows that vertex $s_3$ has to be into the region delimited by the horizontal line through $t_2$, the polygonal line $[t_1, s_2]$, and the polygonal line $[s_2, t_2]$; hence $y(s_2) < y(s_3)$. Repeating the argument, visiting vertices in clockwise order we obtain $y(s_1) \leq y(s_2) < y(s_3) < \cdots < y(s_n) < y(s_1)$. This is a contradiction. □

For small and big angles in the external region we give the same definition we have given for small and big angles in the internal region.

LEMMA 3. *An upward drawing of a circuit with* $2n$ *switches has exactly* $n - 1$ *big angles in the internal region and exactly* $n + 1$ *big angles in the external region.*

PROOF. (By induction on $n$.) The proof is trivial for $n = 1$ and $n = 2$. Suppose now that the thesis holds for $n$, we show that it holds for $n + 1$. Consider a drawing $\Gamma$ of $C$. Due to Lemma 2, $\Gamma$ has at least one big angle in the internal region. Let $t_i$ be the switch associated with such big angle; suppose $t_i$ is a sink-switch (the proof is analogous if $t_i$ is a source-switch). There is at least one vertex $v$ of $\Gamma$, over the horizontal line through $t_i$ that çan be joined with $t_i$ by using a segment, without crossing $\Gamma$ (see Figure 5). Let $C_1$ be the circuit $t_i$, $s_{i+1}, \ldots, v$ and let $C_2$ be the circuit $t_i$, $v, \ldots, s_i$. Suppose $v$ is a sink-switch with a small angle (the other cases

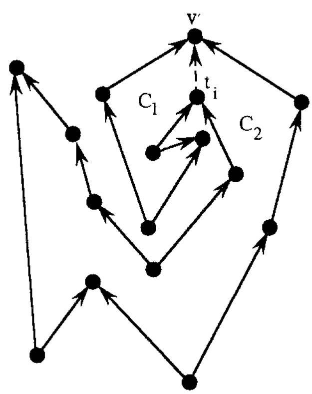


**Fig. 5.** Decomposition of $\Gamma$ in the proof of Lemma 3.

are analogous). Consider the two upward drawings $\Gamma_1$ and $\Gamma_2$ of $C_1$ and $C_2$ obtained with the above construction. Let $2n_1$ and $2n_2$ be the number of switches of $C_1$ and $C_2$, respectively; $n_1 \neq 0$ and $n_2 \neq 0$. We have that $2n = 2n_1 + 2n_2$, in fact, $t_i$ is not a sink-switch in $C_1$ and $C_2$, but $v$ is a sink-switch in both of them. Due to the inductive hypothesis, the number of big angles in $\Gamma_1$ ($\Gamma_2$) is equal to $n_1 - 1$ ($n_2 - 1$). Now $\Gamma$ can be obtained by gluing $\Gamma_1$ and $\Gamma_2$ and by removing segment $t_i$, $v$. So, the number of big angles of $\Gamma$ is given by the number of big angles of $\Gamma_1$ plus the number of big angles of $\Gamma_2$ plus the big angle associated with $t_i$, i.e., $n_1 + n_2 - 1$. Since $n = n_1 + n_2$, it follows the first part of the thesis. The second part is trivially proved by observing that big angles in the internal region correspond to small angles in the external region and vice versa. □

From Lemma 3 we can derive the following, purely geometrical, corollary. Let $P$ be a simple polygon and let $d$ be any oriented straight line of the plane. We call a vertex $v$ of $P$ such that its incident segments are both over or both under the line through $v$ orthogonal to $d$ a *peak with respect to d*. □

COROLLARY 1. *The number of peaks with respect to d that have a convex angle in the internal region of P is equal to the number of peaks with respect to d that have a concave angle in the internal region of P plus* 2.

## 4. Upward Drawings and Upward Embeddings.

Let $\Gamma$ be an upward drawing of a digraph $G$. Consider a face $f$ of $\Gamma$. Since $f$ is a circuit, we have that the drawing $\Gamma_f$ obtained from $\Gamma$ by deleting all points and segments that do not belong to $f$ is an upward drawing of a circuit.

Let $2n_f$ be the number of switches of $f$, due to Lemma 3 we have that the number of big angles in the internal region of $\Gamma_f$ is equal to $n_f - 1$. Moreover, the same condition holds for any simple circuit of $G$. So, it raises naturally the question whether the condition of Lemma 3 can be used in some way to decide if a digraph has an upward drawing. In order to answer that question we give a deeper characterization of the upward drawings. Moreover, we introduce the new concepts of *upward embedding* and *upward embedded digraph*, the directed couterparts of the concepts of planar embedding and a planarly embedded graph for undirected planar graphs.

### 4.1. Properties of Upward Drawings.

Let $\Gamma$ be an upward drawing of a digraph $G$. Let $v$ be a vertex of $G$. We have the following properties (see Figure 6):

PROPERTY 1. *The outgoing (incoming) edges of v are drawn in $\Gamma$ entirely over (under) the horizontal line through v.* (See vertex 1 in Figure 6.)

PROPERTY 2. *Suppose v is a source or a sink and consider the faces of $\Gamma$ that share v; we have that v is a switch in all of them.* (Vertex 2 in Figure 6 is a switch in faces $A$, $B$, and $C$.)

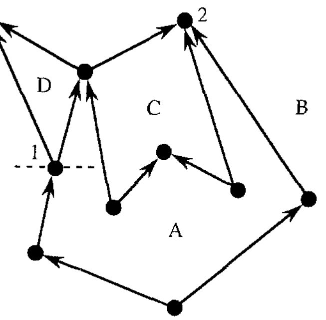


**Fig. 6.** An example for Properties 1–4.

PROPERTY 3. *Let $v$ be an internal vertex (a vertex which is not a source or a sink) with total degree $n$; $v$ is a switch in exactly $n - 2$ faces.* (Vertex 1 in Figure 6 is a switch in face $D$ and is not a switch in faces $A$ and $B$.)

PROPERTY 4. *Consider the angles defined by pairs of consecutive outgoing (incoming) edges of $v$. If $v$ is a source (sink) exactly one of them is a big angle; if $v$ is an internal vertex all of them are small.* (Vertex 2 in Figure 6 has exactly one big angle in face $B$ while vertex 1 has no big angles.)

Given an upward drawing, Property 1 allows us to identify for each vertex $v$ *two linear lists of edges*, obtained by visiting from left to right the outgoing and the incoming edges. If $v$ is a source, the list of incoming edges is empty. If $v$ is a sink, the list of outgoing edges is empty.

Let $v$ be a source or a sink of $G$ and let $f$ be one of the faces of $\Gamma$ that share $v$. We say that $v$ is *assigned* to $f$ if $v$ has a big angle in $f$. We say that $v$ is *not assigned* to all the other faces. Observe that if $v$ is a source (sink), the first and the last element of the list of the outgoing (incoming) edges belong to the face to which $v$ is assigned.

An *upward face* of an upward drawing is a face where a label in $\{B, S\}$ is associated with each switch. Label $B$ is associated with switches that are sources or sinks of $G$ assigned to $f$; label $S$ is associated with all the other switches, namely, to switches that are sources or sinks of $G$ that are not assigned to $f$ and to switches that are internal vertices of $G$. The upward faces of the upward drawing of Figure 1 are shown in Figure 7.

The upward face that is the boundary of the unbounded region is called the *external upward face.*

*4.2. Upward Embeddings.* Two upward drawings of a digraph are *equivalent* if, for each vertex, they define the same two linear lists of incoming and outgoing edges. In this way the upward drawings of a digraph are grouped into equivalence classes.

Given a digraph $G$, a *2-lists-embedding* arranges the incident edges of each vertex

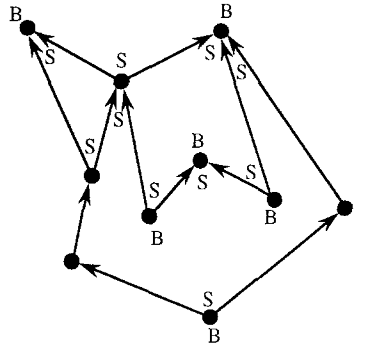


**Fig. 7.** Labels $B$ and $S$ in the upward faces of the upward drawing of Figure 1.

of $G$ into two linear ordered lists: a list of incoming edges and a list of outgoing edges. A digraph with a given 2-lists-embedding is a *2-lists-embedded digraph.*

Given a digraph $G$ that has at least one upward drawing, we associate to each equivalence class $\Gamma$ of upward drawings of $G$ an *upward embedded digraph,* i.e., digraph $G$ with the 2-lists-embedding (*upward embedding*) that is given by the two linear lists of edges incident on each vertex of $\Gamma$.

It is easy to show that all the upward drawings of an upward embedded digraph have the same set of upward faces. Moreover, all the upward drawings of an upward embedding have the same external upward face.

The planarly embedded digraph resulting from an upward embedded digraph $\Phi$ by considering the clockwise ordering obtained concatenating the two lists of edges defined on each vertex is called the *underlying planarly embedded digraph* of $\Phi$.

We now rephrase Lemma 3 in the framework of upward embeddings as follows.

Let $\Phi$ be an upward embedded digraph, let $h$ be its external upward face, and let $f$ be an internal upward face. Denote with $2n_h$ and $2n_f$ the number of switches of $h$ and $f$, respectively.

LEMMA 4. *The number of sources and sinks of* $\Phi$ *assigned to* $f$ *is equal to* $n_f - 1$. *The number of sources and sinks of* $\Phi$ *assigned to* $h$ *is equal to* $n_h + 1$.

From Property 4 and from Lemma 4 we have that if $\Phi$ is an upward embedded digraph, then:

(a) Each source or sink is assigned to exactly one of the faces it belongs to.
(b) For each face the condition of Lemma 4 is satisfied.

*4.3. Candidate Embeddings.* Another important step toward our characterization of upward embeddings is the concept of *candidate embedding.* A planarly embedded acyclic digraph $\Psi$ is a *candidate embedded digraph* if, for each vertex $v$, we have that all the outgoing (incoming) edges appear consecutively in the list of the edges incident on $v$. In this case we say that the planar embedding of $\Psi$ is a *candidate embedding.* See the example in Figure 8.

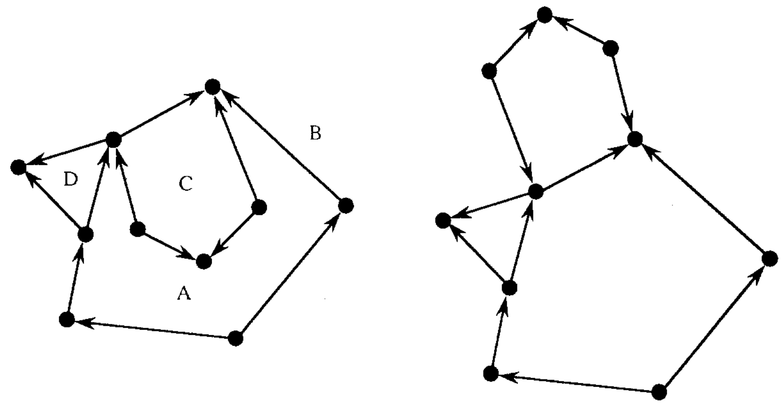


**Fig. 8.** Two planar drawings of the digraph of Figure 1: the one on the left corresponds to a candidate embedding, while the one on the right does not correspond to any candidate embedding.

Given a candidate embedded digraph we choose one of its faces as the *candidate external face* and call the remaining faces *internal.*

Let $\Psi$ be a candidate embedded digraph and let $h$ be a candidate external face of $\Psi$. We define the *capacities* of the faces of $\Psi$. According to Lemma 4 we define the capacity $c_f$ of a face $f \neq h$ as

$$c_f = n_f - 1 = |S_f| - 1 = |T_f| - 1,$$

where $S_f$ ($T_f$) is the set of source-switches (sink-switches) of $f$. Similarly, we define the capacity $c_h$ of the candidate external face $h$ as

$$c_h = n_h + 1 = |S_h| + 1 = |T_h| + 1.$$

It is interesting to observe that candidate embedded digraphs have Properties 2 and 3 that have been shown for upward drawings. Another property of candidate embedded digraphs is given in the following lemma.

LEMMA 5. *Let $\Psi$ be a candidate embedded digraph. Let $S$ and $T$ be the sets of sources and sinks of $\Psi$. Let $F$ be the set of faces of $\Psi$ and let $h \in F$ be a candidate external face. We have that*

$$\sum_{f \in F} c_f = |S| + |T|.$$

PROOF.

$$\sum_{f \in F} c_f = \sum_{f \in (F - \{h\})} (n_f - 1) + n_h + 1 = \sum_{f \in F} n_f - |F| + 2,$$

where $2n_f$ is the number of switches of face $f$. By applying Properties 2 and 3 we have

$$2 \sum_{f \in F} n_f = |E_S| + |E_T| + |E_I| - 2|I| = 2|E| - 2|I|,$$

where $E$ is the set of edges of $G$, $E_S$ $(E_T)$ is the set of edges outgoing (incoming) from (into) the sources (sinks) of $G$, $E_I$ is the set of edges incident on the internal vertices of $G$, and $I$ is the set of the internal vertices.

From Euler's theorem we have

$$2(|E| - |I|) = 2(|S| + |T| + |F| - 2)$$

and, hence,

$$\sum_{f \in F} c_f = |S| + |T| + |F| - 2 - |F| + 2 = |S| + |T|. \qquad \square$$

As an example of Lemma 5, consider the candidate embedded digraph on the left of Figure 8. If we consider face $B$ as the candidate external face we have $c_A = 2$, $c_B = 3$, $c_C = 1$, and $c_D = 0$; the number of sources and sinks is six.

*4.4. Upward-Consistent Assignments.* Given a candidate embedded digraph, a candidate external face, and an assignment of sources and sinks to faces, and consequently an assignment of labels $S$ and $B$ to switches of faces, we call this assignment *upward-consistent* if it satisfies conditions (a) and (b) of Section 4.2. See the three examples of upward-consistent assignments for the same candidate embedded digraph, where face $B$ is choosen as the candidate external face, shown in Figure 9.

It is clear that an upward-consistent assignment forces a candidate embedded digraph in a 2-lists-embedded digraph.

The concept of an upward-consistent assignment can be generalized in terms of an assignment of sources and sinks to any circuit of the candidate embedded digraph by means of the concept of a *derived assignment*. Let $\Psi$ be a candidate

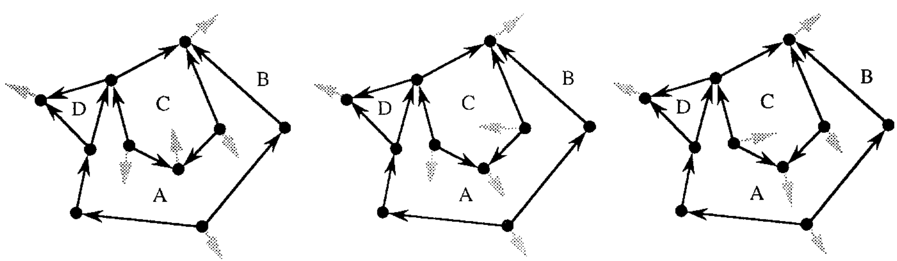


**Fig. 9.** Three upward-consistent assignments.

embedded digraph and let $h$ be a candidate external face of $\Psi$. Let $A$ be an upward-consistent assignment for $\Psi$. Consider any simple circuit $\gamma$ of $\Psi$.

Circuit $\gamma$ splits $\Psi$ into two parts. One of them contains $h$. Consider the planarly embedded digraph $\Psi_\gamma$ obtained from $\Psi$ by removing edges and vertices of $\Psi$ that are in the part of $\Psi$ that does not contain $h$. Observe that $\gamma$ is a face of $\Psi_\gamma$ and observe that $\Psi_\gamma$ is itself a candidate embedded digraph. Moreover, all the faces of $\Psi_\gamma$, that were faces of $\Psi$, have the same capacity they had in $\Psi$ and all the new sources and sinks created by the removal of edges lie on face $\gamma$.

We define for $\Psi_\gamma$ the derived assignment $A_\gamma$ "based" on $A$ and show that it is an upward-consistent assignment for $\Psi_\gamma$. Let $s$ be a source or a sink of $\Psi_\gamma$. We call the *derived assignment* for $\Psi_\gamma$, and denote if by $A_\gamma$, the assignment obtained from $A$ in the following way:

1. If $s$ was assigned in $A$ to a face still appearing in $\Psi_\gamma$ it is assigned to the same face in $A_\gamma$.
2. Otherwise $s$ is assigned to $\gamma$.

LEMMA 6. *Let $\Psi$ be a candidate embedded digraph and let $A$ be an upward-consistent assignment for $\Psi$. Let $\gamma$ be a circuit of $\Psi$. The derived assignment $A_\gamma$ is an upward-consistent assignment for $\Psi_\gamma$.*

PROOF. From the definition of an upward-consistent assignment we have to show the following condition: for each face $f$ of $\Psi_\gamma$ the number of sources and sinks assigned to $f$ is equal to $c_f$.

The capacity $c_f$ of a face $f \neq \gamma$ of $\Psi_\gamma$ is the same as in $\Psi$ and the same number of sources (sinks) as in $A$ is assigned to $f$ in $A_\gamma$; hence, for such faces, the condition is satisfied.

Observe that all the sources and sinks not assigned in $A_\gamma$ to a face $f \neq \gamma$ lie on $\gamma$. The number of such sources and sinks is exactly $|S_\gamma| + |T_\gamma| - \sum_{f \neq \gamma} c_f$, where $S_\gamma$ and $T_\gamma$ are the set of sources and sinks of $\Psi_\gamma$. Moreover, such sources and sinks are all assigned to $\gamma$.

Since $\Psi_\gamma$ is a candidate embedded digraph, by Lemma 5 we have that $c_\gamma = |S_\gamma| + |T_\gamma| - \sum_{f \neq \gamma} c_f$. □

*4.5. A Characterization of Upward Embeddings.* The following theorem gives a complete characterization of the upward embeddings and is the key for the algorithms of Section 5.

THEOREM 3. *Given a candidate embedded biconnected digraph $\Psi$ and a candidate external face $h$ of $\Psi$, the 2-lists-embedding $\Phi$ obtained from the assignment $A$ of sources and sinks to faces is an upward embedding with external face $h$ if and only if $A$ is an upward-consistent assignment.* (See, in Figure 10, the three upward drawings obtained from the upward-consistent assignments of Figure 9.)

PROOF. The only-if-part follows immediately from Lemma 4.

In the proof of the if-part we use the results of Theorems 1 and 2. Namely,

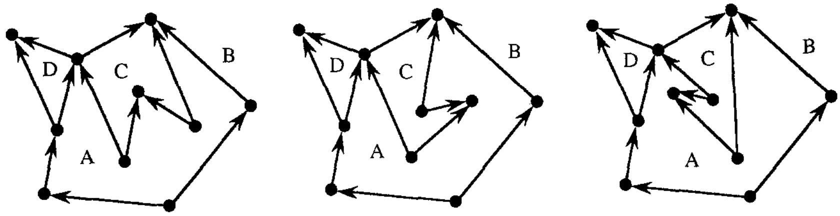


**Fig. 10.** Three upward drawings obtained from the upward-consistent assignments of Figure 9.

we exploit the upward-consistent assignment $A$ to construct a planarly embedded $st$-digraph that includes $\Psi$.

We give a procedure that, for each face $f$, adds edges between pairs of switches of $f$. We show that each edge insertion preserves planarity and acyclicity. We also show that the new planar embedding that we obtain after each edge insertion is itself a candidate embedding and we define for it a new assignment that is an upward-consistent assignment. Moreover, we show that after all edge insertions have been performed, the resulting planarly embedded digraph has exactly one source $s$ and one sink $t$ that stay on the same face. Hence, it is possible to add the edge $(s, t)$ to obtain a planarly embedded $st$-digraph that includes $\Psi$.

*A procedure for inserting edges.* We associate to each face $f$ a circular sequence $\sigma_f$ of symbols obtained by traversing $f$ in clockwise order and assigning $s_B$ and $t_B$ ($B$-symbols) to source-switches and sink-switches labeled $B$ in $f$, and $s_S$ and $t_S$ ($S$-symbols) to source-switches and sink-switches labeled $S$ in $f$. If $f$ is an internal face, $\sigma_f$ contains $c_f$ $B$-symbols and $c_f + 2$ $S$-symbols. We now concentrate on the edge insertions in an internal face. Afterwards we consider the external face.

The procedure for edge insertion in an internal face works as follows. It looks in $\sigma_f$ for *canonical* subsequences. When one such canonical subsequence is found, an edge is added to $\Psi$ in $f$; $f$ is split into two new faces $f'$ and $f''$, and $\sigma_f$ is split into two new sequences $\sigma_{f'}$ and $\sigma_{f''}$. Figure 11 shows an example of the behavior of procedure SaturateFace. Observe that this technique is structurally similar to the "symbolic decomposition" of rectilinear polygons given in [26].

**Procedure** SaturateFace($f$)
**begin**
**If** $f$ has exactly one source-switch and one sink-switch (i.e., $|\sigma_f| = 2$)
**then stop**
**else begin**
    Find a canonical subsequence $(x, y, z)$ in $\sigma_f$ composed by one $B$-symbol followed by two consecutive $S$-symbols
    InsertEdge($f$, $x$, $y$, $z$, $f'$);
    SaturateFace($f'$)
  **end**
**end** (procedure SaturateFace)

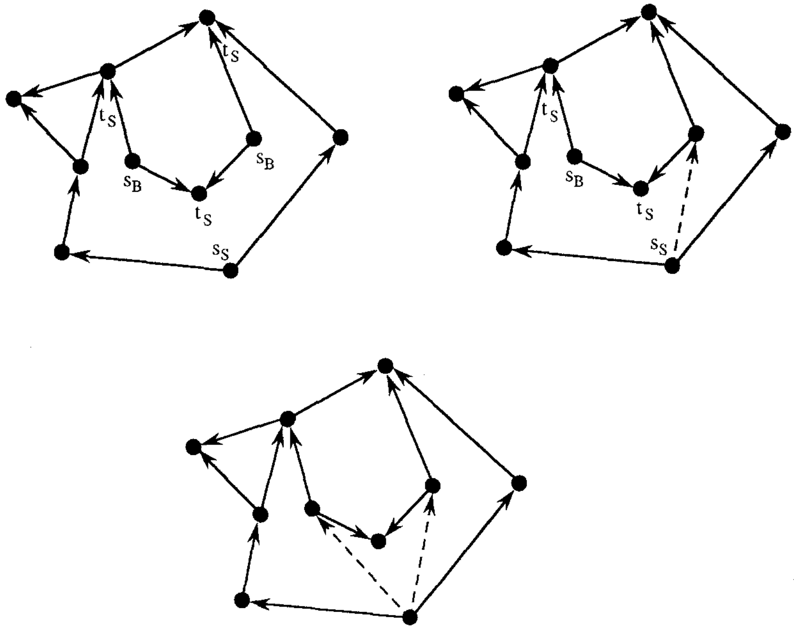


**Fig. 11.** An example for procedure SaturateFace.

Procedure InsertEdge($f, x, y, z, f'$) is defined as follows (observe that such a procedure returns only face $f'$, since $f''$ does not need further processing).

**Procedure** InsertEdge($f, x, y, z, f'$)
**begin**
Let $v_x$, $v_y$, and $v_z$ be the vertices associated with symbols $x$, $y$, and $z$, respectively;
**case** $(x, y, z)$ **of** (in both cases two new faces are created: $f''$ consists of the part of $f$ containing $v_x$, $v_y$, and $v_z$ plus the new edge; $f''$ has only one source and only one sink; the face $f'$ is described for each case)
$(s_B, t_S, s_S)$: Add edge $(v_z, v_x)$. $f'$ consists of the part of $f$ that does not contain $v_y$ plus the new edge $(v_z, v_x)$. Observe that $v_x$ is not a source of the new digraph and $\sigma_{f'}$ can be obtained from $\sigma_f$ by replacing $s_B, t_S, s_S$ with $s_S$;
$(t_B, s_S, t_S)$: Add edge $(v_x, v_z)$. $f'$ consists of the part of $f$ that does not contain $v_y$ plus the new edge $(v_x, v_z)$. Observe that $v_x$ is not a sink of the new digraph and $\sigma_{f'}$ can be obtained from $\sigma_f$ by replacing $t_B$, $s_S, t_S$ with $t_S$;
**end** (procedure InsertEdge)

*Planarity and acyclicity.* As far as planarity is concerned, each edge is inserted by procedure InsertEdge inside a face. Hence, it does not originate crossings.

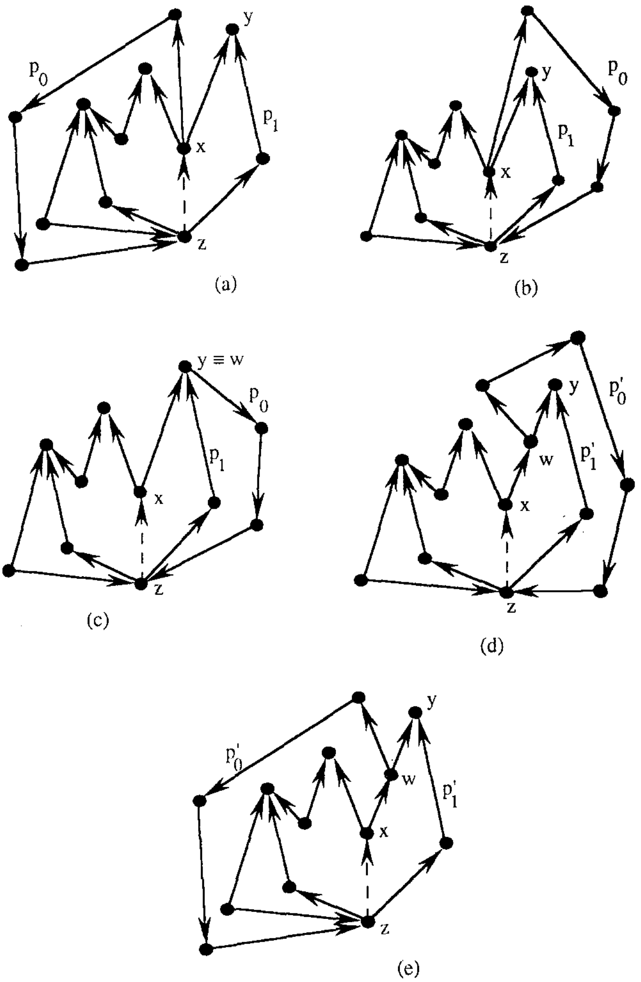


**Fig. 12.** Cycles in the proof of Theorem 3.

Now we prove that each edge insertion preserves the acyclicity of the digraph. The proof is by contradiction and is essentially based on Lemma 6.

Suppose a simple cycle $C$ is obtained after the insertion of edge $(z, x)$ (see Figure 12) in the candidate embedded digraph $\Psi'$, derived from $\Psi$ after a number of sources and sinks have been eliminated by the procedure InsertEdge. Suppose (the other case is analogous) that both $x$ and $z$ are source-switches of $f$. Let $A'$ be the upward-consistent assignment for $\Psi'$, where $z$ is not assigned to $f$ while $x$ is assigned to $f$. Denote by $p_0$ the path of $\Psi'$ from $x$ to $z$ that gives rise to the cycle after the addition of the edge $(z, x)$. In order to have a cycle through $x$ and $z$ we have that $z$ cannot be a source in $\Psi'$.

Now consider the path $p_1$ of $f$ from $x$ to $z$ containing the sink-switch $y$. Two cases are possible.

*Case 1: Paths $p_0$ and $p_1$ are vertex disjoint (except for the endvertices $x$ and $z$).* The concatenation of $p_0$ and $p_1$ is a circuit $\gamma$ with zero capacity, since it contains only the switches $x$ and $y$. If $\gamma$ "contains" $f$ ($f$ is in the part of $\Psi'$ that does not contain the candidate external face, see Figure 12(a)), then, in the derived assignment $A'_\gamma$, $x$ is assigned to $\gamma$, a contradiction. If $\gamma$ does not contain $f$, then $y$ is assigned to $\gamma$ in $A'_\gamma$, a contradiction (see Figure 12(b)).

*Case 2: Paths $p_0$ and $p_1$ share one or more vertices different from $x$ and $z$.* Observe that the common vertices lie all on the directed path from $x$ to $y$, otherwise $C$ would not be simple. Let $w$ be the last common vertex on the directed path from $x$ to $y$; $w$ and $y$ are distinct vertices, otherwise a cycle was already present before the insertion of the edge $(z, x)$ (see Figure 12(c)). Let $p'_0$ be the path from $w$ to $z$ on the cycle and let $p'_1$ be the path from $w$ to $z$ on the face $f$, containing $y$ (see Figure 12(d)). The concatenation $\gamma$ of $p'_0$ and $p'_1$ is a circuit with zero capacity, since it contains only the switches $w$ and $y$. If $\gamma$ does not contain $f$, then $y$ is assigned to $\gamma$ in $A'_\gamma$, a contradiction. If $\gamma$ contains $f$ (see Figure 12(e)), then $w$ becomes a new source in the embedded graph $\Psi'$ that has to be assigned to $\gamma$ in $A'_\gamma$, getting again a contradiction.

*The candidate embedding invariant.* After the $(z, x)$ edge insertion, the resulting embedding is still a candidate embedding. Suppose (the other case is analogous) that $z$ and $x$ are both source-switches. Edge $(z, x)$ is the only incoming edge in the adjacency list of $x$, so, in the adjacency list of $x$, all the outgoing (incoming) edges appear consecutively. Concerning the adjacency list of $z$, edge $(z, x)$ is inserted between two consecutive outgoing edges, thus not altering the bi-partition of the list.

*The upward-consistent assignment invariant.* After the $(z, x)$ edge insertion, the upward-consistent assignment is modified as follows. All the sources and the sinks assigned to $f$ in $A$ that remain sources and sinks after the addition of $(z, x)$ are assigned to $f'$; all the sources and sinks of the rest of the digraph are assigned as in $A$. It is easy to see that the resulting assignment is upward-consistent.

*Single-source–single-sink.* We have to show that after all edge insertions have been performed the resulting digraph has exactly one source and one sink and remains planar after the addition of an edge between them.

To do that we first have to show that after procedure SaturateFace is performed on one internal face, all the faces that are obtained from that face contain exactly one source-switch and one sink-switch both labeled $S$. We prove that if an internal face $f$ has more than one source-switch and sink-switch, then it is always possible to find in $\sigma_f$ one of the two canonical subsequences of procedure InsertEdge. Due to the presence of $c_f + 2$ $S$-symbols over the $2c_f + 2$ symbols of $\sigma_f$ it is always possible to find a nonempty set of subsequences of contiguous $S$-symbols composed by at least two elements. Among such subsequences there is at least one that is preceded by one $B$-symbol. Observe that the canonical subsequences are exactly the possible subsequences composed by one $B$-symbol followed by two consecutive $S$-symbols.

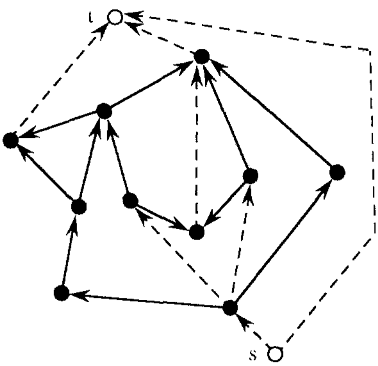


**Fig. 13.** The *st*-digraph that includes the digraph of Figure 8.

Because of the above discussion no vertex $v$ that does not belong to the external face can be a source or a sink after procedure SaturateFace has been performed on all the internal faces. In fact, suppose $v$ is a source or a sink. We have that not one of the faces surrounding $v$ has a label $B$ on $v$, thus contradicting property (a) of the upward-consistent assignments. Hence, after procedure SaturateFace has been performed on all the internal faces all the remaining sources and sinks are on the candidate external face.

When procedure SaturateFace is applied to the candidate external face $h$, since the number of assigned sources and sinks is now $c_h = |S_h| + 1$, the saturate procedure stops when the final circular sequence is composed by $k \geq 0$ $S$-symbols and $k + 2$ $B$-symbols. Since no two $S$-symbols appear consecutively in the sequence, the final sequence has the following structure in terms of $S$- and $B$-symbols: $\sigma_h \equiv B_1, \sigma_1, B_2, \sigma_2$, where $\sigma_1$ and $\sigma_2$ are two alternating sequences of $S$- and $B$-symbols. Each of $\sigma_1$ and $\sigma_2$ starts with an $S$-symbol and ends with a $B$-symbol, and both of them could be empty.

Observe that the $B$-symbols of one of the two alternating subsequences refer to sources while the $B$-symbols of the other subsequence refer to sinks. At this point all the sinks assigned to the external face can be connected to one new sink with edges directed from the original sinks to the new sink. The same can be done for the sources, obtaining the $st$-digraph. It is easy to see that the resulting digraph is acyclic. Moreover, $s$ and $t$ are on the same face. Figure 13 shows a planar $st$-digraph that includes the digraph of Figure 8. □

**5. Upward-Drawing Testing.** Let $G$ be a planar digraph with $n$ vertices and let $S$ and $T$ be the sets of its sources and sinks, respectively. In this section we give the algorithm for testing if $G$ has an upward drawing in the case where it is triconnected.

First, we devise an algorithm that, given a digraph with a fixed planar embedding $\Psi$ and an external face $h$, allows us to test if an upward embedding $\Phi$ exists whose underlying planarly embedded digraph is $\Psi$ and that has $h$ as an external upward face.

Second, we exploit such an algorithm and the fact that a triconnected planar graph has a unique planar embedding to define the algorithm for triconnected graphs.

Finally, we discuss further applications of our techniques.

*5.1. Upward-drawing Testing for Digraphs with a Fixed Embedding and with a Given External Face.* The fixed-embedding algorithm returns **true** or **false** depending on the existence of an upward embedding and works as follows.

**Algorithm** *Fixed-Embedding-Test*

1. Check if $\Psi$ is a candidate embedding. If the check is negative, return **false** and **stop**;
2. Look for an upward-consistent assignment on $\Psi$; if one exists, then return **true**, else return **false**.

The first step can be trivially performed in linear time.

Concerning Step 2, Theorem 3 allows us to test if it is possible to obtain an upward embedding from a candidate upward embedding by solving the perfect $c$-matching [1] problem defined on the bipartite network $N(L_1, L_2, A)$, where $L_1$ and $L_2$ are the two sets of vertices and $A$ is the set of edges. The vertices of $L_1$ are in one-to-one correspondence with the sources and sinks of $G$ and are labeled 1; the vertices of $L_2$ are in one-to-one correspondence with the faces of $\Psi$ and are labeled with their capacities. An edge in $A$ between a vertex $v$ of $L_1$ and a vertex $f$ of $L_2$ exists if and only if $v$ belongs to $f$ in $\Psi$. Observe that, due to Lemma 5, $|L_1| = \sum_{f \in L_2} c_f$. The $c$-matching problem is the following:

$$\sum_{(v,f) \in A} x_{vf} = c_f, \qquad \forall f \in L_2,$$

$$\sum_{(v,f) \in A} x_{vf} = 1, \qquad \forall v \in L_1,$$

where $x_{vf} = 1$ if vertex $v$ is assigned to face $f$ and $x_{vf} = 0$ otherwise.

The $c$-matching problem has a solution if and only if (see [18]) the $st$-network $N'$ defined below has a flow of value $|S| + |T|$. Observe that there are other works that use flow networks for solving related problems on angles in planar drawings [31], [26], [19].

Network $N'$ is obtained from $N$ by introducing two new vertices $s$ and $t$. Vertex $s$ is connected to each vertex $v$ of $L_1$ with an edge with capacity 1, directed from $s$ to $v$. Each face $f$ of $L_2$ is connected to $t$ with an edge with capacity $c_f$ directed from $f$ to $t$. All the edges of $A$ have capacity 1 and are directed from $L_1$ to $L_2$.

Now we analyze the time complexity of Algorithm Fixed-Embedding-Test.

THEOREM 4. *Let $G$ be a biconnected digraph with $n$ vertices, let $r$ be the total number of sources and sinks of $G$. Let $\Psi$ be a planar embedding of $G$ and let $h$ be a face of $\Psi$. Algorithm Fixed-Embedding-Test allows us to test if $G$ has an upward* *embedding whose underlying planar embedding is $\Psi$ and whose external upward face is h is $O(n + r^2)$ time.*

PROOF. The number of vertices in $L_1$ is $r$ and the number of vertices in $L_2$ is bounded by $r$ (see Lemma 5 and take into account that the only meaningful vertices of $L_2$ are those with capacity different from zero). Moreover, since the bipartite network $N$ is planar and since the number of edges added to $N$ to obtain $N'$ is $O(r)$ we have that $N'$ is sparse. Also, since the maximum flow value is $r$, using the Ford–Fulkerson algorithm that performs successive flow augmentations the max-flow problem can be solved in $O(r^2)$ time. □

*5.2. Upward-Drawing Testing for Triconnected Digraphs.* In this section we always refer to triconnected digraphs. Theorem 4 allows us to test if a triconnected digraph has an upward drawing by simply performing Algorithm Fixed-Embedding-Test on all the $n$ possible external faces. A trivial implementation of the above approach leads to an $O(n(n + r^2))$-time algorithm.

The performance can be improved by exploiting the following lemmas.

LEMMA 7. *Let $\Psi$ be a planarly embedded acyclic triconnected digraph. Let $r$ be the number of sources and sinks of $\Psi$. The number of faces of $\Psi$ that have at least one source and one sink is $O(r)$.*

PROOF. Let $\Psi'$ be the digraph defined as follows. The vertices of $\Psi'$ are the sources and sinks of $\Psi$. For each face $f$ of $\Psi$ that contains at least one source and one sink, one of the sources and one of the sinks of $f$ are arbitrarily selected and an edge between them is inserted in the edge set of $\Psi'$.

The number of faces of $\Psi$ that have at least one source and one sink is equal to the number of edges of $\Psi'$.

Since $\Psi$ is triconnected, an edge between two vertices $x$ and $y$ of $\Psi'$ can be inserted only twice, otherwise $x$ and $y$ would result a separating pair of $\Psi$. Hence, $\Psi'$ is a multigraph with at most two edges for each pair of vertices and without self-loops.

Moreover, by construction, $\Psi'$ is planar, hence, it has at most $6r - 12$ edges. □

Observing that the only meaningful external faces are those that have at least one source and one sink, we can conclude that the number of candidate upward external faces that have to be taken into account by our algorithm is $O(r)$.

Now we show that, for each possible external face, instead of solving a complete max-flow problem, we can simply evaluate an augmenting flow.

Consider again the max-flow problem stated above. We can define a new network $N''$ obtained from $N'$ by a slight modification of the capacities $c_f$, i.e., $c''_f = n_f - 1$ for all $f \in L_2$; intuitively this corresponds to not having yet identified an external face.

Let $maxflow(N'')$ be equal to the flow that is pumped by the solution of the maxflow problem in the sink of $N''$.

LEMMA 8. *If $maxflow(N'') < |S| + |T| - 2$, then $G$ has no upward embedding.*

PROOF. Suppose an upward embedding of $G$ exists and let $h$ be its upward external face and suppose that $maxflow(N'') < |S| + |T| - 2$.

Let $N'$ be the network associated with the feasible embedding, where the capacity of face $h$ is $c_h = n_h + 1$. We have that $maxflow(N') = |S| + |T|$. Now, consider the network $N'''$ obtained from $N''$ by changing only the capacity $c''_h$, such that $c'''_h = c''_h + 2 = c_h$. Obviously $N''' \equiv N'$, and then $maxflow(N''') = |S| + |T|$. However, $maxflow(N''') \leq maxflow(N'') + 2 < |S| + |T|$, a contradiction. □

Suppose, now, the above necessary condition is satisfied and let $\mathbf{x}$ be the optimum flow of $N''$, with value $|S| + |T| - 2$. Clearly, $\mathbf{x}$ is also feasible for $N'$ which is the network associated with a particular choice of the external face. In order to decide if $N'$ corresponds to an upward embedding, we have to prove that $maxflow(N') = |S| + |T|$. To do that we can simply try to augment the feasible flow $\mathbf{x}$ of exactly two units.

Moreover, for any candidate external face $h$ we have a candidate embedding and then a different network $N'_h$ (i.e., with different values of the capacities). In the worst case, in order to prove that $G$ admits an upward embedding, we have to look for the augmenting flow for every possible candidate upward external face, whose number is $O(r)$. Since the computation of flow $\mathbf{x}$ can be performed in $O(r^2)$ time (the argument is analogous to the one of Theorem 4) and since the search for a unitary augmenting flow can be performed in $O(r)$ time, we can claim the following theorem:

THEOREM 5. *Let $G$ be a triconnected digraph with $n$ vertices and let $r$ be the total number of sources and sinks of $G$. An algorithm exists that allows us to test if $G$ has an upward drawing in $O(n + r^2)$ time.*

*5.3. Applications.* Concerning the possibility of using the results of the algorithm to produce an upward drawing, we note that the proposed algorithm produces as output an upward-consistent assignment. We can exploit procedure SaturateFace in the proof of Theorem 3 to produce a planar *st*-digraph that includes $G$ starting from such an assignment. We can use the *st*-digraph as input to any of the drawing algorithms for such kind of digraphs (many such algorithms have been listed in the introduction).

The time complexity of procedure SaturateFace is linear. A trivial way to obtain such a linear-time behavior is to traverse the circular sequence of symbols of each face by using a stack for the $B$-symbols. In this way we decompose each internal face $f$ into $c_f + 1$ faces and the external face $h$ into $c_h + 3$ faces in time proportional to the number of $B$-symbols assigned to the face. From the above considerations and by Lemma 5 we have:

THEOREM 6. *Let $\Phi$ be an upward-embedded digraph with n vertices. An algorithm exists that allows us to construct an st-digraph that includes $\Phi$ in $O(n)$ time.*

As a final remark, we observe that our approach allows us also to represent implicitly all the possible upward embeddings of a given triconnected graph. In fact, such upward embeddings are in one-to-one correspondence with the solutions of the perfect $c$-matching (max-flow) problem that we have presented.

**6. Open Problems.** We have presented a polynomial-time algorithm for testing if a triconnected digraph has an upward drawing. Our results are based on an algorithm that allows us to test if a biconnected digraph with a given planar embedding has an upward drawing that preserves the embedding. Although this is out of the scope of this paper, it is interesting to observe that our algorithm for digraphs with a fixed embedding can be extended to the case of connected digraphs by slightly changing the definition of an upward-consistent assignment. Related problems that remain open are the following:

1. Find a polynomial-time algorithm for upward-drawing testing in biconnected digraphs or prove the NP-completeness. Observe that the problem has been solved for digraphs with one source in [13] and that several other problems that have been solved for planar graphs with a fixed embedding are still open in the case of variable embedding (see, for instance, [26]).
2. Characterize the class of digraphs that have a unique upward embedding (the equivalent of triconnected graphs for planar graphs).

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## References

[1] C. Berge, *Graphs*, North-Holland, Amsterdam, 1985.

[2] K. Booth and G. Lueker, Testing for the Consecutive Ones Property, Interval Graphs, and Graph Planarity Using PQ-Tree Algorithms, *J. Comput. System Sci.*, vol. 13, pp. 335–397, 1976.

[3] N. Chiba, T. Nishizeki, S. Abe, and T. Ozawa, A Linear Algorithm for Embedding Planar Graphs Using PQ-Trees, *J. Comput. System Sci.*, vol. 30, pp. 54–76, 1985.

[4] H. de Fraysseix and P. Rosenstiehl, A Depth First Characterization of Planarity, *Ann. Discrete Math.*, vol. 13, pp. 75–80, 1982.

[5] G. Di Battista, W. P. Liu, and I. Rival, Bipartite Graphs, Upward Drawings, and Planarity, *Inform. Process. Lett.*, vol. 36, pp. 317–322, 1990.

[6] G. Di Battista and R. Tamassia, Algorithms for Plane Representations of Acyclic Digraphs, *Theoret. Comput. Sci.*, vol. 61, pp. 175–198, 1988.

[7] G. Di Battista and R. Tamassia, Incremental Planarity Testing, *Proc. 30th IEEE Symposium on Foundations of Computer Science*, pp. 436–441, 1989.

[8] G. Di Battista, R. Tamassia, and I. G. Tollis, Area Requirement and Symmetry Display of Planar Upward Drawings, *Discrete Comput. Geom.*, vol. 7, pp. 381–401, 1992.

[9] P. Eades and R. Tamassia, Algorithms for Drawing Graphs: An Annotated Bibliography, Tech. Report No. CS-89-09, Brown University, 1989.

[10] P. Eades and L. Xuemin, How To Draw a Directed Graph, *Proc. IEEE Workshop on Visual Languages*, pp. 13–17, 1989.

[11] S. Even, *Graph Algorithms*, Computer Science Press, Rockville, MD, 1979.

[12] J. Hopcroft and R. E. Tarjan, Efficient Planarity Testing, *J. Assoc. Comput. Mach.*, vol. 21, no. 4, pp. 549–568, 1974.

[13] M. D. Hutton and A. Lubiw, Upward Planar Drawing of Single Source Acyclic Digraphs, *Proc. 2nd ACM–SIAM Symposium on Discrete Algorithms*, pp. 203–211, 1991.

[14] R. Jégan, R. Nowakowski, and I. Rival, The Diagram Invariant Problem for Planar Lattices, *Acta Sci. Math. (Szeged)*, vol. 51, pp. 103–121, 1987.

[15] D. Kelly, Fundamentals of Planar Ordered Sets, *Discrete Math.*, vol. 63, pp. 197–216, 1987.

[16] D. Kelly and I. Rival, Planar Lattices, *Canad. J. Math.*, vol. 27, pp. 636–665, 1975.

[17] A. Lempel, A. Even, and I. Cederbaum, An Algorithm for Planarity Testing of Graphs, *Theory of Graphs* (International Symposium, Rome, 1966) (P. Rosenstiehl, ed.), Gordon and Breach, New York, pp. 215–232, 1967.

[18] L. Lovasz and M. D. Plummer, *Matching Theory*, Annals of Discrete Mathematics, vol. 29, North-Holland, Amsterdam, p. 71, 1986.

[19] S. M. Malitz and A. Papakostas, On the Angular Resolution of Planar Graphs, *Proc. ACM Symposium on Theory of Computing*, 1992.

[20] T. Nishizeki and N. Chiba, *Planar Graphs: Theory and Algorithms*, Annals of Discrete Mathematics, North-Holland, Amsterdam, 1988.

[21] C. Platt, Planar Lattices and Planar Graphs, *J. Combin. Theory Ser. B*, vol. 21, pp. 30–39, 1976.

[22] I. Rival, The Diagram, in *Graphs and Orders* (I. Rival, ed.), Reidel, Dordrecht, pp. 103–133, 1985.

[23] I. Rival, Graphical Data Structures for Ordered Sets, in *Algorithms and Orders* (I. Rival, ed.), Kluwer, Boston, pp. 3–31, 1989.

[24] I. Rival and J. Urrutia, Representing Orders on the Plane by Translating Convex Figures, *Order*, vol. 4, pp. 319–339, 1988.

[25] K. Sugiyama, S. Tagawa, and M. Toda, Methods for Visual Understanding of Hierarchical Systems, *IEEE Trans. Systems Man Cybernet.*, vol. 11, pp. 109–125, 1981.

[26] R. Tamassia, On Embedding a Graph in the Grid with the Minimum Number of Bends, *SIAM J. Comput.*, vol. 16, pp. 421–444, 1987.

[27] R. Tamassia, G. Di Battista, and C. Batini, Automatic Graph Drawing and Readability of Diagrams, *IEEE Trans. Systems Man Cybernet.*, vol. 18, pp. 61–79, 1988.

[28] R. Tamassia and I. G. Tollis, A Unified Approach to Visibility Representations of Planar Graphs, *Discrete Comput. Geometry*, vol. 1, pp. 321–341, 1986.

[29] C. Thomassen, Planar Acyclic Oriented Graphs, *Order*, vol. 5, pp. 349–361, 1989.

[30] W. Trotter and J. Moore, Jr., The Dimension of Planar Posets, *J. Combin. Theory Ser. B*, vol. 22, pp. 54–67, 1977.

[31] G. Vijayan, Geometry of Planar Graphs with Angles, *Proc. 2nd ACM Symposium on Computational Geometry*, pp. 116–124, 1986.

[32] H. Whitney, Congruent Graphs and the Connectivity of Graphs, *Amer. J. Math.*, vol. 54, pp. 150–168, 1932.