Arch. Math., Vol. 54, 493–498 (1990)

0003-889X/90/5405-0493 $ 2.70/0

Projective resolution of identity in $C(K)$ spaces

By

M. VALDIVIA *)

For a certain class $\mathscr{A}$ of compact spaces, which is biger than the class of Corson compact spaces, it is proved that if K belongs to $\mathscr{A}$, the Banach space $C(K)$ admits a projective resolution of identity constructed through a family of linear extension operators.

The vector spaces we shall use here are defined over the field H of real or complex numbers. If H is the real field R, we denote by L the field of rational numbers; if H is the complex field C, we mean by L the field of complex numbers $a + bi$ where a and b are rational numbers.

Given a set A, $|A|$ denotes its cardinal number. If α is an ordinal number, $|\alpha|$ is its cardinal number. If E is a topological space, the density character of E, dens E, is the first cardinal α such that E has a dense subset A with $|A| = \alpha$. As it is usual, ω denotes the first infinite ordinal.

Unless the contrary would be specifically stated, $\|\,.\,\|$ is the norm in any Banach space X. If T denotes the identity operator on X and μ is the first ordinal with $|\mu| = \text{dens}\, X$, a projective resolution of identity is a well ordered family

$$\{P_\alpha : \omega \leqq \alpha \leqq \mu\}$$

of projections in X which satisfies the following conditions:

(i) $\|P_\alpha\| = 1$, $\omega \leqq \alpha \leqq \mu$.
(ii) dens $P_\alpha(X) \leqq |\alpha|$, $\omega \leqq \alpha \leqq \mu$.
(iii) $P_\alpha \circ P_\beta = P_\beta = P_\beta \circ P_\alpha$, $\omega \leqq \beta \leqq \alpha \leqq \mu$.
(iv) $\overline{\cup \{P_\beta(X) : \omega \leqq \beta < \alpha\}} = P_\alpha(X)$ whenever α is an ordinal limit.
(v) $P_\mu = T$.

If K is a compact topological space, $C(K)$ denotes the Banach space of continuous functions from K into H endowed with the norm

$$\|f\| = \sup\{|f(x)| : x \in K\} \quad \text{for} \quad f \in C(K).$$

If K is a compact subset of the cube $[0, 1]^I$, we write $K(I)$ to denote the subset of K formed with the elements $(x_i : i \in I)$ such that

$$\{i \in I : x_i \neq 0\}$$

*) Support in part by CAICYT.

is countable. Let us denote by e_i the projection of K on the i-th coordinate. If P is a subset of K, $\bar{P}$ denotes its closure and we write $I(P)$ to denote the subset of I formed by the elements j of I such that for some $(x_i : i \in I)$ in P we have $x_j \neq 0$. If B is a subset of $C(K)$, $(B)^*$ is the self conjugate linear algebra on L generated by B together with the constant functions over K valued on L. $\overline{(B)^*}$, the closure of $(B)^*$ in $C(K)$ is a self conjugate closed subalgebra of $C(K)$.

We shall say that a compact topological space D belongs to the class $\mathscr{A}$ if it is homeomorphic to a compact subset K of $[0, 1]^I$ for some set I and such that $K(I)$ is dense in K.

A compact topological space is a Corson compact if it is homeomorphic to a compact subset of $\sum(R^I)$, [2]. Obviously, the class $\mathscr{A}$ contains the class of Corson compact spaces. Using Theorem 2 it is easy to prove the following result, [1] and [3]: *If K belongs to $\mathscr{A}$ there are a set Γ and an one-to-one bounded operator $T: C(K) \to c_0(\Gamma)$ that is also pointwise to pointwise continuous.*

Lemma. *Let K be a compact subset of $[0, 1]^I$ such that $K(I)$ is dense in K. Let A_0 and B_0 be two infinite subsets of $K(I)$ and $C(K)$ respectively. If λ is a cardinal number such that $|A_0| \leqq \lambda$ and $|B_0| \leqq \lambda$ there exist a subset M of $K(I)$ and a linear extension operator T from $C(\bar{M})$ into $C(K)$ such that*

1) $M \supset A_0$, $|M| \leqq \lambda$,
2) $\|T\| = 1$, dens $C(\bar{M}) \leqq \lambda$, $T(C(\bar{M})) = \overline{(B_0 \cup \{e_i : i \in I(M)\})^*}$.

Proof. We are going to describe the process of construction by recurrence. So, let us start by supposing that for a given nonnegative integer n we have found

$$B_n \subset C(K), \quad A_n \subset K(I), \quad |A_n| \leqq \lambda, \quad |B_n| \leqq \lambda.$$

For each $f \in B_n$ we choose some point $x(f)$ of $K(I)$ such that $|f|$ attains its supremum on it. We set

$$A_{n+1} = A_n \cup \{x(f) : f \in B_n\}.$$

Obviously $|A_{n+1}| \leqq \lambda$. We write

$$B_{n+1} = (B_0 \cup \{e_i : i \in I(A_{n+1})\})^*.$$

Let $M := \bigcup_{n=0}^{\infty} A_n$ and let E be the closure of $\bigcup_{n=0}^{\infty} B_n$ in $C(K)$. We obviously have

$$M \supset A_0, \quad |M| \leqq \lambda, \quad E = \overline{(B_0 \cup \{e_i : i \in I(M)\})^*}.$$

Let us denote by $|||\,.\,|||$ the norm on $C(\bar{M})$. We set

$$Sf = f\,|_{\bar{M}}, \quad f \in E.$$

Given $\varepsilon > 0$ and f in E, we find a positive integer n together with $g \in B_n$ such that $\|f - g\| < \varepsilon$. Since $x(g)$ is a point in A_{n+1} where $|g|$ attains its supremum it now follows that

$$\begin{aligned}\|f\| &\leqq \|g\| + \|f - g\| \leqq |g(x(g))| + \varepsilon = |||Sg||| + \varepsilon \\ &\leqq |||Sg - Sf||| + |||Sf||| + \varepsilon \leqq |||Sf||| + 2\varepsilon\end{aligned}$$

from where we have

$$\|f\| \leqq |||Sf|||$$

and we deduce that S is a linear isometry from E into $C(\bar{M})$. Let us observe that $S(E)$ is a closed and self conjugate subalgebra of $C(\bar{M})$ which contains the constants. Moreover, if $x = (x_i : i \in I)$ and $y = (y_i : i \in I)$ are different points of $\bar{M}$ there is some $j \in I$ such that $x_j \neq y_j$ and so for some positive integer m and $(u_i : i \in I)$ in A_m we see that $u_j \neq 0$. It follows that $e_j \in E$ and $e_j(x) = x_j \neq y_j = e_j(y)$. Consequently, $S(E)$ separates points of $\bar{M}$ and the Stone-Weierstrass theorem assures that $S(E) = C(\bar{M})$. Obviously, dens $C(\bar{M}) \leqq \lambda$. The operator $T = S^{-1}$ is the linear extension operator we are looking for. □

Theorem 1. *Let K be an infinite compact subset of* $[0, 1]I$ *such that $K(I)$ is dense in K. Let μ be the first ordinal number such that its cardinal number coincides with* dens $K(I)$. *Then there exists a family*

$$\{K_\alpha : \omega \leqq \alpha \leqq \mu\}$$

of compact subsets of K with $K_\alpha(I)$ dense in K_α together a family

$$\{T_\alpha : \omega \leqq \alpha \leqq \mu\}$$

of linear extension operators from $C(K_\alpha)$ into $C(K)$ such that

(i) $\|T_\beta\| = 1$, $\omega \leqq \beta \leqq \mu$.
(ii) dens $C(K_\beta) \leqq |\beta|$, $\omega \leqq \beta \leqq \mu$.
(iii) $K_\beta \subset K_\gamma$ *and* $T_\beta(C(K_\beta)) \subset T_\gamma(C(K_\gamma))$, $\omega \leqq \beta \leqq \gamma \leqq \mu$.
(iv) $K_\alpha = \overline{\cup\{K_\beta : \omega \leqq \beta < \alpha\}}$ *and* $T_\alpha(C(K_\alpha)) = \overline{\cup\{T_\beta(C(K_\beta)) : \omega \leqq \beta < \alpha\}}$ *whenever α is an ordinal limit.*
(v) $K_\mu = K$.

Proof. If dens $K(I) = \aleph_0$, we write $K_\omega = K$ and T_ω for the identity mapping on $C(K)$. Let us now suppose that dens $K(I) > \aleph_0$. Let $\{x_\nu : \nu < \mu\}$ be a dense subset of $K(I)$. We can apply the lemma dealing with

$$A_0 = \{x_\nu : \nu < \omega\}, \quad B_0 = \{e_i : i \in I(A_0)\}, \quad \lambda = \aleph_0$$

to obtain a subset A_ω of $K(I)$ and a linear extension operator T_ω from $C(\bar{A}_\omega)$ into $C(K)$ such that

$$A_\omega \supset A_0, \quad |A_\omega| = \aleph_0, \quad \|T_\omega\| = 1, \quad \text{dens } C(\bar{A}_\omega) = \aleph_0$$

and if we write

$$B_\omega = (\{e_i : i \in I(A_\omega)\})^*$$

then

$$T_\omega(C(\bar{A}_\omega)) = \bar{B}_\omega.$$

We set $K_\omega = \bar{A}_\omega$. Let us proceed by transfinite induction. Let us take $\omega < \alpha \leqq \mu$ and suppose we have determined the family

$$\{A_\beta : \omega \leqq \beta < \alpha\}$$

of subsets of $K(I)$ together with the family

$$\{T_\beta : \omega \leqq \beta < \alpha\}$$

of linear extension operators from $C(K_\beta)$ into $C(K)$, where $K_\beta = \bar{A}_\beta$, and the family

$$\{B_\beta : \omega \leqq \beta < \alpha\}$$

of subsets of $C(K)$ such that

$$A_\beta \supset \{x_\nu : \nu < \beta\}, \quad |A_\beta| = |\beta|, \quad \|T_\beta\| = 1, \quad \text{dens}\, C(K_\beta) \leqq |\beta|,$$

$$B_\beta = (\{e_i : i \in I(A_\beta)\})^*, \quad T_\beta(C(K_\beta)) = \bar{B}_\beta.$$

We can now distinguish two cases. If α is not an ordinal limit, we have $\alpha = \gamma + 1$. The idea would be to apply the lemma for the sets

$$A_0 = A_\gamma \cup \{x_\nu : \nu < \alpha\}, \quad B_0 = \{e_i : i \in I(A_0)\}$$

and the cardinal number $\lambda = |\gamma| = |\alpha|$. We would obtain a subset A_α of $K(I)$ and a linear extension operator T_α from $C(\bar{A}_\alpha)$ into $C(K)$ such that

$$A_\alpha \supset A_0, \quad |A_\alpha| = |\alpha|, \quad \|T_\alpha\| = 1, \quad \text{dens}\, C(\bar{A}_\alpha) \leqq |\alpha|,$$

and if we write

$$B_\alpha = (\{e_i : i \in I(A_\alpha)\})^*$$

then

$$T_\alpha(C(\bar{A}_\alpha)) = \bar{B}_\alpha.$$

We set $K_\alpha = \bar{A}_\alpha$ that finishes the construction for the first case. If α is an ordinal limit we put

$$A_\alpha := \cup \{A_\beta : \omega \leqq \beta < \alpha\}, \quad K_\alpha = \bar{A}_\alpha,$$

$$B_\alpha := \cup \{B_\beta : \omega \leqq \beta < \alpha\} = (\{e_i : i \in I(A_\alpha)\})^*.$$

Let us denote by $\|\,.\,\|_\beta$ the norm on $C(K_\beta)$, $\omega \leqq \beta \leqq \alpha$. We set

$$S_\alpha f = f|_{K_\alpha}, \quad f \in \bar{B}_\alpha.$$

Given any $\varepsilon > 0$, and $f \in B_\alpha$, we find an ordinal $\beta, \omega \leqq \beta < \alpha$, and $g \in B_\beta$ such that $\|f - g\| < \varepsilon$. Then

$$\begin{aligned} \|f\| &\leqq \|g\| + \|f - g\| = \|T_\beta(g|_{K_\beta})\| + \|f - g\| \leqq \|g|_{K_\beta}\|_\beta + \varepsilon \\ &\leqq \|g|_{K_\alpha}\|_\alpha + \varepsilon = \|S_\alpha g\|_\alpha + \varepsilon \leqq \|S_\alpha g - S_\alpha f\|_\alpha + \|S_\alpha f\|_\alpha + \varepsilon \\ &\leqq \|S_\alpha f\|_\alpha + 2\varepsilon. \end{aligned}$$

Therefore

$$\|f\| \leqq \|S_\alpha f\|_\alpha$$

from where it follows that S_α is a linear isometry from $\bar{B}_\alpha$ into $C(K_\alpha)$. Moreover, $S_\alpha(\bar{B}_\alpha)$ is a closed subalgebra of $C(K_\alpha)$ which is self conjugated and contains the constants. On the other hand, if $x = (x_i : i \in I)$ and $y = (y_i : i \in I)$ are different points of K_α, there is some

$j \in J$ with $x_j \neq y_j$ from where it follows that for some ordinal number β and $(u_i : i \in I)$ in A_β we have $u_j \neq 0$. Thus, we see that e_j belongs to $\bar{B}_\alpha$ and $e_j(x) = x_j \neq y_j = e_j(y)$. Consequently, $S_\alpha(\bar{B}_\beta)$ separate points of K_α and the Stone-Weierstrass theorem assures that $S_\alpha(\bar{B}_\alpha) = C(K_\alpha)$. We take $T_\alpha = S_\alpha^{-1}$. We now have

$$A_\alpha \supset \{x_\nu : \nu < \alpha\}, \quad |A_\alpha| = |\alpha|, \quad \|T_\alpha\| = 1, \quad \text{dens}\, C(K_\alpha) \leqq |\alpha|,$$

$$B_\alpha = (\{e_i : i \in I(A_\alpha)\})^*, \quad T_\alpha(C(K_\alpha)) = \bar{B}_\alpha.$$

We have finished the construction because

$$\{K_\alpha : \omega \leqq \alpha \leqq \mu\} \quad \text{and} \quad \{T_\alpha : \omega \leqq \alpha \leqq \mu\}$$

verifies the conclusion of our theorem. □

Theorem 2. *Let K be an infinite compact of the class $\mathscr{A}$. Let μ be the first ordinal with $|\mu| = \text{dens}\, K$. Then there is a family*

$$\{K_\alpha : \omega \leqq \alpha \leqq \mu\}$$

of compact subsets of K with K_α in the class $\mathscr{A}$, together with a family

$$\{T_\alpha : \omega \leqq \alpha \leqq \mu\}$$

of linear extension operators from $C(K_\alpha)$ into $C(K)$ such that if

$$P_\alpha f = T_\alpha(f|_{K_\alpha}), \quad f \in C(K),$$

then

$$\{P_\alpha : \omega \leqq \alpha \leqq \mu\}$$

is a projective resolution of identity in $C(K)$.

Proof. We can suppose that K is compact in some cube $[0, 1]^I$. Let

$$\{K_\alpha : \omega \leqq \alpha \leqq \mu\} \quad \text{and} \quad \{T_\alpha : \omega \leqq \alpha \leqq \mu\}$$

be the families of compact subsets of K and linear extension operators constructed in the former theorem. Let us take f in $C(K)$ and $\omega \leqq \beta \leqq \alpha \leqq \mu$. Then

$$\|P_\alpha f\| = \|T_\alpha(f|_{K_\alpha})\| \leqq \|f|_{K_\alpha}\|_\alpha \leqq \|f\|$$

and so, $\|P_\alpha\| = 1$. We also gave

$$(P_\alpha \circ P_\beta) f = P_\alpha(T_\beta(f|_{K_\beta})) = T_\alpha(T_\beta(f|_{K_\beta})|_{K_\alpha}) = T_\beta(f|_{K_\beta}) = P_\beta f,$$

$$(P_\beta \circ P_\alpha) f = P_\beta(T_\alpha(f|_{K_\alpha})) = T_\beta(T_\alpha(f|_{K_\alpha})|_{K_\beta}) = T_\beta(f|_{K_\beta}) = P_\beta f$$

and thus

$$P_\alpha \circ P_\beta = P_\beta = P_\beta \circ P_\alpha.$$

P_μ obviously coincides with the identity in $C(K)$. Since

$$P_\alpha(C(K)) = T_\alpha(C(K_\alpha))$$

we have

$$\text{dens}\, P_\alpha(C(X)) = \text{dens}\, C(K_\alpha) \leqq |\alpha|.$$

Finally, if α is a ordinal limit, $\omega \leqq \alpha \leqq \mu$, it follows that

$$\overline{\cup \{P_\beta(C(K)) : \omega \leqq \beta < \alpha\}} = \overline{\cup \{T_\beta(C(K_\beta)) : \omega \leqq \beta < \alpha\}}$$
$$= T_\alpha(C(K_\alpha)) = P_\alpha(C(K))$$

and everything in the definition of projective resolution of identity has been checked. □

If we apply a result of Troyanski [4] and Zizler [5] on renorming theory, from the former theorem we obtain:

Corollary. *If K is the continuous image of a compact space of the class $\mathscr{A}$, then $C(K)$ admits an equivalent norm which is locally uniformly rotund. Particularly, the diadic compact spaces satisfies this property.*

References

[1] S. ARGYROS, S. MERCOURAKIS and S. NEGREPONTIS, Functional-analytic properties of Corson-compact spaces. Studia Math. **89**, 197–229 (1988).
[2] H. H. CORSON, Normality in subsets and product spaces. Amer. J. Math. **81**, 785–796 (1959).
[3] S. P. GUL'KO, On properties of function spaces (in Russian). In: Seminar on General Topology, P. S. Aleksandrov (ed.), Izdat. Moscow Univ., 8–41, Moscow 1981.
[4] S. TROYANSKI, On locally uniformly convex and differentiable norms. Studia Math. **37**, 173–180 (1971).
[5] V. ZIZLER, Locally uniformly rotund renorming and decomposition of Banach spaces. Bull. Australian Math. Soc. **29**, 259–265 (1984).

Eingegangen am 13. 10. 1988 *)

Anschrift des Autors:

M. Valdivia
Facultad de Matemáticas
Dr. Moliner 50
46100 Burjasot/Valencia
Spain

*) Eine Neufassung ging am 19. 5. 1989 ein.