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Concentration–compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents

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Abstract. We formulate the concentration–compactness principle at infinity for both subcritical and critical case. We show some applications to the existence theory of semilinear elliptic equations involving critical and subcritical Sobolev exponents.

Mathematics Subject Classification: 35J, 49J

1. Introduction

The concentration–compactness principles, discovered by P.L. Lions [15], [16], have proved to be very effective methods in variational problems involving a critical Sobolev exponent. In the current literature these principles are referred to as the first and second concentration–compactness principles (CCP1, CCP2) (Lemma 1 in [16] p. 115 and Lemma I.1 in [15] p. 158). The proofs of both concentration–compactness principles can also be found in the monograph by Struwe [17]. Both concentration–compactness principles are used to examine the behaviour of weakly convergent sequences in Sobolev spaces in situations where the lack of compactness occurs either due to the appearance of a critical Sobolev exponent or due to the unboundedness of a domain. The application of these principles helps to find level sets of a given variational functional for which the Palais–Smale condition holds.

The main purpose of this article is to formulate a variant of these two principles, namely, the concentration–compactness principle at infinity (CCP∞) for both critical and subcritical case. This variant, in the critical case, has already been introduced in [7] and subsequently has been used in [4] to improve the results of paper [6]. The most interesting feature of this variant at infinity is that it can be used instead of the first variational principle. The CCP2 , roughly speaking, is only concerned with a possible concentration of a weakly convergent sequence at finite points and it does not provide any information about the loss of mass of a sequence at infinity. To overcome this difficulty, the CCP1 is used to show that a sequence is tight and that the so-called vanishing and dichotomy cannot occur. In general, the use of the CCP1 is cumbersome and very technical. In this article we show how one can avoid the use of the first concentration–compactness principle by applying a version at infinity of the second principle (Proposition 2 in Sect. 3). In a subcritical case the situation is different. Due to the Sobolev compact embedding theorem a possible loss of mass of a weakly convergent sequence can only occur at infinity. Proposition 3 of Sect. 3 expresses this phenomenon in quantitative terms. In a subcritical case, a number of the existence results have been obtained by thorough examination of the Palais–Smale sequences (see [1], [5], [16] and [20] and references given there). One of the deepest results in this direction (Lemma 3.1 [5], Lemma on p.5 in [3] or Theorem 4.1 in [20]) explains how the Palais–Smale sequence fails to be relatively compact. Each term of such a sequence can be split into the same finite number of terms which are related to a possible loss of mass at infinity. This allows to estimate level sets of a given variational functional for which the Palais–Smale condition holds. The application of this procedure (the use of variants of Lemma 3.1 in [5]) can be eliminated by the use of Proposition 3 from Sect. 3.

The paper is organized as follows. Section 3 contains versions of concentration-compactness principle at infinity for critical and subcritical case. The rest of the paper is devoted to applications of Propositions 2 and 3 from Sect. 3. As an application of Proposition 2 we consider in Sect. 4 the following problem

$$(1)\qquad \begin{cases} -\Delta u - \lambda k(x)u = K(x)|u|^{2^*-2}u \ \text{ in } \mathbb{R}_N, \\ u(x) > 0 \ \text{ on } \mathbb{R}_N \ \text{ and } u \in D^{1,2}(\mathbb{R}_N), \end{cases}$$

where $2^* = \frac{2N}{N-2}$, $N \geq 3$, and λ is a positive parameter. We impose on functions k and K conditions ensuring that this problem can be written in a variational form. We aim to show that there exists $\lambda_\circ > 0$ such that for each $0 < \lambda < \lambda_\circ$, problem (1) has at least one solution. The constant $\lambda_\circ$ is determined as the first eigenvalue of the following problem

$$(2)\qquad \begin{cases} -\Delta u - \lambda k(x)u = 0 \ \text{ in } \mathbb{R}_N \\ u \not\equiv 0, \ u \in D^{1,2}(\mathbb{R}_N). \end{cases}$$

To prove the existence of the first eigenvalue to problem (2), we derive in Sect. 2 a compact embedding of the space $D^{1,2}(\mathbb{R}_N)$ into a weighted Lebesgue space. Problem (1), which originates in differential geometry (the Yamabe problem), has attracted a considerable interest in recent years. The method, that we use to solve problem (1), relies on the min–max principle of the mountain pass type. It is quite natural to apply both concentration–compactness principles to determine level sets for which a variational functional for problem (1) satisfies the Palais–Smale condition. We avoid the use the CCP1 by applying Proposition 2. Applications of Proposition 3 to subcritical cases are discussed in Sects. 5 and 6. In Sect. 5 we show that the problem

$$(3) \qquad -\Delta u + \lambda u = |u|^{p-2}u \ \text{ in } \Omega,$$

where $\lambda > 0$ is a constant, $2 < p < 2^*$ and Ω is a periodic domain, has a nontrivial solution. We reprove the existence result from paper [20] (Theorem 4.2); however, we believe that our approach is simpler because we avoid the use of a global compactness result of type Lemma 3.1 in [5] or Theorem 4.1 in [20]. We also give some improvements of Corollaries 3.2 and 3.4 from paper [5]. Finally, Sect. 6 is devoted to the study of the existence of a solution to the problem

$$(4) \qquad -\Delta u + u = \lambda b(x)|u|^{p-2}u + c(x)|u|^{q-2}u \ \text{ in } \mathbb{R}_N,$$

where $2 < p, q < 2^*$ and $\lambda > 0$ is a parameter. The existence theorem of this section is related to the existence result contained in paper [10] (Theorem 2.4). Again, we believe that our approach is simpler and under slightly weaker condition on c than in [10].

2. Preliminaries, compact embeddings of $D^{1,2}(\mathbb{R}_N)$ and $W^{1,2}(\mathbb{R}_N)$

In a given Banach space X, we denote by "$\to$" and "$\rightharpoonup$" strong and weak convergence, respectively.

By $D^{1,2}(\mathbb{R}_N)$ we denote the closure of $C_0^\infty(\mathbb{R}_N)$ with respect to the norm

$$\|u\|^2 = \int_{\mathbb{R}_N} |Du|^2\,dx.$$

The dual space to $D^{1,2}(\mathbb{R}_N)$ is denoted by $D^{-1,2}(\mathbb{R}_N)$, that is, $D^{1,2}(\mathbb{R}_N)^* = D^{-1,2}(\mathbb{R}_N)$. It is well known that the space $D^{1,2}(\mathbb{R}_N)$ is not compactly embedded in $L^2(\mathbb{R}_N)$. However, this embedding is compact if we replace the space $L^2(\mathbb{R}_N)$ by a weighted Lebesgue space $L^2_r(\mathbb{R}_N)$ under a suitable assumption on a weight r. Let $r \in L^1_{\text{loc}}(\mathbb{R}_N)$ and $r \geq 0$, $r \not\equiv 0$ on $\mathbb{R}_N$, we define $L^p_r(\mathbb{R}_N)$ by

$$L^p_r(\mathbb{R}_N) = \{u \in L^p_{\text{loc}}(\mathbb{R}_N);\ \int_{\mathbb{R}_N} |u(x)|^p r(x)\,dx < \infty\}$$

equipped with the norm

$$\|u\|^p_{r,p} = \int_{\mathbb{R}_N} |u(x)|^p r(x)\,dx.$$

By $Q(x,l)$, $l > 0$, we denote the cube of the form

$$Q(x,l) = \{y \in \mathbb{R}_N;\ |y_j - x_j| < \frac{l}{2},\ j = 1,\dots,N\}.$$

Lemma 1. *Suppose that* $r \in L^1(\mathbb{R}_N) \cap L^{\frac{p}{p-2}}_{loc}(\mathbb{R}_N)$, *for some* $2 < p < 2^*$, $r \geq 0$ *and* $r \not\equiv 0$ *on* $\mathbb{R}_N$ *and moreover*

$$\lim_{|x|\to\infty} \int_{Q(x,l)} r(y)^{\frac{p}{p-2}}\, dy = 0 \tag{5}$$

for some $l > 0$. *Then* $D^{1,2}(\mathbb{R}_N)$ *is compactly embedded into* $L^2_r(\mathbb{R}_N)$.

Proof. Without loss of generality we may assume that $l = 1$. It suffices to show that for every $\delta > 0$ there exists $j > 0$ such that

$$\|f - f\chi_{Q(0,j)}\|_{r,2} < \delta \tag{6}$$

for all $f \in D^{1,2}(\mathbb{R}_N)$, such that $\|f\| \leq 1$, where $\chi_{Q(0,j)}$ is a characteristic function of the cube $Q(0,j)$. Indeed, let $\{f_m\}$ be bounded sequence in $D^{1,2}(\mathbb{R}_N)$. We may assume that $\|f_m\| \leq 1$ for all $m \geq 1$ and that there exists $f \in D^{1,2}(\mathbb{R}_N)$ such that $f_m \to f$ in $L^p(Q(0,R))$ for each $R > 0$ and $Df_m \rightharpoonup Df$ in $L^2(\mathbb{R}_N)$. It follows from (6) that

$$\int_{\mathbb{R}_N - Q(0,j)} |f_m - f|^2 r\, dx \leq 2\delta.$$

Since $f_m \to f$ in $L^p(Q(0,j))$, the last inequality implies that $f_m \to f$ in $L^2_r(\mathbb{R}_N)$. To prove (6) we cover $\mathbb{R}_N$ with cubes $Q(z,1)$, $z \in \mathbb{Z}_N$. For $\eta > 0$ we use (5) to find $j \in \mathbb{N}$ such that $\int_Q r(y)^{\frac{p}{p-2}}\, dy \leq \eta$ for every $Q = Q(z,1)$ outside $Q(0,j)$ and $\int_{\mathbb{R}_N - Q(0,j)} r(y)\, dy \leq \eta$. If Q is any such cube then by the Sobolev and Hölder inequalities we have

$$\int_Q f^2\, dx \leq \left(\int_Q |f|^{2^*}\, dx\right)^{\frac{2}{2^*}} \leq S^{-1}\|Df\|_2^2, \tag{7}$$

where S denotes the best Sobolev constant. Also, by the Sobolev inequality we have

$$\int_Q |f(x) - f_Q|^p\, dx \leq C\left[\int_Q |Df|^2\, dx\right]^{\frac{p}{2}}, \tag{8}$$

where $f_Q = \int_Q f(x)\, dx$, for some constant $C > 0$ and for all $2 \leq p \leq 2^*$. It follows from (7) and (8) that

$$\begin{aligned}
\int_Q f^2 r\, dx &\leq 2\left[\int_Q |f - f_Q|^2 r\, dx + \int_Q \left(\int_Q |f(x)|\, dx\right)^2 r(y)\, dy\right] \\
&\leq 2\left[\left(\int_Q |f - f_Q|^p\, dx\right)^{\frac{2}{p}} \left(\int_Q r(y)^{\frac{p}{p-2}}\, dy\right)^{\frac{p-2}{p}} \right. \\
&\qquad \left. + \left(\int_Q |f|^{2^*}\, dx\right)^{\frac{N-2}{N}} \left(\int_Q r(y)\, dy\right)\right]
\end{aligned}$$

$$\leq C_1 \left(\int_Q |Df|^2 dx\right) \left(\int_Q r^{\frac{p}{p-2}} dx\right)^{\frac{p-2}{p}} + 2S^{-1} \left(\int_Q |Df|^2 dx\right) \left(\int_Q r\, dx\right),$$

where $C_1 > 0$ is a constant independent of η. We now add all these inequalities over $Q(z, 1)$ outside $Q(0,j)$ to get

$$\int_{\mathbb{R}_N - Q(0,j)} f^2 r\, dx \leq C_1 \eta^{\frac{p-2}{p}} + 2S^{-1} \int_{\mathbb{R}_N - Q(0,j)} r\, dx \leq C_1 \eta^{\frac{p-2}{p}} + 2S^{-1}\eta.$$

Inequality (5) follows by taking η so small that

$$C_1 \eta^{\frac{p-2}{p}} + 2S^{-1}\eta \leq \delta.$$

Lemma 2. *Suppose that $r \in L_{loc}^{\frac{q+\epsilon}{\epsilon}}(\mathbb{R}_N)$ for some ϵ and q satisfying $2 < q < q + \epsilon < 2^*$, $r \geq 0$, $r \not\equiv 0$ and that*

$$\lim_{|x| \to \infty} \int_{Q(x,l)} r(y)^{\frac{q+\epsilon}{\epsilon}} dy = 0$$

for some $l > 0$. Then $W^{1,2}(\mathbb{R}_N)$ is compactly embedded into $L_r^q(\mathbb{R}_N)$.

The proof is similar to that of Lemma 1 and is omitted (see also [11]).

We now use Lemma 1 to establish the existence of the first eigenvalue to problem (2).

Lemma 3. *Suppose that $k \in L^1(\mathbb{R}_N) \cap C(\mathbb{R}_N)$, $k \not\equiv 0$, k is somewhere positive and that*

$$\lim_{|x| \to \infty} \int_{Q(x,l)} |k(y)|^{\frac{p}{p-2}} dy = 0$$

for some $l > 0$ and $2 < p < 2^*$. Then there exists a positive function $v \in D^{1,2}(\mathbb{R}_N)$ such that

$$0 < \lambda_\circ = \inf\{\|Du\|_2^2;\ \|u\|_{k,2} = 1,\ u \in D^{1,2}(\mathbb{R}_N)\} = \|Du\|_2^2.$$

Proof. Let $\{u_m\}$ be a minimizing sequence. Since $\{u_m\}$ is bounded in $D^{1,2}(\mathbb{R}_N)$ we may assume that $u_m \rightharpoonup v$ in $D^{1,2}(\mathbb{R}_N)$. Consequently, $\int_{\mathbb{R}_N} v^2 k\, dx = 1$ and

$$\int_{\mathbb{R}_N} |Dv|^2 dx \leq \lim_{m \to \infty} \int_{\mathbb{R}_N} |Dv_m|^2 dx = \lambda_\circ$$

and the result follows.

3. Concentration–compactness principle at infinity

We recall the second concentration–compactness principle of P.L. Lions [15].

Proposition 1. *Let $\{u_m\}$ be a weakly convergent sequence to u in $D^{1,2}(\mathbb{R}_N)$ such that $|u_m|^{2^*} \overset{\star}{\rightharpoonup} \nu$ and $|Du_m|^2 \overset{\star}{\rightharpoonup} \mu$ in the sense of measures. Then, for some at most countable index set J we have*

(i) $\nu = |u|^{2^*} + \sum_{j\in J} \nu_j \delta_{x_j}, \quad \nu_j > 0,$
(ii) $\mu \geq |Du|^2 + \sum_{j\in J} \mu_j \delta_{x_j}, \quad \mu_j > 0,$
(iii) $S\nu_j^{\frac{2}{2^*}} \leq \mu_j,$

where S is the best Sobolev constant, $x_j \in \mathbb{R}_N$, δ_{x_j} are Dirac measures at x_j and μ_j, ν_j are constants.

This result does not provide any information about a possible loss of mass at infinity. Proposition 3 below expresses this fact in quantitative terms.

Proposition 2. *Let $\{u_m\}$ be weakly convergent sequence in $D^{1,2}(\mathbb{R}_N)$ and define*

(i) $\nu_\infty = \lim_{R\to\infty} \limsup_{m\to\infty} \int_{|x|>R} |u_m|^{2^*}\,dx,$
(ii) $\mu_\infty = \lim_{R\to\infty} \limsup_{m\to\infty} \int_{|x|>R} |Du_m|^2\,dx.$

The quantities ν_∞ and μ_∞ exist and satisfy

(iii) $\limsup_{m\to\infty} \int_{\mathbb{R}_N} |u_m|^{2^*}\,dx = \int_{\mathbb{R}_N} d\nu + \nu_\infty$, $\limsup_{m\to\infty} \int_{\mathbb{R}_N} |Du_m|^2\,dx = \int_{\mathbb{R}_N} d\mu + \mu_\infty$, *and*
(iv) $S\nu_\infty^{\frac{2}{2^*}} \leq \mu_\infty.$

The quantities μ_∞ and ν_∞ were introduced in [7], where the proof of (iv), based on the Sobolev inequality can be found. A very simple proof of relations (iii) can be found in [4]. Obviously, Proposition 2 remains true in the space $D_\circ^{1,2}(\Omega)$, where Ω is an unbounded domain in $\mathbb{R}_N$ and $D_\circ^{1,2}(\Omega)$ is the closure of $C_\circ^\infty(\Omega)$ with respect to the norm

$$\|Du\|_2^2 = \int_\Omega |Du(x)|^2\,dx.$$

We now shift our attention to a subcritical case in $W_\circ^{1,2}(\Omega)$, where $\Omega \subset \mathbb{R}_N$ is unbounded domain. Let $\lambda > 0$ and $2 < p < 2^*$ and we set

$$\text{(M)} \qquad 0 < \alpha_\lambda(\Omega) = \inf\{\int_\Omega (|Du|^2 + \lambda u^2)\,dx;\ \|u\|_p = 1\}.$$

Proposition 3. *Let $\{u_m\}$ be a weakly convergent sequence to u in $W_\circ^{1,2}(\Omega)$ and define*

(a) $\alpha_\infty = \lim_{R\to\infty} \limsup_{m\to\infty} \int_{\Omega\cap(|x|>R)} |u_m|^p\,dx,$
(b) $\beta_\infty = \lim_{R\to\infty} \limsup_{m\to\infty} \int_{\Omega\cap(|x|>R)} (|Du_m|^2 + \lambda u_m^2)\,dx.$

These quantities are well defined and satisfy

(c) $\limsup_{m\to\infty} \int_\Omega |u_m|^p\,dx = \int_\Omega |u|^p\,dx + \alpha_\infty$,
(d) $\limsup_{m\to\infty} \int_\Omega (|Du_m|^2 + \lambda u_m^2)\,dx \geq \int_\Omega (|Du|^2 + \lambda u^2)\,dx + \beta_\infty$

and

(e) $\alpha_\lambda \alpha_\infty^{\frac{2}{p}} \leq \beta_\infty$.

Proof. Let $\phi_R \in C^1(\mathbb{R}_N)$ be such that $\phi_R(x) = 0$ for $|x| < R$ and $\phi_R(x) = 1$ for $|x| > R + 1$ and $0 \leq \phi_R(x) \leq 1$ on $\mathbb{R}_N$. It follows from (M) that

$$\alpha_\lambda \|\phi_R u_m\|_p^2 \leq \int_\Omega (|D(u_m\phi_R)|^2 + \lambda u_m^2 \phi_R^2)\,dx.$$

Since

$$\lim_{R\to\infty} \limsup_{m\to\infty} \int_\Omega |\phi_R u_m|^p\,dx = \alpha_\infty,$$

$$\lim_{R\to\infty} \limsup_{m\to\infty} \int_\Omega (|Du_m|^2 + \lambda u_m^2)\phi_R^2\,dx = \beta_\infty,$$

$$\lim_{R\to\infty} \lim_{m\to\infty} \int_\Omega Du_m D\phi_R \phi_R u_m\,dx = 0$$

and

$$\lim_{R\to\infty} \lim_{m\to\infty} \int_\Omega |u_m D\phi_R|^2\,dx = 0,$$

the inequality (e) follows. To show (c) we write

$$\limsup_{m\to\infty} \int_\Omega |u_m|^p\,dx = \int_{\Omega\cap(|x|<R)} |u|^p\,dx + \limsup_{m\to\infty} \int_{\Omega\cap(|x|>R)} |u_m|^p\,dx.$$

Letting $R \to \infty$, relation (c) follows. By a similar argument, using the lower semicontinuity of the L^2-norm with respect to the weak convergence, we deduce (d).

4. Existence result for problem (1)

Throughout this section it is assumed that

(A) $k \in C(\mathbb{R}_N) \cap L^1(\mathbb{R}_N)$, $k \not\equiv 0$ on $\mathbb{R}_N$ and

$$\lim_{|x|\to\infty} \int_{Q(x,l)} |k(y)|^{\frac{p}{p-2}}\,dy = 0$$

for some $l > 0$ and $2 < p < 2^*$.

(B) $K \in C(\mathbb{R}_N) \cap L^\infty(\mathbb{R}_N)$ and there exist $x_\circ \in \mathbb{R}_N$, constants $\sigma > 0$, $C_1 > 0$, $C_2 > 0$ and $R > 0$ such that

$$|K(x_\circ) - K(x)| \le C_1 |x - x_\circ|^2 \text{ for } N = 4,$$

$$|K(x_\circ) - K(x)| \le C_1 |x_\circ - x|^{2+\sigma} \text{ for } N \ge 5,$$

$k(x_\circ) > 0$ and $|k(x_\circ) - k(x)| \le C_2 |x - x_\circ|$ for all $|x - x_\circ| < R$ and $K(x_\circ) = \|K\|_\infty$.

We define a functional $F : D^{1,2}(\mathbb{R}_N) \to \mathbb{R}$ by

$$F(u) = \frac{1}{2} \int_{\mathbb{R}_N} |Du|^2 \, dx - \frac{\lambda}{2} \int_{\mathbb{R}_N} k(x) u^2 \, dx - \frac{1}{2^*} \int_{\mathbb{R}_N} K(x) |u|^{2^*} \, dx.$$

It follows from Lemma 1 that functional F is well defined. Critical points of this functional are solutions of problem (1).

We commence by finding level sets of the functional F for which the Palais–Smale condition holds.

We recall that $\{u_m\} \subset D^{1,2}(\mathbb{R}_n)$ satisfies the Palais - Smale condition at level c (the $(PS)_c$ condition for short) if $F(u_m) \to c$ and $F'(u_m) \to 0$ in $D^{-1,2}(\mathbb{R}_N)$ imply that $\{u_m\}$ possesses a convergent subsequence in $D^{1,2}(\mathbb{R}_N)$.

We need the truncated Talenti extremal function [19]

$$u_\epsilon(x) = \frac{\phi(x)}{\left(\epsilon + |x - x_\circ|^2\right)^{\frac{N-2}{2}}}, \quad \epsilon > 0,$$

where $\phi \in C^1_\circ(\mathbb{R}_N)$, $0 \le \phi(x) \le 1$ on $\mathbb{R}_N$, $\phi(x) = 1$ for $|x - x_\circ| < \frac{R}{2}$ and $\phi(x) = 0$ for $|x - x_\circ| > R$.

Lemma 4. *For $\epsilon > 0$ sufficiently small we have*

$$\sup_{t \ge 0} F(t v_\epsilon) < \frac{S^{\frac{N}{2}}}{N K(x_\circ)^{\frac{N-2}{2}}} = \frac{S^{\frac{N}{2}}}{N \|K\|_\infty^{\frac{N-2}{2}}},$$

where $v_\epsilon(x) = \frac{u_\epsilon(x)}{\|u_\epsilon\|_{2^*}}$.

Proof. We only outline the main steps of the proof and for details we refer to the paper [9]. According to formulae (1.11), (1.12) and (1.13) in [8] we have

$$\|u_\epsilon\|_2^2 = \frac{K_1}{\epsilon^{\frac{N-2}{2}}} + O(1), \quad N \ge 3,$$

$$\|u_\epsilon\|_{2^*}^2 = \frac{K_2}{\epsilon^{\frac{N-2}{2}}} + O(\epsilon), \quad N \ge 3$$

and

$$\|u_\epsilon\|_2^2 = \begin{cases} \frac{K_3}{\epsilon^{\frac{N-4}{2}}} + O(1), & N \ge 5 \\ K_3 |\log \epsilon| + O(1), & N = 4, \\ K_3 + O(\epsilon^{\frac{1}{2}}), & N = 3, \end{cases}$$

where K_1, K_2 and K_3 are positive constants depending only on N, and $S = \frac{K_1}{K_2}$. We now observe that there exists a unique point $t_\epsilon \in (0, \infty)$ such that

$$F(t_\epsilon v_\epsilon) = \max_{t \geq 0} F(tv_\epsilon) = \frac{X_\epsilon^{\frac{N}{2}}}{N V_\epsilon^{\frac{N-2}{2}}}$$

where

$$X_\epsilon = \int_{\mathbb{R}_N} (|Dv_\epsilon|^2 - \lambda k v_\epsilon^2)\, dx \text{ and } V_\epsilon = \int_{\mathbb{R}_N} K v_\epsilon^{2^*}\, dx.$$

To estimate X_ϵ we write

$$\lambda \int_{\mathbb{R}_N} k v_\epsilon^2\, dx = \frac{\lambda k(x_\circ)\|u_\epsilon\|_2^2}{\|u_\epsilon\|_{2^*}^2} + \frac{1}{\|u_\epsilon\|_{2^*}^2}\left[\lambda \int_{\mathbb{R}_N} \frac{(k(x) - k(x_\circ))}{(\epsilon + |x - x_\circ|^2)^{N-2}} \phi(x)^2\, dx\right]$$

and we get

$$-\lambda \int_{\mathbb{R}_N} k v_\epsilon^2\, dx \leq \begin{cases} -\frac{\lambda \epsilon k(x_\circ) K_3}{K_2} + O(\epsilon^{\frac{N-2}{2}}) + O(\epsilon^{\frac{3}{2}}), & N \geq 5, \\ -\frac{\lambda k(x_\circ) K_3 \epsilon |\log \epsilon|}{K_2} + O(\epsilon), & N = 4. \end{cases}$$

If $N = 4$, then $|K(x) - K(x_\circ)| \leq C_1 |x - x_\circ|^2$ for $|x - x_\circ| < R$ and this gives the following estimate

$$V_\epsilon \geq K(x_\circ) - \frac{C\epsilon^{1-\frac{N}{2}}}{\|u_\epsilon\|_{2^*}^{2^*}} = K(x_\circ) + O(\epsilon)$$

for some constant $C > 0$. If $N \geq 5$ we use the inequality $|K(x) - K(x_\circ)| \leq C_1 |x - x_\circ|^{2+\sigma}$ to obtain

$$V_\epsilon \geq K(x_\circ) + O(\epsilon^{1+\frac{\sigma}{2}}).$$

Combining these estimates we get

$$F(t_\epsilon v_\epsilon) \leq \begin{cases} \dfrac{S^{\frac{N}{2}}\left[1 - \frac{\lambda k(x_\circ) K_3 \epsilon}{K_2 S} + O(\epsilon^{\frac{3}{2}})\right]^{\frac{N}{2}}}{N K(x_\circ)^{\frac{N-2}{2}}\left[1 + O(\epsilon^{1+\frac{\sigma}{2}})\right]^{\frac{N-2}{2}}} & \text{for } N \geq 5, \\ \dfrac{S^{\frac{N}{2}}\left[1 - \frac{k(x_\circ) K_3 \epsilon |\log \epsilon|}{K_2 S} + O(\epsilon)\right]^{\frac{N}{2}}}{N K(x_\circ)(1 + O(\epsilon))} & \text{for } N = 4 \end{cases}$$

and taking $\epsilon > 0$ small the assertion follows. It can be checked that the case $N = 3$ is inconclusive.

We now are in a position to formulate the first existence result. It will follow from the proof of Theorem 1 below that the $(PS)_c$ condition holds for $c \in \left(0, \frac{S^{\frac{N}{2}}}{N \|K\|_\infty^{\frac{N-2}{2}}}\right)$.

Theorem 1. *Let $0 < \lambda < \lambda_\circ$ and $N \geq 4$. Then problem (1) has a solution.*

Proof. We define a functional $F_* : D^{1,2}(\mathbb{R}_N) \to \mathbb{R}$ by

$$F_*(u) = \frac{1}{2}\int_{\mathbb{R}_N} |Du|^2\,dx - \frac{\lambda}{2}\int_{\mathbb{R}_N} k\left(u^+\right)^2 dx - \frac{1}{2^*}\int_{\mathbb{R}_N} K\left(u^+\right)^{2^*} dx,$$

where $u^+ = \max(0, u)$. It follows from Lemma 3 and the Sobolev inequality that

$$F_*(u) \geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_\circ}\right)\|Du\|_2^2 - \frac{\|K\|_\infty}{2^* S^{\frac{2^*}{2}}}\|Du\|_2^{2^*}$$

for all $u \in D^{1,2}(\mathbb{R}_N)$. Hence there exists constants $\alpha > 0$ and $\rho > 0$ such that

$$F_*(u) \geq \alpha \text{ for } \|u\| = \rho.$$

We also have $F_*(0) = 0$ and $F_*(t_\circ v_\epsilon) = F(t_\circ v_\epsilon) < 0$ for $t_\circ > 0$ sufficiently large, where $\epsilon > 0$ is chosen so that Lemma 4 holds. Let

$$\Gamma = \{g \in C([0,1], D^{1,2}(\mathbb{R}_N));\ g(0) = 0 \text{ and } g(1) = t_\circ v_\epsilon\}$$

and we put

$$c = \inf_{g\in\Gamma}\sup_{t\in[0,1]} F_*(g(t)). \tag{9}$$

It is clear that $\alpha \leq c$ and by Lemma 4 we have

$$c < \frac{S^{\frac{N}{2}}}{N\|K\|_\infty^{\frac{N-2}{2}}}. \tag{10}$$

By the mountain pass principle without the (PS) condition (see Theorem 2.2 in [8]) there exists a sequence $\{u_m\} \subset D^{1,2}(\mathbb{R}_N)$ such that $F_*(u_m) \to c$ and $F_*'(u_m) \to 0$ in $D^{-1,2}(\mathbb{R}_N)$. We now check that $\{u_m\}$ is bounded in $D^{1,2}(\mathbb{R}_N)$. Indeed, we have

$$\begin{aligned}\langle F_*'(u_m), u_m^-\rangle &= \int_{\mathbb{R}_N} Du_m Du_m^-\,dx - \lambda\int_{\mathbb{R}_N} ku_m^+u_m^-\,dx \\ &\quad - \int_{\mathbb{R}_N} K\left(u_m^+\right)^{2^*-1}u_m^-\,dx = -\int_{\mathbb{R}_N}|Du_m^-|^2\,dx.\end{aligned}$$

Since $F_*'(u_m) \to 0$ in $D^{-1,2}(\mathbb{R}_N)$, we see that $u_m^- \to 0$ in $D^{1,2}(\mathbb{R}_N)$ and hence $F(u_m^+) = F_*(u_m^+) \to c$. On the other hand we have

$$\langle F_*'(u_m), u_m^+\rangle = \int_{\mathbb{R}_N}|Du_m^+|^2\,dx - \lambda\int_{\mathbb{R}_N} k\left(u_m^+\right)^2 dx - \int_{\mathbb{R}_N} K\left(u_m^+\right)^{2^*} dx \to 0.$$

Consequently,

$$\begin{aligned}c + o(1) &= F(u_m^+) - \frac{1}{2^*}\langle F_*(u_m), u_m^+\rangle = \frac{1}{N}\int_{\mathbb{R}_N}|Du_m^+|^2\,dx \\ &\quad -\frac{\lambda}{N}\int_{\mathbb{R}_N} k\left(u_m^+\right)^2 dx \geq \frac{1}{N}\left(1 - \frac{\lambda}{\lambda_\circ}\right)\int_{\mathbb{R}_N}|Du_m^+|^2\,dx\end{aligned}$$

and the claim easily follows. Since $\{u_m\}$ is bounded in $D^{1,2}(\mathbb{R}_N)$ we may assume that $u_m \rightharpoonup u$ in $D^{1,2}(\mathbb{R}_N)$, $u_m \to u$ a. e. and $u_m^+ \rightharpoonup u^+$ in $L^{2^*}(\mathbb{R}_N)$. By Lemma 1 $\int_{\mathbb{R}_N} k\left(u_m^+\right)^2 dx \to \int_{\mathbb{R}_N} k\left(u^+\right)^2 dx$; taking a subsequence if necessary. This means that

$$-\Delta u - \lambda k u^+ = K\left(u^+\right)^{2^*} \text{ in } D^{-1,2}(\mathbb{R}_N).$$

Taking as a test function u^- we get

$$\int_{\mathbb{R}_N} |Du^-|^2 \, dx = 0$$

which implies that $u^- \equiv 0$ on $\mathbb{R}_N$, that is, $u \geq 0$ on $\mathbb{R}_N$. We now show that $u \not\equiv 0$ and consequently $u > 0$ on $\mathbb{R}_N$ by the Harnack inequality. Towards this end we follow the argument [8]. From the fact that $F_*(u_m) \to c$ we easily deduce that

$$l = \lim_{m\to\infty} \int_{\mathbb{R}_N} |Du_m|^2 \, dx > 0.$$

By Lemma 3 have

$$\lim_{m\to\infty} \lambda \int_{\mathbb{R}_N} k\left(u_m^+\right)^2 dx = \lambda \int_{\mathbb{R}_N} ku^2 \, dx.$$

We now show that $\lambda \int_{\mathbb{R}_N} ku^2 \, dx \not= 0$. Otherwise $l = \lim_{m\to\infty} \int_{\mathbb{R}_N} K\left(u_m^+\right)^{2^*} dx > 0$ and $c = \frac{l}{2} - \frac{l}{2^*} = \frac{l}{N}$. On the other hand we have

$$\begin{aligned} l &= \lim_{m\to\infty} \|Du_m\|_2^2 \geq S \lim_{m\to\infty} \|u_m\|_{2^*}^2 \\ &\geq S\|K\|_\infty^{\frac{2-N}{N}} \lim_{m\to\infty} \left[\int_{\mathbb{R}_N} K\left(u_m^+\right)^{2^*} dx\right]^{\frac{N-2}{N}} = S\|K\|_\infty^{\frac{2-N}{N}} l^{\frac{N-2}{N}} \end{aligned}$$

which is equivalent to $l \geq S^{\frac{N}{2}} \|K\|_\infty^{\frac{2-N}{2}}$, that is, $c \geq \frac{S^{\frac{N}{2}}}{N} \|K\|_\infty^{\frac{2-N}{2}}$ and this contradicts (10). By virtue of Proposition 1 there exist measures μ and ν, an at most countable set J, sequences of positive constants $\{\mu_j\}$, $\{\nu_j\}, j \in J$, and a sequence $\{x_j\} \subset \mathbb{R}_N$, $j \in J$, such that relations (a), (b) and (c) hold. We now follow an argument from [1], [2]. Let x_k be a singular point of measures μ and ν and let $\phi_\delta \in C_o^1(\mathbb{R}_N)$ be such that $\phi_\delta = 1$ for $|x - x_k| < \delta$, $\phi_\delta(x) = 0$ for $|x - x_k| \geq 2\delta$, $0 \leq \phi_\delta(x) \leq 1$ and $|D\phi_\delta(x)| \leq \frac{2}{\delta}$ in $\mathbb{R}_N$. Since $\langle F_*'(u_m), u_m\phi_\delta\rangle \to 0$ as $m \to \infty$ we get

$$\begin{aligned} \int_{\mathbb{R}_N} \phi_\delta \, d\mu - & \lambda \int_{\mathbb{R}_N} ku^2\phi_\delta \, dx - \int_{\mathbb{R}_N} K\phi_\delta \, d\nu \\ &\leq \limsup_{m\to\infty} \int_{\mathbb{R}_N} |Du_m||u_m||D\phi_\delta| \, dx \\ &\leq \tilde{C}\left(\int_{\mathbb{R}_N} u^2 |D\phi_\delta|^2 \, dx\right)^{\frac{1}{2}}, \end{aligned}$$

where $\tilde{C} = \sup_{m\geq 1} \|Du\|_2$. By virtue of the Hölder inequality we have

$$\int_{\mathbb{R}_N} u^2 |D\phi_\delta|^2\,dx \le \left(\int_{|x_k-x|<2\delta} |u|^{2^*}\,dx\right)^{\frac{2}{2^*}} \left(\int_{|x_k-x|<2\delta} |D\phi_\delta|^N\,dx\right)^{\frac{2}{N}}$$

$$\le 16\omega_N^{\frac{2}{N}} \left(\int_{|x_k-x|<2\delta} |u|^{2^*}\,dx\right)^{\frac{2}{2^*}}, \tag{11}$$

where ω_N is a volume of a unit ball in $\mathbb{R}_N$. Letting $\delta \to 0$ in (11) we deduce that $K(x_k)\nu_k \ge \mu_k$. We may assume that $K(x_k) > 0$, since otherwise there is no singularity at x_k. Combining this with (c) of Proposition 1 we deduce that either (i) $\nu_k = 0$ or (ii) $\nu_k \ge \left(\frac{S}{K(x_k)}\right)^{\frac{N}{2}}$. In order to consider a possible concentration of measures ν and μ at infinity we introduce constants ν_∞ and μ_∞ defined in Proposition 2. Letting $K(\infty) = \limsup_{|x|\to\infty} K(x)$, we see that

$$\lim_{R\to\infty} \limsup_{m\to\infty} \int_{|x|\ge R} K(x)|u_m|^{2^*}\,dx \le K(\infty)\nu_\infty. \tag{12}$$

Also, by Lemma 1 we have

$$\lim_{R\to\infty} \lim_{m\to\infty} \int_{\mathbb{R}_N} k\left(u_m^+\right)^2 \phi_R\,dx = 0, \tag{13}$$

where ϕ_R is a function defined in the proof of Proposition 3. Since $\langle F_*'(u_m), u_m\phi_R\rangle \to 0$ as $m \to \infty$, we deduce, using (12) and (13), that $\mu_\infty \le K(\infty)\nu_\infty$. It now follows from Proposition 2(iv) that either (iii) $\nu_\infty = 0$ or (iv) $\nu_\infty \ge \left(\frac{S}{K(\infty)}\right)^{\frac{N}{2}}$. Here, we assume that $K(\infty) > 0$, since otherwise there is no concentration at infinity. To complete the proof we show that $\nu_\infty = 0$ and $\nu_k = 0$ for each k. Indeed, we have for every continuous function ψ, with $0 \le \psi \le 1$ on $\mathbb{R}_N$, that

$$\begin{aligned} c &= \lim_{m\to\infty} \left[F_*(u_m) - \frac{1}{2}\langle F_*'(u_m), u_m\rangle\right] \\ &= \frac{1}{N} \lim_{m\to\infty} \int_{\mathbb{R}_m} K\left(u_m^+\right)^{2^*} dx \ge \frac{1}{N} \lim_{m\to\infty} \int_{\mathbb{R}_N} \psi K\left(u_m^+\right)^{2^*} dx. \end{aligned}$$

Assuming (ii), we see that the set J must be finite and taking $\psi = \phi_\delta$ we get, letting $\delta \to 0$, that

$$c \ge \frac{1}{N} K(x_k)\nu_k \ge \frac{S^{\frac{N}{2}}}{NK(x_k)^{\frac{N-2}{2}}} \ge \frac{S^{\frac{N}{2}}}{N\|K\|_\infty^{\frac{N-2}{2}}}$$

which is impossible, so $\nu_k = 0$ for each k. Similarly, if (iv) occurs, we take $\psi = \phi_R$ and then let $R \to \infty$ to get

$$c \ge \frac{S^{\frac{N}{2}}}{NK(\infty)^{\frac{N-2}{2}}} \ge \frac{S^{\frac{N}{2}}}{N\|K\|_\infty^{\frac{N-2}{2}}},$$

which is impossible, so $\nu_\infty = \mu_\infty = 0$. It is now routine to show that $u_m \to u$ in $D^{1,2}(\mathbb{R}_N)$.

5. Application to problem (3)

We establish the existence of a solution to problem (3) in unbounded periodic domains. In a final part of this section we discuss the Palais–Smale condition for an exterior domain.

We first consider problem (3) in $\Omega = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}_{N-1}, |x'| < r\}$, $r > 0$. It will be clear that Thereom 2 below can be extended to periodic unbounded domains whose definition will be given later.

We obtain a solution to problem (3) by considering the minimization problem (M). We set

$$G(u) = \frac{1}{2} \int_\Omega (|Du|^2 + \lambda u^2)\, dx - \frac{1}{p} \int_\Omega |u|^p\, dx.$$

As a norm in $W_o^{1,2}(\Omega)$ we take

$$\|u\|_\lambda^2 = \int_\Omega (|Du|^2 + \lambda u^2)\, dx.$$

Theorem 2. *The minimization problem (M) has a solution, that is, problem (3) admits a nontrivial solution.*

Proof. Let $\{u_m\}$ be a minimizing sequence for (M) and we put $v_m = \alpha_\lambda^{\frac{1}{p-2}} u_m$. It follows from Theorem 2.1 in [20] that

$$(14) \qquad G(v_m) = \Big(\frac{1}{2} - \frac{1}{p}\Big)\alpha_\lambda^{\frac{p}{p-2}} + o(1) \text{ and } G'(v_m) = o(1) \text{ in } W^{-1,2}(\Omega).$$

Our objective is to show that $\{v_m\}$ is, up to a translation, relatively compact in $W_o^{1,2}(\Omega)$. Let $0 < s < (\frac{1}{2} - \frac{1}{p})\alpha_\lambda{}^{\frac{p}{p-2}}$ and according to (14) we may assume that $G(v_m) > s$ for all m. Let

$$Q_m = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}_{N-1};\ |x'| < r \text{ and } m \le x_1 < m+1\}$$

and we put

$$d_n = \max_{m \in \mathbb{Z}} \|v_n\|_{L_p(Q_m)}.$$

We now claim that there exists a $\delta > 0$ such that $d_n \ge \delta$ for all $n = 1, 2, \ldots$. Indeed, by (14)

$$\|v_m\|_\lambda^2 = \|v_m\|_p^p + o(1),$$

and hence

$$G(v_m) = \Big(\frac{1}{2} - \frac{1}{p}\Big) \int_\Omega |v_m|^p\, dx + o(1) = \Big(\frac{1}{2} - \frac{1}{p}\Big)\|v_m\|_p^p = o(1).$$

Letting $b = \frac{2p}{p-2}$, we obtain by the Sobolev inequality that

$$\begin{aligned} bG(v_n)+o(1) &= \int_\Omega |v_n|^p\,dx = \sum_{m\in\mathbb{Z}} \|v_n\|^p_{L^p(Q_m)} \\ &\le \max_{m\in\mathbb{Z}} \|v_n\|^{p-2}_{L^p(Q_m)} \sum_{m\in\mathbb{Z}} \|v_n\|^2_{L^p(Q_m)} \\ &\le d_n^{p-2}C\sum_{m\in\mathbb{Z}}\int_{Q_m}(|Dv_n|^2+\lambda v_n^2)\,dx \\ &= d_n^{p-2}C\|v_n\|^2_\lambda = d_n^{p-2}CbG(v_n)+o(1) \end{aligned}$$

for some constant independent of n. Since $G(v_n)\ge s>0$ for $n=1,\ldots,$ we see that there exists a constant $\delta>0$ such that $d_n\ge\delta$ for $n=1,\ldots$. For each n we choose Q_{m_n} such that

$$\|v_n\|_{L^p(Q_{m_n})}\ge\frac{\delta}{2}.$$

This means that for each n there exists an integer $y_n=y_{m_n}$ such that

$$\|v_n\|_{L^p(Q_{m_n})}=\|v_n(\cdot+y_n)\|_{L^p(Q_\circ)}\ge\frac{\delta}{2}. \tag{15}$$

Since by (14) $\{v_n(\cdot+y_n)\}$ is bounded in $W_\circ^{1,2}(\Omega)$ we see that we can select a subsequence of $\{v_n(\cdot+y_n)\}$, which we again relabel as $\{v_n(\cdot+y_n)\}$ such that

$$v_n(\cdot+y_n)\rightharpoonup v \text{ in } W^{1,2}(\Omega),$$

$$v_n(\cdot+y_n)\to v \text{ in } L^p_{\mathrm{loc}}(\Omega)$$

and by (15) we have $\|v\|_{L^p(Q_\circ)}\ge\frac{\delta}{2}$, that is, $v\not\equiv 0$. We put $w_m(x)=v_n(x+y_n)$ and denote by α_∞ and β_∞ the quantities related to $\{w_m\}$ by (a) and (b) of Proposition 3. We write for $R>0$

$$\int_{\Omega\cap(|x|<R)}|w_m(x)|^p\,dx+\int_{\Omega\cap(|x|>R)}|w_m(x)|^p\,dx=\alpha_\lambda^{\frac{p}{p-2}}.$$

Letting $m\to\infty$ and then $R\to\infty$ we get

$$\int_\Omega|v(x)|^p\,dx+\alpha_\infty=\alpha_\lambda^{\frac{p}{p-2}}. \tag{16}$$

To complete the proof it is sufficient to show that $\alpha_\infty=0$. Arguing indirectly, we assume that $\alpha_\infty>0$. Since $v\not\equiv 0$, it follows from (16) that

$$\alpha_\infty<\alpha_\lambda^{\frac{p}{p-2}}. \tag{17}$$

Let ϕ_R be the function from the proof of Proposition 3. Since $G'(w_m)\to 0$ in $W^{-1,2}(\mathbb{R}_N)$ and we have

$$\begin{aligned} o(1) &= \langle G'(w_m),w_m\phi_R\rangle=\int_\Omega(|Dw_m|^2\phi_R+\lambda\phi_R w_m^2)\,dx \\ &\quad+\int_\Omega Dw_m D\phi_R w_m\,dx-\int_\Omega|w_m|^p\phi_R\,dx. \end{aligned}$$

Letting $m \to \infty$ and then $R \to \infty$ we find that $\alpha_\infty = \beta_\infty$. This combined with (e) of Proposition 3 gives $\alpha_\lambda \alpha_\infty^{\frac{2}{p}} \leq \alpha_\infty$, that is, $\alpha_\lambda^{\frac{2}{p-2}} \leq \alpha_\infty$, which contradicts (17).

Following [20] we introduce the definition of a periodic domain. We say that $\Omega \subset \mathbb{R}_N$ is a periodic domain if there exists a partition $\{Q_m\}$ of Ω and sequence of points $\{y_m\} \subset \mathbb{R}_N$ such that

(i) $\{y_m\}$ forms a subgroup of $\mathbb{R}_N$,
(ii) $Q_\circ$ is bounded and $Q_m = Q_\circ + y_m$.

Typical examples of periodic domain are: 1) $\Omega_1 = O \times \mathbb{R}_m$, where O is a bounded domain in $\mathbb{R}_p$ and $N = m + p$, 2) $\Omega_2 = \mathbb{R}_N$, 3) $\Omega_3 = \{(x, y) \in \mathbb{R}_2;\ \sin x < y < 1 + \sin x\}$.

Inspection of the proof of Theorem 2 shows that

Corollary 1. *The problem (3) admits a nontrivial solution on any periodic $\Omega \subset \mathbb{R}_N$ domain.*

Obviously, we have reproved here Theorem 4.2 from [20] whose proof was based on the global compactness Theorem 4.1 of this paper. The first existence result for problem (3) was obtained by Esteban [13] in a special class of symmetric functions. Later, a similar result without symmetry conditions, was established by P.L. Lions ([16], Theorem V.5 on p. 278).

Let us now consider functional G with $\Omega = \mathbb{R}_N - D$, where D is a bounded domain. A starting point is the observation made by Benci and Cerami that problem (M) does not have a solution and that $\alpha_\lambda(\Omega) = \alpha_\lambda(\mathbb{R}_N)$ (see Theorem 2.4 in [5]). Suppose that $G(u_m) \to c$, with $c > 0$ and $G'(u_m) \to 0$ in $W^{-1,2}(\Omega)$. If $\{u_m\}$ has a subsequence $\{u_{m_k}\}$ such that $u_{m_k} \to u$ in $W_\circ^{1,2}(\Omega)$, then

$$c = \left(\frac{1}{2} - \frac{1}{p}\right) \int_\Omega (|Du|^2 + \lambda u^2)\, dx = \left(\frac{1}{2} - \frac{1}{p}\right) \int_\Omega |u|^p\, dx.$$

Obviously, $u \not\equiv 0$ and since $\alpha_\lambda(\Omega)$ is not achieved we must have

$$\alpha_\lambda \left(\int_\Omega |u|^p\, dx \right)^{\frac{2}{p}} < \int_\Omega (|Du|^2 + \lambda u^2)\, dx.$$

The last two relations imply that

$$c > \left(\frac{1}{2} - \frac{1}{p}\right) \alpha_\lambda^{\frac{p}{p-2}}. \tag{18}$$

If $\{u_m\}$ is not relatively compact in $W_\circ^{1,2}(\Omega)$, we may assume that

$$\lim_{m\to\infty} \int_\Omega (|Du_m|^2 + \lambda u_m^2)\, dx = \int_\Omega |u_m|^p\, dx = l = \frac{2p}{p-2} c.$$

It follows from the definition of $\alpha_\lambda(\Omega)$ that $\alpha_\lambda l^{\frac{2}{p}} \le l$, that, is $\alpha_\lambda^{\frac{p}{p-2}} \le l$, This again implies that (18) holds. Consequently, one can expect that the Palais–Smale condition holds for levels c satisfying (18). In fact, it was proved in [5] (see Corollary 3.4 there) that if

$$(\frac{1}{2} - \frac{1}{p})\alpha_\lambda^{\frac{p}{p-2}} < c < 2(\frac{1}{2} - \frac{1}{p})\alpha_\lambda^{\frac{p}{p-2}},$$

then every sequence $\{u_m\}$ of nonnegative functions in $W_o^{1,2}(\Omega)$ such that $G(u_m) \to c$ and $G'(u_m) \to 0$ in $W^{-1,2}(\Omega)$ is relatively compact in $W^{1,2}(\Omega)$. We now establish a complementary result.

Proposition 4. *Suppose that* $\{u_m\} \subset W_o^{1,2}(\Omega)$ *is such that* $G(u_m) \to c$, *with*

$$(\frac{1}{2} - \frac{1}{p})\alpha_\lambda^{\frac{p}{p-2}} < c \le 2(\frac{1}{2} - \frac{1}{p})\alpha_\lambda^{\frac{p}{p-2}}$$

and $G'(u_m) \to 0$ *in* $W^{-1,2}(\Omega)$. *If* $\{u_{m_k}\}$ *is a subsequence of* $\{u_m\}$ *such that* $u_{m_k} \rightharpoonup u$ *in* $W_o^{1,2}(\Omega)$, *with* $u \not\equiv 0$, *then* $u_{m_k} \to u$ *in* $W_o^{1,2}(\Omega)$.

Proof. Our proof is based on Proposition 3. For simplicity we denote again our subsequence $\{u_{m_k}\}$ by $\{u_m\}$. We now observe that $G'(u_m) \to 0$ in $W^{-1,2}(\Omega)$ implies that

$$\int_\Omega (|Du_m|^2 + \lambda u_m^2)\,dx = \int_\Omega |u_m|^p\,dx + o(1)$$

and we may write

$$\begin{aligned} G(u_m) &= (\frac{1}{2} - \frac{1}{p}) \int_\Omega (|Du_m|^2 + \lambda u_m^2)\,dx + o(1) \\ &= (\frac{1}{2} - \frac{1}{p}) \int_\Omega |u_m|^p\,dx + o(1). \end{aligned}$$

Let α_∞ and β_∞ be quantities from Proposition 3 corresponding to the sequence $\{u_m\}$. Again, it is sufficient to show that $\alpha_\infty = 0$. Suppose that $\alpha_\infty > 0$. It follows from Proposition 3(a), (b) and the previous relation that

$$c = (\frac{1}{2} - \frac{1}{p}) \left(\int_\Omega |u|^p\,dx + \alpha_\infty \right) \tag{19}$$

and

$$c \ge (\frac{1}{2} - \frac{1}{p}) \left(\int_\Omega (|Du|^2 + \lambda u^2)\,dx + \beta_\infty \right). \tag{20}$$

Since $\langle G'(u_m), u_m\phi_R\rangle \to 0$, as $m \to \infty$, where ϕ_R is a function from the proof of Proposition 3, we see that $\alpha_\infty = \beta_\infty$. This combined with (d) of Proposition 3 gives

$$\alpha_\lambda^{\frac{p}{p-2}} \le \alpha_\infty. \tag{21}$$

On the other hand (19) and (20) yield that

$$(22)\qquad \int_{\Omega}(|Du|^2+\lambda u^2)\,dx \le \int_{\Omega}|u|^p\,dx.$$

Since $\alpha_\lambda(\Omega)$ is not achieved we have

$$\alpha_\lambda\left(\int_{\Omega}|u|^p\,dx\right)^{\frac{2}{p}} < \int_{\Omega}(|Du|^2+\lambda u^2)\,dx$$

and by virtue of (22) we obtain

$$\alpha_\lambda < \left(\int_{\Omega}|u|^p\,dx\right)^{\frac{p-2}{p}},$$

which is equivalent to

$$\alpha_\lambda^{\frac{p}{p-2}} < \int_{\Omega}|u|^p\,dx.$$

Finally, this combined with (19) and (21) leads to the inequality

$$c > 2\left(\frac{1}{2}-\frac{1}{p}\right)\alpha_\lambda^{\frac{p}{p-2}}$$

which is impossible.

6. Application to problem (4)

We make the following assumption on coefficients b and c of equation (4).

(I) $b \in L^\infty(\mathbb{R}_N)$ with $b \ge 0$, $b \not\equiv 0$ on $\mathbb{R}_N$ and $\lim_{|x|\to\infty} b(x) = \bar{b} > 0$,

(II) $c \in L_{\mathrm{loc}}^{\frac{q+\epsilon}{\epsilon}}(\mathbb{R}_N)$ for some ϵ satisfying $2 \le q < q+\epsilon < 2^*$, $c \ge 0$, $c \not\equiv 0$ on $\mathbb{R}_N$ and

$$\lim_{|x|\to\infty}\int_{Q(x,l)} c(y)^{\frac{q+\epsilon}{\epsilon}}\,dy = 0$$

for some $l > 0$.

According to Lemma 2 $W^{1,2}(\mathbb{R}_N)$ is compactly embedded into the weighted Lebesgue space $L_c^q(\mathbb{R}_N)$.

To write problem (4) in a variational form we introduce a functional H : $W^{1,2}(\mathbb{R}_N) \to \mathbb{R}$ given by

$$H(u) = \frac{1}{2}\int_{\mathbb{R}_N}(|Du|^2+u^2)\,dx - \frac{\lambda}{p}\int_{\mathbb{R}_N} b(x)|u|^p\,dx - \frac{1}{q}\int_{\mathbb{R}_N} c(x)|u|^q\,dx.$$

We consider the constrained minimization problem

$$(23)\qquad I_\lambda = \inf\{H(u);\ u \in M_\lambda\},$$

where

$$M_\lambda = \{u \in W^{1,2}(\mathbb{R}_N);\ u \not\equiv 0 \text{ and } \langle H'(u), u\rangle = 0\}.$$

It follows from the Lagrange multiplier method that every minimizer of (23) is a nontrivial solution to problem (4) (see [10] and [18]). Let

$$H_c(u) = \frac{1}{2}\int_{\mathbb{R}_N} (|Du|^2 + u^2)\,dx - \frac{1}{q}\int_{\mathbb{R}_N} c(x)|u|^q\,dx.$$

By Lemma 2 the functional $f(u) = \frac{1}{q}\int_{\mathbb{R}_N} c(x)|u|^q\,dx$ is weakly continuous on $W^{1,2}(\mathbb{R}_N)$. Therefore, the constrained minimization problem

$$0 < I^* = \inf\{H_c(u);\ u \in M_*\},$$

where

$$M_* = \{u \in W^{1,2}(\mathbb{R}_N);\ \langle H_c'(u), u\rangle = 0,\ u \not\equiv 0\},$$

has a solution which satisfies the equation

$$-\Delta u + u = c(x)|u|^{q-2}u \ \text{ in } \mathbb{R}_N.$$

Consequently, we can check as in Proposition 2.1 in [10] that

$$I_\lambda \le I^* \ \text{ for each } \lambda > 0. \tag{24}$$

As in Sect. 3 we set

$$0 < \alpha_1 = \alpha_1(\mathbb{R}_N) = \inf\{\int_{\mathbb{R}_N} (|Du|^2 + u^2)\,dx;\ \|u\|_p = 1\}.$$

Theorem 3. *Suppose that $I_\lambda < \frac{p-2}{2p}\alpha_1^{\frac{p}{p-2}}\left(\lambda\bar{b}\right)^{-\frac{2}{p-2}}$, then problem (4) has a nontrivial solution.*

Proof. Applying Ekeland's variational principle [12] we can find a minimizing sequence $\{u_m\} \subset M_\lambda$ such that $H(u_m) \to I_\lambda$ and $H'(u_m) \to 0$ in $W^{-1,2}(\mathbb{R}_N)$ (see the proof of Theorem 2.4 in [10]). It is easy to check that $\{u_m\}$ is bounded in $W^{1,2}(\mathbb{R}_N)$ and we may assume that $u_m \rightharpoonup u$ in $W^{1,2}(\mathbb{R}_N)$. Our objective is to show that $\{u_m\}$ is relatively compact in $W^{1,2}(\mathbb{R}_N)$. Towards this end we set

$$\beta_\infty = \lim_{R\to\infty} \limsup_{m\to\infty} \int_{|x|>R} |u_m|^p\,dx$$

and

$$\alpha_\infty = \lim_{R\to\infty} \limsup_{m\to\infty} \int_{|x|>R} (|Du_m|^2 + u_m^2)\,dx.$$

It follows from Proposition 3(e) that

$$\alpha_1 \alpha_\infty^{\frac{2}{p}} \le \beta_\infty. \tag{25}$$

It is sufficient to show that $\alpha_\infty = 0$. Suppose that $\alpha_\infty > 0$ and let ϕ_R be the function defined in the proof of Proposition 3. Since $\langle H'(u_m), u_m\phi_R\rangle \to 0$ as

$m \to \infty$ in $W^{-1,2}(\mathbb{R}_N)$, we easily deduce from this that $\beta_\infty \leq \lambda \bar{b} \alpha_\infty$. This combined with (25) gives

$$\alpha_\infty \geq \left(\frac{\alpha_1}{\lambda \bar{b}}\right)^{\frac{p}{p-2}} \tag{26}$$

On the other hand we have

$$\begin{aligned} H(u_m) &- \frac{1}{2}\langle H'(u_m), u_m\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\lambda \int_{\mathbb{R}_N} b(x)|u_m|^p\,dx + \left(\frac{1}{2} - \frac{1}{q}\right) \int_{\mathbb{R}_N} c(x)|u_m|^q\,dx \\ &\geq \frac{(p-2)\lambda}{2p} \int_{\mathbb{R}_N} b(x)|u_m|^p \phi_R\,dx. \end{aligned}$$

Letting $m \to \infty$ and then $R \to \infty$ we see that

$$I_\lambda \geq \frac{p-2}{2p} \lambda \bar{b} \alpha_\infty.$$

This combined with (26) leads to

$$I_\lambda \geq \frac{p-2}{2p} \left(\frac{\alpha_1}{\lambda \bar{b}}\right)^{\frac{p}{p-2}} \bar{b}\lambda = \frac{p-2}{2p} \alpha_1^{\frac{p}{p-2}} \left(\lambda \bar{b}\right)^{-\frac{2}{p-2}}$$

which is impossible.

A straightforward calculation (see Proposition 2.2 in [10]), based on a uniqueness result for positive radial solutions (see [14]) for equation (4) with $b \equiv \bar{b}$ and $c \equiv 0$, shows that

$$I_\lambda^\infty = \frac{p-2}{2p} \alpha_1^{\frac{p}{p-2}} \left(\lambda \bar{b}\right)^{-\frac{2}{p-2}},$$

where

$$I_\lambda^\infty = \inf\{F_\infty(u);\ u \in M_\lambda^\infty\},$$

with

$$F_\infty(u) = \frac{1}{2} \int_{\mathbb{R}_N} (|Du|^2 + u^2)\,dx - \frac{\lambda}{p} \int_{\mathbb{R}_N} \bar{b}|u|^p\,dx$$

and

$$M_\lambda^\infty = \{u \in W^{1,2}(\mathbb{R}_N);\ u \not\equiv 0 \text{ and } \langle F'_\infty(u), u\rangle = 0\}.$$

The assumption of Theorem 3 takes the form $I_\lambda < I_\lambda^\infty$ which is a condition used in papers [15] and [16].

Finally, taking into account (24) we can reformulate Theorem 3 in the following way

Corollary 2. *Suppose that* $\lambda \in \left(0, \left(\frac{p-2}{2pI^*}\right)^{\frac{p-2}{2}} \alpha_1^{\frac{p}{2}} \bar{b}^{-1}\right)$, *then problem (4) admits a nontrivial solution.*

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