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On Isometries and on a Theorem of Liouville*)

By

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In this paper, it is shown that LIOUVILLE's theorem [Theorem (III)] on conformal mappings of space is valid for (local) transformations of class C^1. This is deduced by a procedure used to prove Theorem (I) dealing with the smoothness of solutions of overdetermined systems of partial differential equations. This procedure depends on the notion of strong L^2-derivatives. A particular case of Theorem (I) implies that if two non-singular Riemann metrics of class C^σ, where $\sigma = 1, 2, \ldots$, are isometric by virtue of a mapping of class C^1, then the mapping is necessarily of class $C^{1+\sigma}$; Theorem (II).

1. Overdetermined systems. The first theorem of this paper concerns smoothness properties of solutions of overdetermined systems of partial differential equations.

(I) *Let $u = (u^1, \ldots, u^d)$ and $v = (v^1, \ldots, v^d)$. Let $f^I(u, v, v_1^1, v_2^1, \ldots, v_d^d)$, where $I = 1, 2, \ldots, \frac{1}{2}d(d+1)$, be functions of class C^σ, $\sigma \geqq 1$, on a $(d+d+d^2)$-dimensional domain. Let $v = v(u)$ be of class C^1 and satisfy the system of partial differential equations*

$$(1) \qquad f^I(u, v, \partial v^1/\partial u^1, \ldots, \partial v^d/\partial u^d) = 0, \qquad \text{where } I = 1, \ldots, \tfrac{1}{2}d(d+1),$$

on some u-domain. In addition, let

$$(2) \qquad \Delta(u) \neq 0,$$

where $\Delta(u)$ is the determinant of the matrix of coefficients of $\partial^2 v^i/\partial u^j \partial u^k$ in the system of $\frac{1}{2}d^2(d+1)$ linear equations obtained by differentiating (1) formally with respect to u^k, $k = 1, \ldots, d$, and assuming $\partial^2 v^i/\partial u^j \partial u^k = \partial^2 v^i/\partial u^k \partial u^j$. Then $v = v(u)$ is of class $C^{1+\sigma}$.

This theorem (with $d = 2$) was proved in [4] under the stronger assumption that $v = v(u)$ is of class C^2 (instead of class C^1). The question as to whether or not the above form of (I) is true was left open.

The proof of (I) below will depend on properties of functions having a finite "modified Dirichlet integral" in which partial derivatives are replaced by difference quotients. Essentially, (I) will first be proved in an L^2-sense for the case $\sigma = 1$.

It will be clear from the proof of (I) that an analogue holds if u and v are vectors of different dimensions, say, u is a d'-vector and v is a d-vector.

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In this case, the system (1) of $\frac{1}{2}d(d+1)$ equations must be replaced by a system of $[\frac{1}{2}d(d'+1)+\frac{1}{2}]$ equations and condition (2) must be replaced by the assumption that the rank of the corresponding $d'[\frac{1}{2}d(d'+1)+\frac{1}{2}]$ by $\frac{1}{2}dd'(d'+1)$ matrix of coefficients of the $\frac{1}{2}dd'(d'+1)$ variables $\partial^2 v^i/\partial u^j \partial u^k = \partial^2 v^i/\partial u^k \partial u^j$ is $\frac{1}{2}dd'(d'+1)$ at every point u under consideration.

2. Isometries. It is clear from [4] that (I) has applications in differential geometry (and, in some cases, leads to refinements of the assertions in [4]). In particular, the considerations of [4] indicate that (I) has the following corollary (cf. Section 8 below):

(II) *Let* $u=(u^1, \ldots, u^d)$, $v=(v^1, \ldots, v^d)$. *Let* $(g_{ik}(u))$, $(h_{ik}(v))$ *be non-singular, symmetric matrices of class* C^σ *in vicinities of* $u=0$, $v=0$, *respectively. Let* $v=v(u)$ *be a mapping of class* C^1 *in a vicinity of* $u=0$ *satisfying* $v(0)=0$ *and transforming the Riemannian metric*

$$ds^2 = g_{ik}\, du^i\, du^k \tag{3}$$

into

$$ds^2 = h_{ik}\, dv^i\, dv^k. \tag{4}$$

Then $v=v(u)$ *is of class* $C^{1+\sigma}$.

If $\sigma=0$, this theorem is trivial. If $\sigma \geq 1$ and if $v=v(u)$ is of class C^2 (instead of class C^1), then (II) follows from the transformation rules for the Christoffel symbols. For these rules show that the second order partial derivatives of v^i can be expressed as polynomials in g^{jk}, h^{jk} and the first order partial derivatives of v^j, g_{jk} and h_{jk}.

Using the theory of geodesics and parallel transport, (II) was proved in [3], pp. 222—226, in the binary ($d=2$), positive definite case. In [5], (II) was proved for $d=2$ without the assumption that (3) is a positive definite metric. The proof in [5] is not applicable to the case of d (>2) dimensions; for example, in the positive definite case, it depends on the Cauchy-Riemann equations.

3. Liouville's theorem. Theorem (II) and the arguments used in the proof of (I) will be used to obtain the following form of LIOUVILLE's theorem.

(III) *Let* $d \geq 3$, $u=(u^1, \ldots, u^d)$, $v=(v^1, \ldots, v^d)$. *Let* $v=v(u)$ *be a (local) mapping of class* C^1 *with non-vanishing Jacobian which maps the Euclidean metric*

$$ds^2 = (dv^1)^2 + \cdots + (dv^d)^2 \tag{5}$$

into a conformal metric

$$ds^2 = \gamma^2[(du^1)^2 + \cdots + (du^d)^2], \qquad \textit{where } \gamma = \gamma(u) > 0. \tag{6}$$

Then $v=v(u)$ *is a Möbius transformation (hence, analytic).*

The proofs of LIOUVILLE's theorem in standard texts require that $v=v(u)$ is of class C^3. A proof in which it is only assumed that $v=v(u)$ is of class C^2 is given in [2]. In (III), there is, of course, no assumption on the smoothness of γ; if γ is assumed to be of class C^1, then (III) follows from [2] and (II).

4. Two lemmas. The proofs of (I), (II), (III) will depend on two simple lemmas. The first is a consequence of a result of SÓLYI [8]. The second is obtained by a crude variation of arguments of MORREY [6] dealing with quasi-conformal maps and generalizations of these arguments by NIRENBERG [7].

The following notation will be used below: The range of a lower case index is to 1 to d, that of I is 1 to $\frac{1}{2}d(d+1)$. Repeated indices denote summation. When there is no possibility for confusion, partial differentiation with respect to the i-th component u^i of u will be denoted by the subscript i. If $g=g(u)$ is of class C^1 on a u-domain R, the Dirichlet integral of g over R will be denoted by $I_R(g)$,

$$I_R(g)=\sum_{i=1}^{d}\int_R |g_i(u)|^2 du,$$

where $du=du^1\dots du^d$. If $g(u)$ is continuous on a domain containing the closure of R, then, for small $|h|>0$, $I_{Rh}(g)$ will denote the modified Dirichlet integral

$$I_{Rh}(g)=h^{-2}\sum_{i=1}^{d}\int_R |\Delta_i g|^2 du, \tag{7}$$

where Δ_i denotes the difference operator

$$\Delta_i g=g(u^1,\dots,u^{i-1},u^i+h,u^{i+1},\dots,u^d)-g(u^1,\dots,u^d). \tag{8}$$

The following is implied by SÓLYI's paper [8] (cf. Section 5 below):

Lemma 1. *Let $g=g(u)$ be a continuous function on a domain containing the closure of a sphere T. Let there exist a constant K satisfying*

$$I_{Th}(g)\le K \tag{9}$$

for small $|h|>0$. Then $g(u)$ has strong L^2-derivatives on compact subsets of T.

In particular, the partial derivatives $g_k(u)$, $k=1,\dots,d$, exist almost everywhere on T. Below, this lemma will be applied to the components $g=v_j^i(u)$ of $v_j=\partial v/\partial v^j$. It will follow that v^i has second order partial derivatives $v_{jk}^i=v_{kj}^i$ almost everywhere and that if $f(u,v,v_1^1,\dots,v_d^d)$ is of class C^1 in its $d+d+d^2$ variables, then $f(u,v(u),v_1^1(u),\dots,v_d^d(u))$ can be differentiated by the chain rule almost everywhere with respect to u^k, $k=1,\dots,d$.

Lemma 2. *Let $z=z(u)=(z^1(u),\dots,z^d(u))$ be a vector function of class C^1 on a bounded u-domain R. Let there exist constants C_1, C_2, C_{ijkm} such that $|z^i(u)|\le C_1$ and*

$$\sum_{i=1}^{d}\sum_{j=1}^{d}|z_j^i|^2\le C_{ijkm}\,\partial(z^i,z^j)/\partial(u^k,u^m)+C_2. \tag{10}$$

Then, for any compact subset T of R, there exists a constant K, depending only on R, T, C_1, C_2, C_{ijkm}, such that the Dirichlet integrals of z^j satisfy

$$\sum_{j=1}^{d} I_T(z^j)\le K. \tag{11}$$

5. Proof of Lemma 1. In order to make clear the meaning of "strong L^2-derivatives" and the validity of the remarks following Lemma 1, a verification of Lemma 1 will be indicated.

In this verification, there is no loss of generality in supposing that $g(u^1, \ldots, u^d)$ is of period 2π in each variable u^j and that (9) holds for the parallelopiped T: $|u^j| \leq \pi$; [8], pp. 53—54. Let $g(u)$ have the Fourier development

$$g(u) \sim \sum a_n e^{in \cdot u},$$

where $n = (n_1, \ldots, n_d)$, $n \cdot u = n_1 u^1 + \cdots + n_d u^d$ and $n_j = 0, \pm 1, \ldots$. Let $g^N(u)$ denote the (finite) Fejér sum

$$g^N(u) = \sum (1 - |n_1|/N) \ldots (1 - |n_d|/N)\, a_n e^{in \cdot u}, \qquad \text{where } |n_j| < N.$$

The inequality (9) implies the convergence of the d-fold series

$$\sum |a_n|^2 |n|^2 = \sum |a_n|^2 (n_1^2 + \cdots + n_d^2) < \infty;$$

[8], pp. 49—50.

From the continuity of g, it follows by FEJER's theorem that $g^N \to g$ uniformly as $N \to \infty$. Finally, from the last formula line, it follows that, for $j = 1, \ldots, d$, the sequence $\partial g^1/\partial u^j, \partial g^2/\partial u^j, \ldots$ converges in the L^2-mean on T: $|u^k| \leq \pi$, to

$$g_j \sim \sum i n_j a_n e^{in \cdot u}.$$

6. Proof of Lemma 2. Without loss of generality, it can be supposed that T is a sphere, say $|u| \leq a$, and that R contains the slightly larger sphere $T(\theta)$: $|u| \leq a/\theta$, where θ is a fixed number on the range $0 < \theta < 1$. Let $e(r)$ be a non-negative function of class C^∞ for $r \geq 0$ satisfying $e = 1$ for $0 \leq r \leq a$ and $e = 0$ for $r \geq \frac{1}{2}(a + a/\theta)$.

An integration by parts gives

$$\int_{T(\theta)} e^2(r)\,(z_k^i z_m^j - z_m^i z_k^j)\, du = -2 \int_{T(\theta)} e e' z^i (z_m^j r_k - z_k^j r_m)\, du,$$

where $r = |u|$ and $e' = de/dr$. This integration by parts is valid even though z is only of class C^1 (rather, than of class C^2).

If $\varepsilon > 0$, then the absolute value of the last integral is majorized by

$$\varepsilon \int_{T(\theta)} e^2 \left(|z_m^j|^2 + |z_k^j|^2\right) du + \varepsilon^{-1} C_1^2 \int_{T(\theta)} e'^2(r)\, du,$$

since $|ab| \leq \frac{1}{2}\varepsilon a^2 + \frac{1}{2}\varepsilon^{-1} b^2$, $|z^i| \leq C_1$ and $|r_k|, |r_m| \leq 1$. Thus (10) implies that

$$\sum_{i=1}^{d} \sum_{j=1}^{d} \int_{T(\theta)} e^2 |z_j^i|^2 du \leq \varepsilon \sum_{i=1}^{d} |C_{ijkm}| \int_{T(\theta)} e^2 \left(|z_m^j|^2 + |z_k^j|^2\right) du + \tfrac{1}{2} K,$$

where K is a constant depending only on $C_1, C_2, C_{ijkm}, \varepsilon, R, T$ and the function $e(r)$. Hence if ε is chosen so small that $2d^2 \varepsilon |C_{ijkm}| \leq \frac{1}{2}$, then

$$\sum_{i=1}^{d} \sum_{j=1}^{d} \int_{T(\theta)} e^2 |z_j^k|^2 du \leq K.$$

Since $e \geq 0$ for $r \geq 0$ and $e = 1$ for $r \leq a$, the inequality (11) follows.

7. Proof of (I). Consider first the case $\sigma = 1$. In view of the local nature of the assertion of (I), it is sufficient to prove it for a vicinity of a given point, say $u = 0$, in the domain of definition of $v = v(u)$.

Let i be a fixed integer, $1 \leq i \leq d$, and h be a number such that $|h| > 0$ is small. Put

$$z = h^{-1} \Delta_i v, \tag{12}$$

where Δ_i is the difference operator defined in (8). Thus $z = z(u, h) = (z^1, \ldots, z^d)$ is a vector function of class C^1 for small $|u|$ and $|h| > 0$. It is a continuous function, even for $h = 0$, if $z(u, 0)$ is defined to be $\partial v / \partial u^i$.

Apply the difference operator $h^{-1} \Delta_i$ to the equation (1). By the mean value theorem of differential calculus, the result can be written as

$$(\partial f^I / \partial u^i)_0 + (\partial f^I / \partial v^j)_0 z^j + (\partial f^I / \partial v^j_n)_0 z^j_n = 0,$$

where the subscript 0 means that the partial derivative is evaluated at a suitable intermediary point. By virtue of the continuity assumptions, the last equation can be written as

$$d_j^{In} z_n^j = \sum_{j=1}^{d} \sum_{n=1}^{d} o(1) |z_n^j| + g^I(u, h), \tag{13}$$

where d_j^{In} is the value of $\partial f^I / \partial v_n^j$ at the point $(u, v, v_1^1, \ldots, v_d^d) = (0, v(0), v_1^1(0), \ldots, v_d^d(0))$; $o(1)$ refers to $|u| \to 0$ and $0 < |h| \to 0$; and $g^I(u, h)$ denotes a continuous function for small $|u|, |h|$ (≥ 0).

Multiply (13) by z_m^k and write the resulting equation in the form

$$\sum_{j=1}^{d} \left[\sum_{n=1}^{m} d_j^{In} z_n^j z_m^k + \sum_{n=m+1}^{d} d_j^{In} z_m^j z_n^k \right]$$
$$= \sum_{j=1}^{d} \sum_{n=m+1}^{d} d_j^{In} (z_m^j z_n^k - z_n^j z_m^k) + \sum_{j=1}^{d} \sum_{n=1}^{d} o(1) |z_n^j|^2 + g^I z_m^k,$$

where $I = 1, \ldots, \frac{1}{2} d(d+1)$, $m = 1, \ldots, d$ and k is fixed. This gives an inhomogeneous, linear system of $\frac{1}{2} d^2 (d+1)$ equation in the $\frac{1}{2} d^2 (d+1)$ variables $z_n^j z_m^k$, where $1 \leq j \leq d$ and $1 \leq n \leq m \leq d$. The definition of $\Delta(u)$ in (2) shows that the determinant of this system is $\Delta(0) \neq 0$. Hence, $z_n^j z_m^k$, where $1 \leq n \leq m \leq d$, can be written in the form

$$c_{pqrs} \partial(z^p, z^q) / \partial(u^r, u^s) + \sum_{p=1}^{d} \sum_{q=1}^{d} o(1) |z_q^p|^2 + c_I^q g^I z_q^k,$$

where c_{pqrs}, c_I^q are constants. In particular, $|z_m^k|^2$, when $m = 1, \ldots, d$, is of this form. Since k is arbitrary on the range $1 \leq k \leq d$, an application of an inequality of the type $|g^I z_q^k| \leq \frac{1}{2} \varepsilon |z_q^k|^2 + \frac{1}{2} \varepsilon^{-1} |g^I|^2$ and the boundedness of g^I imply that

$$\sum_{k=1}^{d} \sum_{m=1}^{d} \left(1 - \tfrac{1}{2} \varepsilon + o(1)\right) |z_m^k|^2 \leq C_{pqrs} \partial(z^p, z^q) / \partial(u^r, u^s) + C \varepsilon^{-1},$$

for suitable constants C_{pqrs} and C.

It follows from Lemma 2 that there exists a constant K such that (11) holds for small $|h|>0$ if T is a fixed small sphere $|u|\leq a$. The definitions of I_T and $z=h^{-1}\Delta_i v$ show, therefore, that

$$h^{-2}\sum_{k=1}^{d}\sum_{m=1}^{d}\int_T |\Delta_i v_m^k|^2\,du\leq K$$

for every $i=1,\ldots,d$. Hence, $I_{Th}(v_m^k)\leq Kd$.

By virtue of Lemma 1, v_m^k has strong L^2-derivatives and $\partial^2 v^k/\partial u^j\,\partial u^m = \partial^2 v^k/\partial u^m\,\partial u^j$ exist almost everywhere. Furthermore, $f^I(u, v(u), v_1^1(u),\ldots,v_d^d(u))$ can be differentiated by the chain rule almost everywhere with respect to u^k, $k=1,\ldots,d$. The resulting set of $\frac{1}{2}d^2(d+1)$ equations,

$$\partial f^I/\partial u^k + (\partial f^I/\partial v^i)\,v_k^i + (\partial f^I/\partial v_j^i)\,v_{jk}^i = 0,$$

is linear in the variables $v_{jk}^i = v_{kj}^i$ and can be solved for these variables, by virtue of (2). The continuity of the functions

$$(14)\qquad \partial f^I/\partial u^k,\ (\partial f^I/\partial v^i)\,v_k^i,\quad \partial f^I/\partial v_j^i$$

shows that $\partial^2 v^i/\partial u^j\,\partial u^k$ can be extended to continuous functions. Thus, v has continuous, second order partial derivatives. This proves the case $\sigma=1$ of (I).

If $\sigma=2$, it follows that the functions (14) are of class C^1 and, hence, $\partial^2 v^i/\partial u^j\,\partial u^k$ is of class C^1; that is, v is of class C^3. Clearly, (I) follows for any $\sigma\geq 1$ by induction.

8. Proof of (II). The transformation rule for metric tensors shows that the equivalence ot (3) und (4) means that the $\frac{1}{2}d(d+1)$ equations

$$(15)\qquad g^{ij}(u)\,v_i^k v_j^m - h^{km}(v) = 0$$

hold, where (g^{ij}), (h^{km}) are the (symmetric) matrices inverse to (g_{ij}), (h_{km}), respectively. The function of $(u, v, v_1^1,\ldots,v_d^d)$ on the left of (15) is of class C^σ with respect to its $d+d+d^2$ variables, by the assumptions of (II). If $v=v(u)$ and $v_i^k=\partial v^k(u)/\partial u^i$, formal differentiation of (15) with respect to u^j, where $j=1,\ldots,d$, leads to a linear system of equations for the second order partials of v. Furthermore, the determinant of the matrix of coefficients of these derivatives does not vanish; cf. the transformation rule for the Christoffel symbols. Hence (II) is the particular case of (I) in which (15) plays the role of (1).

9. Proof of (III). The equivalence of (5) and (6) means that

$$(16)\qquad \sum_{k=1}^{d} v_k^i v_k^j = \gamma^2\delta^{ij},$$

where δ^{ij} is 1 or 0 according as $i=j$ or $i\neq j$. After an affine change of variables, it can be supposed that $u=0$ is a point of the u-domain under consideration and that

$$(17)\qquad v_k^j(0)=\delta^{jk}.$$

Let i, h, Δ_i, and z have the same significance as in (12). Then (16) and (17) show that

$$z^i_k + z^k_j = (h^{-1}\Delta_i\gamma^2)\,\delta^{jk} + \sum_{m=1}^{d}\sum_{n=1}^{d} o(1)\,|z^m_n|, \tag{18}$$

where the $o(1)$ refers to $|u|\to 0$ and $0<|h|\to 0$.

If $j=k$, then the last relation becomes

$$z^j_j = \tfrac{1}{2}(h^{-1}\Delta_i\gamma^2) + \sum_{m=1}^{d}\sum_{n=1}^{d} o(1)\,|z^m_n|. \tag{19}$$

This formula and the corresponding one for z^k_k show that

$$z^j_j - z^k_k = \sum_{m=1}^{d}\sum_{n=1}^{d} o(1)\,|z^m_n|. \tag{20}$$

Hence, by (18) and (20), where $j\neq k$,

$$z^j_j z^k_k - z^j_k z^k_j = \tfrac{1}{2}\left(|z^j_j|^2 + |z^k_k|^2 + |z^j_k|^2 + |z^k_j|^2\right) + \sum_{m=1}^{d}\sum_{n=1}^{d} o(1)\,|z^m_n|^2. \tag{21}$$

Clearly, this implies an inequality of the type (10).

The proof of (I) implies, therefore, that the function $v=v(u)$ (of class C^1) has strong, second order, L^2-derivatives in a vicinity of $u=0$; in particular, $\partial^2 v^i/\partial u^j\partial u^k$ exists and equals $\partial^2 v^i/\partial u^k\partial u^j$ almost everywhere. It follows from (16) and $\gamma>0$ that γ has strong, first order, L^2-derivatives.

It will now be verified that a standard proof of LIOUVILLE's theorem ([*1*], pp. 460—462) involving the Lamé equations can be carried out under these differentiability conditions on $v=v(u)$ and $\gamma=\gamma(u)$.

The derivation formulae for (16) (that is, the transformation rules for the Christoffel symbols) are valid almost everywhere and give

$$v_{ij} = \gamma^{-1}(\gamma_j v_i + \gamma_i v_j) \qquad \text{if } i\neq j, \tag{22}$$

$$v_{ii} = \gamma^{-1}\Big(\gamma_i v_i - \sum_{k\neq i}\gamma_k v_k\Big). \tag{23}$$

Let $y_i = y_i(u)$ denote the unit vector

$$y_i = \lambda v_i, \qquad \text{where } \lambda = \gamma^{-1}. \tag{24}$$

(Note that the subscript i on y_i does not denote partial differentiation.) The vectors (24) have strong L^2-derivatives and (22), (23) imply

$$\partial y_i/\partial u^j = -\lambda_i v_j \qquad \text{if } i\neq j, \tag{25}$$

$$\partial y_i/\partial u^i = \sum_{k\neq i}\lambda_k v_k \tag{26}$$

almost everywhere.

Let the i-th coordinate u^i be fixed, not on a certain 1-dimensional zero set. Then (25) gives

$$\int_J \lambda_i\, dv = 0, \tag{27}$$

if J is, for example, the boundary of a rectangle $a^j \leqq u^j \leqq b^j$, $a^k \leqq u^k \leqq b^k$, $u^m = \text{const}$ if $m \neq j, k$; $i \neq j, k$; and the numbers a^j, b^j, a^k, b^k do not belong to certain 1-dimensional zero sets. Obvious modifications of the proof of Theorem 2, [2], p. 328, show that λ_i does not depend on u^j if $j \neq i$; that is, $\lambda_i = \partial\lambda/\partial u^i$ which is defined almost everywhere is a function only of the i-th component u^i of u. (The proof in [2] remains essentially valid for the present situation since the property of having strong L^2-derivatives is invariant under C^1 changes of the independent variables.)

If $i, j\ (\neq i)$ are fixed, the relations (25), (26) imply

$$\int_J \lambda_i v_j \, du^j - \Big(\sum_{k \neq i} \lambda_k v_k\Big) du^i = 0, \tag{28}$$

if J is the boundary of a rectangle $a \leqq u^i \leqq b$, $\alpha \leqq u^j \leqq \beta$, $u^k = \text{const}$ if $k \neq i, j$, and a, b, α, β do not belong to a certain 1-dimensional zero set. Since λ_k depends only on u^k, (28) gives

$$\lambda_i(b)\, s(b) - \lambda_i(a)\, s(a) - \sum_{k \neq i} \lambda_k(u^k)\, [t_k(\beta) - t_k(\alpha)] = 0, \tag{29}$$

where

$$\left\{ \begin{aligned} s(u^i) = s(u^i, \alpha, \beta) = v(u^1, \ldots, u^{j-1}, \beta, u^{j+1}, \ldots, u^d) - \\ - v(u^1, \ldots, u^{j-1}, \alpha, u^{j+1}, \ldots, u^d) \end{aligned} \right. \tag{30}$$

and

$$t_k(u^j) = t_k(u^j, a, b) = \int_a^b v_k(u)\, du^i. \tag{31}$$

Rewrite (29) in the form

$$s(b)\, [\lambda_i(b) - \lambda_i(a)] = -\lambda_i(a)\, [s(b) - s(a)] + \sum_{k \neq i} \lambda_k(u^k)\, [t_k(\beta) - t_k(\alpha)]. \tag{32}$$

Consider a, α, β, u^k for $k \neq i$ to be fixed and b to be a variable. Then $s(b)$ is of class C^1. Also $t_k(u^j, a, b)$ is of class C^1, as a function of b; cf. (31). Thus, by (32), $s(b)[\lambda_i(b) - \lambda_i(a)]$ is of class C^1. The vector $s(b)$ does not vanish if α and β are suitably chosen (and fixed), for v cannot be a constant on any u^j-interval; cf. (16).

It follows that $\lambda_i(u^i) = \partial\lambda/\partial u^i$ can be extended to a continuous function and so, λ has continuous partial derivatives with respect to u^i, $i = 1, \ldots, d$. Thus λ (hence γ) is of class C^1. By (II), $v = v(u)$ is of class C^2.

Divide (32) by $b - a$ and let $b \to a$. It follows that λ_i has a derivative λ_{ii} with respect to u^i and that

$$s(a)\, \lambda_{ii}(a) = -\lambda_i(a)\, \{v_i\} + \sum_{k \neq i} \lambda_k(u^k)\, \{v_k\},$$

where $\{g\}$ denotes the difference of the values of g when $u^i = a$, $u^j = \beta$ and $u^i = a$, $u^j = \alpha$. Consequently, λ has continuous second derivatives, λ_{ii} and $\lambda_{ij} = 0$ if $i \neq j$. Hence, by (II), $v = v(u)$ is of class C^3. The proof of (III) can now be completed as in [1], pp. 460–462.

References

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