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Supersolvable Automorphism Groups of Solvable Groups*

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Introduction

A long-standing question is the following:

Conjecture. Let AG be a finite solvable group and assume $G \lhd AG$, $(|A|, |G|) = 1$ and $C_G(A) = 1$. Then the Fitting height of G is bounded above by the composition length of A (i.e. the number of primes dividing $|A|$, counting multiplicities).

In 1973 Berger [9] (using [2–5, 7, 8]) proved the conjecture for A nilpotent and $\mathbb{Z}_p \wr \mathbb{Z}_p$ free for all primes p (here $\mathbb{Z}_p \wr \mathbb{Z}_p$ is the wreath product). The conjecture is also known in some other cases [1, 10]. For more details on the history of the problem before 1973 see [9]. The main result of the following work is the proof (Theorem 4.7) of the conjecture if A is supersolvable and every proper subgroup of A is $\mathbb{Z}_r \wr \mathbb{Z}_s$ free (for all r, s) and $G(\varepsilon, p^n, q)$ free (for $G(\varepsilon, p^n, q)$ not nilpotent). $G(\varepsilon, p^n, q)$ is defined in 1.1 and some of its properties are given in Proposition 1.2. As special cases of this theorem we get all the known cases of the conjecture (see note after Definition 4.4).

The conjecture is naturally translated into a representation-theoretic question. Namely: if AG is as in the conjecture, the Fitting height of G is equal to the composition length of A and M is a faithful irreducible kAG-module (k an algebraically closed field, $\operatorname{char}(k) \nmid |A|$), can $C_M(A) = 0$?

Since $C_G(A) = 1$, M is always induced from N_1 a kA_0G-module where $A_0 \subset A$. Now, if N_1 is induced from a kA_0G_0-module $N_2 (G_0 \subset G)$, we need to study the permutation representation of A_0 on the cosets of A_0G_0 in A_0G. Since G is solvable this essentially reduces to the study of the permutation structure of $\mathbb{F}_pA_0$-modules (p a prime, $p \mid |G:G_0|$). We prove (Theorem 2.2) that under our hypothesis if W is a $\mathbb{F}_pA_0$-module then there is a vector $v \in W$ such that $C_{A_0}(v) = \ker(W)$ (we say that $A_0/\ker W$ has a regular orbit on W). This generalizes a result of Berger [8].

So we get to the situation where N_2 is a primitive faithful kA_1G_1-module (A_1 and G_1 quotients of A_0 and G_0 respectively). At this point there is an ex-

* This is essentially the author's doctoral dissertation at the University of Chicago

traspecial p-group $P \subseteq G_1$ $P \lhd A_1 G_1$ such that $Z(P) \subseteq Z(A_1 G_1)$ and A_1 acts faithfully on $P/Z(P)$. If G_1 is nilpotent we get information easily, so we may assume that there is a subgroup $H \lhd A_1 G_1, P \subseteq H \subseteq G_1$ such that $[H, P] = P$ and $H/C_H(P/P')$ is elementary Abelian. Under these hypothesis we show (Theorem 3.4) that $N_2|_{A_1}$ contains a regular direct summand (a copy of kA_1). This generalizes a result of Berger [8].

We need to have the subgroup H in order to apply Theorem 3.4, so we define the subgroup $\mathrm{supp}_A(G)$ of G (Definition 4.2) to be the appropriate part of $F(G)$ (the Fitting subgroup of G). $\mathrm{supp}_A(G)$ essentially behaves as the Fitting subgroup (see Proposition 4.3). We obtain Theorem 4.6 which easily implies the main result and would be false with $F(G)$ in place of $\mathrm{supp}_A(G)$.

The work is divided into four sections. The last is the proof that Theorem 2.2 and Theorem 3.4 are sufficient to prove the conjecture in our case. The first three sections are a proof of Theorem 2.2 and Theorem 3.4.

To prove Theorem 2.2 we consider two cases: either W is induced and we apply induction and Proposition 1.6; or it is not in which case the structure is very tight (Proposition 2.1) and we have necessary and sufficient conditions for the existence of regular orbits (Proposition 1.4).

To prove Theorem 3.4 we consider P/P' as an $A_1 H$-module. There are essentially two cases: either it is induced (then $N_2|_{A_1}$ is "tensor induced") and again we apply Proposition 1.6; or it is not induced in which case the structure is very tight (Proposition 3.1) and we describe $N_2|_{A_1 H}$ in great detail (Theorem 3.2). In my view Theorem 3.2 is likely to find applications in other problems in finite solvable group theory. This is why, although only the non-modular case is used, it is proved in slightly more general form. In any case, Theorem 3.2 allows us to apply (again) Proposition 1.4 which now gives sufficient conditions for the existence of regular modules.

In view of possible applications and to clarify the method, I have tried to work with slightly more general groups than supersolvable ones. We assume only some consequences of supersolvability as needed for the various arguments. We do need, however, A to be supersolvable in the final Theorem 4.7.

1. Basic Definitions and Results

Definitions and Notation 1.1. For p^n a power of a prime p and $\varepsilon = -1$ or a power of 2 such that $\varepsilon | n$ define:

$F_\varepsilon(p^n) = \mathbb{F}_{p^n}^*$ (the multiplicative group of the field of p^n elements) if $\varepsilon > 0$;
$\quad$ = subgroup of order $p^n + 1$ of $\mathbb{F}_{p^{2n}}^*$ if $\varepsilon = -1$.

$\mathrm{Gal}(1, p^n) = \mathrm{Gal}(\mathbb{F}_{p^n} : \mathbb{F}_p)$ (the Galois group of $\mathbb{F}_{p^n}$).

$\mathrm{Gal}(-1, p^n) = O_{2'}(\mathrm{Gal}(\mathbb{F}_{p^{2n}} : \mathbb{F}_p))$.

If $\varepsilon \neq \pm 1$ $\mathrm{Gal}(\varepsilon, p^n) = \langle \varphi \rangle \times \mathrm{Gal}(-1, p^{(n/2)})$ where φ is the automorphism of $F_\varepsilon(p^n)$: $x \to x^{-p^{(n/\varepsilon)}}$.

$G(\varepsilon, p^n) = \mathrm{Gal}(\varepsilon, p^n) \ltimes F_\varepsilon(p^n)$ (the semi-direct product).

$o(x)$ = order of x.

If $\sigma \in \mathrm{Gal}(\varepsilon, p^n)$ and $y \in F_\varepsilon(p^n)$ we denote N_σ the norm map i.e.

$$N_\sigma(y) = \prod_{\tau \in \langle \sigma \rangle} \tau(y) = y \cdot \sigma(y) \dots \sigma^{(o(\sigma)-1)}(y).$$

Suppose that q is a prime and divides $|\mathrm{Gal}(\varepsilon, p^n)|$. Define

$$GN(\varepsilon, p^n, q) = A \ltimes N \subseteq G(\varepsilon, p^n)$$

where A is the subgroup of $\mathrm{Gal}(\varepsilon, p^n)$ or order q, and

$$N = \{x \in F_\varepsilon(p^n) \colon \prod_{\sigma \in A} \sigma(x) = 1\}$$

the set of all elements of $F_\varepsilon(p^n)$ of norm 1 under a non-trivial element of A.

Proposition 1.2. *For any ε, p, n, q such that $GN(\varepsilon, p^n, q)$ is defined take $N = GN(\varepsilon, p^n, q) \cap F_\varepsilon(p^n)$. We have the following:*

1) *N is a cyclic normal subgroup of $GN(\varepsilon, p^n, q)$ of index q and every element of $GN(\varepsilon, p^n, q)$ not in N has order q.*

2) *If q is odd or $s > 1$, $GN(2^s, p^n, q) = GN(1, p^n, q)$.*

3) *If $\varepsilon = \pm 1$, $|N| = (p^n - \varepsilon)/(p^{(n/q)} - \varepsilon)$.*

4) *If $\varepsilon = 2 = q$, $|N| = p^{(n/2)} - 1$.*

5) *For any prime r dividing $|N|$, we have $r \geqq q$ and either $r = q$ or $r \equiv 1\,(q)$.*

6) *If $GN(\varepsilon, p^n, q)$ is nilpotent one of the following is satisfied:*

a) *$q = 3$, $p = 2$, $\varepsilon = -1$, $n = 3$, $GN(\varepsilon, p^n, q)$ is elementary Abelian of order 9 and $F_\varepsilon(p^n)$ is a 3-group.*

b) *$q = 2$, $p = 3$, $\varepsilon = 2$, $n = 2$, $GN(\varepsilon, p^n, q)$ is elementary Abelian of order 4 and $F_\varepsilon(p^n)$ is a 2-group.*

c) *$q = 2$, p is a Mersenne prime, $\varepsilon = 1$, $n = 2$ and $GN(\varepsilon, p^n, q)$ is non-Abelian dihedral of order $2(p+1)$.*

d) *$q = 2$, p is a Fermat prime greater than 3, $\varepsilon = 2$, $n = 2$ or $p = 3$, $\varepsilon = 2$, and $n = 4$. In any case $GN(\varepsilon, p^n, q)$ is non-Abelian dihedral of order $2(p^{(n/2)} - 1)$.*

7) *If $GN(\varepsilon, p^n, q)$ is Abelian, then $\varepsilon \neq 1$ and $F_\varepsilon(p^n)$ is a q-group. Any nilpotent non-Abelian $GN(\varepsilon, p^n, q)$ contains the dihedral group of order 8.*

Proof. 1) Take $\sigma \in \mathrm{Gal}(\varepsilon, p^n)\, x \in F_\varepsilon(p^n)$, then $(\sigma x)^{o(\sigma)} = N_\sigma(x)$. So we have 1).

2) is clear from the definition.

3) and 4) see Proposition 1.3 a).

5) Clearly if $x \in N$ and $x \in Z(GN(\varepsilon, p^n, q))$, then $x^q = 1$, so if $r \mid |N|$ and $r \neq q$ is a prime $q \mid r - 1$.

6) Let $GN(\varepsilon, p^n, q)$ be nilpotent. This implies that N is a q-group. So assume that N is a q-group.

Suppose first that $q \neq 2$. We may assume then that $\varepsilon = \pm 1$ by 2). Now if $\dfrac{p^n - \varepsilon}{p^{(n/q)} - \varepsilon} = q$ we have a). So suppose that $q^2 \,\Big|\, \dfrac{p^n - \varepsilon}{p^{(n/q)} - \varepsilon}$. Set $\dfrac{n}{q} = m$ and we have

$$\frac{p^n - \varepsilon}{p^m - \varepsilon} = p^{m(q-1)} + \varepsilon p^{m(q-2)} + \dots + \varepsilon^{q-1}.$$

Clearly $p^m \equiv \varepsilon + aq\ (q^2)$ for some $a \in \mathbb{Z}$, so

$$
\begin{aligned}
(\operatorname{mod} q^2)\ |N| &= p^{m(q-1)} + \varepsilon p^{m(q-2)} + \ldots + \varepsilon^{q-1}\\
&\equiv (\varepsilon + aq)^{(q-1)} + \varepsilon(\varepsilon + aq)^{(q-2)} + \ldots + \varepsilon^{q-1}\\
&\equiv \varepsilon^{q-1} + (q-1)\,aq\varepsilon^{q-2} + \varepsilon^{q-1} + (q-2)\,aq\varepsilon^{q-2} + \ldots + \varepsilon^{q-1}\\
&= q\varepsilon^{q-1} + \frac{q(q-1)}{2}\,aq\varepsilon^{q-2} \equiv q\varepsilon^{q-1} \text{ since } q \neq 2.
\end{aligned}
$$

This is a contradiction.

So assume now that $q=2$. This means that $GN(\varepsilon, p^n, q)$ is either Abelian or a dihedral 2-group. We also have $\varepsilon \neq -1$. If $\varepsilon = 1$,

$$|N| = \frac{p^n - 1}{p^{(n/2)} - 1} = p^{(n/2)} + 1$$

so that we get c). By 2), since $n=2$, this excludes the possibility $\varepsilon > 2$. If $\varepsilon = 2$,

$$|N| = p^{(n/2)} - 1$$

and we get b) and d). This completes the proof.

7) This follows from 6).

The first part of the next proposition can be viewed as a generalization of Hilbert's Theorem 90.

Proposition 1.3. a) *Let $\sigma \in \mathrm{Gal}(\varepsilon, p^n)$ be of order s and set*

$$N = \{x \in F_\varepsilon(p^n) \colon N_\sigma(x) = 1\}.$$

Then we have that $x \in N$ iff $x = \frac{\sigma(y)}{y}$ for some $y \in F_\varepsilon(p^n)$. Furthermore we have

$$
\begin{aligned}
|N| &= \frac{p^n - \varepsilon}{p^{(n/s)} - \varepsilon} && \varepsilon = \pm 1,\\
&= \frac{p^n - 1}{p^{(n/s)} - 1} && \varepsilon > 1\ \varepsilon \nmid s,\\
&= \frac{p^n - 1}{p^{(n/s)} + 1} && \varepsilon > 1\ \varepsilon \mid s.
\end{aligned}
$$

b) *Let A be a subgroup of $G(\varepsilon, p^n)$. Set $B = A \cap F_\varepsilon(p^n)$. There exists a conjugate A_1 in $G(\varepsilon, p^n)$ of A such that for any $x \in A \setminus B$ of prime order q_x, there is $\sigma \in A_1 \cap \mathrm{Gal}(\varepsilon, p^n)$ $\sigma \neq 1$ such that $\sigma^{q_x} = 1$.*

Proof. a) Clearly $N_\sigma\left(\frac{\sigma(y)}{y}\right) = 1$, so that $f(y) = \frac{\sigma(y)}{y}$ is a homomorphism of $F_\varepsilon(p^n)$ into N.

If for some r, $z = \frac{\sigma^r(y)}{y}$ we have

$$z = \frac{\sigma(\sigma^{(r-1)}(y)\,\sigma^{(r-2)}(y) \ldots y)}{\sigma^{(r-1)}(y)\,\sigma^{(r-2)}(y) \ldots y},$$

so that we may replace σ by σ^r provided that $(r,s)=1$. So we may suppose

$$\sigma(x)=x^{\alpha p^{(\beta n/s)}} \qquad x\in F_\varepsilon(p^n)$$

where $\alpha=-1$ if $\varepsilon>1$ and $\varepsilon|s$, and $\alpha=1$ otherwise, and $\beta=2$ if $\varepsilon=-1$, and $\beta=1$ otherwise.

$$N_\sigma(x)=x\cdot x^{\alpha p^{(\beta n/s)}}\dots x^{\alpha^{(s-1)}p^{(\beta n/s)(s-1)}}=x^\nu$$

where $\nu=1+\alpha p^{(\beta n/s)}+\dots+\alpha^{(s-1)}p^{(\beta n/s)(s-1)}=\dfrac{p^{\beta n}-1}{\alpha p^{(\beta n/s)}-1}$. So

$$|N|=\left(\frac{p^{\beta n}-1}{\alpha p^{(\beta n/s)}-1},\,|F_\varepsilon(p^n)|\right).$$

Now $f(x)=x^{\alpha p^{(\beta n/s)}-1}$, so that

$$|\ker f|=(\alpha p^{(\beta n/s)}-1,\,|F_\varepsilon(p^n)|).$$

If $\varepsilon\neq-1$,

$$|N|=\left(\frac{p^n-1}{\alpha p^{(n/s)}-1},\,p^n-1\right)=\frac{p^n-1}{|\alpha p^{(n/s)}-1|},$$
$$|\ker f|=(\alpha p^{(n/s)}-1,\,p^n-1)=|\alpha p^{(n/s)}-1|.$$

Hence a) is satisfied in this case.

If $\varepsilon=-1$, then s is odd and

$$|N|=\left(\frac{p^{2n}-1}{p^{(2n/s)}-1},\,p^n+1\right)=\frac{p^n+1}{p^{(n/s)}+1},$$
$$|\ker f|=(p^{(2n/s)}-1,\,p^n+1)=p^{(n/s)}+1.$$

Hence we have a) in all cases.

b) Write $|\mathrm{Gal}(\varepsilon,p^n)|=q_1^{\varepsilon_1}\dots q_s^{\varepsilon_s}q_{s+1}^{\varepsilon_{s+1}}\dots q_r^{\varepsilon_r}$ a product of primes such that $A\backslash B$ has elements of order q_i $i=1,\dots,s$ but not of order $q_{s+1},\dots,q_r$. Assume also that $q_1<q_2<\dots<q_s$.

Take A^* a conjugate of A which contains elements $\sigma_1,\dots,\sigma_{i-1}\in\mathrm{Gal}(\varepsilon,p^n)$ of order $q_1,\dots,q_{i-1}$ respectively, with i as large as possible. Assume $i\leqq s$.

Take $x\in A^*\backslash B$ of order q_i. We may write $x=\sigma_i\alpha$ with $\sigma_i\in\mathrm{Gal}(\varepsilon,p^n)$ of order q_i and $\alpha\in F_\varepsilon(p^n)$. We have $1=(\sigma_i\alpha)^{q_i}=N_{\sigma_i}(\alpha)$.

Since $\sigma_j\in A^*$ for $j=1,\dots,i-1$ we have $[\sigma_j,x]\in A^*$ and $[\sigma_j,x]=[\sigma_j,\sigma_i\alpha]=\sigma_j^{-1}\alpha^{-1}\sigma_i^{-1}\sigma_j\sigma_i\alpha=[\sigma_j,\alpha]$. On the other hand σ_j normalizes $\langle\alpha\rangle$, and since $N_{\sigma_i}(\alpha)=1$, any prime dividing $|\langle\alpha\rangle|$ is at least as large as q_i (Proposition 1.2, 5). Therefore $q_j\nmid|\langle\alpha\rangle|$ and

$$\langle\alpha\rangle=C_{\langle\alpha\rangle}(\sigma_j)\times[\sigma_j,\langle\alpha\rangle].$$

Since $[\sigma_j,\langle\alpha\rangle]=\langle[\sigma_j,\alpha]\rangle\subseteq A^*$ and $\sigma_i\alpha\in A^*$, there is $y\in C_{\langle\alpha\rangle}(\sigma_j)$ such that $\sigma_i y\in A^*$.

This process actually gives $y \in C_{\langle\alpha\rangle}(\sigma_1, \ldots, \sigma_{i-1})$ such that $\sigma_i y \in A^*$ and (of course) $N_{\sigma_i}(y)=1$.

Now if $i=1$, or $i>1$ and σ_1 is a field automorphism, $y \in F_\varepsilon(p^{n/(q_1 \cdots q_{i-1})})$ and by a) there is $z \in F_\varepsilon(p^{n/(q_1 \cdots q_{i-1})})$ such that $y = \frac{\sigma_i^{-1}(z)}{z}$. If $i>1$ and σ_1 is not a field automorphism then $q_1=2$, $\varepsilon=2$ and

$$C_{F_2(p^{n/(q_2 \cdots q_{i-1})})}(\sigma_1) = F_{-1}(p^{n/(q_1 \cdots q_{i-1})}),$$

so that by a) in any case there is $z \in C_{F_\varepsilon(p^n)}(\sigma_1, \ldots, \sigma_{i-1})$ such that $y = \frac{\sigma_i^{-1}(z)}{z}$.

Take A^{*z}. Then, since $z \in C_{F_\varepsilon(p^n)}(\sigma_1, \ldots, \sigma_{i-1})$, $\sigma_j \in A^{*z}$ for $j=1, \ldots, i-1$.

$$\sigma_i = \sigma_i \sigma_i^{-1}(z^{-1})\, y z = z^{-1} \sigma_i y z \in A^{*z},$$

a contradiction. So we have b).

Propositon 1.4. *$G(\varepsilon, p^n)$ acts on the left in a natural way on $F_\varepsilon(p^n)$, with $F_\varepsilon(p^n)$ acting by multiplication and $\mathrm{Gal}(\varepsilon, p^n)$ in its natural way. Let $A \subseteq G(\varepsilon, p^n)$ and $B = A \cap F_\varepsilon(p^n)$. Then the following are equivalent:*

A) *A has a regular orbit on $F_\varepsilon(p^n)$.*

B) *For any prime q, $GN(\varepsilon, p^n, q)$ is not conjugate in $G(\varepsilon, p^n)$ to a subgroup of A.*

C) *For any prime q such that $A \setminus B$ has an element of order q, if we take $\sigma_q \in \mathrm{Gal}(\varepsilon, p^n)$ of order q and $N_q = \{x \in F_\varepsilon(p^n)\colon N_{\sigma_q}(x)=1\}$. Then $N_q \nsubseteq B$.*

Proof. Set $F = F_\varepsilon(p^n)$ and $\pi = \{q$ a prime: $A \setminus B$ has an element of order $q\}$. By Proposition 1.3 b), we may assume that, for any $q \in \pi$, $\sigma_q \in \mathrm{Gal}(\varepsilon, p^n) \cap A$ is an element of order q. Now B) and C) are equivalent.

In this proposition C_F denotes the fixed points in F under the action described in the statement of the proposition. We get $\bigcup_{a \in A^{\#}} C_F(a) = \bigcup C_F(a)$ where a runs through the elements of prime order of $A \setminus B$. So

$$\bigcup_{a \in A^{\#}} C_F(a) = \bigcup_{q \in \pi} \bigcup_{b \in B} C_F(b \cdot \sigma_q).$$

But $\bigcup_{b \in B} C_F(b \cdot \sigma_q) = \left\{ x \in F\colon \frac{\sigma_q(x)}{x} \in B \right\}$ is a subgroup of F and is proper iff $N_q \nsubseteq B$, by Proposition 1.3, a). Since F is cyclic,

$$F = \bigcup_{q \in \pi} \bigcup_{b \in B} C_F(b \sigma_q) \quad \text{iff} \quad \bigcup_{b \in B} C_F(b \cdot \sigma_q) = F$$

for some $q \in \pi$, or equivalently $N_q \subseteq B$ for some $q \in \pi$. This shows Proposition 1.4.

Definition 1.5. We shall say that a maximal subgroup A of G is relevant iff there is an Abelian normal subgroup $N \lhd G$ with a subgroup $N_1 \subseteq N$ such that $N_G(N_1) \subseteq A$.

Proposition 1.6. *Let G be a solvable group that does not contain any section of the form $\mathbb{Z}_r \wr \mathbb{Z}_s$ (r and s possible equal; $\mathbb{Z}_r \wr \mathbb{Z}_s$ being the wreath product) and* *such that for every chief factor N/M of G, $G/C_G(N/M)$ has a regular orbit on the elements of N/M. Suppose A is a maximal subgroup of G and K is a normal subgroup of A such that $\bigcap_{x\in G} K^x = 1$ and A/K operates on the left by permutations on some given set Ω with a regular orbit and at least one other orbit.*

Then G permutes $\Omega^G = 1 \otimes \Omega \times x_1 \otimes \Omega \times \ldots \times x_{n-1} \otimes \Omega$ where $\times$ denotes cartesian product and $x_0 = 1$, $x_1, \ldots, x_{n-1}$ is a system of representatives of $\{xA: x\in G\}$. Moreover G has at least three orbits on Ω^G and if any of the following is satisfied, then G has a regular orbit on Ω^G.

1) $2 \nmid |G:A|$, and, if $2 \mid |K|$, G has a normal Sylow 2-subgroup;

or 2) $2 \nmid |G:A|$, A/K is a cyclic 2-group of order at least 4, and if $8 \mid |G|$, G is supersolvable;

or 3) A/K has more than two orbits on Ω;

or 4) A is relevant;

or 5) $A \lhd G$;

or 6) $K = 1$.

Proof. Let M be maximal with respect to the properties $M \subseteq A$ and $M \lhd G$. Let N/M be a chief factor of G. Then $N \cap A \lhd AN = G$. But $M \subseteq N \cap A \subseteq A$ and therefore $M = N \cap A$.

(*) Also $C_A(N/M) \lhd AN = G$ and $C_A(N/M) = M$.

Let $x_0 = 1$ and $x_1, \ldots, x_{n-1}$ be representatives of the classes N/M. We label x_1 a representative of a class in a regular orbit under $G/C_G(N/M)$ on N/M. We have $n = |N/M|$. Then x_i $i = 0, 1, \ldots, n-1$ is a system of representatives of the classes xA in G since if $x_i A = x_j A$, then $x_j^{-1} x_i \in A \cap N = M$ and $i = j$, and if $g \in G = N \cdot A$, $g = n \cdot a$, $n \in N, a \in A$, and $n \in x_i M$ for some i so that $g \in x_i MA = x_i A$.

Set $B = A/K$. We have the Frobenius homomorphism $G \to S_n \wr B$ defined by $g \to (s(g), b(g))$, where $s(g)$ is the permutation $x_i A \to g x_i A$ taken to be of $\{0, 1, \ldots, n-1\}$ and

$$b(g): \{0, 1, \ldots, n-1\} \to B,$$
$$b(g)(i) = x_{s(g)(i)}^{-1} g x_i K.$$

Since $g x_i \in x_{s(g)(i)} A$, $x_{s(g)(i)}^{-1} g x_i \in A$ and $x_{s(g)(i)}^{-1} g x_i K \in B$. This gives a homomorphism since by definition of $S_n \wr B$,

$$(\pi_1, f_1)(\pi_2, f_2) = (\pi_1 \circ \pi_2, (f_1 \circ \pi_2) f_2),$$

and we have for $g, g' \in G$,

$$(s(g), b(g))(s(g'), b(g')) = (s(gg'), b(gg')),$$

since of course $s(gg') = s(g) \circ s(g')$ and

$$\begin{aligned}[(b(g) \circ s(g')) \cdot b(g')](i) &= b(g)[s(g')(i)] \cdot b(g')(i) \\ &= x_{s(g)(s(g')(i))}^{-1} \cdot g \cdot x_{s(g')(i)} \cdot x_{s(g')(i)}^{-1} \cdot g' \cdot x_i \cdot K \\ &= x_{s(gg')(i)}^{-1} \cdot g \cdot g' \cdot x_i \cdot K.\end{aligned}$$

It is clear that the kernel of this homomorphism is contained in K and is therefore 1. From now on we consider G contained in $S_n \wr B$.

$S_n \wr B$ has a natural action on $1 \otimes \Omega \times x_1 \otimes \Omega \times \ldots \times x_{n-1} \otimes \Omega$ defined by

$$(\pi, f)((x_i \otimes \lambda_i)) = (x_{\pi(i)} \otimes f(i)\lambda_i)$$

and the action of G on Ω^G is the restriction to G of this action.

We denote the elements of Ω^G as n-tuples of elements of Ω. Note that if $m \in M$ then $s(m)=1$. If $g \in G$, then $g \in A$ iff $s(g)(0)=0$. If $a \in A$ then $a x_i A = a x_i a^{-1} A$ and therefore $s(a)$ corresponds to the action $xM \to axa^{-1}M$ of A on N/M. If $g \in G$ and $s(g)(0)=0$ and $s(g)(1)=1$ then $g \in A$, and since $x_1 M$ generates a regular orbit of $G/C_G(N/M)$, by (*), $g \in C_A(N/M)=M$ and $s(g)=1$.

Take ω to be a representative of a regular orbit of A/K on Ω and v a representative of another orbit of A/K on Ω. It is clear that at least $(\omega, \omega, \ldots, \omega)$, $(v, v, \ldots, v)$ and $(v, \omega, \ldots, \omega)$ belong to different G-orbits, so that G has at least three orbits on Ω^G. Hence we suppose that G has no regular orbit on Ω^G.

Then we have (it is possible that $n=2$ and some of the final letters do not actually appear):

a) $1 \neq C_G((\omega, \omega, \ldots, \omega))$ and $1 \neq \sigma_1 \in S_n \cap G$;

b) $1 \neq C_G((v, \omega, \ldots, \omega))$ and $1 \neq (\sigma_2, (\lambda_0, 1, \ldots, 1)) \in G$ with $\sigma_2(0)=0$;

c) $1 \neq C_G((\omega, v, \omega, \ldots, \omega))$ and $1 \neq (\sigma_3, (1, \lambda_1, 1, \ldots, 1)) \in G$ with $\sigma_3(1)=1$;

d) $1 \neq C_G((v, v, \omega, \ldots, \omega))$ and $1 \neq (\tau, (\lambda_2, \lambda_3, 1, \ldots, 1)) = t \in G$ where either $\tau=1$, or $\tau(0)=1$ and $\tau(1)=0$.

If $A \lhd G$ then $A=M$ and $\sigma_2=1$ and $1 \neq (\lambda_0, 1, \ldots, 1) \in G$. But in this case $\sigma_1(0) \neq 0$ and $\langle \sigma_1, (\lambda_0, 1, \ldots, 1) \rangle$ has a section isomorphic to $\mathbb{Z}_r \wr \mathbb{Z}_s$. If $K=1$ $C_G((\omega, v, \ldots, v))=1$. Hence one of 1)–4) is satisfied.

Suppose $(\alpha, \beta, 1, \ldots, 1) \in G$ with $\alpha \neq 1$. Then if $\sigma_3(0) \neq 0$ we have a section of

$$\langle (\alpha, \beta, 1, \ldots, 1), (\sigma_3, (1, \lambda_1, 1, \ldots, 1)) \rangle (1 \times B \times 1 \times \ldots \times 1)/(1 \times B \times 1 \times \ldots \times 1)$$

isomorphic to $\mathbb{Z}_r \wr \mathbb{Z}_s$. Therefore $\sigma_3=1$ and $1 \neq (1, \lambda_1, 1, \ldots, 1) \in G$. If $(\alpha', \beta', 1, \ldots, 1) \in G$ with $\beta' \neq 1$, in the same way we get $1 \neq (\lambda_0, 1, \ldots, 1) \in G$. Hence if $1 \neq (\lambda'_2, \lambda'_3, 1, \ldots, 1) \in G$, we get $1 \neq (\lambda_0, 1, \ldots, 1) \in G$ and $1 \neq (1, \lambda_1, 1, \ldots, 1) \in G$. But σ_1 does not fix both 0 and 1 and so a section of G is $\mathbb{Z}_r \wr \mathbb{Z}_s$. Hence we have

(**) $(\lambda'_2, \lambda'_3, 1, \ldots, 1) \in G$ implies $\lambda'_2 = \lambda'_3 = 1$.

Now we get $\tau \neq 1$ and $t^2=1$ since $\tau^2=1$. If 1) is satisfied $2 \nmid |G:A|$, so t is conjugate in G to an element of A. This element will act trivially on ω, so that $2 \mid |K|$. On the other hand t acts non-trivially on N/M so that either $\mathbb{Z}_2 \wr \mathbb{Z}_2$ is involved in G or G does not have a normal Sylow 2-subgroup. Hence 1) is not satisfied.

If 2) is satisfied again we get $2 \mid |K|$ and G is supersolvable. Now $n=|N/M|$ is an odd prime and $|A/M| \mid n-1$. Since B is a cyclic 2-group, if P is a Sylow n-subgroup $|P|=n$ and $P \subseteq G'$ since A acts non-trivially on N/M. Because G is supersolvable, P centralizes any G-chief factor of M, and hence P centralizes M. Since B is Abelian, this means that if $(\mu_0, \ldots, \mu_{n-1}) \in M$ we have $\mu_0 = \mu_1 = \ldots = \mu_{n-1}$. So M is cyclic and $M \subseteq Z(G)$. Since A/M is cyclic we get that A is Abelian.

Consider all elements of the form F:

$$(v, \mu_1 \omega, \mu_2 \omega, \ldots, \mu_{n-1} \omega)$$

with $\mu_i \in B$. There are $|B|^{n-1}$ of them. For each $q_1=2, q_2, \ldots, q_\gamma$ the distinct primes dividing $|A/M|$, take a coset Γ_i in A/M of order q_i $i=1, \ldots, \gamma$. Now

$$C_G((v, \mu_1 \omega, \ldots, \mu_{n-1} \omega)) \subseteq (A \setminus M) \cup \{1\},$$

so that for some $i=1, \ldots, \gamma$,

$$\Gamma_i \cap C_G((v, \mu_1 \omega, \ldots, \mu_{n-1} \omega)) \neq \emptyset,$$

i.e. contains an element of order q_i.

Now there are at most 2 elements of order 2 in Γ_1 and at most 1 element of order q_j in Γ_j for $j=2, \ldots, \gamma$, because A is Abelian and M is a cyclic 2-group. On the other hand each element of prime order in Γ_i fixes the first coordinate and acts with orbits of length q_i on the others. It can be conjugated by an element of $B \times \ldots \times B$ to an element of S_n, so it centralizes $|B|^{(n-1)/q_i}$ elements of the form F. Therefore we get $|B|^{n-1} \leqq 2|B|^{(n-1)/2} + \ldots + |B|^{(n-1)/q_\gamma}$. The right hand side has at most $n-2$ summands and we get $|B|^{n-1} \leqq (n-2)\, 2|B|^{(n-1)/2}$. Therefore we get $|B|^{(n-1)/2} \leqq 2(n-2)$ and $|B| \geqq 4$ $n \geqq 3$, which is a contradiction. Hence 2) is not satisfied.

If 3) is satisfied, let v_1 be a representative of a third orbit of B on Ω. We get $1 \neq C_G((v, v_1, \omega, \ldots, \omega))$ and $1 \neq (\lambda'_2, \lambda'_3, 1, \ldots, 1) \in G$, which contradicts (**). Hence 3) is not satisfied.

Hence 4) is satisfied, i.e. A is relevant. There is $V \lhd G$ Abelian and $V_1 \subseteq V$ such that $t \notin N_G(V_1)$. Let $v \in V_1$ be such that $v^t \notin V_1$. Then $\langle v, v^t, t\rangle / \langle v\rangle \cap \langle v^t\rangle$ is isomorphic to $\mathbb{Z}_2 \wr \mathbb{Z}_r$ for some r. This contradiction completes the proof of Proposition 1.6.

2. The Regular Orbit Theorem

Proposition 2.1. *Let A be a finite group and $C \lhd A$ be a cyclic group such that $C_A(C)=C$. Suppose that M is an irreducible faithful $\mathbb{F}_p A$-module such that $M|_C$ is homogeneous. Then we may identify M with $\mathbb{F}_{p^n}^+$ ($n=\dim_{\mathbb{F}_p}(M)$) and $A \subseteq G(1, p^n)$ with $A \cap F_1(p^n)=C$ and the action of A on M is given by the natural action of $G(1, p^n)$ on $\mathbb{F}_{p^n}^+$. $M|_C$ is irreducible.*

Proof. Let $k \supseteq \mathbb{F}_p$ be a splitting field for A and C. Then $k \otimes_{\mathbb{F}_p} M$ is a direct sum of faithful irreducible modules all distinct and conjugate under the Galois group $\mathrm{Gal}(k:\mathbb{F}_p)$ (since $\mathrm{char}\, k \neq 0$ the Schur index is trivial, see for example [12] Theorem 9.21). But each is induced from a faithful irreducible kC-module and therefore $k \otimes_{\mathbb{F}_p} M|_C$ is a sum of non-isomorphic kC-modules. Since $M|_C$ is homogeneous as a $\mathbb{F}_p C$-module, this means that M is irreducible as a $\mathbb{F}_p C$-module. Let e be the primitive idempotent of $\mathbb{F}_p C$ such that $eM \neq 0$. Then $p \nmid |C|$ and, by Schur's lemma, $e\mathbb{F}_p C$ is a field, say $e\mathbb{F}_p C = F \simeq \mathbb{F}_{p^n}$.

Let $v \in M$ $v \neq 0$, then define:

$$m: F \to M,$$
$$m(f) = fv.$$

This is an isomorphism as $\mathbb{F}_p C$-modules.

Every element $a \in A$ centralizes e and so normalizes F. $f \to f^{a^{-1}}$ is an automorphism of F, let us denote it by $\sigma(a)$. Define $\psi(a) \in F^*$ by $a \cdot v = \psi(a)^{a^{-1}} \cdot v$. Then if $a, b \in A$,

$$a \cdot b \cdot v = a \cdot \psi(b)^{b^{-1}} v = \psi(b)^{b^{-1}a^{-1}} av$$
$$= \psi(b)^{b^{-1}a^{-1}} \psi(a)^{a^{-1}} v$$

and

$$\psi(a \cdot b) = [\psi(b)^{b^{-1}a^{-1}} \psi(a)^{a^{-1}}]^{a \cdot b}$$
$$= \psi(b) \psi(a)^b = \psi(b) \cdot \sigma(b^{-1})(\psi(a)),$$

so that $a \to (\sigma(a), \psi(a))$ is an injective homomorphism of A into $G(1, p^n)$ and the action of A on $\mathbb{F}_{p^n}^+$ is the natural one:

$$a \cdot \lambda = \sigma(a)(\psi(a) \cdot \lambda).$$

Theorem 2.2. *Let G be a finite group and W an $\mathbb{F}_p G$-module (p a prime) which is the direct sum of irreducible $\mathbb{F}_p G$-modules. Assume for all sections G^* of G we have:*

1) *$\mathbb{Z}_r \wr \mathbb{Z}_s$ is not isomorphic to G^*;*

2) *$GN(1, p^{q^\alpha}, q)$ (q a prime, $\alpha \geqq 1$) is not isomorphic to G^*;*

3) *If all Abelian normal subgroups of G^* are cyclic then G^* has a cyclic self-centralizing normal subgroup;*

4) *If N/M is a chief-factor of G^* then $G^*/C_{G^*}(N/M)$ has a regular orbit on N/M.*

Then G has a regular orbit on W.

Note. Conditions 3) and 4) are always satisfied if G is supersolvable. In view of Proposition 1.2, 7), if G is nilpotent, Condition 1) implies Condition 2).

Proof of Theorem. Assume false and choose a counter-example which minimizes $|G| + \dim_{\mathbb{F}_p}(W)$. Then if $W = W_1 \oplus W_2$ a direct sum of $\mathbb{F}_p G$-modules, if we set $K_i = \ker W_i$ ($i = 1, 2$), by the choice of G and W there are $v_i \in W_i$ such that $C_G(v_i) = K_i$. Then $C_G(v_1 + v_2) = 1$, a contradiction. So W is irreducible.

Suppose $A \lhd G$ is Abelian. If A is not cyclic $W|_A$ is not homogeneous. Let C be a homogeneous component of $W|_A$. Consider G_1 a maximal subgroup of G such that $G_1 \supseteq N_G(C_A(C)) \supseteq N_G(C)$. Then there is a $\mathbb{F}_p G_1$-module V such that $W = V^G$.

Now set $K = \ker V$. We have that G_1/K has a regular orbit and the orbit $\{0\}$ on V. By Proposition 1.6, 4), since G_1 is relevant, G has a regular orbit on W, a contradiction. So A is cyclic.

So by Condition 3), G has a self-centralizing cyclic normal subgroup C. Consider $W|_C$. If $W|_C$ is not homogeneous, let V_1 be a homogeneous component of $W|_C$, then we may set $G_1 \supseteq N_G(V_1)$ a maximal subgroup, W_1 a $\mathbb{F}_p G_1$-

module such that $W=W_1^G$ and $K_1=\ker(W_1)$. We see that $K_1\cap C=1$ and therefore $[K_1,C]=1$ and so $K_1=1$. So G_1 has a regular orbit on W_1 and Proposition 1.6, 6) gives that G has a regular orbit on W. So $W|_C$ is homogeneous.

Hence we may apply Proposition 2.1 and $G\subseteq G(1,p^n)$ and the non-zero elements of W can be identified with $F_1(p^n)$. Now if G contains $GN(1,p^n,q)$ it also contains $GN(1,p^{q^\alpha},q)$ where q^α is the q-part of n. So Proposition 1.4 shows that G has a regular orbit. This concludes the proof of the theorem.

3. The Regular Module Theorem

Proposition 3.1. *Let A be a finite group and $C\lhd A$ a cyclic subgroup such that $C_A(C)=C$. Let M be an irreducible faithful $\mathbb{F}_p A$-module of dimension $2n$ with an A-invariant non-singular symplectic form $\langle\,,\rangle$. Then*

1) If $M|_C$ is homogeneous, we may identify A with a subgroup of $G(1,p^{2n})$ and M with $\mathbb{F}_{p^{2n}}^+$ and the form is given by $\langle x,y\rangle=\mathrm{Tr}(\mu(x\varphi(y)-\varphi(x)y))$ where Tr: *$\mathbb{F}_{p^n}\to\mathbb{F}_p$ is the trace, φ is the field automorphism of $\mathbb{F}_{p^{2n}}$ of order 2, $\mu\in\mathbb{F}_{p^{2n}}^*$, $\mu+\varphi(\mu)=0$ and for any $\sigma\in\mathrm{Gal}(-1,p^n)$ $\sigma(\mu)=\mu$.*

Further $C\subseteq F_{-1}(p^n)$, $G(-1,p^n)$ fixes $\langle\,,\rangle$ and for any subgroup $N\subseteq C$, if either $|N|>2$ or $|A:C|$ is odd, we have $C_A(N)\subseteq G(-1,p^n)$.

In addition if $p=2$ and M is endowed with an A-invariant quadratic form $Q\colon M\to\mathbb{F}_2$ such that $Q(v+w)=Q(v)+\langle v,w\rangle+Q(w)$ for all $v,w\in M$, then after the previous identification, $Q(x)=\mathrm{Tr}(x\varphi(x))$ for every $x\in M$ and Q is invariant under $G(-1,p^n)$.

2) If $M|_C$ is the sum of two distinct totally isotropic homogeneous components, set $G=G(1,p^n)$, $V=\mathbb{F}_{p^n}^+$ and let τ act on G by centralizing $\mathrm{Gal}(1,p^n)$ *and inverting $F_1(p^n)$ and let $\tau^2=-1\in F_1(p^n)$. We may identify A with a subgroup of $\langle\tau\rangle G$ and M with $V\oplus V^*$ (where V^* is the dual of V) with form*

$$\langle(v,f),(v',f')\rangle=f'(v)-f(v')$$

and the action of G is the natural one and

$$\tau(v,f)=(-t^{-1}(f),t(v))$$

where

$$t\colon V\to V^*$$

is the isomorphism

$$t(\lambda)\colon \mathbb{F}_{p^n}\to\mathbb{F}_p\qquad t(\lambda)(x)=\mathrm{Tr}(\lambda x)\quad x\in\mathbb{F}_{p^n},$$

and $\mathrm{Tr}\colon\mathbb{F}_{p^n}\to\mathbb{F}_p$ *the trace map. $\langle\tau\rangle G$ fixes $\langle\,,\rangle$.*

Also if $2^s|n$ with $s>0$, we take $\zeta\in\mathrm{Gal}(1,p^n)$ of order $2^s\cdot n_{2'}$ ($n_{2'}$ the odd part of n) and $\alpha\in\mathbb{F}_{p^n}$ such that $\dfrac{\zeta(\alpha)}{\alpha}=-1$. Then we identify $G(2^s,p^n)$ with $\langle\tau\zeta\alpha\rangle\ltimes F_1(p^n)\subseteq\langle\tau\rangle G$. The set $G(2^s,p^n)\subseteq\langle\tau\rangle G$ is independent of ζ and α.

Finally $C\subseteq F_1(p^n)$ and, if $N\subseteq C$ is a subgroup and $|N|>2$, then $C_A(N)\subseteq G(\varepsilon,p^n)$ for some $\varepsilon|n$ a power of 2.

Proof. 1) Since $M|_C$ is homogeneous we may identify $M \simeq \mathbb{F}^+_{p^{2n}}$ and $A \subseteq G(1, p^{2n})$ with $C \subseteq F_1(p^{2n})$ by Proposition 2.1.

Consider the automorphism of $C: x \to x^{-1}$, it extends to an automorphism φ of $\mathbb{F}_p C$. If e is the unique primitive idempotent of $\mathbb{F}_p C$ such that $eM \neq 0$, then since $0 \neq \langle eM, M \rangle = \langle M, \varphi(e) M \rangle$ we have $\varphi(e) = e$. So φ gives an automorphism of $\mathbb{F}_{p^{2n}} = e\mathbb{F}_p C$ of order 2 (since $|C| > 2$). $\mathbb{F}_{p^{2n}}$ has a unique automorphism of order 2. Now for all $v, w \in M$, $\langle v, w \rangle = \langle v \cdot 1, w \rangle = \langle 1, \varphi(v) w \rangle = -\langle \varphi(v) w, 1 \rangle = -\langle 1, \varphi(w) v \rangle$. First we show: (*) There is $\mu \in \mathbb{F}^*_{p^{2n}}$ such that $\mu(v\varphi(w) - \varphi(v) w) \in \mathbb{F}_{p^n}$, $\mu + \varphi(\mu) = 0$ and $\langle v, w \rangle = \mathrm{Tr}(\mu(v\varphi(w) - \varphi(v) w))$ for all $v, w \in M$.

If $p = 2$, there is $\mu \in \mathbb{F}^*_{p^{2n}}$ such that $\langle 1, x \rangle = \mathrm{tr}(\mu x)$ for all $x \in M$, where $\mathrm{tr}: \mathbb{F}_{p^{2n}} \to \mathbb{F}_p$ is the trace map (different from Tr). But $\langle 1, \varphi(v) w \rangle = \langle 1, v\varphi(w) \rangle$ for all $v, w \in M$, and $\mathrm{tr}(\mu x) = \mathrm{tr}(\mu\varphi(x)) = \mathrm{tr}(\varphi(\mu) \cdot x)$ for all $x \in M$, so that $\varphi(\mu) = \mu$ and $\mu + \varphi(\mu) = 0$ and $\langle v, w \rangle = \mathrm{tr}(\mu \cdot \varphi(v) w) = \mathrm{Tr}(\mu(\varphi(v) w + v\varphi(w)))$. Hence we have (*) in this case.

If $p \neq 2$ $\langle v, w \rangle = \langle \frac{1}{2}(v\varphi(w) - \varphi(v) w), 1 \rangle$ for any $v, w \in M$. Let $\mu_1 \in \mathbb{F}^*_{p^{2n}}$ such that $\mu_1 + \varphi(\mu_1) = 0$. Then there is $\mu_2 \in \mathbb{F}^*_{p^n}$ such that $\mathrm{Tr}(\mu_2 x) = \langle \frac{1}{2} \mu_1^{-1} x, 1 \rangle$ for all $x \in \mathbb{F}_{p^n}$. But now $\langle v, w \rangle = \mathrm{Tr}(\mu_2 \mu_1 (v\varphi(w) - \varphi(v) w))$, since $\mu_1(v\varphi(w) - \varphi(v) w) \in \mathbb{F}_{p^n}$. Setting $\mu = \mu_1 \mu_2$ we get (*).

If we choose $\lambda \in \mathbb{F}^*_{p^{2n}}$ and set $\psi(v) = \lambda^{-1} v$ for all $v \in M$, we get a new identification of A with a subgroup of $G(1, p^{2n})$ and M with $\mathbb{F}^+_{p^{2n}}$ with form given by

$$\begin{aligned} \langle v, w \rangle &= \mathrm{Tr}(\mu(\lambda v \varphi(\lambda w) - \varphi(\lambda v) \lambda w)) \\ &= \mathrm{Tr}(\mu\lambda\varphi(\lambda)(v\varphi(w) - \varphi(v) w)), \end{aligned}$$

i.e. we may choose μ to be any element of $\mathbb{F}^*_{p^{2n}}$ provided $\mu + \varphi(\mu) = 0$. In particular we may choose μ so that for any $\sigma \varepsilon \mathrm{Gal}(-1, p^n)$ we have $\sigma(\mu) = \mu$ and, if $p = 2$, $\mu = 1$.

The subset of elements of $F_1(p^{2n})$ which leave $\langle\ ,\ \rangle$ invariant is $F_{-1}(p^n)$ and, from $\sigma(\mu) = \mu$ above, we get that $G(-1, p^n)$ leaves $\langle\ ,\ \rangle$ invariant. Since $C \subseteq F_1(p^{2n})$ and fixes $\langle\ ,\ \rangle$, we have $C \subseteq F_{-1}(p^n)$. If $N \subseteq C$ and $|N| > 2$, we get that $|C_A(N): C|$ is odd, since φ inverts $F_{-1}(p^n)$; so $C_A(N) \subseteq G(-1, p^n)$.

Finally assume that $p = 2$ and Q is a quadratic form as in the last sentence of 1). Then clearly if $c \in C$ we have $Q(c) = Q(c^{-1})$ since Q is A-invariant. Let $a, b \in M$ and suppose $Q(a) = Q(\varphi(a))$ and $Q(b) = Q(\varphi(b))$. Then

$$Q(a+b) = Q(a) + Q(b) + \langle a, b \rangle = Q(\varphi(a)) + Q(\varphi(b)) + \langle \varphi(b), \varphi(a) \rangle = Q(\varphi(a+b))$$

(the middle equality follows from $\langle a, b \rangle = \langle 1, \varphi(a) b \rangle = \langle \varphi(b), \varphi(a) \rangle$, C being Abelian). Hence $Q(a) = Q(\varphi(a))$ for any $a \in M$, because C linearly spans M. Now if $\lambda \in M$, $Q(\lambda + \varphi(\lambda)) = \langle \lambda, \varphi(\lambda) \rangle = \mathrm{Tr}((\lambda + \varphi(\lambda))^2)$ ($\mu = 1$ in this case). So $Q(1) = \mathrm{Tr}(1\varphi(1)) = Q(c) = \mathrm{Tr}(c\varphi(c))$ for any $c \in C$, since Q is A-invariant and $c\varphi(c) = 1$. Notice that $\mathrm{Tr}((a+b)\varphi(a+b)) = \mathrm{Tr}(a\varphi(a)) + \mathrm{Tr}(b\varphi(b)) + \langle a, b \rangle$; but then since $Q(c) = \mathrm{Tr}(c\varphi(c))$ for $c \in C$ and C linearly spans M we get $Q(v) = \mathrm{Tr}(v\varphi(v))$ for all $v \in M$. From this, one gets 1).

2) Suppose now that $M|_C$ is the sum of two distinct totally isotropic homogeneous components. Let V be one of them. Let $B = N_A(V)$ be its inertia group.

By Clifford's Theorem V is an irreducible B-module. We may apply Proposition 2.1 to B and V and we may assume $B \subseteq G(1,p^n)$, $C \subseteq F_1(p^n)$ and $V = \mathbb{F}_{p^n}^+$. We also know that V is totally isotropic.

Let W be the other homogeneous component of $M|_C$ so that $M|_C = V \oplus W$. Let V^* be the dual of V. For each $f \in V^*$ there is a unique $w \in W$ such that $\langle v, w \rangle = f(v)$ for all $v \in V$ (there is one because $\langle\ ,\ \rangle$ is non-singular and if $\langle v, w \rangle = \langle v, w' \rangle$ for all $v \in V$, then $w - w' \in V \cap W = 0$). Therefore we may identify $W = V^*$ and B acts on V^* by the dual action to that on V (hence for $b \in B, f \in V^*$ we have $(bf)(v) = f(b^{-1}v)$ for $v \in V$). Furthermore we have $\langle (v,f), (v',f') \rangle = f'(v) - f(v')$ for $v, v' \in V$, $f, f' \in V^*$. Set S to be the subgroup of the linear group of $V \oplus V^*$ which fixes the form. Let $\bar{G} \subseteq S$ be the representation of G on $V \oplus V^*$ and set $\bar{\ }$ for the representation of any subgroup of G. $\bar{G}$ normalizes $\bar{C}$ and $\bar{G} \subseteq N_S(\bar{C}) \cap N_S(V)$. Since $C_S(\bar{C}) = \overline{F_1(p^n)}$ (because C is irreducible on V), $N_S(\bar{C}) \cap N_S(V)$ has a normal cyclic self-centralizing subgroup and acts faithfully on V so that Proposition 2.1 shows that $|N_S(\bar{C}) \cap N_S(V)| \leqq |G(1,p^n)| = |\bar{G}|$. So we have $\bar{G} = N_S(\bar{C}) \cap N_S(V)$.

Define τ as acting on $V \oplus V^*$ as in the statement of the proposition. For any $v, v' \in V, f, f' \in V^*$ we have

$$\begin{aligned}\langle \tau(v,f), \tau(v',f') \rangle &= \langle (-t^{-1}(f), t(v)), (-t^{-1}(f'), t(v')) \rangle \\ &= t(v')(-t^{-1}(f)) - t(v)(-t^{-1}(f')) \\ &= -\mathrm{Tr}(v' \cdot t^{-1}(f)) + \mathrm{Tr}(v \cdot t^{-1}(f')) \\ &= -f(v') + f'(v) = \langle (v,f), (v',f') \rangle.\end{aligned}$$

So $\tau \in S$. It is clear that τ^2 is multiplication by -1 on M and that if $\lambda \in F_1(p^n)$ $\lambda t(v) = t(\lambda^{-1} v)$ for all $v \in V$, and if $\sigma \in \mathrm{Gal}(1,p^n)$ $\sigma \cdot t(v) = t(\sigma(v))$. Therefore τ normalizes $\bar{C}$, inverts $\overline{F_1(p^n)}$ and centralizes $\overline{\mathrm{Gal}(1,p^n)}$, and interchanges V and V^*. Now since $\bar{G} = N_S(\bar{C}) \cap N_S(V)$ has index at most 2 in $N_S(\bar{C})$ we have $\langle \tau \rangle \bar{G} = N_S(\bar{C})$. It is clear that the representation $\bar{A} \subseteq S$ of A normalizes $\bar{C}$ so $\bar{A} \subseteq \langle \tau \rangle \bar{G}$.

If $2^s | n$ with $s > 0$, we take $\zeta \in \mathrm{Gal}(1,p^n)$ of order $2^s \cdot n_{2'}$. Clearly $N_\zeta(-1) = 1$ so that by Proposition 1.3 a), there is $\alpha \in F_1(p^n)$ with $\alpha^{-1}\zeta(\alpha) = -1$. Now $(\tau\zeta\alpha)^2 = \tau\zeta\alpha\tau\zeta\alpha = \tau^2\zeta^2\zeta^{-1}(\alpha^{-1})\alpha = \zeta^2$, so that $o(\tau\zeta\alpha) = o(\zeta) = 2^s \cdot n_{2'}$ and $\langle \tau\zeta\alpha \rangle$ acts by conjugation on $F_1(p^n)$ as $\mathrm{Gal}(2^s, p^n)$, so we may identify $\langle \tau\zeta\alpha \rangle \ltimes F_1(p^n)$ with $G(2^s, p^n)$. Clearly the set $G(2^s, p^n) \subseteq \langle \tau \rangle G$ does not depend on $\alpha \in F_1(p^n)$ and contains $\tau\zeta'$ for any $\zeta' \in \mathrm{Gal}(1,p^n)$ of order $2^s \cdot n_{2'}$. So $G(2^s, p^n) \subseteq \langle \tau \rangle G$ is independent of ζ.

Finally we know that $C \subseteq F_1(p^n)$. Let N be a subgroup of C and assume $|N| > 2$. Denote by F_0 the subfield of $\mathbb{F}_{p^n}$ generated by N. Clearly $\mathrm{Gal}(\mathbb{F}_{p^n}:F_0) F_1(p^n) = C_G(N)$. If $C_{\langle\tau\rangle G}(N) \subseteq G = G(1,p^n)$ we take $\varepsilon = 1$ and we are done. So assume there is $\tau\zeta \in C_{\langle\tau\rangle G}(N)$ a 2-element with $\zeta \in \mathrm{Gal}(1,p^n)$ and set $o(\tau\zeta) = 2^s$. Now $s > 0$ and $(\tau\zeta)^2 = \zeta^2(-1) \in C_G(N)$. Since $|N| > 2$, ζ is non-trivial on F_0 so that ζ^2 generates a Sylow 2-subgroup of $\mathrm{Gal}(\mathbb{F}_{p^n}:F_0)$ and $2^s | n$. Take ζ_0 a generator of a Hall 2'-subgroup of $\mathrm{Gal}(\mathbb{F}_{p^n}:F_0)$, then $\langle \tau\zeta\zeta_0 \rangle F_1(p^n) = C_{\langle\tau\rangle G}(N) \subseteq G(2^s, p^n)$, as we wanted to show. This completes the proof of the proposition.

Theorem 3.2. *Let p and q be distinct primes $n>0$ a natural number. Let*

$$A=O_{\{p,q\}'}(C_{\mathrm{Gal}(\varepsilon,p^n)}(O_q(F_\varepsilon(p^n))))\ltimes F_\varepsilon(p^n)$$

(for some ε) act on an extraspecial p-group P with the actions and the symplectic form on P/P' given by Proposition 3.1, 1) for $\varepsilon=-1$, and Proposition 3.1, 2) for $\varepsilon>0$. Let k be an algebraically closed field of characteristic q. Then any faithful irreducible kP-module M can be extended to a kAP-module in such a way that there is a one-to-one function

$$v\colon F_\varepsilon(p^n)\to M$$

with the following properties:

a) *If $x\cdot\lambda$ ($x\in A$, $\lambda\in F_\varepsilon(p^n)$) denotes the action of A on $F_\varepsilon(p^n)$ where $F_\varepsilon(p^n)$ acts by multiplication, then $x(v(\lambda))=v(x\cdot\lambda)$ for any $x\in A$, $\lambda\in F_\varepsilon(p^n)$.*

b) *Either $v(F_\varepsilon(p^n))$ is linearly independent, or $\varepsilon=-1$ and every proper subset of $v(F_\varepsilon(p^n))$ is linearly independent.*

Note. If $q\nmid|G(\varepsilon,p^n)|$ then $A=O_{p'}(G(\varepsilon,p^n))$.

Proof of Theorem. Fix the data of the proposition. Since $(|A|,|P|)=1$, we may extend M to a kAP-module in such a way that $\det_M(a)\in\{+1,-1\}$ for any $a\in A$ (see [12] (6.28) for this result of Gallagher).

Let R be the full ring of algebraic integers in $\mathbb{C}$ and take I a maximal ideal of R such that $q\in I$. If we identify R/I with a subfield of k we can define consistently Brauer characters (see for example [12] Chapt. 15 for a brief description of Brauer characters). Denote by B the Brauer character of AP on M.

It follows from results of Glauberman ([12] Chap. 13), or more explicitly Isaacs [13] or Berger [7], that we have:

A) For every q-regular $a\in A$, $B(a)\in\mathbb{Z}$, and $B(a)^2$ is the number of fixed points in P/P' under a. Choose γ a generator of $\mathrm{Gal}(\varepsilon,p^n)\cap A$ and we choose M such that:

B) If γ_2 is the 2-part of $\gamma, B(\gamma_2)\geqq 0$.

Set $F=F_\varepsilon(p^n)$. The next result follows from Hall-Higman [11] and Shult [14]. For a uniform proof in the case $p\neq 2$ see Berger [6].

C) $M|_F\simeq kF/S$ if $\varepsilon=-1$, and

$$\simeq kF\oplus S \qquad \text{if } \varepsilon>0,$$

where S is a 1-dimensional sub-kF-module of kF generated by $\sum_{x\in F}\mu_x x$ where $\mu_x\in\{1,-1\}$ for $x\in F$ and $\alpha(\sum_{x\in F}\mu_x x)=\sum_{x\in F}\mu_x x$ for every automorphism α of F.

Set $H=O_{q'}(F)$ and $Q=O_q(F)$. Take $M|_H=N_0\oplus N_1\oplus\ldots\oplus N_r$ the sum of the distinct homogeneous components, with N_0 the component of $S|_H$ (note: $\dim_k N_0\geqq 0$ and $\dim_k N_i>0$ $i=1,\ldots,r$). $\langle\gamma\rangle$ permutes the sets $N_0,\ldots,N_r$ fixing N_0. Set χ_i the irreducible H-character corresponding to N_i, for $i=0,\ldots,s$. Then if $\xi\in\langle\gamma\rangle$, $\xi N_i=N_i$ iff $\ker\chi_i\supseteq\left\{\frac{\xi(x)}{x}\colon x\in H\right\}=\left\{\frac{\xi(x)}{x}\colon x\in F\right\}$, since ξ centralizes Q.

By Proposition 1.3a) we know the order of $\left\{\frac{\xi(x)}{x}: x\in F\right\}$. Let $d(\xi)=\dim_k \sum N_i$ where i runs through $i\in\{0,1,\dots,r\}$ such that $\xi N_i=N_i$. Then by the above and C) we have:

D) $d(\xi)=p^{n/o(\xi)}+2$ if $\varepsilon>1$ and $\varepsilon|o(\xi)$, and
$\quad =p^{n/o(\xi)}$ otherwise.

If $\varepsilon>1$, let τ, t, ζ and α be as in Proposition 3.1, 2). Then $(\tau\zeta\alpha)^2=\zeta^2$, so the number of fixed points of $\xi\in\langle\gamma\rangle$ on P/P' is known except if $\varepsilon>1$ and $\varepsilon|o(\xi)$. In this case $\xi=\tau\zeta^s\alpha$ for some odd s. Let $\zeta_0=\zeta^s$, we have $\xi^2=\zeta_0^2$. If $\xi(x,f)=(x,f)\in V\oplus V^*$, we get $\zeta_0^2(x)=x$ and $f=t(\zeta_0(\alpha x))$. Conversely, if $\zeta_0^2(x)=x$ and $f=t(\zeta_0(\alpha x))$, we have $f=\zeta_0\alpha^{-1}t(x)$ (see proof of Proposition 3.1, 2)) and therefore $-t^{-1}(\zeta_0\alpha\cdot f)=-t^{-1}(\zeta_0^2\zeta_0^{-1}(\alpha)\alpha^{-1}\cdot t(x))=\zeta_0^2(x)=x$, because $\zeta_0^{-1}(\alpha)\alpha^{-1}=-1$ since s is odd. Hence ξ fixes exactly $p^{2n/o(\xi)}$ elements of $V\oplus V^*$. Furthermore in this case both p and q are odd. Take g_0 a generator of H. We get $2|p^{n/o(\xi)}+1$ and $N_\xi(g_0)=(\xi g_0)^{o(\xi)}\neq 1$ by Proposition 1.3, a). Hence ξg_0 fixes exactly one element of $V\oplus V^*$. We obtain by A) and Proposition 1.3, a):

E) For any ε and $\xi\in\langle\gamma\rangle$ we have $B(\xi)=\pm p^{n/o(\xi)}$; if $\varepsilon>1$ and $\varepsilon|o(\xi)$ then $B(\xi g_0)=\pm 1$ and $g_0\notin\left\{\frac{\xi(x)}{x}: x\in F\right\}$.

Suppose x a q-regular element acts on a k-vector space N. Denote by $B_N(x)$ the Brauer character of x on N. Suppose $\xi_1\in\langle\gamma\rangle$ and $\xi_1^2=\xi$. Assume, if possible, that $B_{N_i}(\xi)=-\dim_k N_i$ for all i such that $\xi N_i=N_i$. Then $B_{N_i}(\xi_1)$ is an integer multiple of $\sqrt{-1}$ for each i such that $\xi_1 N_i=N_i$. $B(\xi_1)=\sum B_{N_i}(\xi_1)$ where i runs through those with $\xi_1 N_i=N_i$, so that $B(\xi_1)$ is also a multiple of $\sqrt{-1}$ which contradicts E). Now $B(\xi)=\sum B_{N_i}(\xi)$ where i runs through those with $\xi N_i=N_i$ and $B(\xi)=\pm d(\xi)$ by D) and E). So we get, since $B_{N_i}(\xi)$ is the sum of $\dim_k N_i$ roots of unity,

F) If ξ is a square in $\langle\gamma\rangle$, $B_{N_i}(\xi)=\dim_k N_i$ for every ξ-invariant N_i.

Suppose $\varepsilon>1$ and $\varepsilon|o(\xi)$. Recall that γ_2 is the 2-part of γ. Then $d(\gamma_2)=p^{n/o(\gamma_2)}+2$ and $B(\gamma_2)=p^{n/o(\gamma_2)}$ by D), E) and B). Let L_{-1} and L_1 be the -1 and $+1$ respectively eigenspaces of γ_2 on the sum of the γ_2-invariant N_i's. By F) we get $\dim_k L_{-1}=1$ and $\dim_k L_1=p^{n/o(\gamma_2)}+1$. Since $L_{-1}\subseteq N_i$ for some i, let $\psi_0=\chi_i$. By E) we have $\pm 1=\sum B_{N_i}(\gamma_2 g_0)$ where N_i runs through the γ_2-invariant N_i's. So, using C), $\pm 1=\chi_0(g_0)-2\psi_0(g_0)+\sum|Q|\chi_i(g_0)$. Now $\sum\chi_i(g_0)=0$ since by E) $g_0\notin\left\{\frac{\gamma_2(x)}{x}: x\in F\right\}$. By C) $\chi_0(g_0)=\pm 1$, so from $\psi_0(g_0)=\frac{\chi_0(g_0)\pm 1}{2}$ we get $(\langle g_0\rangle=H)$ $\psi_0=\chi_0$. Define $L_0=L_1+\sum_{i=1}^{r} N_i$. We have, using F),

G) If $\varepsilon>1$ and $\varepsilon|o(\gamma)$, $L_0|_H$ is isomorphic to $|Q|$ copies of the regular H-representations and for any $\xi\in\langle\gamma\rangle$ and N_i such that $\xi N_i=N_i$, ξ acts trivially on $L_0\cap N_i$.

Now since Q centralizes $\langle\gamma\rangle H$, it stabilizes L_{-1} and $N_i\cap L_0$ (of dimension $|Q|$) for $i=0,\dots,r$. On the other hand by C), $M|_Q$ is the direct sum of free kQ-

modules (each indecomposable) and a linear kQ-module. Hence $N_i \cap L_0$ is a free kQ-module for each $i=0,\dots,s$. This shows

H) If $\varepsilon>1$ and $\varepsilon|o(\gamma)$, $L_0|_F$ is a free kF-module.

If $\varepsilon=1$ or $\varepsilon\nmid o(\gamma)$ we define L_0 to be the sum of a kF-direct factor of N_0 free for Q (which exists by C)) and $\sum_{i=1}^{s} N_i$. Clearly L_0 is isomorphic to kF as kF-modules. If $\varepsilon=-1$ take $L_0=M$, $L_0\simeq kF/S$ then. In either case by D), E) and B) we have that if $\gamma_2 N_i=N_i$ then $B_{N_i}(\gamma_2)=\dim_k N_i$. Hence we obtain from F) and G)

I) For any ε and $\xi\in\langle\gamma\rangle$, if $\xi N_i=N_i$ then ξ acts trivially on $N_i\cap L_0$.

Take L' to be kF/S if $\varepsilon=-1$ and kF if $\varepsilon\neq-1$. $\langle\gamma\rangle F$ acts naturally on L' with F by multiplication and γ naturally. We have $L_0|_F\simeq L'|_F$. From I) one sees that in fact $L'\simeq L_0$ as $\langle\gamma\rangle F$-modules. So now the proposition follows from the definition of L' and the fact that $S=k\sum_{x\in F}\mu_x x$ with $\mu_x\neq 0$ for all $x\in F$.

Proposition 3.3. *Let $G=AN$ be a finite solvable group with $N\neq 1$ a cyclic r-group for some prime r, $N\lhd AN$ $A\neq 1$ and $(|A|,|N|)=1$. Let k be an algebraically closed field such that* $\operatorname{char}(k)\nmid|C_A(N)N|$ *and M be a kG-module. Suppose that M contains the image of the $kC_A(N)N$-module $kC_A(N)N$ under the $kC_A(N)N$-map φ with kernel of k-dimension at most 1. Then $M|_A$ has a submodule kA-isomorphic to kA.*

Furthermore if [ ] denotes the image in the corresponding Grothendieck ring, and $[M|_A]-[kA]$ has A-composition length at most 1, then r is odd, $|N|=r$, $|A/C_A(N)|=r-1$, $C_A(N)$ is either elementary Abelian or 1, $[M|_{C_A(N)N}]-[kC_A(N)N]$ has no non-trivial representative module, and A acts transitively on the non-trivial irreducible characters of $C_A(N)$.

Proof. Clearly we may assume that $[kC_A(N)N]-[\varphi(kC_A(N)N)]$ is G-invariant. If $\operatorname{char}(k)\nmid|A|$, A has at most two irreducible characters only if $|A|=2$. So if $C_A(N)=A$ we get the result, since $(|N|,|A|)=1$ in this case. So we assume $A\neq C_A(N)$. So r is odd.

Let ψ be a $C_A(N)$ irreducible character and ω a non-trivial irreducible character of N. Since $\psi\otimes\omega$ is not G-invariant, $M|_{C_A(N)N}$ contains the direct sum of $\psi(1)$ copies of $\psi\otimes\omega$. Now $\psi\otimes\omega|^{AN}$ is G-irreducible and projective and therefore $M|_A$ contains the direct sum of $\psi(1)$ modules isomorphic to Ξ^A, where Ξ is a $kC_A(N)$-module with character ψ. So kA is isomorphic to a kA-module of $M|_A$.

Suppose $[M|_A]-[kA]$ has A-composition length at most 1. The above can be repeated for each non-trivial irreducible character of N, so $A/C_A(N)$ has only one regular orbit on N and $|N|=r$, r is odd and $|A/C_A(N)|=r-1$. Let M_0 be the G-submodule sum of all trivial kN-submodules of M. Then $M_0=0$ or is irreducible under A. So all classes of $C_A(N)$-irreducibles of M_0 are conjugate under A. Hence either $C_A(N)=1$, or $C_A(N)$ is elementary Abelian and $[M|_{C_A(N)N}]-[kC_A(N)N]$ has no representative module. If $C_A(N)=1$, A is cyclic and every irreducible A-module is one dimensional so $1\geqq\dim_k M-|A|$ $=\dim_k M-(|N|-1)$. We get $|N|\geqq\dim_k M$. Hence we have the result.

Theorem 3.4. *Let AG be a finite group with $G \lhd AG$, $(|A|,|G|)=1$ and $P \subseteq G$ $P \lhd AG$ an extraspecial p-group (for some prime p) such that $Z(P) \subseteq Z(AG)$ and A is faithful on $P/Z(P)$. Let M be a kAG-module, where k is an algebraically closed field with* $\mathrm{char}(k) \nmid |A| p$, *such that Z(P) is not trivial on M.*

Suppose $R \subseteq G$ is an A-invariant r-group for some prime r and $R/C_R(P/Z(P))$ is elementary Abelian. Assume further:

1) *R acts non-trivially on every non-identity R-invariant section of $P/Z(P)$.*
2) *If S is a section of $AR/C_R(P/Z(P))$ where all Abelian normal subgroups are cyclic, then S has a self-centralizing cyclic normal subgroup.*
3) *No section of A is isomorphic to $\mathbb{Z}_s \wr \mathbb{Z}_{s'}$ or to $GN(\varepsilon, p^n, q)$ whenever $\varepsilon = -1$ or a power of 2, n is a positive integer, q is a prime dividing $(|A|, |\mathrm{Gal}(\varepsilon, p^n)|)$, and if $\varepsilon \neq 1, r \mid |F_\varepsilon(p^n)|$.*
4) *For any section S of A and any chief factor X of $S, S/C_S(X)$ has a regular orbit on X.*
5) *If $rp = 15$ any chief 2-factor of A is cyclic and, if further $8 \mid |A|$, then either A is supersolvable or has a normal Sylow 2-subgroup.*

Then $M|_A$ has a proper A-regular direct summand and if the A-composition length of $[M|_A]-[kA]$ is 1 and $A \neq 1$, A is a cyclic group of order 4 or 8 and $rp = 15$.

Note. If A is supersolvable conditions 2), 4) and 5) are always satisfied. If A is nilpotent and $\mathbb{Z}_s \wr \mathbb{Z}_s$ free for all primes s, in view of Proposition 1.2, 7), conditions 2) to 5) are always satisfied.

Proof of Theorem. Assume false. Let (AG, M) be a counterexample with $|AG| + \dim_k M$ minimal. We split the proof into a series of steps.

Step 1. $A \neq 1$, $G = RP$, $r \neq p$, $M|_P$ is kP-irreducible and R is elementary Abelian and faithful on P/P'.

Proof. Clearly ARP and M satisfy the hypothesis of the theorem but not its conclusion. Hence $G = RP$. Since R is not trivial on an irreducible R-invariant section of P/P', $r \neq p$. Take M_i/M_{i+1} a kAG-chief factor of M such that $Z(P)$ acts non-trivially on M_i/M_{i+1}. Such a factor exists since $p \neq \mathrm{char}(k)$. If $\dim_k(M_i/M_{i+1}) < \dim_k M$, then $(M_i/M_{i+1})|_A$ properly contains the regular representation, so that M is kAG-irreducible.

Since $Z(P) \subseteq Z(AG)$, now $M|_P$ is homogeneous. Take N an irreducible kP-module isomorphic to a submodule of $M|_P$. Since $(|AR|, p) = 1$, N can be extended to an ARP-module N in a unique way subject to the condition that if $x \in AR$ then $\det_N(x) = 1$. We know that there exists a $kARP/P$-module L such that $M \simeq N \otimes L$ (see [12] (6.17) for this result of Gallagher). Now if $|ARP/\ker N| + \dim_k N < |AG| + \dim_k M, N|_A$ properly contains a regular A-representation and so the theorem is satisfied for $M|_A$, a contradiction. So $\ker N = 1$ and $\dim_k N = \dim_k M$. So, $M|_P$ is irreducible and since $\ker N = C_{AR}(P)$, R is faithful on P/P' and therefore by hypothesis elementary Abelian.

Step 2. AR is irreducible on P/P'.

Proof. Set $\bar{P} = P/P'$ and $f\colon \bar{P} \times \bar{P} \to P'$ $f(x, y) = [x, y]$, f is a non-singular symplectic form. Suppose first that there is $\bar{P}_1 \subset \bar{P}$ a non-trivial AR-submodule of $\bar{P}$ where

f is non-singular. We set $\bar{P}_2 = \bar{P}_1^{\perp}$ the orthogonal complement. Then $\bar{P} = \bar{P}_1 \oplus \bar{P}_2$. Set $P_i \supseteq P'$ such that $P_i/P' = \bar{P}_i$ and $\hat{P} = P_1 \times P_2$ the direct product of P_1 and P_2 considered as abstract groups. Then P_i is an extraspecial group and we take N_i to be an irreducible kP_i-module such that $N_i|_{P'} \oplus M|_{P'}$ is homogeneous. Now AR acts in a natural way on $\hat{P} = P_1 \times P_2$ so we can form the semidirect product $AR\hat{P}$. We view N_i as a $\hat{P}/P_j$ module for $i \neq j$. So N_i can be extended to an $AR\hat{P}$ module N_i such that $\det_{N_i}(x) = 1$ for $x \in AR$. Define

$$\varphi: AR \ltimes (P_1 \times P_2) \to ARP$$
$$\varphi(x, (\alpha, \beta))) = x\,\alpha\,\beta.$$

This is a homomorphism onto ARP and $\ker\varphi \subseteq \ker(N_1 \otimes N_2)$. Therefore we view $N_1 \otimes N_2$ as a ARP-module and $M \simeq N_1 \otimes N_2 \otimes L$ for some linear ARP/P-module L. We get $M|_A \simeq N_1|_A \otimes N_2|_A \otimes L|_A$. Now since $\det_{N_i}(x) = 1$ for any $x \in AR$, $C_{AR}(\bar{P}_i)$ acts trivially on $N_i (i = 1, 2)$. By induction, therefore, $N_i|_{A/C_A(\bar{P}_i)}$ properly contains $k[A/C_A(\bar{P}_i)]$. Since $C_A(\bar{P}_1) \cap C_A(\bar{P}_2) = 1$, $M|_A$ contains kA and the A-composition length of $[M|_A] - [kA]$ is at least 2, a contradiction. So f is singular on every non-trivial proper AR-submodule of $\bar{P}$.

Assume $\bar{B}$ is an irreducible proper AR-submodule of $\bar{P}$. $\bar{B}$ is totally isotropic by the above. Let $\bar{C}$ be an irreducible AR-submodule of $\bar{P}$ such that $\bar{C} \nsubseteq \bar{B}^{\perp}$. Then $\bar{C} \cap \bar{B}^{\perp} = 0$ so that f is non-singular on $\bar{B} + \bar{C}$, hence $\bar{B} + \bar{C} = \bar{P}$ and $\bar{C}$ is "dual" to $\bar{B}$. Therefore AR is faithful on both $\bar{B}$ and $\bar{C}$. Let $B \supseteq P'$ be such that $B/P' = \bar{B}$. Now B is Abelian, $B \lhd ARP$ and $B = [R, B] \times C_B(R) = [R, B] \times P'$. Take θ to be the character of B which extends the irreducible character of $M|_{P'}$ by being trivial on $[R, B]$. It is well known that θ is the character of a B-submodule of $M|_B$. Now let $I(\theta)$ be the inertia group of θ in ARP. Clearly $I(\theta) = ARB$. By Clifford's theorem, there is an ARB-module N such that $M \simeq N^{ARP}$. By Mackey $M|_A = \sum N^x|_{(ARB)^x \cap A}|^A$, where x runs through a set of representatives of the double cosets $ARBxA$ of AG. Clearly the set of representatives can be chosen inside P and then they are representatives for the orbits of A on P/B. We know that A is faithful on P/B, so by Theorem 2.2 we may take $x_0 \in P$ to generate an A-regular orbit in P/B. Suppose $a \in (ARB)^{x_0} \cap A$, then $[a, x_0^{-1}] \in P \cap ARB = B$ so that $a = 1$. Hence $(ARB)^{x_0} \cap A = 1$ and $M|_A$ properly contains the regular A-representation. Hence we get $|A| + 1 = |P/B|$. But $|P/B| \equiv 1\ (r)$ and this is impossible. This completes step 2.

Step 3. Take $B \lhd AR$ Abelian. Then $(P/P')|_B$ is either homogeneous or the direct sum of two totally isotropic homogeneous components.

Proof. Let B_1 be the r'-Hall subgroup B. Then $[B_1, R] \subseteq B_1 \cap R = 1$, so that $B_1 R$ is Abelian and contains B. Hence we assume that $R \subseteq B$ and step 3 is not satisfied for $(P/P')|_B$.

Take $I \subseteq P/P'$ B-irreducible. If I is isomorphic to its dual, we set V the sum of all B-submodules of P/P' isomorphic to I. Clearly f is non-singular on V. If I is not isomorphic to its dual take V_0 the sum of all B-submodules of P/P' isomorphic to I. Clearly V_0 is totally isotropic. If we now take V_1 to be the sum of all B-submodules of P/P' isomorphic to the dual of I, V_1 is totally isotropic and we set $V = V_0 \oplus V_1$. f is non-singular on V. In all cases we have $0 \neq V \neq P/P'$ and V is a non-singular B-module.

Take A_1 a maximal subgroup of A such that $A_1 \supseteq N_A(V)$. Step 2 and Clifford theory give that there is an $A_1 R$-irreducible submodule X of P/P' such that $P/P' \simeq X^{AR}$ as AR-modules. Furthermore f is non-singular on X. Let P_1 be the preimage of X in P. Take $x_1 = 1, x_2, \ldots, x_n \in A$ cosets representatives for xA_1 in A, and set $C = A_1 R/\ker X$. We have a Frobenius map

$$F: AR \to S_n \ltimes (C \times \ldots \times C) = S_n \wr C$$

where $n = |A : A_1|$, see Proposition 1.6. Since AR is faithful on P/P' by step 1, F is injective and we may assume that $AR \subseteq S_n \wr C$. Now set

$$\hat{G} = (S_n \wr C) \ltimes (P_1 \times \ldots \times P_1)$$

where there are n copies of P_1 and $C \times \ldots \times C$ acts by components on $P_1 \times \ldots \times P_1$ and S_n permutes the components. Let N be an $A_1 RP_1$-module such that $N|_{P_1}$ is irreducible, $\det_N(x) = 1$ for $x \in A_1 R$ and $N|_{Z(P_1)} \oplus M|_{Z(P)}$ is homogeneous. Take

$$\tilde{M} = N \otimes \ldots \otimes N \quad (n \text{ times})$$

and let $\hat{G}$ act on $\tilde{M}$ with $(C \times \ldots \times C) \ltimes (P_1 \times \ldots \times P_1)$ by components and S_n permuting the components of the elementary tensors (if $s \in S_n$ and $v_1 \otimes \ldots \otimes v_n \in N \otimes \ldots \otimes N$, we have

$$s(v_1 \otimes \ldots \otimes v_n) = v_{s^{-1}(1)} \otimes \ldots \otimes v_{s^{-1}(n)}).$$

Now $\tilde{M}$ is a $\hat{G}$-module (tensor induced from N) such that $\tilde{M}|_{P_1 \times \ldots \times P_1}$ is irreducible. Define

$$\varphi: AR(P_1 \times \ldots \times P_1) \to ARP$$
$$\varphi(a\,r(y_1, \ldots, y_n)) = a\,r\,y_1^{x_1^{-1}} \ldots y_n^{x_n^{-1}}.$$

φ is a homomorphism onto ARP and $\ker \varphi \subseteq \ker(\tilde{M}|_{AR(P_1 \times \ldots \times P_1)})$. So we may consider $\tilde{M}$ as an ARP-module and $\tilde{M}|_P \simeq M|_P$. By Gallagher's theorem there is L a linear ARP/P-module such that

$$M \simeq \tilde{M}|_{ARP} \otimes L.$$

Set $K = C_{A_1}(P_1)$. By induction, $N|_{A_1}$ contains a proper A_1-direct summand which contains an A_1/K-regular orbit of linearly independent vectors Ω_1. Choose A_1-irreducible submodules N_i of N so that $N = k\Omega_1 \oplus N_1 \oplus \ldots \oplus N_s$. We know $s \geqq 1$. Set $\Omega = \Omega_1 \cup \{N_1, \ldots, N_s\}$. A_1 acts naturally on Ω with one regular A_1/K-orbit and s other (trivial) orbits. Consider the action of A on Ω^A as in Proposition 1.6. We now show that A has a regular orbit on Ω^A and at least two other orbits. If $s > 1$, this follows from Proposition 1.6, 3). If $s = 1$ and $A_1 = K$, then $|P_1| = 8$ and $2 \nmid |A|$ so the result follows from Prop. 1.6, 1). If $s = 1$ and $A_1 \neq K$, then $rp = 15$ and A_1/K is cyclic of order 4 or 8, so by 5) any chief 2-factor of A is cyclic. So if $2 \mid |A : A_1|$ we have $|A : A_1| = 2$ and $A_1 \lhd A$ and by Proposition 1.6, 5), we get the result. Finally if $2 \nmid |A : A_1|$ we may apply either Proposition 1.6, 1) or 1.6, 2) to get the claim in all cases.

Define a function $f: \Omega \to N$, with $f(v)=v$ for $v \in \Omega_1$ and $0 \neq f(N_i) \in N_i$ ($i = 1, \dots, s$). Suppose $(v_1, \dots, v_n)$ generates a regular orbit in Ω^A, then $f(v_1) \otimes \dots \otimes f(v_n)$ generates an orbit of linearly independent vectors in $\tilde{M}$ under A, and different orbits generate linearly independent A-submodules of $\tilde{M}$. Hence $\tilde{M}|_A$ properly contains the regular A-module and the A-composition length of $[\tilde{M}|_A]-[kA]$ is at least 2. The same is true for $M|_A \simeq \tilde{M}|_A \otimes L|_A$. This contradiction completes step 3.

Step 4. We have the proposition.

Proof. From step 1 and step 3 we get that every Abelian normal subgroup of AR is cyclic, so by 2) AR contains a self-centralizing cyclic normal subgroup. So again step 3 gives that we may apply Proposition 3.1, and for some ε and n we have

$$C_{AR}(R) \subseteq O_{p'}(G(\varepsilon, p^n)).$$

By Proposition 3.1 we may assume that $O_{p'}(G(\varepsilon, p^n))$ acts on P. By the Fong-Swan theorem we may assume that $\mathrm{char}(k) \nmid p|G(\varepsilon, p^n)|$. By Theorem 3.2 there is a linear $O_{p'}(G(\varepsilon, p^n)) \ltimes P/P$-module L and a function $v: F_\varepsilon(p^n) \to M \otimes L$ satisfying properties a) and b) of that theorem. Now $r \,|\, |F_\varepsilon(p^n)|$, so by 3) $GN(\varepsilon, p^n, q)$ is not conjugate to a subgroup of $C_A(R)$ for any q a prime dividing $|\mathrm{Gal}(\varepsilon, p^n)|$. On the other hand if $\alpha \in C_{AR}(R) \setminus F_\varepsilon(p^n)$ is of prime order, $\alpha \in C_A(R)$ and $\Pi x^{\alpha^i} = x^{o(\alpha)} \neq 1$ if $x \in R$ and $x \neq 1$. So no conjugate of a $GN(\varepsilon, p^n, q)$ in $G(\varepsilon, p^n)$ lies in $C_{AR}(R)$. Now by Proposition 1.4 we get that $M|_{C_{AR}(R)} \otimes L|_{C_{AR}(R)}$ contains the $C_{AR}(R)$-homomorphic image of the regular $C_{AR}(R)$-module under a $C_{AR}(R)$-map with kernel of dimension at most 1. The same is true of $M|_{C_{AR}(R)}$.

By Proposition 3.3 $M|_A$ has an A-submodule isomorphic to kA. Hence the A-composition length of $[M|_A]-[kA]$ is at most 1. We use Proposition 3.3 in what follows. We get that $[M|_{C_A(R)R}]-[k\,C_A(R)R]$ has no non-trivial representative module, so by condition b) of Theorem 3.2 we get that $\varepsilon = -1$ and $|C_A(R)\,R| = p^n+1$. We also know that r is odd and $|A/C_A(R)| = r-1$ is even. It is easy to see that if $x \in G(1, p^{2n}) \setminus F_{-1}(p^n)$ and $x^2=1$ then x does not fix the form of Proposition 3.1, 1). So we have $2 \,|\, |A \cap F_{-1}(p^n)|$. We know that A is transitive on the non-trivial irreducible character of $C_A(R)$, so $C_A(R) \subseteq F_{-1}(p^n)$ and $|C_A(R)|=2$. We have $|A/C_A(R)| \,|\, 2n$, so if we set $p_1 = p^{2n/(r-1)}$, since p is odd, p_1 is an odd integer and $p_1^{(r-1)/2}+1 = 2r$ (the last equation follows from $|C_A(R)\,R| = p^n+1$). Since r is odd we deduce that either $p=3$, $r=5$ and $|A/C_A(R)|=4$ and A is cyclic of order 8, or $p=5$, $r=3$ and $|A/C_A(R)|=2$ and A is cyclic of order 4. This concludes the proof of step 4 of the theorem.

4. An Application

Proposition 4.1. *Suppose AG is a finite group with $G \lhd AG$ and $(|A|, |G|)=1$. Let V be a kAG-module with* $\mathrm{char}(k) \nmid |A|$ *such that $C_V(A)=0$ and $C_V(A_0) \neq 0$ for every $A_0 \lhd A$, $A_0 \neq A$. Let $G_1 \subseteq G$ be A-invariant and W_1 be a kAG_1-module such that $V = W_1^{AG}$. Set $G_0 = \bigcap_{x \in AG} G_1^x$. Assume*

1) *G/G_0 is solvable;*

2) *For any elementary Abelian A-invariant section X of G/G_0 irreducible under A, $A/C_A(X)$ has a regular orbit on X.*

Then $G=G_0 C_G(A)$.

Proof. Assume false. Choose a counterexample such that $|AG|$ is minimal. Clearly $|G: G_1|>1$. Take $M \supseteq G_1$ a maximal A-invariant proper subgroup of G. Take $K \subseteq M$, $K \lhd AG$ maximal with these properties. Take $H \subseteq G$ such that H/K is an AG-chief factor. Then, since $K \supseteq G_0$, H/K is Abelian and $H \cap M \lhd MHA=AG$, so that $H \cap M=K$ and $HM=G$.

Choose $x_0=1,\dots,x_l$ representatives for xK in H such that $C_A(x_i)=C_A(x_iK):=\{a\in A: [a,x_i]\in K\}$, which is possible because $(|A|,|G|)=1$. Set numbering so that x_1K is a generator for a regular $A/C_A(H/K)$ orbit on H/K, and $x_0,\dots,x_s$ are coset representatives for $AMxA$ in AG.

We show that for $i=0,\dots,l$ we have $(AM)^{x_i}\cap A=C_A(x_i)$. Clearly $C_A(x_i)\subseteq (AM)^{x_i}\cap A$. Suppose $a\in (AM)^{x_i}\cap A$, then $a\in A$ and $a=(a_0\cdot m_0)^{x_i}$ with $a_0\in A$ and $m_0\in M$. Since $(|A|,|M|)=1$ there is $m\in M$ $a_1\in A$ such that $a_0 m_0=a_1^m$. Now $a=a_1^{mx_i}$ and

$$[a,x_i]=a^{-1}x_i^{-1}a\,x_i=(a_1^{-1})^{mx_i}x_i^{-1}a\,x_i=x_i^{-1}m^{-1}a_1^{-1}m\,a\,x_i\in (AM)^{x_i}.$$

So $[a,x_i]\in (AM)^{x_i}\cap H=(AM\cap H)^{x_i}=K$. By the choice of x_i in x_iK we have $a\in C_A(x_i)$ and so $C_A(x_i)=(AM)^{x_i}\cap A$, as desired.

Now set $W=W_1^{AM}$. We have $V=W^{AG}$ and by Mackey

$$V|_A=W^{AG}|_A=\sum_{i=0}^{s} x_i^{-1}\otimes W|_{C_A(x_i)}|^A.$$

Set $N=C_A(H/K)=C_A(x_1)$. Clearly $N\subseteq C_A(x_i)$ for $i=0,\dots,s$, and $N\lhd A$. If $V|_N$ contains the trivial representation, for some $i_0, 0\leqq i_0\leqq s$, we have that $x_{i_0}^{-1}\otimes W|_{C_A(x_i)}|^A|_N$ contains the trivial representation and by Mackey, for some $a\in A$, $a^{-1}\otimes x_{i_0}^{-1}\otimes W|_N$ contains the trivial representation. Since $N\lhd A$, $x_{i_0}^{-1}\otimes W|_N\supseteq 1_N$ and since $x_i\in C_G(N)$ $(i=0,\dots,s)$, $x_1^{-1}\otimes W|_N\supseteq 1_N$ and therefore $V|_A\supseteq 1_A$, a contradiction. So $V|_N \not\supseteq 1_N$, and hence by hypothesis since $N\lhd A$, $N=A$. So $C_A(H/K)=A$ and $G=MH=MC_G(A)$.

Now $V|_A=(l+1)(W|_A)$, so we may apply the proposition to A, M, W, G_1 and W_1: we get a fortiori $M=G_1\cdot C_M(A)$ and hence $G=G_1\cdot C_G(A)$. Now $[G,A]\subseteq G_1$ and $[G,A]\lhd AG$, so $G_0=\bigcap G_1^x\supseteq [G,A]$ and so $G=G_0\cdot C_G(A)$. This completes the proof of the proposition.

Definition 4.2. Let G be a solvable group and A act on G. A subgroup G is called a generating A-support subgroup of G if:

1) $P\lhd AG$, $P\subseteq G$ and P is a p-group for some prime p.
2) There are AG-invariant subgroups P_1 and H such that
   A) $P_1\subseteq Z(P)$, P/P_1 is elementary Abelian and AG-completely reducible,
   B) $H\subseteq C_G(P_1)$,
   C) $H/H\cap C_G(P/P_1)$ is elementary Abelian for some prime r,
   D) H acts non-trivially on each H-chief factor of P/P_1.

We call the A-support of G (denoted $\operatorname{supp}_A(G)$) the subgroup generated by all subgroups $S \subseteq G$ such that $S \lhd AG$ and either S is Abelian or a generating A-support subgroup of G.

Note. This is related but not equivalent to Berger's notion of L-support.

Proposition 4.3. *Let G be a solvable group and A act on G. Then we have the following:*

1) $\bigcap C_G(X) \subseteq F(G)$ *(where X runs through the AG-chief factors of* $\operatorname{supp}_A(G)$). *In particular* $C_G(\operatorname{supp}_A(G)) \subseteq F(G)$.

2) *If* $N \subseteq G$ *and* $N \lhd AG$, $\operatorname{supp}_A(G)N/N \subseteq \operatorname{supp}_A(G/N)$.

3) *If* $B \subseteq A$ *and* $(|A|, |G|) = 1$, $\operatorname{supp}_B(G) \supseteq \operatorname{supp}_A(G)$.

4) $C_A(\operatorname{supp}_A(G)) \subseteq C_A(G/F(G))$.

Proof. 1) Suppose false. Then if $Y = \bigcap C_G(X)$, then $Y \lhd G$ so that $F(Y) \subseteq F(G)$ and there is $x \in Y$ an r-element acting non-trivially on $O_p(G)$ for r and p distinct primes. Take $P \lhd AG$, $P \subseteq O_p(G)$ minimal such that x acts non-trivially on P. Now P is not Abelian; let $P_1 \subset P$ be such that $P_1 \lhd AG$. We have $x \in C_G(P_1)$ and hence $[C_G(P_1), P] \subseteq P$ and x is not trivial on $[C_G(P_1), P] \lhd AG$. So $[C_G(P_1), P] = P \subseteq C_G(P_1)$ and $P_1 \subseteq Z(P)$. So we have that $P/Z(P)$ is elementary Abelian and AG-irreducible.

We also have

$$C_G(Z(P)) \supset C_G(Z(P)) \cap C_G(P/Z(P)),$$

so we may take $H \lhd AG$,

$$C_G(Z(P)) \supseteq H \supset C_G(Z(P)) \cap C_G(P/Z(P)),$$

minimal with these properties. Now since G is solvable $H/(C_G(Z(P)) \cap C_G(P/Z(P)))$ is elementary Abelian. Consider $(P/Z(P))|_H$. By Clifford's Theorem it is the direct sum of H-irreducibles none trivial. So P is a generating A-support subgroup, and x acts non-trivially on the AG-chief factor $P/Z(P)$, a contradiction.

2) is clear from the definition.

3) If P is a generating A-support subgroup of G, $(P/P_1)|_G$ is totally reducible and hence, since $(|G|, |A|) = 1$, $(P/P_1)|_{BG}$ is also. The rest is clear.

4) Set $S = \operatorname{supp}_A(G)$ and $A_0 = C_A(S)$. We have $[S, A_0, G] = 1$ and $[G, S, A_0] = 1$, so that $[A_0, G] \subseteq C_G(S) \subseteq F(G)$ by 1). This shows 4).

Definition 4.4. Let A be a finite group and π a set of primes. We say that A is π-regular if:

1) $\pi(A) \cap \pi = \varnothing$;

2) For any $p \in \pi$ and any elementary Abelian p-group H on which A acts and any section S of AH, if all Abelian normal subgroups of S are cyclic, S has a self-centralizing cyclic normal subgroup;

3) For any section S of A and any chief-factor X of S, $S/C_S(X)$ has a regular orbit on X;

4) If $\{3, 5\} \subseteq \pi$, any chief 2-factor of A is cyclic and if further $8 \mid |A|$, either A is supersolvable or it has a normal Sylow 2-subgroup;

5) No section of A is isomorphic to $\mathbb{Z}_r \wr \mathbb{Z}_s$ (any $r,s>1$) or to $GN(\varepsilon, p^n, q)$ where $p\in\pi, n\geqq 1$ is an integer, $q \mid |\mathrm{Gal}(\varepsilon, p^n)|$ is a prime and if $\varepsilon \neq 1$ $\pi(F_\varepsilon(p^n)) \cap \pi \neq \emptyset$.

Note. Conditions 2), 3) and 4) are always satisfied if A is supersolvable. If A is nilpotent, satisfies 1) and is $\mathbb{Z}_p \wr \mathbb{Z}_p$-free for all p, in view of Prop. 1.2, 7), A is π-regular.

Proposition 4.5. *Let AG be a finite group where $G \lhd AG$ is solvable and V a kAG-module. Assume the following:*

1) *k is a splitting field for all subgroups of AG;*

2) *$V|_G$ is homogeneous and faithful* (i.e. *it is the direct sum of isomorphic faithful irreducible kG-modules*);

3) *$C_V(A)=0$;*

4) *A is $\{\mathrm{char}(k)\}\cup\pi(G)$-regular.*

Then

$$C_V(C_A(\mathrm{supp}_A(G)))=0.$$

Proof. Assume false. Take a counterexample with $|AG|+\dim_k V$ minimal. Set $S = \mathrm{supp}_A(G)$.

Step 1. V is irreducible and for each $A_1 \subset A$ we have $C_V(A_1) \neq 0$.

Proof. If V is not irreducible, take V^* any kAG-chief factor of V. Since $\mathrm{char}(k) \nmid |A|$, we may replace V by V^* and still have the hypothesis of the theorem and, since $\dim_k V^* < \dim_k V$, we have $C_V(C_A(S))=0$, a contradiction. So V is irreducible.

If $A_1 \subset A$ is such that $C_V(A_1)=0$, we have by induction and Proposition 4.3, 3)

$$0 = C_V(C_{A_1}(\mathrm{supp}_{A_1}(G))) \supseteq C_V(C_A(\mathrm{supp}_A(G))),$$

which is a contradiction.

Step 2. For any Abelian subgroup $C \subseteq G$ such that $C \lhd AG$ we have $[A, C]=1$.

Proof. Since $V|_G$ is homogeneous and $(|A|,|G|)=1$, by a well known lemma of Glauberman ([12], (13.9)), there is a homogeneous component $V_1 \subseteq V|_C$ such that $N_{AG}(V_1) \supseteq A$. Set $G_1 = N_G(V)$, we have $N_{AG}(V_1) = AG_1$. Now since k is a splitting field, A centralizes the action of C on V_1. By Theorem 2.2, Proposition 4.1 and Clifford's Theorem we have

$$V = \sum_i x_i^{-1} \otimes V_1 \qquad \text{with } x_i \in C_G(A).$$

Hence

$$[A, C] = [A, C^{x_i}] = [A, C]^{x_i} \subseteq (\ker V_1)^{x_i},$$

and so $[A, C]=1$, as desired.

Now A does not centralize all generating A-support subgroups of G. Let P be a generating A-support subgroup of G not centralized by A, minimal with these properties. Choose p, P_1, r and H according to the definition. It is clear that A centralizes every proper AG-invariant subgroup of P.

Step 3. P is extraspecial, $Z(P) \subseteq Z(AG)$ and $P/Z(P)$ is AG-irreducible.

Proof. We know that $A \subseteq C_{AG}(Z(P))$. Set $G_1 = C_G(P_1)$ and assume $G_1 \neq G$. We have $C_{AG}(P_1) = AG_1$. Now $AG_1 \lhd AG$ and so $[A, G] \subseteq G_1$ and $G = G_1 \cdot C_G(A)$.

Choose W a homogeneous component of $V|_{G_1}$ such that $N_{AG}(W) \supseteq A$. Set $K = \ker_{G_1}(W)$. By induction we have

$$C_W(C_A(\mathrm{supp}_A(G_1/K))) = 0.$$

But $V = \sum_i y_i^{-1} \otimes W$ with $y_i \in C_G(A)$ and we get

$$C_V(C_A(\mathrm{supp}_A(G_1/K))) = 0. \tag{1}$$

If we set $A_1 = \{x \in A: [x, P] \subseteq K\}$ we have by Proposition 4.3, 2) and the fact that $P \subseteq \mathrm{supp}_A(G_1)$,

$$A_1 \supseteq C_A(\mathrm{supp}_A(G_1/K)). \tag{2}$$

For any $y \in C_G(A)$ we have

$$[A_1, P] = [A_1, P^y] = [A_1, P]^y \subseteq K^y,$$

so, since $\bigcap_{y \in C_G(A)} K^y = 1$, we have $[A_1, P] = 1$ and

$$A_1 \subseteq C_A(P). \tag{3}$$

Now (1) (2) and (3) give $C_V(C_A(P)) = 0$, so by step 1, $C_A(P) = A$. This contradiction shows that $P_1 \subseteq Z(AG)$.

Since V is irreducible this gives that P_1 is cyclic. Since $P/Z(P)$ has exponent p and $P' \subseteq P_1$ we get $|P'| = p$. Suppose $P' \neq P_1$. Let C be a cyclic subgroup of P containing P_1 maximal with those properties. Then there is $K \subseteq P$ such that $K \cap C = P'$ and $\langle K, C \rangle = P$. Now if $x \in K$ we have $x^p \in K \cap C = P'$ so $x^p = z^p$ for some $z \in P_1$. Hence $xz^{-1} \in \Omega_1(P)$ and $P = \Omega_1(P)\, C$. If $p \neq 2$, A acts non-trivially on $\Omega_1(P)$ and, if $p = 2$, $P/\Omega_1(P)$ is cyclic and A acts trivially on it, so in any case $P = \Omega_1(P)$. Let $h \in C_H(P/P_1)$ and $x \in P$ with $x^p = 1$. Then $x^h = x \cdot z$, $z \in P_1$, with $(x \cdot z)^p = z^p = 1$. So $z \in P'$ and $C_H(P/P_1) = C_H(P/P')$. $P = [H, P]\, P_1$, so A acts non-trivially on $[H, P]$ and $[H, P] = P$. Now $p \nmid |H/C_H(P/P')|$ so $P_1 \subseteq C_P(H) \subseteq P'$, a contradiction. Hence $P_1 = P'$.

Now since P/P' is AG-completely reducible P/P' is AG-irreducible and $Z(P) = P'$ and P is extraspecial as desired. This completes step 3.

Step 4. We have the proposition.

Proof. By step 3 $V|_P$ is homogeneous. Let U be the irreducible P-module of $V|_P$. Since AG is irreducible on P/P', $\det_U(x) = 1$ for any $x \in P$. Extend U to an AP-module such that $\det_U(x) = 1$ for $x \in AP$. Further extend U to a projective AG-representation with $\det_U(x) = 1$ for $x \in AG$. This shows that there is a finite central extension $A\hat{G}$ of AG by a p-group such that U is an ordinary $A\hat{G}$-module. Hence $\pi(\hat{G}) = \pi(G)$. Let R be an A-invariant Sylow r-subgroup of the corresponding extension of H. Since A is $\pi(G) \cup \{\mathrm{char}(k)\}$-regular, setting $A_0 = A \cap \ker(U) = C_A(P/P')$, we have, by Theorem 3.4,

(*) $U|_A$ contains an A/A_0 regular direct summand.

On the other hand $V=U\otimes M$ where M is an irreducible $A\hat{G}$-module. So

$$V|_A = U|_A \otimes M|_A,$$

and since $A_0 \subset A$, we have by step 1,

$$1_{A_0} \subseteq V|_{A_0} = U|_{A_0} \otimes M|_{A_0}.$$

$A_0 \lhd A$ and $A_0 \subseteq \ker U$, so there is some A-irreducible module I in $M|_A$ such that $I|_{A_0}$ is the sum of irreducible conjugate modules containing the trivial module, i.e. $A_0 \subseteq \ker I$. Take I^* the dual module of I. $I^* \subseteq U|_A$ by (*), and therefore $I^* \otimes I \subseteq V|_A$, i.e. $C_V(A) \neq 0$. This contradiction completes the proof of the proposition.

Theorem 4.6. *Let AG be a finite group with $G \lhd AG$ solvable and $(|A|, |G|)=1$. Let M be an irreducible kAG-module with k a field and set $\pi=\pi(G)\cup\{\mathrm{char}(k)\}$. Assume*

1) *M is faithful for G;*
2) *$B_1 \supset B_2$ are normal subgroups of A with $|B_1/B_2|$ a prime;*
3) *$C_M(B_1)=0$ and $C_M(B_2)\neq 0$;*
4) *Either* A) *or* B):

A) *There is $1 \neq C \subseteq G$, $C \lhd AG$ Abelian, $C_C(A)=1$ and for any $A_1 \subset A$, A_1 is π-regular;*

B) *B_1 is π-regular.*

Then if we set $S=\mathrm{supp}_A(G)$, we have

$$C_S(B_1)=C_S(B_2) \quad \text{and} \quad C_{G/F(G)}(B_1)=C_{G/F(G)}(B_2).$$

Proof. There is no loss in assuming that k is a splitting field for all subgroups of AG, so we do. Take a counterexample with $|AG|$ minimum.

Step 1. $M|_G$ is homogeneous.

Proof. Suppose not. Let $A_1 \subset A$ and N be an irreducible A_1G-module such that $M=N^{AG}$. Let $K=\ker_G(N)$. Clearly $\bigcap_{a\in A} K^a=1$. Write $\bar{G}=G/K$ and use the barr convention. For $i=1,2$, by Mackey, we choose $a_{ij}\in A$ with

$$M|_{B_i} = N^{AG}|_{B_i} = \sum_j N^{a_{ij}}|_{A_1^{a_{ij}}\cap B_i}|^{B_i} = \sum_j N^{a_{ij}}|_{(A_1\cap B_i)^{a_{ij}}}|^{B_i},$$

since $B_i \lhd A$. So by Frobenius reciprocity $C_M(B_i)=0$ iff $C_N(A_1\cap B_i)=0$. This shows that $A_1\cap B_1 \nsubseteq B_2$ and that we may apply induction, which together with Proposition 4.3, 2) gives:

$$(*) \qquad C_{SK/K}(B_1\cap A_1)=C_{SK/K}(B_2\cap A_1) \quad \text{and}$$
$$C_{\bar{G}/F(\bar{G})}(B_1\cap A_1)=C_{\bar{G}/F(\bar{G})}(B_2\cap A_1).$$

Now

$$[C_S(B_2), B_1]=[C_S(B_2), (A_1\cap B_1)B_2] \subseteq [C_S(B_2\cap A_1), A_1\cap B_1] \subseteq K.$$

Since $[C_S(B_2), B_1]$ is A-stable, we have $C_S(B_2) = C_S(B_1)$. Next take $F/K = F(\bar{G})$.

$$\begin{aligned}[C_{G/F(G)}(B_2), B_1] &= [C_{G/F(G)}(B_2), (B_1 \cap A_1) B_2] \\ &\subseteq [C_{G/F(G)}(B_2 \cap A_1), B_1 \cap A_1] \subseteq F/F(G).\end{aligned}$$

$\bigcap_{a \in A} F^a$ is a nilpotent normal subgroup of G (since $\bigcap_{a \in A} K^a = 1$) and therefore $\bigcap_{a \in A} F^a = F(G)$. Since $[C_{G/F(G)}(B_2), B_1]$ is A-stable, we have $C_{G/F(G)}(B_2) = C_{G/F(G)}(B_1)$. A contradiction. This shows step 1.

Step 2. We have the proposition.

Proof. If 4) A) holds, the fact that $(|A|, |G|) = 1$ and $M|_G$ is homogeneous gives that A centralizes a non-trivial quotient of $C = [A, C]$, a contradiction. So 4) B) holds. By step 1 we may apply Proposition 4.5 to get

$$C_M(C_{B_1}(\mathrm{supp}_{B_1}(G))) = 0,$$

or a fortiori, by Proposition 4.3, 3),

$$C_M(C_{B_1}(S)) = 0.$$

Since $C_M(B_2) \neq 0$, this means $B_1 = B_2\, C_{B_1}(S)$.

$$[C_S(B_2), B_1] = [C_S(B_2), B_2\, C_{B_1}(S)] = 1,$$

so $C_S(B_1) = C_S(B_2)$.

By Proposition 4.3, 4), if we set $H = G/F(G)$ we have $C_{B_1}(S) \subseteq C_{B_1}(H)$. We get

$$[C_H(B_2), B_1] = [C_H(B_2), B_2\, C_{B_1}(H)] = 1,$$

so $C_H(B_2) = C_H(B_1)$. This concludes the proof of the theorem.

Theorem 4.7. *Let AG be a group where $G \lhd AG$ is solvable, $(|A|, |G|) = 1$ and $C_G(A) = 1$. Assume that A is supersolvable and every proper subgroup $B \subset A$ is $\pi(G)$-regular (Definition 4.4). Let $A_0 = 1 \subset A_1 \subset \ldots \subset A_n = A$ be a chief series of A. Set h the Fitting height of G. Then $h \leqq n$.*

Proof. By Proposition 4.3, 1) and the fact that if $G \neq 1$ then $\mathrm{supp}_A(G) \neq 1$, if $h > 0$ we may choose an AG chief factor of F_1 of $\mathrm{supp}_A(G)$ such that $G/C_G(F_1)$ has Fitting height $h-1$. If $h-1 > 0$, we may choose F_2 a chief factor of $\mathrm{supp}_A(G/C_G(F_1))$ such that $G/C_G(F_2)$ has Fitting height $h-2$. Continuing this process we get AG chief factors of G

(*) $F_1, F_2, \ldots, F_h$ such that, for $i, j = 1, \ldots, h$ and $j > i$, either F_j is a factor of $\mathrm{supp}_A(G/C_G(F_i))$ or F_j is a factor of $(G/C_G(G_i))/F(G/C_G(F_i))$.

Since $C_{F_i}(A) = 1$, we may define a map

$$f\colon \{1, \ldots, h\} \to \{1, \ldots, n\}$$
$$f(i) = \text{smallest } k \text{ such that } C_{F_i}(A_k) = 1.$$

If $k=f(i)=f(j)$ and, say, $j>i$, we have A_k/A_{k-1} of prime order (since A is supersolvable) and therefore Theorem 4.6, the fact that $(|A|,|G|)=1$ and (*) give

$$C_{F_j}(A_k)=C_{F_j}(A_{k-1}),$$

a contradiction. So f is one-to-one and $h \leqq n$ as required.

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