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# On Confluence of One-Rule Trace-Rewriting Systems*

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**Abstract.** The paper presents combinatorial criteria for confluence of one-rule trace-rewriting systems. The criteria are based on self-overlaps of traces, which are closely related to the notion of conjugacy of traces, and can be tested in linear time. As a special case, we reobtain the corresponding results for strings.

## 1. Introduction

In this paper we consider the property of confluence for one-rule systems over traces, that is, over strings modulo a partial commutation relation (which specifies that certain pairs of letters commute). One value of one-rule systems in the study of rewriting techniques is that they can illuminate the differences between types of systems and the complexity of questions about rewriting. For example, any one-relator group has a decidable word problem, while the same question for monoids (that is, for one-rule Thue systems) remains open. Another set of contrasts is given by questions of termination. It was shown by Dauchet that termination is undecidable for one-rule term-rewriting systems [8], as it is for finite string-rewriting (Thue) systems. For one-rule Thue systems, it is not known whether termination is decidable. Of course, a length-reducing system must always be terminating; for the other possibilities, see, for example, the work of Métivier [18] and of Kurth [15].

Confluence is a desirable computational property for rewriting systems; one advantage it gives is that equivalence-proofs for pairs of objects may be restricted to

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the finding of common descendants. In the setting of trace rewriting, confluence corresponds to the preperfect property for Thue systems, and to a weak form of confluence modulo an equivalence relation for term-rewriting systems. For trace-rewriting systems in general, confluence is undecidable even if the systems are finite and length-reducing [20]. In contrast, for finite and length-reducing Thue systems, confluence can be decided in polynomial time [4]. The results here show the decidability of confluence for one-rule trace-rewriting systems in which at least one (nonempty) component of the left-hand side of the rule is not a factor of the right-hand side. For a rule with a connected and nonempty left-hand side, the results apply also to systems satisfying a certain condition (which will always hold for strings) on the letters in the rule and the partial commutation relation. In particular, confluence is decidable for all length-reducing one-rule trace-rewriting systems, and for all terminating one-rule systems in which the left-hand side of the rule is "connected"; however, the proofs make no assumption of termination.

The form of our results for connected left-hand sides is a generalization of the characterization of confluence for one-rule Thue systems [15], [23], [26], and is based on comparing the (self-) overlaps of the two sides of the rule. An overlap of a trace $t$ some trace $s$ that is both an initial and a final factor of $t$; such an overlap $s$ is also a conjugator of the parts of $t$ that remain when it is removed. This relationship between overlaps and conjugacy allows us to make use of the information developed by Duboc and others on conjugation of traces. When the left-hand side of the rule is not connected and some component of the left-hand side of the rule is not a factor of the right-hand side, the confluence property puts severe restrictions on the structure of the two sides of the rule (Theorem 5.1). For example, if the single rule is $(u, v)$, the left-hand side $u$ is not connected and no component of $u$ is a factor of $v$, then the system can only be confluent if $u$ lies in a (totally) commutative submonoid.

The criteria developed here for confluence are simple enough that, for a fixed partially commutative alphabet, they can be tested in linear time (in the lengths of the traces making up the rule). The tests involve pattern matching for traces and, in particular, calculation of overlaps of traces.

The combinatorial analysis underlying the results here is more complicated than that for strings, due to the influence of the partial commutation relation on letters. In the situations to which our analysis applies, confluence is equivalent to the "strong confluence" property, which can be established by local tests without assuming termination. This equivalence between confluence and strong confluence also holds for one-rule Thue systems; it does not hold for multirule systems, nor for every one-rule trace-rewriting system.

The complications introduced by commuting letters can be seen clearly when the left-hand side of the single rule is empty. A Thue system with empty left-hand sides is always trivially confluent. In the trace case we are able to give a characterization of strong confluence based on interactions between the right-hand side of the rule and commuting pairs of letters (Theorem 6.1). Strong confluence is not equivalent to confluence in this situation: Otto [22] has given an example of a one-rule trace-rewriting system with an empty left-hand side that is confluent but not strongly confluent.

Section 2 contains a review of definitions and basic results about traces and about rewriting systems, and Section 3 gives the development of two necessary conditions for confluence of a one-rule trace-rewriting system. Section 4 contains characterizations of confluence when the left-hand side of the single rule is connected and every letter independent of the left-hand side commutes with the right-hand side. Section 5 is concerned with systems for which the left-hand side is nonempty and not connected, and Section 6 is concerned with systems for which the left-hand side is empty. Section 7 summarizes the results and presents the remaining cases.

## 2. Preliminary Definitions and Results

### 2.1. *Traces*

The reader should consult, if necessary, the first chapter of [17] for notation and basic results on strings (elements of a free monoid). Here we use $e$ to denote the empty string. Also, $\Psi$ denotes the *Parikh mapping*, which takes a string $x \in A^*$ to the sequence of nonnegative integers $\Psi(x) = (|x|_a | a \in A)$. Parikh images are operated on and compared component by component.

The setting for the following work is a (finitely generated) free partially commutative monoid $M(A,I)$, which is determined by an alphabet $A$ and a symmetric and irreflexive *independence* relation $I \subseteq A \times A$. The import of placing a pair $(a,b)$ into $I$ is that we choose not to distinguish between $ab$ and $ba$. The formal definition of $M(A,I)$ is thus as a quotient $A^*/\equiv$ of the free monoid $A^*$ by the congruence relation $\equiv$ generated by $\{ab \equiv ba | (a,b) \in I\}$. Following Mazurkiewicz, a congruence class in this quotient is called a *trace*, and the quotient is called a *trace monoid*. Note that elements of a congruence class have the same Parikh image. We use representative strings to name congruence classes. Let $D = (A \times A) - I$ denote the (symmetric and reflexive) *dependence* relation of the trace monoid. Both the independence and the dependence relations are undirected graphs on vertex set $A$. For simplicity in the examples, they are given by listing symmetric pairs just once and not listing reflexive pairs.

Traces can also be profitably viewed as labeled acyclic graphs that take the dependence relation into account. If a trace is given as (the congruence class of) a string $a_1 \cdots a_n$, then the associated graph is the labeled (multi-) set $\{a_1, \ldots, a_n\}$ with an arc from $a_i$ to $a_j$ whenever $i < j$ and $(a_i, a_j) \in D$. The notions of connectedness and decomposition into connected components are most easily seen using this point of view.

Traces $x, y$ are *independent*, denoted by $x \perp y$, if $\text{alph}(x) \times \text{alph}(y)$ is a subset of $I$. Similarly, call subsets $B$ and $C$ of the alphabet $A$ independent if $B \times C \subseteq I$. If a trace $t$ satisfies $t \equiv xy$ with $x \perp y$, then $t \equiv yx$ also; in Sections 5 and 6, we use the notation $t = x \oplus y$ as a shorthand for "$t \equiv xy$ and $x \perp y$," with the obvious extension to more than two factors.

A trace is *connected* if its alphabet forms a connected subgraph of the dependence graph $(A, D)$; turning this around, a trace $t$ is *not connected* if $\text{alph}(t) = B \cup C$ with $B$, $C$ nonempty and independent. A trace $x$ is a (connected) *component* of a trace $t$ if $x$ is nonempty and connected, and, for some $y$, $t = x \oplus y$. The empty trace is thus connected but has no components; every other trace is the nonempty product of its components.

Some additional notation is useful. For a trace $x$, let $I(x) = \{a \in A | a \perp x\} = \{a \in A | \{a\} \times \text{alph}(x) \subseteq I\}$ denote the set of letters that are independent of (every letter in) $x$, and let $D(x) = A - I(x)$. Let $COM(x) = \{a \in A | ax \equiv xa\}$ denote the set of letters that commute with $x$ in $M(A, I)$; note that $COM(x) = I(x) \cup \{a \in A | D(a) \cap \text{alph}(x) = \{a\}\}$. The sets $I(x)$, $D(x)$, and $COM(x)$ depend only on the alphabet of $x$, not on its form.

The three following propositions review well-known properties of traces. The first gives a division property for traces that generalizes that for strings; it is used here to analyze equations involving traces.

**Proposition 2.1** (Division of Traces) [7, Proposition 1.3]. *For all* $x, y, w, z \in A^*$, *if* $xy \equiv wz$, *then* $p, q, r, s \in A^*$ *exist such that* $x \equiv pq, y \equiv rs$, $w \equiv pr$, $z \equiv qs$, *and* $q$ *is independent of* $r$.

A second very useful property of traces is that they can be represented (faithfully) by tuples of strings. For a subset $B$ of $A$, the *projection* on $B$ is the homomorphism $h : A^* \to B^*$ determined by defining $h(b) = b$ for $b \in B$ and $h(c) = e$ for $c \notin B$. Proofs of the following proposition may be found, for example, in [5] and [7]. One of its consequences is that strings can be tested for congruence (i.e., whether they represent the same trace) in linear time for a fixed alphabet and independence relation [3]. Projection is used here to show that traces are the same and to translate questions about traces into questions about strings.

**Proposition 2.2** (Projection of Traces). *For* $a, b \in A$, *let* $\pi_{ab}$ *denote the projection on* $\{a, b\}$. *For all* $x, y \in A^*$, $x \equiv y$ *if and only if, for all pairs* $(a, b) \in D$, $\pi_{ab}(x) = \pi_{ab}(y)$.

It follows from either of the two previous propositions that trace monoids are cancellative.

A trace $x$ is *imprimitive* if some trace $y$ and integer $k \geq 2$ exist such that $x \equiv y^k$; otherwise, $x$ is *primitive*. Each nonempty trace $x$ has a (primitive) root: there is some $y$ and some $k \geq 1$ such that $x \equiv y^k$, and $y$ is unique modulo the congruence $\equiv$. The following fact can be easily derived from the work of Duboc [12] and Cori and Métivier [6] on commuting traces.

**Proposition 2.3.** *If* $x$ *is connected and* $xy \equiv yx$, *then* $y \equiv st$ *where* $s \perp x$ *and* $t$ *is a power of the root of* $x$.

For a trace $x$, let $OVL(x) = \{y \in M(A, I) | p, q \text{ exist such that } x \equiv py \equiv yq\}$ denote the set of traces that are (self-) overlaps of $x$. Under this definition, the empty trace is an overlap of every trace, and each trace is an overlap of itself. A fact about overlaps that is used frequently (and that follows easily from Proposition 2.2) is that if $x \equiv py \equiv yq$ where $x$ is connected and $p$ is nonempty, then

$\mathrm{alph}(p) = \mathrm{alph}(q) = \mathrm{alph}(x)$; that is, the part left over when a proper overlap is removed from a connected trace contains occurrences of all the letters of the trace.

A pair of traces $(x, y)$ is *conjugate* if there is some $z$ (called a conjugator) such that $xz \equiv zy \equiv w$. Such a conjugator $z$ is both a prefix and a suffix of the product $w$, that is, it is an overlap of $w$. Conjugacy is an equivalence relation on $M(A, I)$. The following lemma expresses other connections between conjugation and overlaps, as well as a structural relationship between a connected trace and its overlaps.

**Lemma 2.4.** *Suppose $u$ is a nonempty, connected trace. There are traces $p$, $q$, $r$, $t$ with the following properties:*

1. *$t$ is the maximum proper overlap of $u$: any proper overlap of $u$ is an overlap of $t$.*
2. *$r$ is the minimum conjugator of $(p, q)$: $pr \equiv rq$ and, for any trace $x$, if $px \equiv xq$, then $x \equiv p^k ry$ for some $k \geq 0$ and some trace $y$ that is independent of $u$.*
3. *$u \equiv p^{m+1}r$ and $t \equiv p^m r$ for some $m \geq 0$, $p$, $q$ are primitive, and $\mathrm{alph}(p) = \mathrm{alph}(q) = \mathrm{alph}(u)$.*
4. *$OVL(u) = OVL(pr) \cup \{p^k r \mid 1 \leq k \leq m+1\}$.*

*Proof.* See also Theorem 4 of [24]. The existence of maximum overlaps of connected, nonempty traces has been demonstrated elsewhere [25, Theorem 3.3]; also, if $y, z \in OVL(x)$, then $w \in OVL(x)$ exists such that $\Psi(w) = \max\{\Psi(y), \Psi(z)\}$. (More precisely, the overlaps of a trace form a lattice under "Parikh order" [25].) Note further that if $y$, $z$ belong to $OVL(x)$ and $\Psi(y) \leq \Psi(z)$, then $y \in OVL(z)$.

Suppose now that $u$ is a connected, nonempty trace and write $u \equiv pt \equiv tq$ where $t$ is the maximum proper overlap of $u$. Since $u$ is connected and $p$, $q$ are nonempty, $\mathrm{alph}(p) = \mathrm{alph}(q) = \mathrm{alph}(u)$; hence, in addition, $p$ and $q$ are connected.

Since $p$ and $q$ are conjugate traces, it follows from the work of Duboc [12] that they have a minimum conjugator $r$; indeed, any shortest conjugator has property 2. In addition, $p$ is not a prefix of $r$ and $\Psi(r) \not\geq \Psi(p)$. Let $p_0$, $q_0$ be the roots of $p$ and $q$, respectively; then $p_0 r \equiv r q_0$ and, for some $n \geq 1$, $p \equiv p_0^n$ and $q \equiv q_0^n$ [12, Proposition 3.5].

Since $t$ is a conjugator of $(p, q)$, $t \equiv p^m ry$ for some $m \geq 0$ and some $y \perp u$; but $y$ must be empty because $\mathrm{alph}(t) \subseteq \mathrm{alph}(u)$. Thus, $t \equiv p^m r$ and $u \equiv pt \equiv p^{m+1} r$ for some $m \geq 0$. The traces $p$ and $q$ must be primitive, since otherwise $u$ would have a strictly longer overlap than $t$. (To see this, suppose $p$ or $q$ is imprimitive. As noted above, $p$ and $q$ have the same exponent, so $p \equiv p_0^n$ and $q \equiv q_0^n$ for some $n \geq 2$; but then $t' = p_0^{n-1} t$ is an overlap of $u$: $u \equiv p_0^n t \equiv p_0 t' \equiv p_0^{n-1} p_0 p^m r \equiv t' q_0$.) As is the case for strings, $p$ is the prefix of $u$ belonging to the conjugacy class of primitive traces $x$ such that $u$ is a factor of a power of $x$; however, this fact is not needed here.

Finally, property 4 certainly holds if $m = 0$, so suppose $m$ is positive. It is enough to show that, for any proper overlap $s$ of $u$, if $\Psi(s) \geq \Psi(p)$, then $s \equiv p^k r$ for some $k$, $1 \leq k \leq m$, and otherwise $\Psi(s) \leq \Psi(pr)$. Since $t$ is the maximum proper overlap of $u$, $s$ is a suffix of $t$: write $t \equiv xs$, and note that $u \equiv pxs \equiv xsq$. If $\Psi(s) \geq \Psi(p)$, then $s \equiv py$ for some $y$ (since both $p$ and $s$ are prefixes of $u$), so

$u \equiv pxs \equiv xpyq$ and therefore $px \equiv xp$ and it follows that $ps \equiv pyq \equiv sq$. Since $s$ is thus a conjugator of $(p, q)$ and $\text{alph}(s) \subseteq \text{alph}(u)$, indeed $s$ has the form $p^k r$ for some $k$; $k$ is at least 1 because $\Psi(s) \geq \Psi(p)$ and $\Psi(r) \ngeq \Psi(p)$, and $k$ is at most $m$ because $s$ is a prefix of $t$. Now suppose $\Psi(s) \ngeq \Psi(p)$. Both $s$ and $pr$ are overlaps of $u$, so there is some $w \in OVL(u)$ such that $\Psi(w) = \max\{\Psi(s), \Psi(pr)\} \geq \Psi(pr)$. From the previous case, $w \equiv p^k r$ for some $k \geq 1$. However, $k \geq 2$ would imply $\Psi(s) = \Psi(w) \geq \Psi(p)$; hence $k = 1$ and $\Psi(s) \leq \Psi(pr)$, completing the proof. □

### 2.2. *Reduction and Rewriting Systems*

An *(abstract) reduction system* $S$ on a set $U$ is a set of pairs of elements of $U$. For such a system $S \subseteq U \times U$, and a pair $(u_1, u_2) \in S$, write $u_1 \longrightarrow u_2$; we say that "$u_1$ reduces to $u_2$ in one step." The *reduction relation* $\overset{*}{\longrightarrow}$ determined by $S$ is the transitive, reflexive closure of the one-step reduction relation $\longrightarrow$. The equivalence relation determined by $S$ is the symmetric, transitive, and reflexive closure of $\longrightarrow$.

If there is some $u_2$ such that $u_1 \longrightarrow u_2$, then $u_1$ is *reducible*, and otherwise $u_1$ is *irreducible*. If $u_1 \overset{*}{\longrightarrow} u_2$, then $u_1$ is an *ancestor* of $u_2$, and $u_2$ is a *descendant* of $u_1$. A pair of elements $(u_1, u_2)$ is *joinable* if they have a common descendant, that is, if there is some $u_3$ such that $u_1 \overset{*}{\longrightarrow} u_3$ and $u_2 \overset{*}{\longrightarrow} u_3$.

A reduction system is *confluent* if, whenever $z \overset{*}{\longrightarrow} x$ and $z \overset{*}{\longrightarrow} y$, there is some $w$ such that $x \overset{*}{\longrightarrow} w$ and $y \overset{*}{\longrightarrow} w$. In other words, a system is confluent if every pair of elements with a common ancestor has a common descendant (that is, the pair is joinable). Confluent reduction systems have the Church–Rosser property: every pair of equivalent elements is joinable. A reduction system is *terminating* (or Noetherian) if there is no infinite chain of reductions $x_1 \longrightarrow x_2 \longrightarrow x_3 \longrightarrow \cdots$. A system that is both confluent and terminating is called a *complete* (or convergent) system.

A reduction system is *locally confluent* if, whenever $z \longrightarrow x$ and $z \longrightarrow y$, the pair $(x, y)$ is joinable. While any locally confluent and terminating system is confluent, we use here a more restricted notion of local confluence, one that allows proofs of the confluence property for nonterminating systems. A pair of elements $(u_1, u_2)$ is *strongly joinable* if they have a common descendant that can be obtained from each in at most one step: either $u_1 = u_2$ or $u_1 \longrightarrow u_2$ or $u_2 \longrightarrow u_1$ or there is some $u_3$ such that $u_1 \longrightarrow u_3$ and $u_2 \longrightarrow u_3$. A reduction system is *strongly confluent* if, whenever $z \longrightarrow x$ and $z \longrightarrow y$, the pair $(x, y)$ is strongly joinable. (This definition of strong confluence is the one used by Dershowitz and Jouannaud [9], which is slightly stronger than that of Huet [13].) The following fact can be easily proved by induction on the length of reduction sequences.

**Proposition 2.5.** *If a reduction system is strongly confluent, then it is confluent.*

Suppose $M = M(A, I) = A^*/\equiv$ is a trace monoid. A rewriting system on $M$ is a set of rules $R \subseteq M \times M$. The system $R$ determines a reduction system on $M$ by defining $x \longrightarrow y$ if $x \equiv sut$ and $y \equiv svt$ for some traces $s$, $t$ and some rule $(u, v) \in R$. A (semi-) Thue system is a rewriting system on a free monoid $A^* = M(A, \emptyset)$. A rewriting system $R$ on a trace monoid $M(A, I)$ gives rise to a Thue system on $A^*$ by fixing particular strings to represent the rules in $R$ and adding a set $S$ of rules of the form $(ab, ba)$ for pairs $(a, b) \in I$ to express the commutation of independent letters. Further definitions for Thue systems may be found in [1] and [14].

A length-reducing trace-rewriting system $R$ is confluent exactly when its associated Thue system $T = R \cup S$ is "preperfect"; the corresponding notion for term-rewriting systems is that "$R/S$ is confluent mod $S$." The preperfect property is undecidable for finite Thue systems [19]; it remains undecidable for systems $T = R \cup S$ in which $R$ is finite and $S$ consists of just one rule of the form $(ab, ba)$ [20], that is, for trace-rewriting systems on a trace monoid $M(A, I)$ where $I$ contains just one pair of independent letters. By contrast, the results here concern systems $T = R \cup S$ in which $S$ is finite and $R$ has size one. A stronger property that has also been studied for Thue systems is that of "almost-confluence." This is a decidable property for finite Thue systems; for term-rewriting systems, it corresponds to "$R$ is confluent modulo $S$." Suppose $R = \{(u, v)\}$ is a one-rule trace-rewriting system with $|u| > |v|$, and (viewing the traces as strings) $T = \{(u, v)\} \cup S$ is its associated Thue system. If $|u| = 1$, then $T$ is almost confluent. When $|u| > 1$, $T$ is almost confluent if and only if it is preperfect and, for all letters $a$, $b$, no commutation rule applies to $aub$. In effect, for one-rule trace-rewriting systems, the property of almost-confluence requires the single rule and the independence relation to be disjoint.

## 3. Necessary Conditions for Confluence

This section begins with a demonstration (Lemma 3.1) that, for a one-rule Thue system, critical pairs (which are derived from overlaps of the left-hand side of the rule) can only be joined in a restricted manner. In Lemma 3.2 we apply that information to one-rule trace-rewriting systems in general, to draw conclusions from joinability under certain conditions. For example, if $R = \{(u, v)\}$ is a confluent trace-rewriting system and $u$ is connected, then any proper overlap of $u$ with Parikh image bounded above by that of $v$ must be an overlap of $v$. Example 3.3 shows the limitations of the technique used in deriving Lemma 3.2 from Lemma 3.1.

One source of additional complexity of trace rewriting over string rewriting is the possible existence of letters that are independent of the left-side of a rule. Such letters must interact in some way with the other rules of the system to ensure confluence. For a one-rule system $R = \{(u, v)\}$, the interaction is simple to analyze as long as $u$ is not a factor of $v$: any letter that is independent of $u$ must commute with $v$ (Lemma 3.4).

The idea behind the following lemma is due to Kurth [15].

**Lemma 3.1.** *Suppose $T = \{(u, v)\}$ is a Thue system on $A^*$ and $u = ps = sq$ with $p$, $q$ nonempty. If $pv$ and $vq$ have a common descendant with respect to $T$, then one of the following conditions holds:*

(i) $|v| < |s|$, *$vq = pv$, and $v \in OVL(s)$.*
(ii) $|v| \geq |s|$, *$vq$ and $pv$ have a common one-step descendant, and $s \in OVL(v)$.*

*Proof.* Suppose $vq$ and $pv$ have a common descendant. If $|v| < |s|$, then $vq$ and $pv$ are both shorter than $u$, so they must be irreducible and hence $vq = pv$. The equations $ps = sq$ and $pv = vq$, together with the inequality $|v| < |s|$, imply that $s = p_0^k v = vq_0^k$ for some $k \geq 1$, where $p_0$ and $q_0$ are the roots of $p$ and $q$, respectively; hence $v \in OVL(s)$.

Now suppose $|v| \geq |s|$. If $s \in OVL(v)$, with, say, $v = v_1 s = sv_2$, then $vq$ and $pv$ have the common one-step descendant $v_1 s v_2$. If $s \notin OVL(v)$, then in fact $vq$ and $pv$ can have no common descendant. To see this, note first that if $s \notin OVL(v)$, then either $s$ is not a prefix of $v$ or $s$ is not a suffix of $v$; the arguments are symmetric, so suppose $s$ is not a prefix of $v$. Since $|v| \geq |s|$, strings $s_0, s_1, v_1$ and distinct letters $a, b$ exist such that $s = s_0 a s_1$ and $v = s_0 b v_1$. Then $s_0$ and $ps_0$ are both prefixes of $u$, so $ps_0 = s_0 p'$ for some $p'$. Since $s_0$ is a prefix of all the strings $u$, $v$, $vq = s_0 b v_1 q$ and $pv = s_0 p' b v_1$, a simple argument will show that descendants of $vq$ and $pv$ with respect to $T$ correspond to descendants of $bv_1 q$ and $p' b v_1$ with respect to the Thue system $\{(asq_1, bv_1)\}$. Thus, we may assume that $s_0$ is empty.

Write, therefore, $v = bv_1$, $s = as_1$, and $u = pas_1 = as_1 q$ where $a \neq b$ and $p$ begins with $a$. Every descendant of $vq = bv_1 q$ begins with $b$ (since the right-hand side of the rule begins with $b$). On the other hand, every descendant of $pv$ begins with $a$, so $vq$ and $pv$ can have no common descendant, as desired. To see the claim concerning $pv$, consider the "dictionary ordering" that puts $a$ before $b$: $x$ appears before $y$ in the dictionary if, for some $z$, $za$ is a prefix of $x$ and $zb$ is a prefix of $y$. (This is a transitive and irreflexive relation on strings.) Since $u$ begins with $a$, application of the rule is increasing for this ordering; hence no descendant of $pv = pbv_1$ can have $pas_1 = u$ as a prefix, and it follows that the first letter, $a$, of $pv$ persists in every descendant of $pv$. □

The following fact is derived by combining Lemma 3.1 with the technique of the projection of traces given in Proposition 2.2.

**Lemma 3.2.** *Suppose $R = \{(u, v)\}$ is a rewriting system on $M(A, I)$, $s$ is a proper overlap of $u$ with $u \equiv ps \equiv sq$, $\Psi(v) \geq \Psi(s)$, and $pv$ and $vq$ have a common descendant with respect to $R$. If $u$ is connected or if* $\mathrm{alph}(s) \subseteq \mathrm{alph}(p)$, *then $s \in OVL(v)$.*

*Proof.* Suppose $pv$ and $vq$ have the common descendant $z$. If $u$ is connected, then $\mathrm{alph}(p) = \mathrm{alph}(u)$, so $\mathrm{alph}(s) \subseteq \mathrm{alph}(p)$. (The only way the inclusion could fail to hold is if $u$ is not connected and some component of $u$ is also a component of $s$.) To see that $\mathrm{alph}(s) \subseteq \mathrm{alph}(p)$ implies $s \in OVL(v)$, consider any pair $(a, b) \in D$; it is enough to argue that $s' = \pi_{ab}(s)$ belongs to $OVL(v')$ for $v' = \pi_{ab}(v)$. If $s' = e$, then surely $s' \in OVL(v')$. If $s' \neq e$, then $\pi_{ab}(p) \neq e$, so $s'$ is a proper overlap of $u' = \pi_{ab}(u) = \pi_{ab}(ps) = \pi_{ab}(sq)$. Let $R_{ab}$ be the Thue system on $\{a, b\}^*$ with the single rule $(u', v')$. Since $pv$ and $vq$ both reduce to $z$ with respect to $R$, the strings $\pi_{ab}(p)v'$ and $v'\pi_{ab}(q)$ both reduce to $\pi_{ab}(z)$ with respect to $R_{ab}$. Since $\Psi(v) \geq \Psi(s)$, the string $v'$ must be at least as long as $s'$, and it follows from Lemma 3.1 that $s' \in OVL(v')$. □

In the proof of Lemma 3.2, joinability of specific traces with respect to a trace-rewriting system $R$ was used to establish joinability of specific strings relative to a Thue system that was found by "projecting" $R$. The following examples show that it is not possible to use projection to lift a characterization of confluence directly from Thue systems to trace-rewriting systems.

**Example 3.3.** Let $A = \{a,b,c,d\}$, $I = \{ac, ad, bd\}$, and $D = \{ab, bc, cd\}$. Let $x$ be the trace $ab^2a^2bc^2dc^2ba$. The trace-rewriting system $R = \{(x,a)\}$ is confluent; this follows from Theorem 4.2, since $x$ is connected, $I(x)$ is empty, and the maximum proper overlap of $x$ is $a$. However, none of the three projected systems is confluent: $R_{ab} = \{(ab^2a^2b^2a, a)\}$ is not confluent on $\{a,b\}^*$, $R_{bc} = \{(b^3c^4b, e)\}$ is not confluent on $\{b,c\}^*$, and $R_{cd} = \{(c^2dc^2, e)\}$ is not confluent on $\{c,d\}^*$. These projected systems are not confluent because the projections of $x$ have more overlaps than $x$ does. For example, $\underline{ccdcc}dcc \rightarrow dcc$ and $ccd\underline{ccdcc} \rightarrow ccd$ with respect to $R_{cd}$, but $dc^2$ and $c^2d$ are irreducible and unequal.

The trace-rewriting system $S = \{(a^2b, acd)\}$ is not confluent, since $a^2bd \rightarrow acdd$ and $a^2bd \equiv da^2b \rightarrow dacd \equiv adcd$ but the traces $acdd$ and $adcd$ are irreducible and distinct. On the other hand, the three projected systems $S_{ab} = \{(a^2b, a)\}$, $S_{bc} = \{(b,c)\}$, and $S_{cd} = \{(e, cd)\}$ are confluent.

The trace-rewriting system $S$ in Example 3.3 could not be confluent because of the letter $d$: $d$ was independent of all the letters in the left-hand side of the rule but did not commute with all of those in the right-hand side. Such a situation cannot occur for Thue systems (with nonempty left-hand sides); the following simple lemma shows that it prevents a terminating one-rule trace-rewriting system from being confluent.

**Lemma 3.4.** *Suppose $R = \{(u,v)\}$ is a rewriting system on $M(A,I)$. If $R$ is confluent and $u$ is not a factor of $v$, then every trace independent of $u$ commutes with $v$.*

*Proof.* Suppose $z$ is independent of $u$ but $vz \not\equiv zv$. Then $uz \rightarrow vz$ and $uz \equiv zu \rightarrow zv$ so, since $R$ is confluent, $vz$ and $zv$ have a common proper descendant. Hence $vz$ (and $zv$) must be reducible: $vz \equiv sut$ for some traces $s$, $t$. However, since $u$ and $z$ have no letter in common, the Division Property applied to $vz \equiv sut$ implies that $u$ is a factor of $v$, a contradiction. □

Recall that $I(x) = \{a \in A | x \perp a\}$ denotes the set of letters independent of $x$, and $COM(x) = \{a \in A | ax \equiv xa\}$ denotes the set of letters that commute with $x$. Every trace independent of $x$ commutes with $y$ if and only if $I(x) \subseteq COM(y)$; hereafter the set inclusion is freely used as an abbreviation for the phrase. Both $I(x)$ and $COM(x)$ can be found easily from the independence relation $I$ and the set alph$(x)$. Note also that when alph$(y) = \{b\}$ consists of a single letter, $COM(y) = I(b) \cup \{b\}$.

The condition in Lemma 3.4 is not a necessary one for confluence of a one-rule trace-rewriting system when the left-hand side of the rule is a factor of the right-hand side. For example, if $A = \{a, b, c\}$ and $I = \{ab, ac\}$, then the system $\{(a, ab)\}$ is confluent, but $a \perp c$ and $(ab)c \not\equiv c(ab)$. For multirule systems, even a complete trace-rewriting system might fail to have the property that traces independent of the left-hand side of some rule commute with the right-hand side of the rule; Otto has given the following example [21, Example 3]: if $A = \{a, b, c\}$ and $I = \{ac\}$, then the system $\{(a, bb), (bbc, cbb)\}$ is confluent and terminating, but $a \perp c$ and $(bb)c \not\equiv c(bb)$.

Returning to the question of applying projection to trace-rewriting systems, suppose $R = \{(u, v)\}$ is a rewriting system on $M(A, I)$ for which $u$ is connected and nonempty. Any feasible combination of confluence or nonconfluence of $R$ and of its projections can occur, as long as the question of whether $I(u) \subseteq COM(v)$ is disregarded. However if $I(u) \not\subseteq COM(v)$, then at least one projected system will be confluent, since it will have an empty left-hand side. Also, it can be shown (using Corollary 4.4 and Theorem 4.2) that if $I(u) \subseteq COM(v)$ and every projected system of $R$ is confluent, then $R$ is confluent; this is in contrast to the system $S$ of Example 3.3.

## 4. When the Left-Hand Side of the Rule Is Connected

This section deals with trace-rewriting systems $R = \{(u, v)\}$ for which the left-hand side $u$ is connected and certain other conditions hold. The principal condition is that traces independent of $u$ commute with $v$ (that is, $I(u) \subseteq COM(v)$); from Lemma 3.4, this is also a necessary condition for confluence when $u$ is not a factor of $v$. Theorem 4.2, the main result of the paper, presents a characterization of confluence in terms of the structure of $u$, $v$ and their overlaps when $u$ is connected and $I(u) \subseteq COM(v)$. In addition, it states that confluence and strong confluence are equivalent for such systems, and identifies a finite set of critical pairs for confluence. Corollary 4.3 restates Theorem 4.2 under the assumptions that $u$ is connected and is not a factor of $v$, and hence characterizes confluence for all terminating one-rule systems with a connected left-hand side. Corollary 4.4 gives the (known) characterization of confluence for Thue systems as a special case of trace-rewriting systems.

Finally, in Corollary 4.5, we show that confluence under these conditions is a decidable property for a fixed trace monoid: it can be tested in linear time in the size of the given system.

The following lemma gives a description of critical pairs for strong confluence under the assumption $I(u) \subseteq COM(v)$. The critical pairs are similar to those for a one-rule Thue system, with a slight complication introduced by traces independent of some overlap of $u$.

**Lemma 4.1.** *Suppose $R = \{(u, v)\}$ is a rewriting system on $M(A, I)$ for which $u$ is connected and $I(u) \subseteq COM(v)$. Let*

$$CP(u, v) = \{(pzv, vzq) \mid u \equiv ps \equiv sq \text{ with } p,\ q,\ s \text{ nonempty and } s \perp z\}.$$

The system $R$ is strongly confluent if and only if every pair in $CP(u, v)$ is strongly joinable.

*Proof.* If $(pzv, vzq) \in CP(u,v)$, then $pzu \to pzv$ and $pzu \equiv pzsq \equiv pszq \equiv uzq \to vzq$; hence if $R$ is strongly confluent, then in particular every pair in $CP(u,v)$ is strongly joinable.

The reverse implication follows from the analyses done by Diekert [10], [11] and by Otto [21] of critical pairs for multirule trace-rewriting systems. In this simpler case it can also be seen directly, as follows.

For strong confluence, we must show that if $x_1ux_2 \equiv y_1uy_2$, then, for $x = x_1vx_2$ and $y = y_1vy_2$, either $x \equiv y$, or one of $x, y$ reduces in one step to the other, or $x$ and $y$ have a common one-step descendant. It is clearly sufficient to show this when $x_1, y_1$ have no nonempty common prefix and $x_2$, $y_2$ have no nonempty common suffix; from the Division Property, it follows that $x_1$ is independent of $y_1$, and $x_2$ is independent of $y_2$, and hence $\Psi(x_1) = \Psi(y_2)$ and $\Psi(x_2) = \Psi(y_1)$. A projection argument then establishes that $x_1u \equiv uy_2$, $ux_2 \equiv y_1u$, and $\mathrm{alph}(x_1)$, $\mathrm{alph}(x_2)$, and $\mathrm{alph}(u) - \mathrm{alph}(x_1x_2)$ are pairwise independent. Up to this point, no assumptions about $u$ and $v$ have been needed. Now, however, since $u$ is connected, we have (by symmetry) two cases: either $\mathrm{alph}(u)$ is disjoint from $\mathrm{alph}(x_1x_2)$ or $\mathrm{alph}(u) \subseteq \mathrm{alph}(x_1)$. In the first case $u$ is independent of $x_1y_1x_2y_2$, so $x_1 \equiv y_2$, $x_2 \equiv y_1$, $x_1$ and $x_2$ commute with $v$, and hence $x = x_1vx_2 \equiv y_1vy_2 = y$. In the second case $x_2$ is independent of $u$, so $x_2 \equiv y_1$ and $x_2$ commutes with $v$. Therefore, $x = (x_1v)x_2$ and $y = y_1vy_2 \equiv x_2vy_2 \equiv (vy_2)x_2$, and we may restrict attention to $x_1v$ and $vy_2$. Since $\mathrm{alph}(u) \subseteq \mathrm{alph}(x_1)$ and $x_1u \equiv uy_2$, the Division Property implies that $u \equiv ps \equiv sq$, $x_1 \equiv pz$, $y_2 \equiv zq$ for some traces $p$, $q$, $s$, $z$ such that $s \perp z$ and $\mathrm{alph}(s) \subseteq \mathrm{alph}(p)$. If $s$ is empty, then $x_1 \equiv uz$, $y_2 \equiv zu$, and so $x_1v$ and $vy_2$ have the common one-step descendant $vzv$. If $s$ is nonempty, then so must $p$ and $q$ be, so $(x_1v, vy_2) \equiv (pzv, vzq)$ belongs to $CP(u,v)$ and, by assumption, is strongly joinable. □

The following theorem presents three properties that are equivalent to confluence for certain one-rule trace-rewriting systems $\{(u,v)\}$. The third property means that either every proper overlap of $u$ is an overlap of $v$, or $v$ splits into a proper overlap of $u$ and a part independent of $u$ (and the other overlaps of $u$ are restricted). The fourth property is a description of a set of critical pairs for confluence of the system, which (as is the case for Thue systems) is a finite set determined solely by the overlaps of $u$.

**Theorem 4.2.** *Suppose $R = \{(u,v)\}$ is a rewriting system on $M(A,I)$ for which $u$ is nonempty and connected. Let $t$ be the maximum proper overlap of $u$. If $I(u) \subseteq COM(v)$, then the following statements are equivalent.*

(1) *$R$ is confluent.*
(2) *$R$ is strongly confluent.*
(3) *Either* (a) *$t \in OVL(v)$ or* (b) *traces $x \neq e$, $y$ and $z$, and integer $k \geq 2$ exist such that $z \perp u$, $u \equiv x^k y$, $v \equiv yz$, and $OVL(u) = OVL(y) \cup \{x^i y | 1 \leq i \leq k\}$.*
(4) *Every pair in the set $\{(pv, vq) | u \equiv ps \equiv sq$ with $p$, $q$, $s$ nonempty$\}$ is joinable.*

*Proof.* Write $u \equiv p_0 t \equiv t q_0 \equiv p_0^{m+1} r$ where (as in Lemma 2.4) $r$ is the minimum conjugator of $(p_0, q_0)$.

Clearly, (2) implies (1), and (1) implies (4).

That (3) implies (2) follows from Lemma 4.1, since we are given that $I(u) \subseteq COM(v)$. To see this, suppose that $(pwv, vwq) \in CP(u,v)$ with $u \equiv ps \equiv sq$ where $p, q, s$ are nonempty and $s \perp w$. If $s \in OVL(v)$ with, say, $v \equiv v_1 s \equiv s v_2$, then $pwv$ and $vwq$ have the common one-step descendant $vwv_2 \equiv v_1 wv$. If $s \notin OVL(v)$, then necessarily clause (3b) holds with $s \in OVL(u) - OVL(v)$: therefore $v \equiv yz, z \perp u, u \equiv x^k y$, and $s \equiv x^i y$ for some $i$, $1 \leq i \leq k-1$, and hence (using $u \equiv ps \equiv sq$) $p \equiv x^{k-i}$ and $py \equiv x^{k-i} y \equiv yq$. From the form of $s$, we see that $\text{alph}(s) = \text{alph}(u)$, so $u \perp w$ and therefore $w$ commutes with $v$; a simple calculation now shows that $pwv \equiv vwq$.

For the final implication, suppose that (4) holds. If $t$ is empty, then it belongs to $OVL(v)$, so suppose that $t \neq e$. Consider the pair $(p_0 v, v q_0)$ where, again, $u \equiv p_0 t \equiv t q_0 \equiv p_0^{m+1} r$; by assumption, $p_0 v$ and $v q_0$ have a common descendant. If $\Psi(v) \geq \Psi(t)$, then (from Lemma 3.2) $t \in OVL(v)$, so suppose $\Psi(v) \not\geq \Psi(t)$. Both $p_0 v$ and $v q_0$ must then be irreducible, so $p_0 v \equiv v q_0$. Since $r$ is the minimum conjugator of $(p_0, q_0)$, it follows that $v \equiv yz$ with $z \perp u$ and $y \equiv p_0^j r$ for some $j \geq 0$; also $j \leq m-1$ since otherwise $\Psi(v) \geq \Psi(t)$. Thus, for $k = m+1-j \geq 2$, $u \equiv p_0^k y$, $v \equiv yz$, and $z \perp u$, and certainly $OVL(y) \cup \{p_0^i y | 1 \leq i \leq k\} \subseteq OVL(u)$. For the reverse inclusion, it is enough, from Lemma 2.4, to show that any nonempty proper overlap of $p_0 r \equiv r q_0$ belongs to $OVL(y)$. This is clearly true if $j > 0$, so suppose $j = 0$, that is, $v \equiv rz$. If $p_0 r \equiv p_1 s \equiv s q_1$ with $p_1$ and $s$ nonempty, then $u \equiv (p_0^m p_1)s \equiv s(q_1 q_0^m)$, so, by assumption, $p_0^m p_1 v$ and $v q_1 q_0^m$ have a common descendant. If either $p_0^m p_1 v$ or $v q_1 q_0^m$ is reducible, then $\Psi(p_0^m p_1 v) = \Psi(v q_1 q_0^m) \geq \Psi(u)$ so $\Psi(p_1 rz) \geq \Psi(p_0 r)$ and hence (since $z \perp p_0 r$) $\Psi(p_1) \geq \Psi(p_0)$. It follows that $\Psi(r) \geq \Psi(s)$, so, since $r$ and $s$ belong to $OVL(u)$, $s \in OVL(r) = OVL(y)$. If $p_0^m p_1 v$ and $v q_1 q_0^m$ are irreducible, then $p_0^m p_1 v \equiv v q_1 q_0^m$ and so $p_0^m p_1 r \equiv r q_1 q_0^m$. We have $p_1 p_0^{m+1} r \equiv p_1 u \equiv p_1 r q_0^{m+1} \equiv p_1 s q_1 q_0^m \equiv p_0 r q_1 q_0^m \equiv p_0^{m+1} p_1 r$, so $p_1 p_0^{m+1} \equiv p_0^{m+1} p_1$. Since $p_0$ is primitive and $p_1$ is nonempty with $\text{alph}(p_1) \subseteq \text{alph}(u) = \text{alph}(p_0)$, the trace $p_1$ must therefore be a positive power of $p_0$, so again $\Psi(p_1) \geq \Psi(p_0)$, $\Psi(r) \geq \Psi(s)$, and $s \in OVL(r)$. □

In the situation treated in Theorem 4.2, the system $\{(u,v)\}$ is confluent if and only if it is confluent as a system on the submonoid of $M(A,I)$ generated by $\text{alph}(uv)$. This equivalence is not true in general when $I(u) \not\subseteq COM(v)$. Theorem 4.2 (and the following corollaries) do apply to some nonterminating systems; for example, when $ab \in D$, the system with rule $(ab, bbaa)$ is (strongly) confluent, but it is not terminating since $abb \to bbaab \to bb(abb)aa$.

The following fact can be easily derived from Theorem 4.2 and Lemma 3.4.

**Corollary 4.3.** *Suppose $R = \{(u,v)\}$ is a rewriting system on $M(A,I)$ for which $u$ is nonempty and connected. Let $t$ be the maximum proper overlap of $u$.*

A. *When $\Psi(v) \geq \Psi(t)$ and $u$ is not a factor of $v$: the system $R$ is confluent if and only if $I(u) \subseteq COM(v)$ and $t \in OVL(v)$.*

B. *When* $\Psi(v) \not\geq \Psi(t)$: *the system R is confluent if and only if* $I(u) \subseteq COM(v)$ *and traces* $x \neq e$, *y and z, and integer* $k \geq 2$ *exist such that* $z \perp u$, $u \equiv x^k y$, $v \equiv yz$, *and* $OVL(u) = OVL(y) \cup \{x^i y | 1 \leq i \leq k\}$.

A series of results [2], [23], [15] (summarized in [26]) has led to a characterization of confluence for all one-rule Thue systems. That characterization for systems with a nonempty left-hand side is given in the following corollary; it is an immediate consequence of Theorem 4.2, as is the fact that confluence and strong confluence are equivalent for one-rule Thue systems.

**Corollary 4.4.** *Suppose* $R = \{(u, v)\}$ *is a Thue system on* $A^*$ *with* $u \neq e$, *and let t be the longest proper overlap of u. The system R is confluent if and only if either* (a) $t \in OVL(v)$ *or* (b) *string* $x \neq e$ *and integer* $k \geq 2$ *exist such that* $u = x^k v$ *and* $OVL(u) = OVL(v) \cup \{x^i v | 1 \leq i \leq k\}$.

The finite set of critical pairs identified in Theorem 4.2 makes it clear that confluence is a decidable property for the one-rule trace-rewriting systems to which it applies. The combinatorial characterization gives another avenue for testing confluence, and, in fact, it can be checked very easily for a fixed trace monoid.

**Corollary 4.5.** *For a fixed partially commutative alphabet* $(A, I)$, *given strings representing traces u, v with u connected and not a factor of v, it can be tested in time linear in* $|uv|$ *whether the trace-rewriting system* $\{(u, v)\}$ *is confluent.*

*Proof.* Note first that, given $u$ and $v$, whether $u$ is connected can be tested by simply examining the alphabet of $u$, and whether $u$ is a factor of $v$ in $M(A, I)$ can be tested in time linear in $|uv|$ [16].

If $u$ is connected, let $t$ be the maximum proper overlap of $u$. Let $p$, $q$ be strings such that $u \equiv pt \equiv tq$, let $r$ be the minimum conjugator of $(p, q)$ in $M(A, I)$, and let $s$ be the maximum proper overlap of $pr$. (The trace $pr$ must be connected since $\text{alph}(p) = \text{alph}(u)$.) The following statement follows easily from Corollary 4.3:

> When $u$ is connected and not a factor of $v$, the system $\{(u, v)\}$ is confluent on $M(A, I)$ if and only if $I(u) \subseteq COM(v)$ and either
>
> (a) $\Psi(v) \geq \Psi(t)$ and $t$ is both a prefix and a suffix of $v$, or
> (b) $\Psi(v) \not\geq \Psi(t)$, $pv \equiv vq$, and either $p$ is a prefix of $v$ or $|s| \leq |r|$.

Given a connected trace $u$, its maximum proper overlap $t$ can be found in time linear in $|u|$ [25]. The Parikh image of a prefix $y$ of a trace $x$ determines the prefix, which can be formed by erasing from $x$ the last $|x|_a - |y|_a$ occurrences of each letter $a$. Therefore, strings representing $p$, $q$, and $r$ can be found easily from $u$ and $t$: $\Psi(p) = \Psi(q) = \Psi(u) - \Psi(t)$ and $\Psi(r) = \Psi(t) - m\Psi(p)$ for $m = \min\{\lfloor |t|_a / |p|_a \rfloor | a \in A\}$. The maximum overlap $s$ of $pr$ can be found in time linear in $|pr|$, and $|pr| \leq |u|$.

As noted in Section 3, whether $I(u)$ is a subset of $COM(v)$ can be tested in constant time once the alphabets of $u$ and $v$ are known. The other functions needed for the statement above, comparing lengths and Parikh images of strings and testing whether traces are the same or one is a prefix or a suffix of the other, can also be performed in time linear in their lengths. □

## 5. When the Left-Hand Side of the Rule Is Not Connected

The information developed in the previous section for trace-rewriting systems follows closely the pattern of that for string-rewriting systems. When the left-hand side of the single rule is not connected, the situation becomes more complex. We have obtained a characterization of confluence, given in the following theorem, only in case at least one component of the left-hand side is not a factor of the right-hand side. When at least two components of the left-hand side are not factors of the right-hand side, any confluent system must be terminating.

**Theorem 5.1.** *Suppose $R = \{(u, v)\}$ is a rewriting system on $M(A, I)$ for which at least one component of $u$ is not a factor of $v$. Let $u = u_1 \oplus \cdots \oplus u_m$, $m \geq 1$, be the decomposition of $u$ into its connected components with, say, $u_1$ not a factor of $v$. The system $R$ is confluent if and only if:*

(i) *The trace $v$ can be decomposed as $v = w \oplus v_1 \oplus \cdots \oplus v_m$ where $w \perp u$, $u_1 v_1$ is connected and $v_1 \perp u_2 \cdots u_m$.*
(ii) *For all $i \geq 2$, if $v_i \not\equiv u_i$ then there is a letter $a_i$ such that $u_i$, $v_i$ belong to $a_i^*$ and, for all $j$ such that $u_j$ is not a factor of $v$, $I(u_j) \subseteq I(a_i) \cup \{a_i\}$.*
(iii) *If at least two components of $u$ are not factors of $v$, then there is a letter $a_1$ such that $u_1$, $v_1 \in a_1^*$ and, for all $j$ such that $u_j$ is not a factor of $v$, $I(u_j) \subseteq I(a_1) \cup \{a_1\}$.*
(iv) *For all $j$ such that $u_j$ is not a factor of $v$, $I(u_j) \subseteq COM(w)$.*
(v) *The system $\{(u_1, v_1)\}$ is confluent.*

Although the statement of Theorem 5.1 includes the possibility that the left-hand side $u$ is connected ($m = 1$), it gives no new information in that case. Under the assumptions of the theorem, it can be seen from the proof that the system is confluent if and only if it is strongly confluent.

Condition (v) in the statement only comes into play when all the components of $u$ except $u_1$ are factors of $v$; otherwise it is superseded by condition (iii). Condition (iii), together with the assumption that $u_1$ is not a factor of $v$, implies that a confluent system in which at least two components of $u$ are not factors of $v$ must be terminating: application of the rule reduces the number of occurrences of the letter $a_1$. Otherwise, as in the case of a connected left-hand side, a confluent system need not be terminating.

*Proof of Theorem 5.1.* Let $NF = \{i | u_i \text{ is not a factor of } v\}$ be the set of indices of components of $u$ that are not factors of $v$; in particular, $1 \in NF$.

First, assume that $R$ is confluent. Taking $u' = u_2 \cdots u_m$, we have $uu' \rightarrow vu'$ and $uu' \equiv u'u \rightarrow u'v$. Both $vu'$ and $u'v$ are irreducible (since $u_1$ is not a factor of $v$)

so $vu' \equiv u'v$. Using projection and Proposition 2.3, it follows that $v = v' \oplus v_2 \oplus \cdots \oplus v_m$ where $v' \perp u'$ and each $v_i$ ($i \geq 2$) is a power of the root of $u_i$. Examination of the components of $\mathrm{alph}(u_1 v')$ allows us to split $v'$ as $v' = w \oplus v_1$ with $\mathrm{alph}(u_1 v_1)$ connected and $w \perp u$. This establishes property (i). If $|NF| > 1$, then an analogous argument shows that, in addition, $v_1$ is a power of the root of $u_1$. In any event, $v_1 \not\equiv u_1$ since $u_1$ is not a factor of $v$.

To see that properties (ii)–(iv) and also $I(u_1) \subseteq COM(v_1)$ hold, first consider any index $i$ such that $v_i \not\equiv u_i$ and $NF$ contains an index other than $i$ (which will in particular be true if $i \geq 2$), and any letter $a \in \mathrm{alph}(u_i)$. The traces $u_i a v$ and $v a u_i$ are then irreducible but have the common ancestor $u_i a u \equiv u a u_i$, so $u_i a v \equiv v a u_i$. From the form established in the previous paragraph for $v$, this implies that $u_i a v_i \equiv v_i a u_i$. Since in this situation $v_i$, $u_i$ are different powers of a common root and $a$ occurs in $u_i$, it follows that $u_i, v_i \in a^*$. Thus, when $v_i \not\equiv u_i$ and $NF - \{i\}$ is nonempty, the root of $u_i$ is a single letter $a_i$. Now suppose $b \perp u_j$ for some $j \in NF$, and let $z = u_1 \cdots u_{j-1} u_{j+1} \cdots u_m$. Then $vbz$ and $zbv$ are irreducible and have the common ancestor $ubz \equiv zbu$, so $vbz \equiv zbv$. A projection argument then shows that $wb \equiv bw$ (establishing (iv)), $v_j b \equiv b v_j$ and, for all $k \neq j$, $v_k b u_k \equiv u_k b v_k$. The second equation for $j = 1$ establishes $I(u_1) \subseteq COM(v_1)$. If $i = j$, then certainly $b \in I(a_i) \cup \{a_i\}$; if $i \neq j$, then the equation $v_i b u_i \equiv u_i b v_i$ implies that $b$ commutes with the root of $u_i$, so $b \in I(a_i) \cup \{a_i\}$.

It remains only to show that the system $R_1 = \{(u_1, v_1)\}$ is confluent. (Note that this is trivially true if some component other than $u_1$ is not a factor of $v$.) Since $I(u_1) \subseteq COM(v_1)$ and $u_1$ is connected, it is enough (from Theorem 4.2) to show that if $u_1 \equiv ps \equiv sq$, then the pair $(pv_1, v_1 q)$ is joinable by $R_1$. In this situation $pu \equiv uq$ so, since $R$ is confluent, $pv$ and $vq$ have a common $R$–descendant $y$. Let $\pi$ denote the projection on $\mathrm{alph}(u_1 v_1)$; it is easy to see (using induction on the length of the reduction sequences) that $\pi(y)$ is a common $R_1$–descendant of $pv_1 = \pi(pv)$ and $v_1 q = \pi(vq)$.

For the reverse implication, we show that the systems are strongly confluent. Note first that conditions (i)–(v) imply that $I(u) \subseteq COM(v)$ and that $\mathrm{alph}(v_i) \subseteq \mathrm{alph}(u_i)$ for $i \geq 2$.

For $1 \leq i \leq m$, let $B_i = D(u_i v_i) = \{b \in A | (a, b) \in D \text{ for some } a \in \mathrm{alph}(u_i v_i)\}$, let $B_0 = A - \mathrm{alph}(uv_1)$, and let $\Pi_i$ denote the projection of $A$ onto $B_i$, $0 \leq i \leq m$. The projection $\Pi_i$ sends $M(A, I)$ to the trace monoid determined by $B_i$ and $I \cap (B_i \times B_i)$. Because any two dependent letters belong to a common set $B_i$, two easy consequences of the Projection Lemma are that:

(1) $x \equiv y$ if and only if $\Pi_i(x) \equiv \Pi_i(y)$ for $0 \leq i \leq m$.
(2) $x$ is a prefix (or suffix) of $y$ if and only if $\Pi_i(x)$ is a prefix (resp. suffix) of $\Pi_i(y)$ for $0 \leq i \leq m$.

Based on the preliminary analysis in the proof of Lemma 4.1, to establish strong confluence it is enough to show that each pair $(x_1 v x_2, y_1 v y_2)$ is strongly joinable when $x_1 u \equiv u y_2$, $u x_2 \equiv y_1 u$, the three sets $\mathrm{alph}(x_1)$, $\mathrm{alph}(x_2)$, and $\mathrm{alph}(u) - \mathrm{alph}(x_1 x_2)$ are pairwise independent, and $u$ is not independent of both $x_1$ and $x_2$. Using a projection argument and the independence relationships, we find that the components of $u$ can be partitioned into two products $u'$ and $u''$ with

$u = u' \oplus u''$ and $x_1y_2u'$ independent of $x_2y_1u''$; further, $x_1 \equiv p'z'$, $y_2 \equiv z'q'$, $u' \equiv p's' \equiv s'q'$ with $z' \perp s'$, and $y_1 \equiv p''z''$, $x_2 \equiv z''q''$, $u'' \equiv p''s'' \equiv s''q''$ with $z'' \perp s''$. Projecting these equations onto the alphabets of the components of $u$, we obtain overlap relationships $u_i \equiv p_is_i \equiv s_iq_i$ with $z'z'' \perp s_i$, $1 \leq i \leq m$. (That is, if $u_i$ is part of $u'$, then $p_i, s_i, q_i$ are the projections of $p', s', q'$ on $B_i$, and $\Pi_i(p''s''q'') = e$; and, similarly, if $u_i$ is part of $u''$.) Assume, without loss of generality, that $u_1$ is part of $u'$.

We distinguish two cases.

(1) *For some* $i$, $s_i \notin OVL(v_i)$. In this case we will see that $x_1vx_2 \equiv y_1vy_2$.

If $s_i$ is not an overlap of $v_i$, then $v_i \not\equiv u_i$ and $u_i$, $v_i$ cannot be powers of a single letter with $u_i$ shorter than $v_i$; therefore $u_i$ is not a factor of $v$, and, by symmetry, we may assume that $s_1 \notin OVL(v_1)$. From (i) and (v) and the representation given in Theorem 4.2, it follows in this case that $\mathrm{alph}(v_1) \subseteq \mathrm{alph}(u_1) = \mathrm{alph}(s_1)$ and $p_1v_1 \equiv v_1q_1$. Hence, in addition, $z'z'' \perp u_1$ and so $z'$ and $z''$ commute with $w$ and with all $a_i$ such that $i \geq 2$ and $u_i \not\equiv v_i$.

To verify that $x_1vx_2 \equiv y_1vy_2$, it suffices to check their projections onto the alphabets $B_0, B_1, \ldots, B_m$. First, $\Pi_0(x_1vx_2) \equiv \Pi_0(p'z'wv_1v_2 \cdots v_mz''q'') \equiv \Pi_0(z'wz'')$ and $\Pi_0(y_1vy_2) \equiv \Pi_0(z''wz')$, and these projections are the same trace because $w$ commutes with $z'$ and $z''$, and $z' \perp z''$. Next, $\Pi_1(x_1vx_2) \equiv \Pi_1(p_1v_1) \equiv \Pi_1(v_1q_1) \equiv \Pi_1(y_1vy_2)$. Finally, for $i \geq 2$, the problem reduces to showing that $p_iz_iv_i \equiv v_iz_iq_i$ where $z_i \perp s_i$. (That is, if, for example, $u_i$ is part of $u''$, then $\Pi_i(x_1vx_2) \equiv \Pi_i(v_iz''q'')$ and $\Pi_i(y_1vy_2) \equiv \Pi_i(p''z''v_i)$, and $z_i = \Pi_i(z'')$.) If $u_i \equiv v_i$, then $p_iz_iv_i \equiv p_iz_iu_i \equiv p_iz_is_iq_i \equiv p_is_iz_iq_i \equiv u_iz_iq_i \equiv v_iz_iq_i$. If $u_i \not\equiv v_i$, then the traces $u_i, v_i, p_i$, and $q_i$ all belong to $a_i^*$, $p_i = q_i$, and $z_i$ commutes with $a_i$, so again $p_iz_iv_i \equiv v_iz_iq_i$.

(2) *For all* $i, s_i \in OVL(v_i)$. In this case $x_1vx_2$ and $y_1vy_2$ have a common one-step descendant.

We will find that $x_1vx_2 \equiv u\alpha$ and $y_1vy_2 \equiv \beta u$ for some $\alpha$, $\beta$ with $v\alpha \equiv \beta v$; from the remarks above, it is again sufficient to work with the projections of the traces on the alphabets $B_i$. To avoid doing the case analysis twice, the two demonstrations are given simultaneously.

Write $v_i \equiv r_is_i \equiv s_it_i$, $1 \leq i \leq m$.

For the projection on $B_0$, we have $\Pi_0(x_1vx_2) \equiv \Pi_0(z'wz'') \equiv \Pi_0(\alpha) \equiv \Pi_0(u\alpha)$ and $\Pi_0(y_1vy_2) \equiv \Pi_0(z''wz') \equiv \Pi_0(\beta) \equiv \Pi_0(\beta u)$. Since $z''$ is independent of $u_1$, it commutes with $w$, so $\Pi_0(v\alpha) \equiv \Pi_0(wz'wz'') \equiv \Pi_0(z''wz'w) \equiv \Pi_0(\beta v)$.

For the projection on $B_1$, first note that, since $x_2y_1 \perp u_1$, $\Pi_1(x_2) \equiv \Pi_1(y_1) \equiv \Pi_1(z'')$ and $z''$ commutes with $v_1$ and hence with $r_1$ (and $t_1$). Therefore $\Pi_1(x_1vx_2) \equiv \Pi_1(p_1z'v_1z'') \equiv \Pi_1(p_1z's_1t_1z'') \equiv \Pi_1(p_1s_1z't_1z'') \equiv \Pi_1(u_1z't_1z'') \equiv \Pi_1(u_1\alpha)$ and $\Pi_1(y_1vy_2) \equiv \Pi_1(z''v_1z'q_1) \equiv \Pi_1(z''r_1z'u_1) \equiv \Pi_1(\beta u_1)$. Then $\Pi_1(v\alpha) \equiv \Pi_1(v_1z't_1z'') \equiv \Pi_1(r_1s_1z't_1z'') \equiv \Pi_1(r_1z's_1t_1z'') \equiv \Pi_1(r_1z'v_1z'') \equiv \Pi_1(z''r_1z'v_1) \equiv \Pi_1(\beta v)$.

Now, suppose $i \geq 2$ and $u_i$ is part of $u'$. Then $x_2y_1 \perp u_iv_i$, so $\Pi_i(x_1vx_2) \equiv \Pi_i(p_iz'v_i) \equiv \Pi_i(p_iz's_it_i) \equiv \Pi_i(u_iz't_i) \equiv \Pi_i(u_i\alpha)$ and $\Pi_i(y_1vy_2) \equiv \Pi_i(v_iz'q_i) \equiv \Pi_i(r_iz'u_i) \equiv \Pi_i(\beta u_i)$; and then $\Pi_i(v\alpha) \equiv \Pi_i(v_iz't_i) \equiv \Pi_i(r_is_iz't_i) \equiv \Pi_i(r_iz'v_i) \equiv \Pi_i(\beta v)$.

Finally, suppose $i \geq 2$ and $u_i$ is part of $u''$, so $x_1y_2 \perp u_iv_i$. If $u_i \equiv v_i$, then $\Pi_i(x_1vx_2) \equiv \Pi_i(v_ix_2) \equiv \Pi_i(u_i\alpha)$ and $\Pi_i(y_1vy_2) \equiv \Pi_i(y_1v_i) \equiv \Pi_i(\beta u_i)$ and $\Pi_i(v\alpha) \equiv \Pi_i(u_ix_2) \equiv \Pi_i(p_is_iz''q_i) \equiv \Pi_i(p_iz''u_i) \equiv \Pi_i(\beta v)$. If $u_i \not\equiv v_i$, then all the traces $u_i, v_i, s_i, p_i = q_i$ and $r_i = t_i$ belong to $a_i^*$ and $z''$ commutes with $a_i$, so $\Pi_i(x_1vx_2) \equiv \Pi_i(v_iz''q_i) \equiv \Pi_i(s_it_iz''q_i) \equiv \Pi_i(s_iq_it_iz'') \equiv \Pi_i(u_i\alpha)$ and $\Pi_i(y_1vy_2) \equiv \Pi_i(p_iz''r_is_i) \equiv \Pi_i(z''r_ip_is_i) \equiv \Pi_i(\beta u_i)$ and $\Pi_i(v\alpha) \equiv \Pi_i(v_it_iz'') \equiv \Pi_i(z''r_iv_i) \equiv \Pi_i(\beta v)$.

This completes the proof. □

In the hope of making the rather complicated statement of Theorem 5.1 more understandable, we present the form to which it reduces when no component of the left-hand side $u$ is a factor of the right-hand side $v$. In particular, every component of $u$ is a power of a single letter and those letters commute with precisely the same subset of $A$, placing severe restrictions on $u$.

**Corollary 5.2.** *Suppose $R = \{(u,v)\}$ is a rewriting system on $M(A,I)$ for which $u$ is not connected and no component of $u$ is a factor of $v$. Let $u = u_1 \oplus \cdots \oplus u_m, m \geq 2$, be the decomposition of $u$ into its connected components. The system $R$ is confluent if and only if:*

(i) *$v = w \oplus v_1 \oplus \cdots \oplus v_m$ with $w \perp u$.*
(ii) *For all $i, 1 \leq i \leq m$, there is a letter $a_i$ such that $u_i, v_i \in a_i^*$ (with $v_i$ shorter than $u_i$) and $I(a_i) \subseteq COM(w)$.*
(iii) *$COM(a_1) = COM(a_2) = \cdots = COM(a_m)$.*

**Corollary 5.3.** *For a fixed partially commutative alphabet $(A,I)$, given strings representing traces $u, v$, where $u$ is not connected and at least one component of $u$ is not a factor of $v$, it can be tested in time linear in $|uv|$ whether the trace-rewriting system $\{(u,v)\}$ is confluent.*

*Proof.* Finding the decompositions of $u$ and $v$ and performing the tests required for conditions (i)–(iv) of Theorem 5.1 can all be done in time bounded by a constant multiple of $|uv|$ when the partially commutative alphabet is fixed; most of this work depend only on the alphabets of $u$ and $v$, and the rest involves such tasks as checking whether two traces are equal or whether one is a factor of another. (Note that the number of components of a trace is bounded by its alphabet size.) Condition (v) can also be tested in linear time, as shown in Corollary 4.5. □

## 6. When the Left-Hand Side of the Rule Is Empty

The results in the previous sections do not apply to a one-rule trace-rewriting system in which the left-hand side of the rule is empty (except for the trivial system $\{(e,e)\}$). Of course, such a system will not be terminating, but it might or might not be confluent. For a string-rewriting system with empty left-hand sides, there is no question about confluence: the system is necessarily strongly confluent. The additional complexity introduced by allowing partial commutations is especially noticeable in this case, since, as Otto has recently shown [22], it is possible for a trace-rewriting system of the form $\{(e,v)\}$ to be confluent but not strongly confluent.

The following proposition gives a characterization of strong confluence for one-rule trace-rewriting systems with an empty left-hand side; the condition asks whether some pair of independent letters can be connected by a path in the dependence relation through the alphabet of the right-hand side.

**Proposition 6.1.** *A trace-rewriting system* $\{(e,v)\}$ *is strongly confluent on* $M(A,I)$ *if and only if, for every pair* $(a,b) \in I$, *the letters a and b lie in different components of* alph*(abv).*

*Proof.* Both sides of the statement are clearly true if $v$ is empty, so suppose $v \neq e$.

For the implication from right to left, it is sufficient to show that, for any traces $s,t$ such that $s \perp t$, there is some $w$ such that $svt \to w$ and $tvs \to w$. Since $s$ is independent of $t$, the condition in the statement implies that no component of $\text{alph}(stv)$ contains both a letter from $s$ and a letter from $t$. It follows that $v = x \oplus y$ with $sx \perp ty$. Therefore, $svt = sx \oplus yt = yt \oplus sx \to yty \oplus xsx = xsx \oplus yty$ and $tvs = xs \oplus ty \to xsx \oplus yty$.

Now suppose the system $\{(e,v)\}$ is strongly confluent. If there is any pair of independent letters connected by a path in the dependence relation through $\text{alph}(v)$, then there is some (possibly different) pair of independent letters both of which are dependent on some letter in $v$; and this latter condition will lead to a contradiction. Assume, therefore, that (distinct) letters $a$, $b$, $c$ exist such that $c \in \text{alph}(v), (a,b) \in I, (a,c) \in D, \text{and} (b,c) \in D$. Let $n$ be any integer larger than $2|v|_a + 2|v|_b$, and consider the pair of reductions $a^n b^n \to a^n v b^n$ and $a^n b^n \equiv b^n a^n \to b^n v a^n$. Since the system is strongly confluent and application of the rule increases length in increments of $|v|$, either $a^n v b^n \equiv b^n v a^n$, or the traces $a^n v b^n$ and $b^n v a^n$ have a common one-step descendant. The first case leads immediately to a contradiction: the projection of $v$ on $\{a,c\}$ cannot commute with $a^n$.

In the second case consider first *any* one-step descendant $t$ of $a^n v b^n$. From the definition of reduction, traces $x,y$ exist such that $a^n v b^n \equiv xy$ and $t \equiv xvy$. Applying the Division Property to the equation $a^n v b^n \equiv xy$, we find that, for some $i,j, x \equiv a^{n-i} v_1 b^j, y \equiv a^i v_2 b^{n-j}$, and $v \equiv v_1 v_2$, with $v_1 \perp a^i$ and $v_2 \perp b^j$. Since $|v|_c > 0$, either $|v_1|_c > 0$ or $|v_2|_c > 0$. If $|v_1|_c > 0$, then $i = 0$ and $t \equiv xvy \equiv a^n v_1 b^j v v_2 b^{n-j}$, so $a^n$ is a prefix of $t$. Also, since $v_1$ and $v$ contain occurrences of the letter $c$, the longest prefix of $t$ in $b^*$ has length at most $|v_1|_b < n$, and the longest suffix of $t$ in $a^*$ has length at most $|vv_2|_a < n$. If $|v_2|_c > 0$, then $j = 0$ and $t \equiv a^{n-i} v_1 v a^i v_2 b^n$, so $b^n$ is a suffix of $t$, the longest prefix of $t$ in $b^*$ has length at most $|v_1 v|_b < n$, and the longest suffix of $t$ in $a^*$ has length at most $|v_2|_a < n$. Summarizing these remarks:

(1) Either $a^n$ is a prefix of $t$ or $b^n$ is a suffix of $t$.
(2) $b^n$ is not a prefix of $t$ and $a^n$ is not a suffix of $t$.

Interchanging the roles of $a$ and $b$, if $s$ is a one-step descendant of $b^n v a^n$, then:

(1*) Either $b^n$ is a prefix of $s$ or $a^n$ is a suffix of $s$.

Since conditions (2) and (1*) cannot hold simultaneously, $a^n v b^n$ and $b^n v a^n$ can have no common one-step descendant. □

The technique used in the proof of Proposition 6.1 only applies when an *a priori* bound is known for the length of reduction sequences to join pairs with a common one-step ancestor; for strongly confluent systems, the bound is 1. It is not difficult to show that if independent letters $a$, $b$ belong to the same component of $\text{alph}(abv)$ but neither belongs to $\text{alph}(v)$, then the system $\{(e, v)\}$ cannot even be confluent: for a sufficiently large integer $m$, every descendant of $av^m b$ must have the $a$ before the $b$ and every descendant of $bv^m a$ must have the $b$ before the $a$.

## 7. Remaining Cases

The results in the previous sections leave open some questions concerning confluence of a one-rule rewriting system $R = \{(u, v)\}$ on a trace monoid $M(A, I)$.

When the left-hand side $u$ is empty, we have characterized strong confluence but not confluence, and, as Otto has shown, there are confluent systems of this type that are not strongly confluent.

When $u$ is nonempty and connected, we have shown that confluence and strong confluence of $R$ are equivalent and characterized the confluence property based on the structure of $u$ and $v$, *except* when $u$ is a factor of $v$ and some letter independent of $u$ fails to commute with $v$. In particular, the characterization applies to all terminating systems of this type. Within the exception to this characterization, both confluent and non-confluent systems are possible.

**Example 7.1.** Let $A = \{a, b, c\}$.

(i) For $I = \{ab, ac\}$, the rules $(a, ab)$ and $(a, abc)$ give rise to strongly confluent systems. (Clearly, $a$ is a factor of both $ab$ and $abc$; also, $I(a) = \{b, c\}, COM(ab) = \{a, b\}$, and $COM(abc) = \{a\}$.)

(ii) For $I = \{ac\}$, the rule $(a, abc)$ gives rise to a nonconfluent system. (For this independence relation, $I(a) = \{c\}$ and $COM(abc) = \varnothing$.)

When the left-hand side $u$ of the rule $(u, v)$ is not connected, we have shown strong confluence to be equivalent to confluence and obtained a structural characterization of confluence, *except* when every component of $u$ is a factor of $v$. Again, the exception permits both confluent and nonconfluent systems.

**Example 7.2.** Let $A = \{a, b, c\}$.

(i) For $I = \{ab, ac\}$, the rule $(ab, abc)$ gives rise to a strongly confluent system.

(ii) For $I = \{ac\}$, the rules $(ac, abc)$ and $(ac, acb)$ give rise to nonconfluent systems.

In Example 7.2(i), the left-hand side of the rule is a factor of the right-hand side; we have no example of a confluent system of this type in which every component of $u$, but not $u$ itself, is a factor of $v$. Note also that in Example 7.2(ii), the first rule determines a terminating system, and the second rule determines a non-terminating one.

The remaining specific questions about confluence are thus in two groups. First, under what circumstances can a one-rule trace-rewriting system be confluent but not strongly confluent? Second, when is a one-rule trace-rewriting system $(u, v)$ in which every component of $u$ (or $u$ itself) is a factor of $v$ confluent or strongly confluent? More generally, while the characterizations presented here give rise to linear-time algorithms for deciding confluence when they apply, the question of whether confluence is a decidable property for one-rule trace-rewriting systems remains open.

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