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Application of the Krein's Method for Determination of Natural Frequencies of Periodically Supported Beam Based on Simplified Bresse-Timoshenko Equations

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With 12 Figures

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Summary

Free vibration of a periodically supported (multispan) beam *in via a* simplified Bresse-Timoshenko theory is studied by the Krein's method suggested in 1933 for the Bernoulli-Euler beams. Approximate differential equations are utilized with both shear deformations and rotary inertia included, but with the term representing the joint action of these effect omitted. Detailed analytical and numerical analysis are performed for the natural frequencies of beams with different boundary conditions at their ends. Following Krein, the continuity requirements at the intermediate supports are treated as equations in finite differences and solved exactly.

As in the classical Bernoulli-Euler beam, the natural frequencies fall into periodically spaced bands, with each band containing a number of frequencies equal to that of spans. The shear deformations and rotary inertia shift the classical frequency bands to the left, this effect being more pronounced for higher bands.

Extensive numerical results are reported for three-, five- and ten-span beams. Comparison with previously reported results (obtained by straightforwarded analysis) for a three-span beam shows excellent agreement.

Introduction

A periodic structure is one consisting of identical constituent elements (basic substructures) interconnected in an identical manner. Many physical systems are characterized by spacewise periodicity, the simplest example being the single crystal in which identically arranged atoms form an infinite or semi-infinite lattice. Many engineering systems are also intentionally based on this principle in order to reduce cost and/or save time, a typical example being the part of an aircraft fuselage consisting of identical bays joined by identical circumferential frames. Missiles, ships, monorail lines, pontoon bridges and the recently proposed floating airfields, may likewise be regarded as periodic structures. Many mathematical methods have been developed for treatment of periodic structures. Comprehensive reviews of periodic beams, plates and shells were written by Lin and Donaldson [1], McDaniel and Henderson [2], Clarkson and Mead [3] and Sen Gupta [4]. A complete bibliography is given in the above references and we mention here only the papers pertinent to our theme of interest — vibration of periodic beams. One of the first analyses was given in 1928 by Timoshenko [5] who wrote down Darnley's [6] finite-difference equations with the serial number of the beam as subscript. He investigated numerically a beam on three supports and gave a graphical solution for the case of unequal span lengths. For that equal span lengths (the two span periodic beam) he concluded that its spectrum includes that of basic substructure which is simply supported at both ends. He failed, however, to formulate a solution for the general period N-span beam (N being any positive integer). This shortage was remedied in 1933 by Krein [7], who considered such a beam under different end conditions, again using Darnley's three moment equations. Krein established that the natural frequencies fall into periodically-spaced bands (with the number of natural frequencies within each band equal to that of spans), and derived an interesting corollary — that periodically supported beams with simply supported ends, as well as, those with clamped ends, have N-1 common series of natural frequencies.

As often happened in the history of science, this pioneering paper reached only a limited audience (probably because of the relatively obscure periodical it appeared in), and his results were rediscovered by Ayre and Jacobsen [8] and by Miles [9], who presented graphical and analytical solutions, respectively, for a multispan beam with simply supported ends. In the first named work a nomograph of natural frequencies versus number of spans was presented for one-, two-, three-, four- and six-span beams with different boundary conditions; it turned out that all natural frequencies of one-, two- and three-span beams are contained in the spectrum of a six-span one. In the second work a finite-difference technique yielded for a N-span beam with simply-supported ends a frequency equation dentical with that of Krein [7]. Lin [10] generalized both Krein's and Miles' approaches for a multispan beam with intermediate elastic supports, characterized by a displacement spring and torsion spring, and also derived a symmetrical fourth-order difference equation, which reduces to the second-order one of Krein and Miles when one of the spring constants tends to infinity.

More recent developments include the central work of Mead [11] who introduced the concept of complex propagation constants in the context of infinite beam vibration problem. Mead's approach was adopted by Sen Gupta in his comprehensive study of periodically subported beams and plates on rigid [12] and flexible [13] supports. In the first-named work Sen Gupta also presented a convenient graphical method for determination of natural frequencies.

All aforementioned studies are concerned with the classical Bernoulli-Euler beams. However, as is well known even for single-span beams, the influence of shear deformations and rotary inertia may be paramount for high natural frequencies and a refined theory is called for. Although there is ample literature on the vibration of single-span beams with both of these effects taken into account, only two publications were devoted to multi-span beams: the paper by Wang [14] and the comment on it by Cowper [15]. Wang derived the three-moment equation within the framework of the refined theory and presented numerical calculations for three-span beams, and Cowper drew attention to his error in requiring continuity of the first spatial derivative of the deflection across an intermediate, instead of continuity of the function representing rotation of the cross-section. Cowper accordingly put forward the proper three-moment equation and Wang subsequently published appropriately corrected numerical results [16].

In the present study, an attempt is made to complete Wang's work by presenting a general solution which would yield the natural frequencies for beams with an arbitrary number of spans.

Formulation of Problem — Basic Equations

The classical Bernoulli-Euler equation for vibration of uniform beam reads:

$$EI\,\frac{\partial^4 y}{\partial x^4} + \varrho A\,\frac{\partial^2 y}{\partial t^2} = 0, \tag{1}$$

where E is the modulus of elasticity, I the moment of inertia of the cross-section, A the cross-sectional area, ϱ mass density of the beam material, y the deflection, x the spatial coordinate along the beam axis and t — time.

In 1877, Lord Rayleigh [17] refined Eq. (1) by taking into account the rotatory movement of the beam elements in addition to translatory ones. This resulted in the following equation:

$$EI\,\frac{\partial^4 y}{\partial x^4} + \varrho A\,\frac{\partial^2 y}{\partial t^2} + \varrho I\,\frac{\partial^4 y}{\partial x^2\,\partial t^2} = 0. \tag{2}$$

Timoshenko [18] in 1921 further refined the theory by taking into account the shear deformation of the beam, and derived the following set of coupled differential equations in terms of the beam displacement y and rotation ψ of the cross-section:

$$EI\,\frac{\partial^2 \psi}{\partial x^2} + kAG\left(\frac{\partial y}{\partial x} - \psi\right) - \varrho I\,\frac{\partial^2 \psi}{\partial t^2} = 0 \tag{3}$$

$$\varrho A\,\frac{\partial^2 y}{\partial t^2} - kAG\left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial \psi}{\partial x}\right) = 0, \tag{4}$$

where k is the shear coefficient and G the shear modulus. Equations (3) and (4) are usually referred to as the Timoshenko beam equations. Later on it turned out

[19] that an analogous method of taking into account rotary inertia and shear deformations was known earlier to Bresse [20]. Accordingly throughout the present study Eqs. (3) and (4) are referred to as the Bresse-Timoshenko equations. They can be put in decoupled form as follows:

$$EI\,\frac{\partial^4 y}{\partial x^4} + m\,\frac{\partial^2 y}{\partial t^2} - mr^2\left(1 + \frac{E}{kG}\right)\frac{\partial^4 y}{\partial x^2\,\partial t^2} + \frac{m^2r^2}{kAG}\,\frac{\partial^4 y}{\partial t^4} = 0 \tag{5}$$

$$EI\,\frac{\partial^4 \psi}{\partial x^4} + m\,\frac{\partial^2 \psi}{\partial t^2} - mr^2\left(1 + \frac{E}{kG}\right)\frac{\partial^4 \psi}{\partial x^2\,\partial t^2} + \frac{m^2r^2}{kAG}\,\frac{\partial^4 \psi}{\partial t^4} = 0, \tag{6}$$

where $r = (I/A)^{1/2}$ is the radius of gyration, and $m = \varrho A$. It should be noted that the coupling is realized via the boundary conditions. In fact, we will utilize Eq. (5) only, since once $y(x, t)$ has been found, $\psi(x, t)$ is determined through Eq. (3).

It is instructive to consider first the free vibration of a uniform Bresse-Timoshenko beam simply supported at its both ends. The boundary conditions are:

$$y(x, t) = 0, \quad \text{at} \quad x = 0 \quad \text{and} \quad x = a \tag{7}$$

$$\frac{\partial \psi}{\partial x}\,(x, t) = 0, \quad \text{at} \quad x = 0 \quad \text{and} \quad x = a, \tag{8}$$

where a denotes the span of the beam. Boundary conditions (7) and (8) are satisfied by setting:

$$y(x, t) = Be^{i\omega t}\sin\frac{n\pi x}{a}, \qquad \psi(x, t) = Ce^{i\omega t}\cos\frac{n\pi x}{a}. \tag{9}$$

Substituting Eqs. (9) in Eq. (5) we obtain the frequency equation:

$$EI\left(\frac{n\pi}{a}\right)^4 - m\omega^2 - mr^2\left(1 + \frac{E}{kG}\right)\left(\frac{n\pi}{a}\right)^2\omega^2 + \frac{m^2r^2}{kAG}\,\omega^4 = 0. \tag{10}$$

This in turn yields

$$\begin{aligned}\omega_{1,2}^2 = \frac{1}{2m^2r^2/kAG}\Bigg\{& m + mr^2\left(1 + \frac{E}{kG}\right)\left(\frac{n\pi}{a}\right)^2 \\ &\mp \sqrt{\left[m + mr^2\left(1 + \frac{E}{kG}\right)\left(\frac{n\pi}{a}\right)^2\right]^2 - 4EI\left(\frac{n\pi}{a}\right)^4\frac{m^2r^2}{kAG}}\Bigg\}\end{aligned} \tag{11}$$

which indicates two bands of frequencies, denoted by the subscripts 1 (lower) and 2 (higher), respectively. Their existence has been demonstrated experimentally by Barr [21]. Asymptotic representation of the discriminant yields:

$$\omega_1^2 \simeq \frac{EI(n\pi/a)^4}{m + mr^2(1 + E/kG)\,(n\pi/a)^2} \tag{12}$$

$$\omega_2^2 \simeq \frac{m + mr^2(1 + E/kG)\,(n\pi/a)^2}{m^2r^2/kAG} + \frac{EI(n\pi/a)^4}{m + mr^2(1 + E/kG)\,(n\pi/a)^2}. \tag{13}$$

Examination of the ratio ω_2^2/ω_1^2 will enable us to compare the order of magnitude of the two bands:

$$\frac{\omega_2^2}{\omega_1^2} = 1 + \frac{[1 + (1 + E/kG)\,(n\pi r/a)^2]^2}{(E/kG)\,(n\pi r/a)^4}. \tag{14}$$

For a beam of rectangular cross-section, made from conventional materials with $k = 5/6$, $v = 0.3$, we have

$$\frac{E}{kG} = 3.12 \tag{15}$$

whence

$$\frac{\omega_2^2}{\omega_1^2} = 1 + \frac{\left[1 + 41\left(\frac{r}{a}\right)^2\right]^2}{304(r/a)^4} \quad \text{for} \quad n = 1 \tag{16}$$

so that

$$\begin{aligned} \frac{\omega_2}{\omega_1} &\simeq 576 \quad \text{for} \quad \frac{r}{a} = 0.01 \\ \frac{\omega_2}{\omega_1} &\simeq 25 \quad \text{for} \quad \frac{r}{a} = 0.05 \\ \frac{\omega_2}{\omega_1} &\simeq 8 \quad \text{for} \quad \frac{r}{a} = 0.1. \end{aligned} \tag{17}$$

It should be noted that Eq. (12) is obtainable directly from Eq. (10) by omitting the last term in it. Timoshenko [18] was the first to suggest this omission in the *characteristic equation* due to its negligible contribution, and in Refs. [22] and [23] this suggestion was applied to the *differential equation* itself (Eq. (5)). We neglect the last terms in Eqs. (5) and (6). These simplified equations:

$$EI\,\frac{\partial^4 y}{\partial x^4} + m\,\frac{\partial^2 y}{\partial t^2} - mr^2\left(1 + \frac{E}{kG}\right)\frac{\partial^4 y}{\partial x^2\,\partial t^2} = 0 \tag{18}$$

$$EI\,\frac{\partial^4 \psi}{\partial x^4} + m\,\frac{\partial^2 \psi}{\partial t^2} - mr^2\left(1 + \frac{E}{kG}\right)\frac{\partial^4 \psi}{\partial x^2\,\partial t^2} = 0 \tag{19}$$

directly lead on the one hand, to Timoshenko's [18] approximation of the natural frequencies, and on the other to the asymptotic representation (12) of an "exact" natural frequency (i.e. with the last terms in Eqs. (5) and (6) retained). Equation (18) in its simplified form was utilized by Elishakoff and Livshits [22] and by Elishakoff and Lubliner [23] in deriving closed-form solutions for vibration of beams excited by wide-band random loading.

In the present study we apply Eqs. (18) and (19) in their simplified form in investigating free vibration of a multispan beam with a view to gain an insight into the influence of shear deformations and rotary inertia on the natural frequencies.

Vibration Analysis of Multispan Beam Simply-Supported of Both Ends

Free harmonic oscillation with angular frequency ω is represented by

$$y(x, t) = Y(x)\, e^{i\omega t} \tag{20}$$

$$\psi(x, t) = \Psi(x)\, e^{i\omega t} \tag{21}$$

and substituting the above in Eq. (18) and (19) we have

$$\frac{d^4 Y}{dx^4} + p^4(r^2 + b^2)\frac{d^2 Y}{dx^2} - p^4 Y = 0 \tag{22}$$

$$\frac{d^4 \Psi}{dx^4} + p^4(r^2 + b^2)\frac{d^2 \Psi}{dx^2} - p^4 \Psi = 0, \tag{23}$$

where

$$p^4 = \frac{m\omega^2}{EI}, \qquad b^2 = \frac{EI}{kAG}. \tag{24}$$

The general solutions of these equations read

$$Y(x) = \bar{B}_1 \cosh s_1 x + \bar{B}_2 \sinh s_1 x + \bar{B}_3 \cos s_2 x + \bar{B}_4 \sin s_2 x \tag{25}$$

$$\Psi(x) = \bar{C}_1 \cosh s_1 x + \bar{C}_2 \sinh s_1 x + \bar{C}_3 \cos s_2 x + \bar{C}_4 \sin s_2 x \tag{26}$$

with

$$s_1 = \left\{-\frac{p^4(r^2 + b^2)}{2} + \frac{1}{2}\left[p^8(r^2 + b^2)^2 + 4p^4\right]^{1/2}\right\}^{1/2} \tag{27}$$

$$s_2 = \left\{+\frac{p^4(r^2 + b^2)}{2} + \frac{1}{2}\left[p^8(r^2 + b^2)^2 + 4p^4\right]^{1/2}\right\}^{1/2}. \tag{28}$$

The relations between the coefficients in Eqs. (25) and (26) are obtainable by substituting the latter two in the governing set, (3) and (4) bearing in mind Eqs. (20) and (21). The resulting relationships are as follows:

$$\bar{C}_1 = e_1 \bar{B}_2, \qquad \bar{C}_2 = e_1 \bar{B}_1 \tag{29}$$

$$\bar{C}_3 = e_2 \bar{B}_4, \qquad \bar{C}_4 = e_2 \bar{B}_3 \tag{30}$$

$$e_1 = \frac{s_1^2 + p^4 b^2}{s_1}, \qquad e_2 = \frac{s_2^2 - p^4 b^2}{s_2}. \tag{31}$$

Equations (25) and (26) must satisfy the continuity requirements

$$Y|_{x_k=0} = Y|_{x_{k+1}=0} = 0 \tag{32}$$

$$\Psi|_{x_k=0} = \Psi|_{x_{k+1}=0} \tag{33}$$

$$\left.\frac{d\Psi}{dx}\right|_{x_k=0} = \left.\frac{d\Psi}{dx}\right|_{x_{k+1}=0} \tag{34}$$

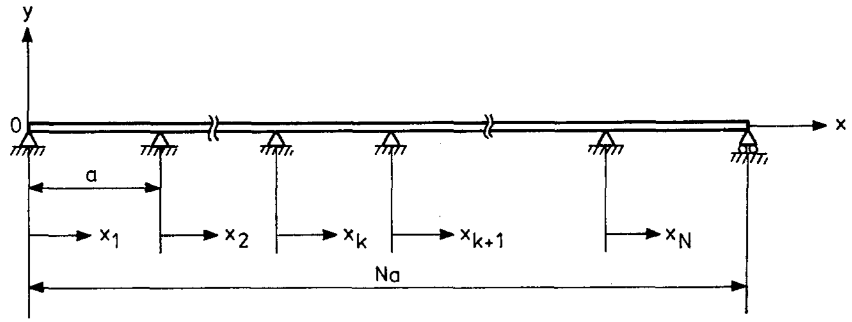

Fig. 1

with regard to the deflections (which are zero), slope and bending moment. (For a discussion of boundary conditions appropriate to Bresse-Timoshenko beams the reader may consult Refs. [15] and [16].)

In Eq. (32) a denotes the span length in the x-direction so that the total length of the beam is Na, N denoting the number of spans (see Fig. 1), and

$$x_\alpha = x - (\alpha - 1)\, a \tag{35}$$

is the coordinate in the local frame of reference associated with span number α.

We now rewrite the general solution given by Eqs. (25) and (26) for each span

$$Y(x_\alpha) = B_1^{(\alpha)} \cosh s_1 x_\alpha + B_2^{(\alpha)} \sinh s_1 x_\alpha + B_3^{(\alpha)} \cos s_2 x_\alpha + B_4^{(\alpha)} \sin s_2 x_\alpha \tag{36}$$

$$\Psi(x_\alpha) = e_1 B_2^{(\alpha)} \cosh s_1 x_\alpha + e_1 B_1^{(\alpha)} \sinh s_1 x_\alpha + e_2 B_4^{(\alpha)} \cos s_2 x_\alpha - e_2 B_3^{(\alpha)} \sin s_2 x_\alpha . \tag{37}$$

Applying the continuity conditions to Eqs. (36) and (37) we obtain:

$$B_1^{(\alpha)} \cosh s_1 a + B_2^{(\alpha)} \sinh s_1 a + B_3^{(\alpha)} \cos s_2 a + B_4^{(\alpha)} \sin s_2 a = 0 \tag{38}$$

$$B_1^{(\alpha+1)} + B_3^{(\alpha+1)} = 0 \tag{39}$$

$$\begin{aligned} e_1 B_2^{(\alpha)} \cosh s_1 a + e_1 B_1^{(\alpha)} \sinh s_1 a + e_2 B_4^{(\alpha)} \cos s_2 a \\ - e_2 B_3^{(\alpha)} \sin s_2 a = e_1 B_2^{(\alpha+1)} + e_2 B_4^{(\alpha+1)} \end{aligned} \tag{40}$$

$$\begin{aligned} e_1 s_1 B_2^{(\alpha)} \sinh s_1 a + e_1 s_1 B_1^{(\alpha)} \cosh s_1 a - e_2 s_2 B_4^{(\alpha)} \sin s_2 a \\ - e_2 s_2 B_3^{(\alpha)} \cos s_2 a = e_1 s_1 B_1^{(\alpha+1)} - e_2 s_2 B_3^{(\alpha+1)} . \end{aligned} \tag{41}$$

Equations (38)—(41) are equations in finite differences with respect to $B_j^{(\alpha)}$. Therefore, the solution for $B_j^{(\alpha)}$ is sought as follows

$$B_j^{(\alpha)} = B_j \lambda^{\alpha-1} . \tag{42}$$

For the particular case $\alpha = 1$ we obtain $B_j^{(1)} = B_j$ and accordingly the integration constants in each span are related to those on the first span as

$$B_j^{(\alpha)} = B_j^{(1)} \lambda^{\alpha-1} . \tag{43}$$

Substituting (43) in (38)—(41), and omitting the common term containing powers of λ, we arrive at

$$B_1^{(1)}C_1 + B_2^{(1)}D_1 + B_3^{(1)}C_2 + B_4^{(1)}D_2 = 0 \tag{44}$$

$$B_1^{(1)} + B_3^{(1)} = 0 \tag{45}$$

$$B_1^{(1)}D_1e_1 + B_2^{(1)}e_1(C_1 - \lambda) - B_3^{(1)}D_2e_2 + B_4^{(1)}e_2(C_2 - \lambda) = 0 \tag{46}$$

$$B_1^{(1)}e_1s_1(C_1 - \lambda) + B_2^{(1)}e_1s_1D_1 - B_3^{(1)}e_2s_2(C_2 - \lambda) - B_4^{(1)}e_2s_2D_2 = 0, \tag{47}$$

where

$$\begin{aligned} C_1 &= \cosh s_1a \\ D_1 &= \sinh s_1a \\ C_2 &= \cos s_2a \\ D_2 &= \sin s_2a. \end{aligned} \tag{48}$$

The nontriviality requirement of the set, (44)—(47) stipulates:

$$\begin{vmatrix} C_1 & D_1 & C_2 & D_2 \\ 1 & 0 & 1 & 0 \\ D_1e_1 & (C_1 - \lambda)\, e_1 & -D_2e_2 & (C_2 - \lambda)\, e_2 \\ s_1(C_1 - \lambda)\, e_1 & s_1D_1e_1 & -s_2(C_2 - \lambda)\, e_2 & -s_2D_2e_2 \end{vmatrix} = 0 \tag{49}$$

or, simply

$$\lambda^2 - 2U\lambda + 1 = 0 \tag{50}$$

with

$$U = \frac{C_2D_1e_2 - C_1D_2e_1}{D_1e_2 - D_2e_1} \tag{51}$$

whence

$$\lambda_1 = U + [U^2 - 1]^{1/2}, \qquad \lambda_2 = [U - U^2 - 1]^{1/2}. \tag{52}$$

Bearing in mind Eq. (52), the general solutions of Eqs. (36) and (37) can be written as ($B_4^{(1)} \equiv B_4$):

$$\begin{aligned} Y(x) = {} & B_{4,1}\lambda_1^{\alpha-1}[F_{1,1} \cosh s_1x + F_{2,1} \sinh s_1x + F_{3,1} \cos s_2x + \sin s_2x] \\ & + B_{4,2}\lambda_2^{\alpha-1}[F_{1,2} \cosh s_1x + F_{2,2} \sinh s_1x + F_{3,2} \cos s_2x + \sin s_2x] \end{aligned} \tag{53}$$

$$\begin{aligned} \Psi(x) = {} & B_{4,1}\lambda_1^{\alpha-1}[F_{1,1}e_1 \sinh s_1x + F_{2,1}e_1 \cosh s_1x - F_{3,1}e_2 \sin s_2x + e_2 \cos s_2x] \\ & + B_{4,2}\lambda_2^{\alpha-1}[F_{1,2}e_1 \sinh s_1x + F_{2,2}e_1 \cosh s_1x - F_{3,2}e_2 \sin s_2x + e_2 \cos s_2x] \end{aligned} \tag{54}$$

where

$$F_{\bar{j},k} \equiv B_{\bar{j},k}/B_{4,k} \qquad (\bar{j} = 1, 2, 3, \quad k = 1, 2)$$
$$F_{1,k} = -F_{3,k} = \frac{D_2}{(C_2 - \lambda_k)}; \qquad F_{2,k} = -\frac{D_2(C_1 - \lambda_k)}{D_1(C_2 - \lambda_k)} \tag{55}$$

are the modal ratios.

Note that Eqs. (53) and (54) already satisfy all continuity conditions and part of the boundary conditions, namely $Y = 0$ at both $x_1 = 0$ and $x_N = a$. In these circumstances, $B_{4,1}$ and $B_{4,2}$ must be such that the remaining boundary conditions $d\Psi/dx = 0$ at both $x_1 = 0$ and $x_N = a$ are satisfied:

$$B_{4,1}(F_{1,1}e_1s_1 - F_{3,1}e_2s_2) + B_{4,2}(F_{1,2}e_1s_1 - F_{3,2}e_2s_2) = 0 \tag{56}$$

$$B_{4,1}\lambda_1^{N-1}(F_{1,1}e_1s_1C_1 + F_{2,1}e_1s_1D_1 - F_{3,1}e_2s_2C_2 - e_2s_2D_2)$$
$$+ B_{4,2}\lambda_2^{N-1}(F_{2,1}e_1s_1C_1 + F_{2,2}e_1s_1D_1 - F_{3,2}e_2s_2C_2 - e_2s_2D_s) = 0. \tag{57}$$

By the nontriviality requirement

$$\frac{D_2^2(s_1^2 + s_2^2)^2\,(\lambda_2^N - \lambda_1^N)}{(C_2 - \lambda_1)\,(C_2 - \lambda_2)} = 0. \tag{58}$$

This equation has two subclasses of solutions, the first of which is

$$D_2 \equiv \sin s_2a = 0. \tag{59}$$

This implies

$$s_2 = \frac{m\pi}{a}, \qquad m = 1, 2, 3, \ldots \tag{60}$$

where m is a positive integer representing the number of half waves in the single-span mode shape.

This series, which is invariant in N, will hereinafter be referred to as "trivial". The second subclass is obtained by zeroing the expression:

$$\lambda_2^N - \lambda_1^N = 0. \tag{61}$$

Substituting

$$U = \cos\theta. \tag{62}$$

Equations (52) become

$$\lambda_1 = \exp(i\theta), \qquad \lambda_2 = \exp(-i\theta) \tag{63}$$

and Eq. (61)

$$\sin N\theta = 0. \tag{64}$$

From Eq. (64), it follows that

$$N\theta = j\pi \tag{65}$$

$j(\gtrless N)$ being any integer indivisible by N. Accordingly U has $N-1$ different values

$$U_j = \cos\frac{j\pi}{N}, \qquad j = 1, 2, \ldots, N-1. \tag{66}$$

Consequently, in addition to the series represented by Eq. (60) this beam has $N-1$ other series of natural frequencies, their total number being equal to that of spans. In other words the spectrum of a N-span beam contains the spectrum of the single-span one.

For insight into the vibration spectrum, we first consider the simplest multi-span case, a two-span beam. Considering the admissible range of j in this case $j \gtrless 2$, we are left with

$$U_1 = \cos\frac{\pi}{2} = 0. \tag{67}$$

Combining Eqs. (51) with (67) we obtain

$$C_1 D_1 e_2 = C_1 D_2 e_1. \tag{68}$$

We shall now show that this transcendental equation is identical to that associated with a single-span beam clamped at one end and simply-supported at the other. Indeed, for the latter, the boundary conditions are:

$$\begin{aligned} Y &= 0, \quad \frac{d\Psi}{dx} = 0 \quad \text{at} \quad x = a \\ Y &= 0, \quad \Psi = 0 \quad \text{at} \quad x = 0 \end{aligned} \tag{69}$$

a being the span of the beam. Under these boundary conditions, we have for $Y(x)$ and $\Psi(x)$ given in Eqs. (36) and (37), with $\alpha \equiv 1$

$$\begin{gathered} B_1 + B_3 = 0 \\ e_1 B_2 + e_2 B_4 = 0 \\ B_1 C_1 + B_2 D_1 + B_3 C_2 + B_4 D_2 = 0 \\ e_1 s_1 B_2 D_2 + e_1 s_1 B_1 C_1 - e_2 s_2 B_4 D_2 - e_2 s_2 B_3 C_2 = 0 \end{gathered} \tag{70}$$

and the nontriviality requirement yields

$$C_2 D_1 e_2 = C_1 D_2 e_1$$

which is identical to Eq. (68) as claimed.

The analytical derivations for a multispan beam clamped at both ends and supported at one end and clamped at the other, are presented in Appendices A and B respectively.

Numerical Results

Equations (66) and (51) are now used (following Sen Gupta [12]) as a basis for an extremely simple graphical method for finding the natural frequencies of a multispan beam simply supported or clamped at both ends. We plot (arccos U) versus the frequency parameter (pa). As j is increased from 1 to N for the simply-supported case of 0 to $N - 1$ for the clamped case, in steps of consecutive integers, the permissible value of (arccos U) increases in equivalent steps of (π/N). Accordingly the ordinate of the (arccos U) vs. (pa) graph is divided into N equal intervals, and the sought frequency is directly given by the abscissae of the intercepts laid off on the graph by horizontal lines drawn through the points of division.

The method is illustrated in Figs. 2 and 3, in which a five-span beam, simply-supported at both ends (within the first five series) is treated by the Bernoulli-Euler and Bresse-Timoshenko approaches, respectively. It is seen that under the latter approach the natural frequencies undergo a relative shift towards the origin, which becomes more pronounced as the frequencies increase.

As an example for comparison with refined theories we consider a standard steel profile 16WF96 with $E = 21000\ \mathrm{kg/mm^2}$; $G = 8400\ \mathrm{kg/mm^2}$; $k = 0.279$. whence $E/kG = 9$. The radius of gyration being $r = \sqrt{I/A}$, Eq. (24) yields $b = 3r$.

Fig. 4 shows the ratio of refined versus classical natural frequencies (actually a correction factor) for a three-span beam simply-supported at both ends.

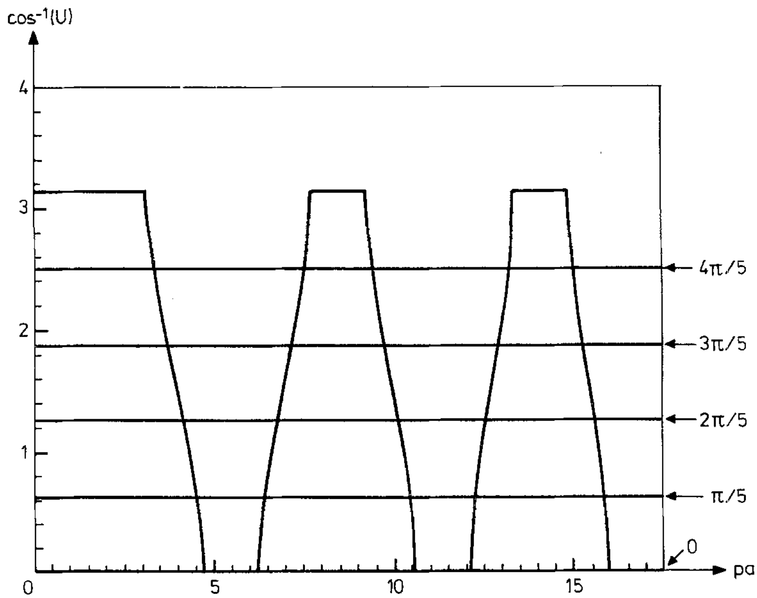

Fig. 2

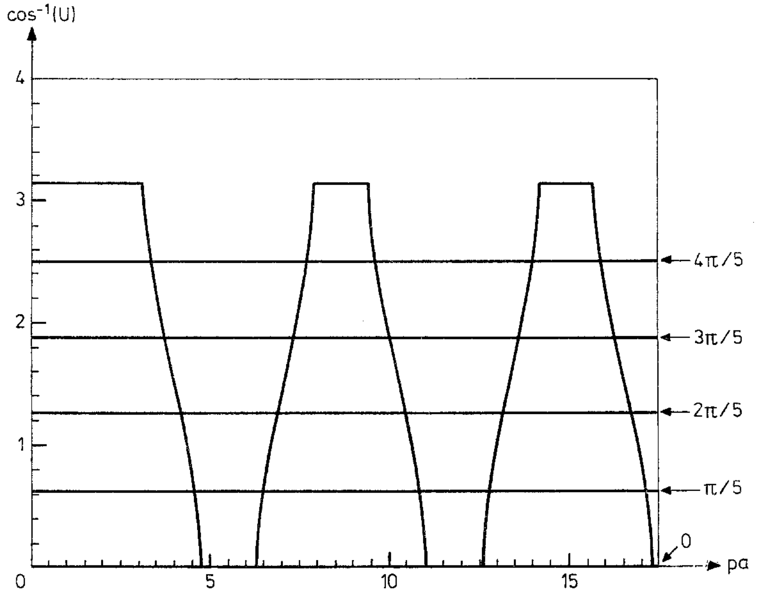
cos⁻¹(U)
4
3
2
1
0
5
10
15
4π/5
3π/5
2π/5
π/5
0
pa

Fig. 3

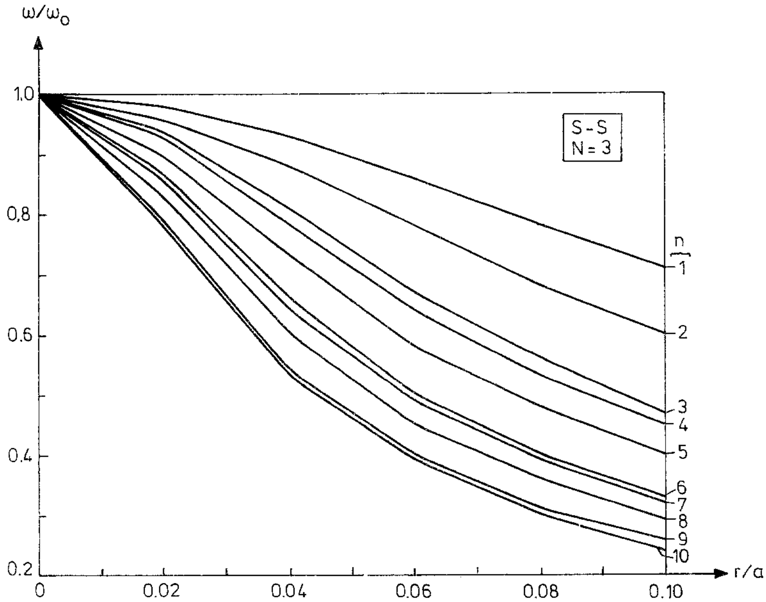
ω/ω₀
1.0
0.8
0.6
0.4
0.2
0
0.02
0.04
0.06
0.08
0.10
r/a
S-S
N = 3
n
1
2
3
4
5
6
7
8
9
10

Fig. 4

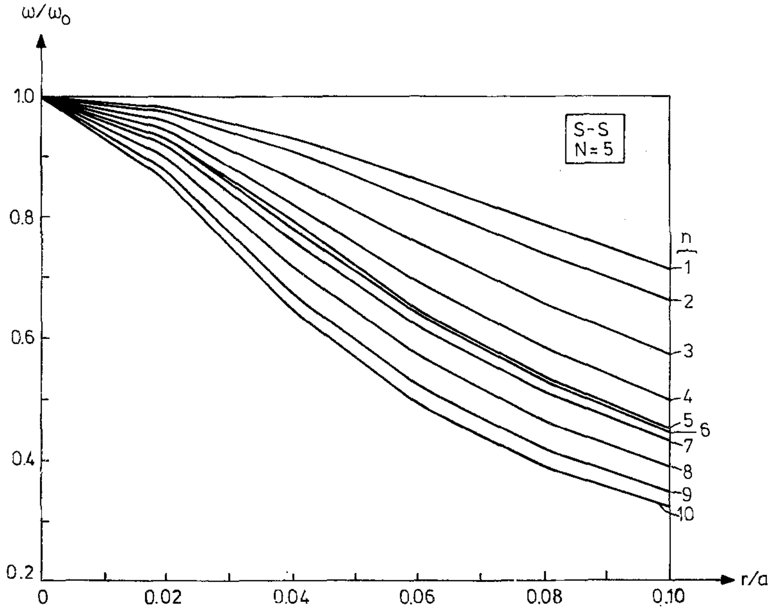
ω/ω0
1.0
0.8
0.6
0.4
0.2
0
0.02
0.04
0.06
0.08
0.10
r/a
S-S
N=5
n
1
2
3
4
5
6
7
8
9
10

Fig. 5

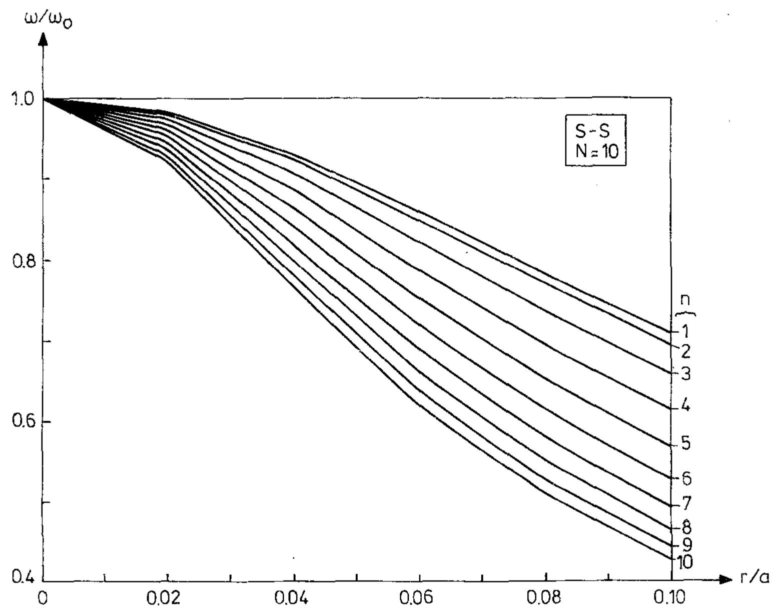
ω/ω0
1.0
0.8
0.6
0.4
0
0.02
0.04
0.06
0.08
0.10
r/a
S-S
N=10
n
1
2
3
4
5
6
7
8
9
10

Fig. 6

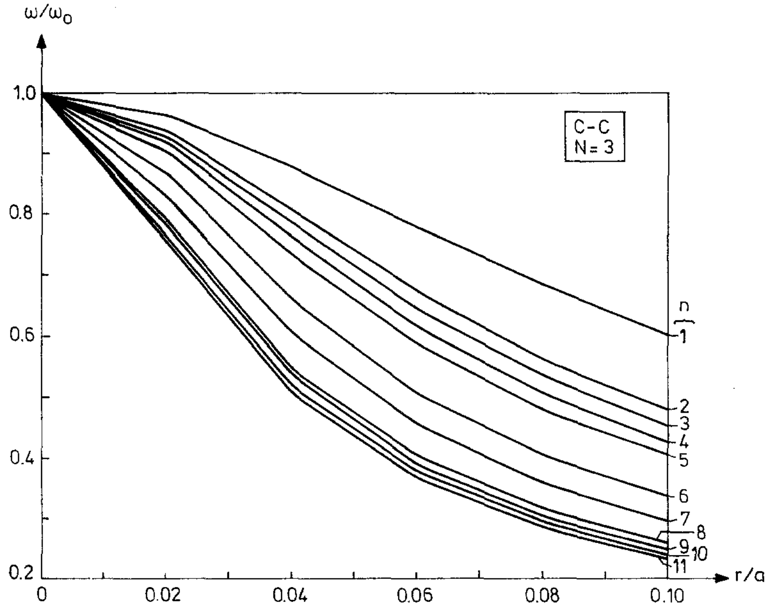
ω/ω0
1.0
0.8
0.6
0.4
0.2
0
0.02
0.04
0.06
0.08
0.10
r/a
C-C
N=3
n
1
2
3
4
5
6
7
8
9
10
11

Fig. 7

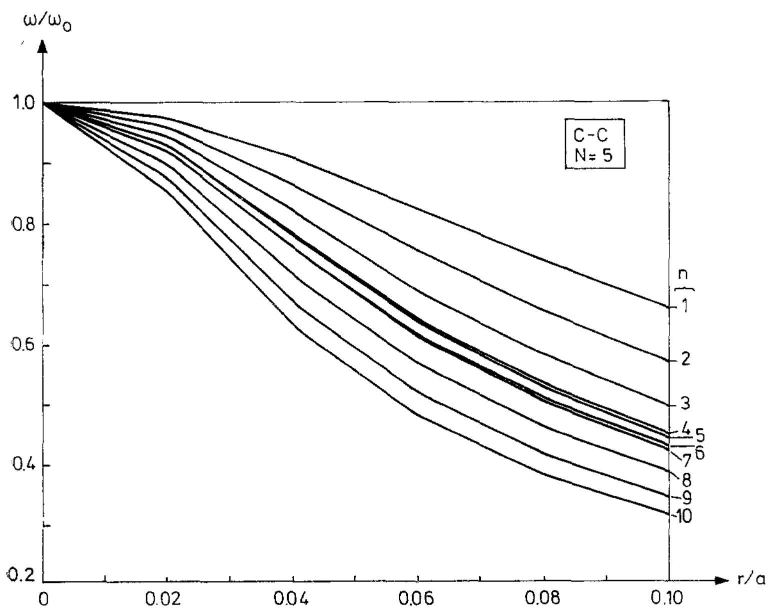
ω/ω0
1.0
0.8
0.6
0.4
0.2
0
0.02
0.04
0.06
0.08
0.10
r/a
C-C
N=5
n
1
2
3
4
5
6
7
8
9
10

Fig. 8

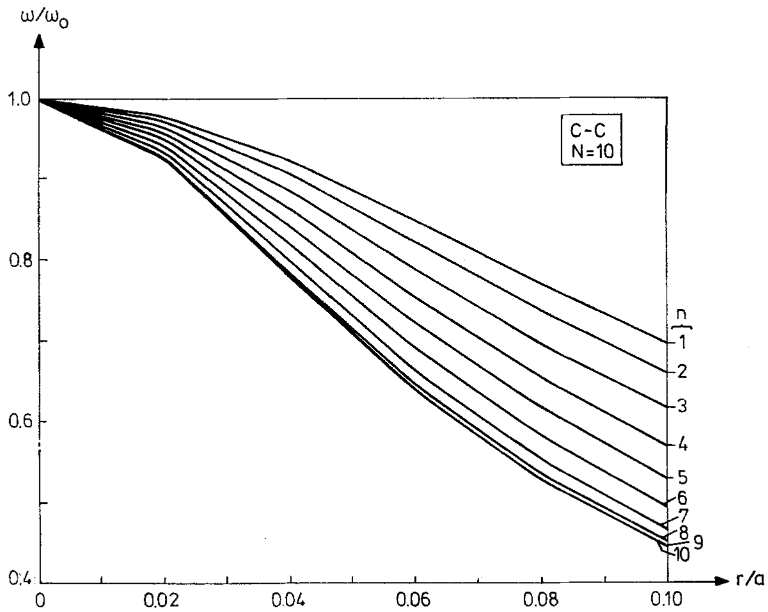
ω/ω0
1.0
0.8
0.6
0.4
0
0.02
0.04
0.06
0.08
0.10
r/a
C-C
N=10
n
1
2
3
4
5
6
7
8
9
10

Fig. 9

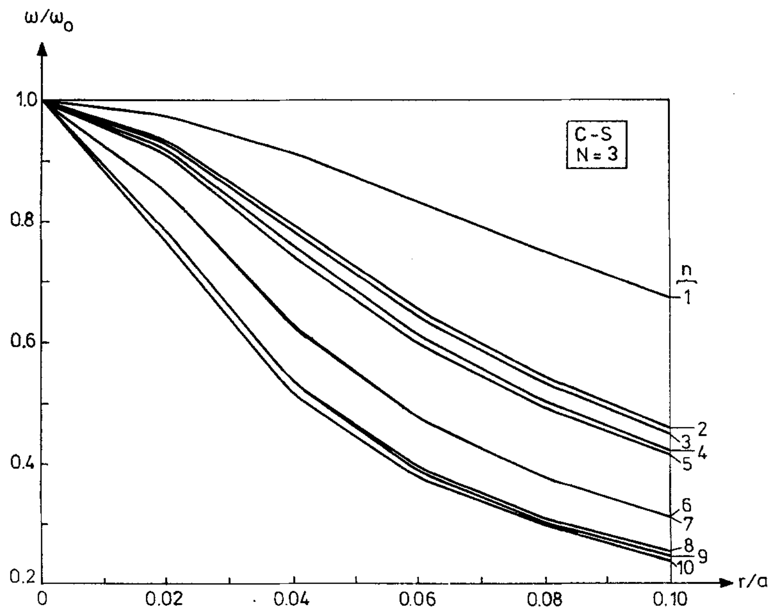
ω/ω0
1.0
0.8
0.6
0.4
0.2
0
0.02
0.04
0.06
0.08
0.10
r/a
C-S
N = 3
n
1
2
3
4
5
6
7
8
9
10

Fig. 10

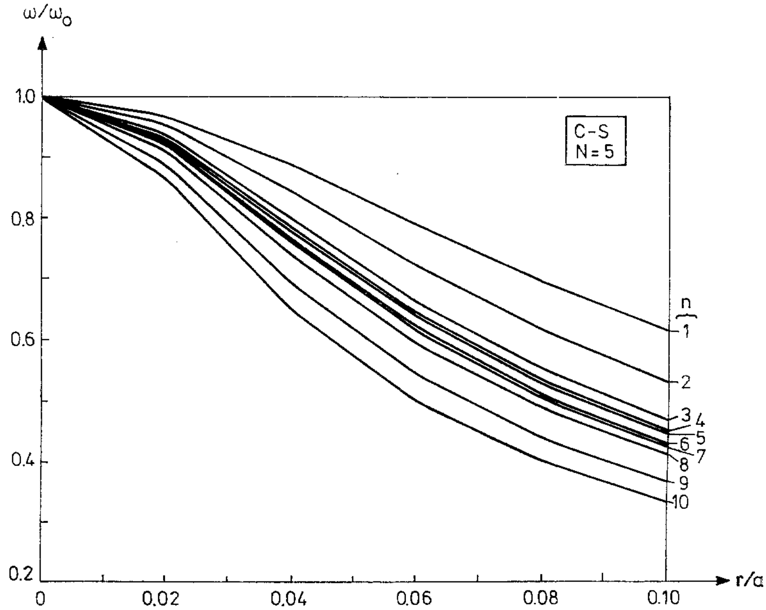
ω/ω0
1.0
0.8
0.6
0.4
0.2
0
0.02
0.04
0.06
0.08
0.10
r/a
C-S
N = 5
n
1
2
3
4
5
6
7
8
9
10

Fig. 11

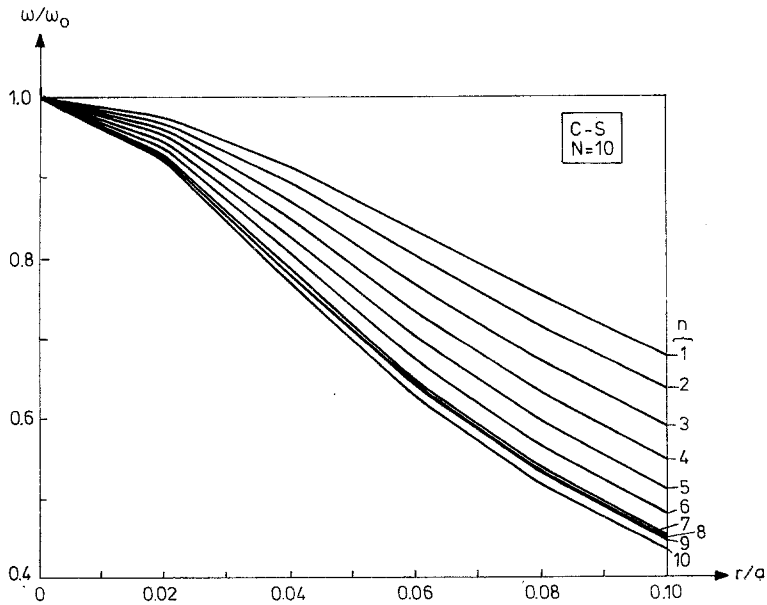
ω/ω0
1.0
0.8
0.6
0.4
0
0.02
0.04
0.06
0.08
0.10
r/a
C-S
N=10
n
1
2
3
4
5
6
7
8
9
10

Fig. 12

Comparison with Wang's results, roughly reproduced from Fig. 2 of Ref. [16], reveals extremely good agreement. For the sake of completeness, the same ratio for a five-span and a ten-span beam simply-supported at both ends is shown in Fig. 5 and 6, respectively.

Figs. 7—12 show the corresponding sets of the correction factor under boundary conditions at the beam ends: the first three — for clamped at both ends, the last three — for one end clamped and the other simply-supported.

As is seen, all the natural frequencies of the latter case are contained in the spectrum of $2N$-span beam simply-supported at both ends (see Appendix B).

Appendix A

Multispan Beam Clamped at Both Ends

In this case $B_{4,1}$ and $B_{4,2}$ in Eqs. (56) and (57) must be such as to satisfy boundary conditions:

$$\Psi = 0 \quad \text{at} \quad x_1 = 0, \qquad x_N = a. \tag{A 1}$$

This yields:

$$B_{4,1}(F_{2,1}e_1 + e_2) + B_{4,2}(F_{2,2}e_1 + e_2) = 0 \tag{A 2}$$

$$\begin{aligned} &B_{4,1}\lambda_1^{N-1}(F_{1,1}e_1D_1 + F_{2,1}e_1C_1 - F_{3,1}e_2D_2 + e_2C_2) \\ &+ B_{4,2}\lambda_2^{N-1}(F_{1,2}e_1D_1 + F_{2,2}e_1C_1 - F_{3,2}e_2D_2 + e_2C_2). \end{aligned} \tag{A 3}$$

The nontriviality requirement yields

$$\frac{(D_2e_1 - D_1e_2)^2\,(\lambda_1 - \lambda_2)}{4D_1^2(C_2 - \lambda_1)\,(C_2 - \lambda_2)}\,[(\lambda_2^2 - 1)\,\lambda_2^{N-1} + (\lambda_1^2 - 1)\,\lambda_1^{N-1}] = 0 \tag{A 4}$$

or

$$\lambda_1^{N-1} + \lambda_2^{N-1} = \lambda_1^{N+1} + \lambda_2^{N+1}. \tag{A 5}$$

Applying Eq. (63), we have

$$\sin(N\theta) = 0 \tag{A 6}$$

whence

$$N\theta = j\pi \tag{A 7}$$

j being any integer.

As for U, it obviously can now have only $N + 1$ different values, namely

$$\begin{aligned} &U_0 = 1, \qquad U_1 = \cos\frac{\pi}{N}, \qquad U_2 = \cos\frac{2\pi}{N}, \ldots, \\ &U_{N-1} = \frac{\cos(N-1)\pi}{N}, \qquad U_N = 1 \end{aligned} \tag{A 8}$$

$$U_j = \cos\frac{j\pi}{N}, \qquad j = 0, 1, 2, \ldots, N. \tag{A 9}$$

It can be shown that the cases $U_0 = 1$ and $U_N = -1$ represent respectively the symmetric and antisymmetric vibration modes of a single-span beam of length a clamped at both bends. For this purpose, we set the origin of the coordinate axes at midspan (so as to take advantage of the symmetry of the system). For the symmetric modes we have.

$$\begin{aligned} Y(x) &= B_1 \cosh s_1 x + B_3 \cos s_2 a \\ \Psi(x) &= e_1 B_1 \sinh s_1 x - e_2 B_3 \sin s_2 x . \end{aligned} \tag{A 10}$$

The boundary conditions being

$$Y\left(\frac{a}{2}\right) = \Psi\left(\frac{a}{2}\right) = 0 \tag{A 11}$$

which yields the frequency equation

$$\begin{vmatrix} \cosh \dfrac{s_1 a}{2} & \cos \dfrac{s_2 a}{2} \\ e_1 \sinh \dfrac{s_1 a}{2} & -e_2 \sin \dfrac{s_2 a}{2} \end{vmatrix} = 0 \tag{A 12}$$

or

$$e_1 \tanh \frac{s_1 a}{2} + e_2 \tan \frac{s_2 a}{2} = 0. \tag{A 13}$$

But

$$\begin{aligned} \tanh \frac{s_1 a}{2} &= \frac{\cosh s_1 a - 1}{\sinh s_1 a} = \frac{C_1 - 1}{D_1} = \frac{D_1}{C_1 + 1} \\ \tan \frac{s_2 a}{2} &= \frac{1 - \cos s_2 a}{\sin s_2 a} = \frac{1 - C_2}{D_2} = \frac{D_2}{C_2 + 1} \end{aligned} \tag{A 14}$$

which yields finally

$$C_2 D_1 e_2 - C_1 D_2 e_1 = D_1 e_2 - D_2 e_1 \tag{A 15}$$

which is equivalent to the case $U_N = 1$.

In complete analogy, for the antisymmetric modes

$$\begin{aligned} Y(x) &= B_2 \sinh s_1 x + B_4 \sin s_2 x \\ \Psi(x) &= e_1 B_2 \cosh s_1 x + e_2 B_4 \cos s_2 x \end{aligned} \tag{A 16}$$

with boundary conditions as per Eq. (A 10). The frequency equation now reads

$$e_2 \tanh \frac{s_1 a}{2} - e_1 \tan \frac{s_2 a}{2} = 0 \tag{A 17}$$

or, by Eq. (A 14),

$$C_2D_1e_2 - C_1D_2e_1 = -(D_1e_2 - D_2e_1)$$

which is equivalent to the case $U_N = -1$, Q.E.D.

Consequently, the subspectrum associated with both $U_0 = 1$ and $U_N = -1$ is the full spectrum of a single-span beam clamped at both ends. It is natural to count this unified subspectrum as a single band. Accordingly it follows from Eq. (A 9) that the N-span clamped beam has $N - 1$ other bands natural frequencies, their total number again being equal to that of spans, as in the case of an N-span beam simply supported at both ends. Checking against the results for a simply-supported beam at $x_1 = 0$ and $x_N = a$, we find $N - 1$ bands in common with the case of a beam clamped at $x_1 = 0$ and $x_N = a$.

Appendix B

Multispan Beam Simply Supported at One End and Clamped at the Other

In this case the boundary conditions are

$$Y = 0 \quad \text{at} \quad x_1 = 0 \quad \text{and} \quad x_N = a \tag{B 1}$$

$$\Psi = 0 \quad \text{at} \quad x_1 = 0, \qquad \frac{d\Psi}{dx} = 0 \quad \text{at} \quad x_N = a. \tag{B 2}$$

Condition (B 1) is satisfied by Eqs. (53) and (54).

Substitution of Eq. (B 2) into Eqs. (53) and (54) yields

$$B_{4,1}(F_{2,1}e_1 + e_2) + B_{4,2}(F_{2,2}e_1 + e_2) = 0 \tag{B 3}$$

$$\begin{aligned} &B_{4,1}\lambda_1^{N-1}(F_{1,1}e_1s_1C_1 + F_{2,1}e_1s_1D_1 - F_{3,1}e_2s_2C_2 - e_2s_2D_2) \\ &+ B_{4,2}\lambda_2^{N-1}(F_{1,2}e_1s_1C_1 + F_{2,2}e_1s_1D_1 - F_{3,2}e_2s_2C_2 - e_2s_2D_2). \end{aligned} \tag{B 4}$$

The nontriviality requirement stipulates

$$\frac{(D_2e_1 - D_1e_2)\, D_2(e_1s_1 + e_2s_s)\, (\lambda_1 - \lambda_2)\, [\lambda_2^N + \lambda_1^N]}{2D_1(C_2 - \lambda_1)\, (C_2 - \lambda_2)} = 0 \tag{B 5}$$

or

$$\lambda_1^N + \lambda_2^N = 0 \tag{B 6}$$

which, using Eq. (63), becomes

$$\cos\,(N\theta) = 0 \tag{B 7}$$

where

$$N\theta = (2j - 1)\,\frac{\pi}{2}, \qquad j = 1, 2, \ldots, N. \tag{B 8}$$

Accordingly, a beam supported at $x_1 = 0$ and clamped at $x_N = a$, also has N bands of natural frequencies, and the trivial series is absent. It is remarkable, that the natural frequencies of a simply-supported-clamped beam coincide with those of a simply supported beam with double the number of spans, $2N$, which correspond to the symmetric modes. This is in perfect analogy with the findings of Krein [7] and Pujara [24] for classical beams.

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