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On the J-integral and energy-release rates in dynamical fracture

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Summary. The apparent contradiction between the Eshelby formulation (consideration of a material force on the material manifold) and the more usual global-dissipation analysis (essentially, the balance of energy about the moving singular point represented by the tip of the crack) of the J-integral and energy-release rate in dynamical fracture is resolved in both pure finite-strain elasticity and Galilean-invariant electrodynamics of electro-magneto-elastic media. The solution uses the notions of mechanical and electromagnetic pseudomomenta (canonical momenta) in finitely deformable continua with cracks.

1 Introduction

We now know that there essentially are two ways to compute the characteristic quantity of fracture mechanics called the energy-release rate G, i.e., in thermodynamical terms, the generalized force conjugate to the extension of a crack. The first of these is a global-dissipation analysis which acknowledges the fact that the global phenomenon (fracture of a material sample) obviously is thermodynamically irreversible while the local mechanical behavior of the bulk material may be fully recoverable (e.g., elastic) – see, e.g., [1, Chapter 7]. The second one is more subtle in that, thought to be "metaphysical" by some authors, it directly involves the computation of the generalized force of a fictitious (i.e., non-Newtonian) type which is acting at the tip of the crack in material space (i.e., not on an infinitesimal element of matter but on a defect). This is the point of view of the theory of defects (material inhomogeneities) and material forces (on singularities) in the spirit of J. D. Eshelby (see [2], [3], and [4] for a synthesis). Mathematically, these two points of view can be summarized in the following formulas; per unit thickness of the sample and considering a straight through crack, either

$$\Phi = G\dot{l} \geqq 0, \quad G = -\frac{\partial \mathscr{W}(l)}{\partial l}, \tag{1.1}$$

or

$$\Phi = \mathscr{F} \cdot \mathscr{U} \geqq 0, \quad G = \boldsymbol{e}_1 \cdot \mathscr{F}, \tag{1.2}$$

where, l being the length of the crack C at time t, $\mathscr{W}(l)$ is the total potential energy of the body $B \backslash C$, $\dot{l}$ is the rate of extension of the crack in direction $\boldsymbol{e}_1$ in matter at time t, and $\mathscr{U}$ is the (material) velocity of matter with respect to the crack, $\mathscr{F}$ is the material force, and Φ is the dissipation resulting from fracture. $\mathscr{W}$ depends on l through the evolving integration volume. In most cases the analyses are carried out in the quasistatic framework (e.g., in brittle fracture), whence $\mathscr{W}$ and $\mathscr{F}$ involve only the strain energy function W.

The question naturally arises of the extension of the above-recalled expressions to dynamical fracture. This last field has indeed become an active field of research and applications [5]. On the conditions that both time rates and stress levels remain reasonable, the elasticity framework can still be envisaged for such a dynamical generalization. Using a direct thermodynamical approach basing on the first law of thermodynamics in physical space favored by G. P. Cherepanov [6], [7] – see also Cherepanov's historical statement in [8] – many authors have straightforwardly shown that the strain energy function W in (1.1) was to be replaced by the total energy density – that we call here the Hamiltonian density $\mathscr{H}$ in the spirit of field theory – i.e., the sum of the potential and kinetic energy [9]–[13][1]:

$$\mathscr{H} = W + \frac{1}{2}\varrho_0 \dot{\boldsymbol{u}}^2 \tag{1.3}$$

per unit undeformed volume. The celebrated path-independent J-integral of fracture (e.g., [1, Chapter 7]) then reads

$$G = J_\Gamma = \int_\Gamma (\mathscr{H} N_1 - \boldsymbol{u}_{,1} \cdot \boldsymbol{T}^d)\, d\Gamma, \tag{1.4}$$

where $_{,1}$ denotes the space derivative along $\boldsymbol{e}_1$, $N_1 = \boldsymbol{N} \cdot \boldsymbol{e}_1$ if $\boldsymbol{N}$ is the outward unit normal to Γ, $\boldsymbol{T}^d$ is the traction at Γ, and Γ is a path encircling the tip of the crack in the counter clockwise sense, starting and ending on the traction-free lips of the crack C. The contour integral in elasticity is independent of Γ and J_Γ is none other than the energy-release rate G to be ultimately compared to a critical value G_c if one desires to apply a criterion of extension to the crack.

The second avenue basing on material forces acting on singularities – here the tip of the crack in its own right – here presents an apparent difficulty which either was remarked upon by some authors without offering a definite answer (e.g., Eischen and Herrmann [15] and Eshelby himself [16]) or completely ignored by others, by mere error in fact ([5, p. 267], see below). Indeed, it seems that the dynamical generalization of (1.2) – according to the analyses of [15] and [3] should be

$$G = J_\Gamma = \int_\Gamma (\boldsymbol{e}_1 \cdot \boldsymbol{b} \cdot \boldsymbol{N})\, dT, \tag{1.5}$$

where $\boldsymbol{b}$ is the so-called Eshelby stress (mixed material) tensor given canonically by

$$\boldsymbol{b} = -(\mathscr{L}\mathbf{1}_R - \boldsymbol{F}^T \cdot \partial\mathscr{L}/\partial\boldsymbol{F}), \tag{1.6}$$

where $\boldsymbol{F}$ is the finite-deformation gradient between the configurations $\mathscr{K}_R$ and $\mathscr{K}_t$, $\mathbf{1}_R$ is the unit dyadic in $\mathscr{K}_R$ and $\mathscr{L}$ is the Lagrangian density per unit volume in $\mathscr{K}_R$, i.e.,

$$\mathscr{L} = \frac{1}{2}\varrho_0 \boldsymbol{v}^2 - W = -\mathscr{H} + \varrho_0 \boldsymbol{v}^2, \tag{1.7}$$

if $\boldsymbol{v}$ is the physical velocity. How can the two approaches be reconciled as (1.4) certainly is the right answer? This is the problem that we address below in Section 3 in pure mechanics after having recalled some fundamental notions of Eshelbian mechanics in Section 2. Sections 4 and 5 are

[1] We cite Nilsson [14] for completeness only as his Laplace-transform formulation for elastodynamic fracture is irrelevant here.

devoted to the solution of the same paradox but in a rather complex situation, that one that prevails in the Galilean electrodynamics of finitely-strained elastic bodies. Section 6 concludes on the case of inelasticity, i.e., for ductile fracture.

2 Elements of Eshelbian mechanics

These elements were given in [3] and will only be briefly reviewed. We consider inhomogeneous anisotropic hyperelasticity (energy-based nonlinear elasticity) in the absence (neglect) of thermal effects and applied body forces. At any regular point X at time t, the balance of linear (physical) momentum is given by

$$\frac{\partial}{\partial t} \boldsymbol{p}_R \bigg|_X - \operatorname{div}_R \boldsymbol{T} = 0, \tag{2.1}$$

where the physical momentum $\boldsymbol{p}_R$ and the first Piola-Kirchhoff stress (a two-point field) $\boldsymbol{T}$ are defined by

$$\boldsymbol{p}_R = \varrho_0(\boldsymbol{X})\, \boldsymbol{v}\big(\chi(\boldsymbol{X}, t), t\big), \qquad \boldsymbol{T} = J_F \boldsymbol{F}^{-1} \cdot \sigma, \tag{2.2.1, 2}$$

where the direct motion is $\boldsymbol{x} = \chi(\boldsymbol{X}, t)$ between the reference configuration $\mathscr{K}_R$ of matter density $\varrho_0(\boldsymbol{X})$ — inertial inhomogeneities — and actual configuration $\mathscr{K}_t$, and σ if the Cauchy stress in $\mathscr{K}_t$. Obviously,

$$\boldsymbol{v} = \frac{\partial \chi}{\partial t}\bigg|_X, \qquad \boldsymbol{F} = \frac{\partial \chi}{\partial \boldsymbol{X}}\bigg|_t, \qquad \boldsymbol{F}^{-1} = \frac{\partial \chi^{-1}}{\partial \boldsymbol{x}}\bigg|_t, \qquad J_F = \det \boldsymbol{F} > 0. \tag{2.3}$$

In the nondissipative isothermal, but inhomogeneous, nonlinear and dynamic case, we can write (a superimposed dot denotes the material time derivative; tr = trace, T = transpose)

$$\mathscr{L} = \mathscr{L}(\boldsymbol{v}, \boldsymbol{F}; \boldsymbol{X}) = \frac{1}{2}\, \varrho_0(\boldsymbol{X})\, \boldsymbol{v}^2 - W(\boldsymbol{F}; \boldsymbol{X}) \tag{2.4}$$

and

$$-\dot{W} + tr\{\boldsymbol{T}\dot{\boldsymbol{F}}\} - 0. \tag{2.5}$$

Equations (2.4) and (2.5), respectively, define a Lagrangian density and reproduce Gibbs' equation for isothermal evolutions. We have

$$\boldsymbol{T} = (\partial W/\partial \boldsymbol{F})^T, \qquad \boldsymbol{f}^{\text{inh}} = (\partial \mathscr{L}/\partial \boldsymbol{X})_{\text{expl}}. \tag{2.6}$$

The latter is a covariant material vector which is the explicit material gradient of the Lagrangian density: it is the material inhomogeneity force such that

$$\boldsymbol{f}^{\text{inh}} = (\nabla_R \varrho_0) \left(\frac{1}{2}\, \boldsymbol{v}^2\right) - (\partial W/\partial \boldsymbol{X})_{\text{expl}}. \tag{2.7}$$

It thus captures both inertial and elastic inhomogeneities.

By left multiplication of (2.1) by $\boldsymbol{F}^T$ and integration by parts while accounting for the following obvious kinematical compatibility condition:

$$\left(\frac{\partial \boldsymbol{F}}{\partial t}\right)^T = \nabla_R \boldsymbol{v}, \tag{2.8}$$

it is shown that there holds the following unbalance of pseudomomentum [3], [4]:

$$\frac{\partial}{\partial t}\mathscr{P}\Big|_X - \mathrm{div}_R\, \boldsymbol{b} = \boldsymbol{f}^{\mathrm{inh}}, \tag{2.9}$$

where we have defined the material (covariant) pseudomomentum $\mathscr{P}$ and the Eshelby (material mixed) stress tensor $\boldsymbol{b}$ by

$$\mathscr{P} = -\varrho_0 \boldsymbol{F}^T \cdot \boldsymbol{v} = \varrho_0 \mathbb{C} \cdot \boldsymbol{V}, \tag{2.10.1, 2}$$

$$\boldsymbol{b} = -(\mathscr{L}\mathbf{1}_R + \boldsymbol{F}^T \boldsymbol{T}). \tag{2.11}$$

Equation (2.10.2) holds on account of the definition of the Green finite-strain tensor $\mathbb{C}$ and of the material (contravariant) velocity $\boldsymbol{V}$ by

$$\mathbb{C} = \boldsymbol{F}^T\boldsymbol{F}, \qquad \boldsymbol{V} = \partial\chi^{-1}/\partial t|_x, \tag{2.12}$$

and the demonstrable relationship (through the chain rule of differentiation)

$$\boldsymbol{v} + \boldsymbol{F}\cdot\boldsymbol{V} = 0. \tag{2.13}$$

Applying now objectivity to W, we obtain, e.g.,

$$W = \bar{W}(\mathbb{E}; \boldsymbol{X}), \qquad \mathbb{E} = \frac{1}{2}(\mathbb{C} - \mathbf{1}_R), \tag{2.14}$$

$$\boldsymbol{b} = -(\mathscr{L}\mathbf{1}_R + \mathbb{C}\cdot\mathbb{S}^E), \qquad \mathbb{S}^E = \boldsymbol{F}^{-1}\cdot\boldsymbol{T}^T = \partial\bar{W}/\partial\mathbb{E}, \tag{2.15.1, 2}$$

where $\mathbb{S}^E$ is the second Piola-Kirchhoff (symmetric, material, contravariant) stress tensor. In both Eqs. (2.10.2) and (2.15.1) we see that $\mathbb{C}$ plays the role of a material (deformed) metric to build covariant or mixed geometric objects from contravariant ones (e.g., $\boldsymbol{V}$ and $\mathbb{S}^E$). In, and only in, elasticity can Eq. (2.9) be also obtained alternately either by (i) applying Noether's theorem for $\boldsymbol{X}$-translations to a variational formulation in which one varies directly the direct motion (keeping $\boldsymbol{X}$ fixed) or (ii) direct variation of the inverse motion χ^{-1} (keeping the actual placement $\boldsymbol{x}$ fixed) [3]. We have called Eshelbian mechanics the consideration in continuum mechanics of fully material balance (or unbalance) laws such as (2.9) on the material manifold $\mathscr{M}^3$ and not in physical space – which is still the case of Eq. (2.1) in spite of it being often referred to as a "material form". For completeness we recall the two forms of the balance of angular momentum:

$$\boldsymbol{F}\boldsymbol{T}^T = \boldsymbol{T}\boldsymbol{F}^T, \qquad \boldsymbol{b}\mathbb{C} = \mathbb{C}\boldsymbol{b}^T, \tag{2.16}$$

of which the second can be verbally expressed as "$\boldsymbol{b}$ is symmetric with respect to the metric $\mathbb{C}$". For further use we also notice that Eq. (2.9) reads in full:

$$\left(\frac{\partial\mathscr{L}}{\partial\boldsymbol{X}}\right)_{\mathrm{expl}} = \nabla_R\mathscr{L} + \mathrm{div}_R\left(\boldsymbol{F}^T\frac{\partial W}{\partial\boldsymbol{F}}\right) + \frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}}{\partial\boldsymbol{V}}\right)\Big|_X, \tag{2.17}$$

as it is readily checked that $\mathscr{L}$ is also written as

$$\mathscr{L} = \frac{1}{2}\, \varrho_0(\boldsymbol{X})\, \boldsymbol{V} \cdot \mathbb{C} \cdot \boldsymbol{V} - W(\boldsymbol{F}; \boldsymbol{X}). \tag{2.18}$$

3 Application to dynamic fracture

Assume now that Eq. (2.9) holds at any point inside the material volume V deprived of a disk-shaped thicknessless region Σ of regular edge $\partial\Sigma$. For further purpose we want to evaluate the global quantity

$$\mathscr{F}^{\text{inh}}(V-\Sigma) = \int_{V-\Sigma} f^{\text{inh}} dV = \int_{V-\Sigma} \left(\left.\frac{\partial \mathscr{P}}{\partial t}\right|_X - \operatorname{div}_R \boldsymbol{b} \right) dV. \tag{3.1}$$

In order to transform the two contributions to the last integral we need appropriate generalizations of the so-called Reynolds (transport) and Green-Gauss (divergence) theorems. Indeed, whereas no specific problem arises for the applications of such theorems in a simply connected region where all fields are assumed to behave regularly everywhere, potential theory and vector analysis [16] tell us that for a nonsimply connected region such as V-Σ one should anticipate on the singular behavior of fields at $\partial\Sigma$ to write down correct generalizations of the sought theorems. In the present case Σ is supposed to be a mathematical crack in the sense that the elastic displacement $\boldsymbol{u}$ is assumed to be discontinuous across the two (traction-free) faces Σ^+ and Σ^- of Σ, and these two faces cannot solder back although they coincide mathematically. Furthermore, the crack Σ may expand and move so that (remember we are in material space) we call $\mathscr{U}$ the material (contravariant) velocity of matter with respect to the crack. Finally, let $S_X(\partial\Sigma)$ a local orthogonal cross section to $\partial\Sigma$ and r a radial coordinate in that section with origin at $\partial\Sigma$ (later on, the tip of the straight through crack). Then the local behavior of the fields depends on the solution of the problem, that we do not know, and that solution depends itself on the material behavior considered. For elastic materials $\boldsymbol{u}$ goes like $\sqrt{r}$ and the stress like $1/\sqrt{r}$ as one approaches $\partial\Sigma$ in $S_X(\partial\Sigma)$. For other (in particular, or in fact, dissipative) behaviours, see for instance [1, pp. 259–261]. Assuming that the elastic solution is the most drastic one from that viewpoint and calling Γ a circuit in $S_X(\partial\Sigma)$ in the counter clockwise sense around $\partial\Sigma$, starting and ending on Σ^- and Σ^+, and $[A] = A^+ - A^-$ the jump in $A(\boldsymbol{X}, t)$ across Σ, where A^+ and A^- are the uniform limits of A in approaching Σ on its two faces along the normal, and $\boldsymbol{N}$ being that normal oriented from the minus to the plus face, we simply record here without proof the required generalizations of the transport and divergence theorems:
Generalized Reynolds theorem:[2]

$$\int_{V-\Sigma} \left.\frac{\partial \mathscr{P}}{\partial t}\right|_X dV = \frac{\partial}{\partial t} \boldsymbol{P}(V-\Sigma) - \int_{\partial V - \Sigma} \mathscr{P}(\mathscr{U} \cdot \boldsymbol{N})\, dA$$
$$+ \int_{\Sigma} [\mathscr{P} \otimes \mathscr{U}] \cdot \boldsymbol{N} d\Sigma - \int_{\partial\Sigma} dL \left\{ \lim_{\Gamma \to 0} \int_{\Gamma} \mathscr{P}(\mathscr{U} \cdot \boldsymbol{N})\, d\Gamma \right\}; \tag{3.2}$$

[2] In [1] whose applications concern quasi-static fracture, the last contribution in (3.2) was overlooked.

Generalized Green-Gauss theorem:

$$\int_{V-\Sigma} (\mathrm{div}_R\, \boldsymbol{b})\, dV = \mathscr{B}_E(V-\Sigma) - \int_{\Sigma} [\boldsymbol{b}] \cdot \boldsymbol{N}\, dA - \int_{\partial\Sigma} dL \left\{ \lim_{\Gamma\to 0} \int_{\Gamma} \boldsymbol{b} \cdot \boldsymbol{N}\, dA \right\}, \tag{3.3}$$

where we have defined the following global quantities:

$$\boldsymbol{P}(V-\Sigma) = \int_{V-\Sigma} \mathscr{P}\, dV, \tag{3.4}$$

$$\mathscr{B}_E(V-\Sigma) = \int_{\partial V-\Sigma} \boldsymbol{b} \cdot \boldsymbol{N}\, dA, \tag{3.5}$$

and assumed that $\mathscr{P}$, which is quadratic in the basic kinematic fields, behaves like $1/r$ as goes to zero in $S_X(\partial\Sigma)$.

On account of (3.2) and (3.3) we find that (3.1) yields

$$\mathscr{F}^{\mathrm{inh}}(V-\Sigma) = \frac{\partial}{\partial t} \boldsymbol{P}(V-\Sigma) - \bar{\mathscr{B}}_E(V-\Sigma) + \int_{\Sigma} [\boldsymbol{b}_{\mathrm{dyn}}] \cdot \boldsymbol{N}\, dA + \int_{\partial\Sigma} dL \left\{ \lim_{\Gamma\to 0} \int_{\Gamma} (\boldsymbol{b}_{\mathrm{dyn}} \cdot \boldsymbol{N})\, d\Gamma \right\}, \tag{3.6}$$

where

$$\bar{\mathscr{B}}_E(V-\Sigma) = \int_{\partial V-\Sigma} \boldsymbol{b}_{\mathrm{dyn}} \cdot \boldsymbol{N}\, dA, \qquad \boldsymbol{b}_{\mathrm{dyn}} = \boldsymbol{b} - \mathscr{P} \otimes \mathscr{U}. \tag{3.7}$$

Equation (3.6) provides the expression of the global material force due to material inhomogeneities and acting on the regular volume $V-\Sigma$ in the presence of the finite crack Σ. But our aim here is to find the expression of a material force which pertains to Σ in an otherwise homogeneous material body. Following [3], to arrive at this result we shall let V itself (or a region V' of V containing the disk-like crack) shrink to Σ. One could then think that the result of this procedure is none other than zero. But the crack is a material inhomogeneity in its own right and the shrinking procedure shows that all contributions disappear with the exception of the one pertaining to $\partial\Sigma$. Thus in the limit we capture a material entity which characterizes the crack through the singular behavior of fields in the immediate vicinity of its edge. We have thus:

$$\mathscr{F}(\Sigma) = \mathscr{F}^{\mathrm{inh}}_{V\to\Sigma}(V-\Sigma) = \int_{\partial\Sigma} dL \left\{ \lim_{\Gamma\to 0} \int_{\Gamma} (\boldsymbol{b}_{\mathrm{dyn}} \cdot \boldsymbol{N})\, d\Gamma \right\}, \tag{3.8}$$

which is the essential result in the spirit of Eshelbian fracture mechanics. For a straight through crack with traction-free faces, $\partial\Sigma$ becomes a straight infinite line and we estimate $\mathscr{F}(\Sigma)$ per unit thickness of the sample. We call $\mathscr{F}(\partial\Sigma)$ this elementary material force which is nothing but

$$\mathscr{F}(\partial\Sigma) = \lim_{\Gamma\to 0} \int_{\Gamma} (\boldsymbol{b}_{\mathrm{dyn}} \cdot \boldsymbol{N})\, d\Gamma. \tag{3.9}$$

The projection of this force onto the unit direction $\boldsymbol{e}_1$ of possible extension of the crack is

$$\mathscr{F}_1(\partial\Sigma) = \boldsymbol{e}_1 \cdot \mathscr{F}(\partial\Sigma) = \lim_{\Gamma\to 0} \int_{\Gamma} \boldsymbol{e}_1 \cdot (\boldsymbol{b} - \mathscr{P} \otimes \mathscr{U}) \cdot \boldsymbol{N}\, d\Gamma. \tag{3.10.1, 2}$$

It remains to evaluate the integrand in the last integral. We recall that

$$\boldsymbol{b}_{\text{dyn}} = -(\mathscr{L}\mathbf{1}_R + \mathscr{P} \otimes \mathscr{U} + \boldsymbol{F}^T\boldsymbol{T}) = -(\mathscr{L}\mathbf{1}_R + \mathscr{P} \otimes \mathscr{U} + \mathbb{C} \cdot \mathbb{S}^E), \tag{3.11}$$

with $\mathscr{L}$ given by Eq. (1.7) and $\mathscr{P}$ by Eq. (2.10). We note that

$$-\boldsymbol{e}_1 \cdot \boldsymbol{b}_{\text{dyn}} \cdot \boldsymbol{N} = \mathscr{L}(\boldsymbol{e}_1 \cdot \boldsymbol{N}) + (\boldsymbol{e}_1 \cdot \mathscr{P})\,(\mathscr{U} \cdot \boldsymbol{N}) + \boldsymbol{e}_1 \cdot \mathbb{C} \cdot \mathbb{S}^E \cdot \boldsymbol{N}, \tag{3.12}$$

and

$$\mathscr{P} = -\varrho_0(\mathbf{1}_R + \nabla_R \boldsymbol{u}) \cdot \dot{\boldsymbol{u}}, \tag{3.13}$$

if $\boldsymbol{u}(\boldsymbol{X}, t) = \boldsymbol{x} - \boldsymbol{X}$ is the displacement field. We shall assume (elasticity) that both $\mathbb{S}^E$ and $\dot{\boldsymbol{u}}$ behave like $1/\sqrt{r}$ so that

$$\lim_{\Gamma \to 0} \int_\Gamma (\boldsymbol{e}_1 \cdot \mathbb{S}^E \cdot \boldsymbol{N})\, d\Gamma = 0, \qquad \lim_{\Gamma \to 0} \int_\Gamma \varrho_0 \dot{\boldsymbol{u}}(\mathscr{U} \cdot \boldsymbol{N})\, d\Gamma = 0. \tag{3.14}$$

For our crack extending in the direction $\boldsymbol{e}_1$ with velocity $-\mathscr{U}$ with respect to matter (see Fig. 1 where both direct and inverse motion descriptions are given), we have $\mathscr{U} = -\dot{l}\boldsymbol{e}_1$ while terms like $\dot{A}$ will give (compare [17])

$$\dot{A} \cong \dot{l}(\boldsymbol{e}_1 \cdot \nabla_R)\, A. \tag{3.15}$$

This allows one to rewrite the relevant critical contribution — the second one — in Eq. (3.12), anticipating the nil contribution in (3.10.2), as $(\dot{l} \neq 0)$

$$(\boldsymbol{e}_1 \cdot \mathscr{P})\,(\mathscr{U} \cdot \boldsymbol{N}) \cong \left\{\dot{l}^{-1}\varrho_0 \dot{\boldsymbol{u}}^2\right\} \cdot \left\{-\dot{l}(\boldsymbol{e}_1 \cdot \boldsymbol{N})\right\} = -\varrho_0 \dot{\boldsymbol{u}}^2(\boldsymbol{e}_1 \cdot \boldsymbol{N}) \tag{3.16}$$

while, anticipating on (3.10.2), the relevant contribution to the last term in (3.12) shall be

$$\boldsymbol{e}_1 \cdot \mathbb{C} \cdot \mathbb{S}^E \cdot \boldsymbol{N} \cong 2\boldsymbol{e}_1 \cdot \mathbb{E} \cdot \mathbb{S}^E \cdot \boldsymbol{N} = (\boldsymbol{e}_1 \cdot \nabla_R)\, \boldsymbol{u} \cdot (\mathbb{S}^E \cdot \boldsymbol{N}). \tag{3.17}$$

Collecting terms from (3.16) and (3.17) we thus obtain

$$-\boldsymbol{e}_1 \cdot \boldsymbol{b}_{\text{dyn}} \cdot \boldsymbol{N} = (\mathscr{L} - \varrho_0 \dot{\boldsymbol{u}}^2)\,(\boldsymbol{e}_1 \cdot \boldsymbol{N}) + (\boldsymbol{e}_1 \cdot \nabla_R)\, \boldsymbol{u} \cdot \boldsymbol{T}^d. \tag{3.18}$$

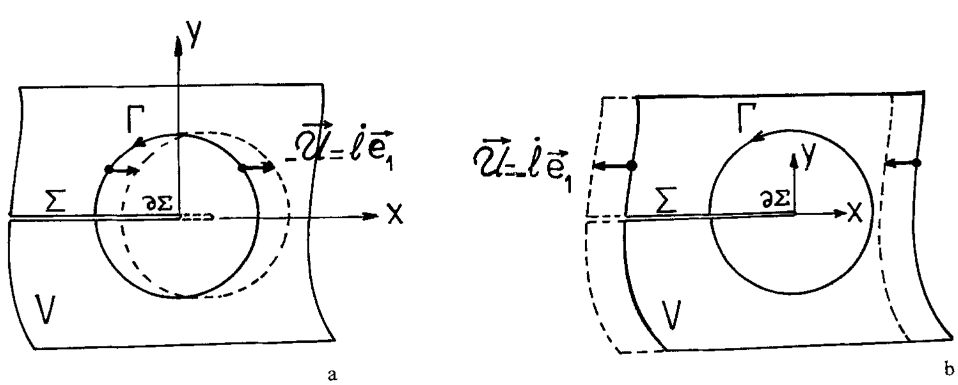

Fig. 1. Straight through crack: **a** direct- and **b** inverse-motion descriptions

On substituting from this into Eq. (3.10), we finally reach the remarkable result

$$\mathscr{F}_1(\partial\Sigma) = \lim_{\Gamma\to 0} J_\Gamma, \tag{3.19}$$

where J_Γ is the dynamic path-independent integral defined in Eq. (1.4). This is the solution to the apparent paradox of passing from the Lagrangian to the Hamiltonian in the dynamic J-integral deduced from the Eshelbian-mechanics framework. According to Freund [10], this was "hypothesized" (quotation marks mine, G.A.M.) by Atkinson and Eshelby [9] – who in fact worked along Cherepanov's line. To be fair, however, a real proof close to the above-given one is sketched out in Eshelby [16] although Eshelby himself refers to his sketchy derivation as a rather "metaphysical one" in which the "elastic field is transported rigidly" along the e_1-direction (compare Part b in our Fig. 1). Two ingredients are central to the above-given derivation: (i) some a priori knowledge of the singularity of the involved fields is required and (ii) taking account of the pseudomomentum contribution is a necessity. When the latter is discarded, passing from the Lagrangian to the Hamiltonian in J_Γ is strictly impossible[3]. In the following two Sections we shall generalize our proof to the rather difficult case of the electrodynamics of continua in the Galilean framework.

4 Reminder on the electrodynamics of continua

In order to proceed to the dynamical electromagneto-elastic case, we need some elements of electrodynamics reformulated in the material framework. Such elements are developed at length in several treatises [19]–[21] to which we refer the reader for details. Here we recall the strict minimum compatible with a good understanding. Only a Galilean invariant approach is considered and obviously is more than sufficient for engineering purposes. Let $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$ and $\boldsymbol{H}$ be the classical electromagnetic fields in the laboratory frame R_L at time t in the actual configuration $\mathscr{K}_t$. Still in $\mathscr{K}_t$ but in a co-moving frame $R_C(\boldsymbol{x}, t)$, the electromotive (field) intensity $\mathscr{E}$, the magnetic field $\mathscr{H}$ and the magnetic induction $\mathscr{B}$ are defined by (c is the velocity of light in vacuum)

$$\mathscr{E} = \boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}, \quad \mathscr{H} = \boldsymbol{H} - \frac{1}{c}\boldsymbol{v}\times\boldsymbol{D}, \quad \mathscr{B} = \boldsymbol{B} - \frac{1}{c}\boldsymbol{v}\times\boldsymbol{E}. \tag{4.1}$$

Material fields are introduced by convection (pull) back to $\mathscr{K}_R$ (the transformation formula depends on the "tensorial variance" and "nature" of the geometrical object):

$$\begin{aligned}
&\mathfrak{B} = J_F \boldsymbol{F}^{-1}\cdot\boldsymbol{B}, && \mathfrak{D} = J_F\boldsymbol{F}^{-1}\cdot\boldsymbol{D},\\
&\bar{\mathfrak{E}} = \boldsymbol{E}\cdot\boldsymbol{F}, && \mathfrak{E} = \mathscr{E}\cdot\boldsymbol{F} = \bar{\mathfrak{E}} - \frac{1}{c}\boldsymbol{V}\times\mathfrak{B},\\
&\bar{\mathfrak{H}} = \boldsymbol{H}\cdot\boldsymbol{F}, && \mathfrak{H} = \mathscr{H}\cdot\boldsymbol{F} = \bar{\mathfrak{H}} + \frac{1}{c}\boldsymbol{V}\times\mathfrak{D},\\
&\bar{\mathfrak{B}} = J_F\boldsymbol{F}^{-1}\cdot\mathscr{B}, && \mathfrak{B} = J_F\boldsymbol{F}^{-1}\cdot\boldsymbol{B} = \bar{\mathfrak{B}} + \frac{1}{c}\boldsymbol{V}\times\bar{\mathfrak{E}},
\end{aligned} \tag{4.2}$$

[3] Thus, Freund's claim in [5, p. 267] that his Eq. (5.6.9) follows from his Eq. (5.6.8) is blatantly wrong as pseudomomentum is discarded altogether.

where V indeed is the material velocity defined in (2.12). At any regular material point X of a nonconducting but magnetized and electrically polarized body in motion, Maxwell's equations then read

$$\nabla_R \times \mathfrak{E} + \frac{1}{C}\frac{\partial}{\partial t}\mathfrak{B}\bigg|_X = 0, \quad \nabla_R \cdot \mathfrak{B} = 0, \tag{4.3.1, 2}$$

$$\nabla_R \times \mathfrak{H} - \frac{1}{C}\frac{\partial}{\partial t}\mathfrak{D}\bigg|_X = 0, \quad \nabla_R \cdot \mathfrak{D} = \mathfrak{Q}_f, \tag{4.3.3, 4}$$

where $\mathfrak{Q}_f$ is the free charge density per unit volume in $\mathscr{K}_R$. Contrary to Eq. (2.1), Eqs. (4.3) are directly written in a completely material form [say, like Eq. (2.9)]. The first couple (4.3.1), (4.3.2) obviously implies the existence of material electromagnetic potentials ϕ (scalar electric potential) and $\mathfrak{A}$ (magnetic vector potential) by

$$\mathfrak{E} = -\left(\nabla_R \phi + \frac{1}{C}\frac{\partial \mathfrak{A}}{\partial t}\bigg|_X\right), \quad \mathfrak{B} = \nabla_R \times \mathfrak{A}. \tag{4.4}$$

Charge conservation requires that

$$\partial \mathfrak{Q}_f / \partial t|_X = 0. \tag{4.5}$$

The second kind of ingredients we need are the notions of electromagnetic Lagrangian and Hamiltonian densities. All interactions with matter are taken care of in the magnetoelastic potential energy $W(F, \bar{\mathfrak{E}}, \mathfrak{B}; X)$ – inhomogeneous bodies – which replaces the W present in (2.4). The remaining contributions of the "free" field E and B (i.e., those fields which exist everywhere including in vacuum) to the electromagnetic Lagrangian and Hamiltonian densities thus read, per unit volume of $\mathscr{K}_R$,

$$\mathscr{L}^F(E, B; F(X, t)) = J_F \cdot \frac{1}{2}(E^2 - B^2) = \frac{1}{2} J_F \bar{\mathfrak{E}} \cdot \mathbf{C}^{-1} \cdot \bar{\mathfrak{E}} - \frac{1}{2} J_F{}^{-1} \mathfrak{B} \cdot \mathbf{C} \cdot \mathfrak{B}, \tag{4.6}$$

and

$$\mathscr{H}^F(E, B; F(X, t)) = J_F \cdot \frac{1}{2}(E^2 + B^2) = \frac{1}{2} J_F \bar{\mathfrak{E}} \cdot \mathbf{C}^{-1} \cdot \bar{\mathfrak{E}} + \frac{1}{2} J_F{}^{-1} \mathfrak{B} \cdot \mathbf{C} \cdot \mathfrak{B}, \tag{4.7}$$

where, following W. Thomson (Lord Kelvin), we may view the electric and magnetic energies as analogs of mechanical kinetic and potential energies, respectively. The transformations yielding the second expressions in Eqs. (4.6) and (4.7) are straightforward on account of the definitions (4.2). Obviously,

$$\mathscr{H}^F = \mathscr{L}^F + J_F \mathfrak{B} \cdot \mathbf{C} \cdot \mathfrak{B} \tag{4.8}$$

or

$$\mathscr{L}^F = -\mathscr{H}^F + J_F \bar{\mathfrak{E}} \cdot \mathbf{C}^{-1} \cdot \bar{\mathfrak{E}}, \tag{4.9}$$

where we recognize in the latter relation something akin to the second part of Eq. (1.7).

The electromagnetomechanical generalization of the motion equation (2.1) reads (cf. [19] or [4])

$$\frac{\partial}{\partial t}(p_R + p_R{}^F)|_X - \mathrm{div}_R (T^E + T^F) = 0, \tag{4.10}$$

wherein

$$\boldsymbol{p}_R = \varrho_0 \boldsymbol{v}, \quad \boldsymbol{p}_R{}^F = J_F(\boldsymbol{E} \times \boldsymbol{B})/C,$$

$$\boldsymbol{T}^E = (\partial W/\partial \boldsymbol{F})^T_{\text{expl}} \tag{4.11}$$

$$\boldsymbol{T}^F = J_F \boldsymbol{F}^{-1} \cdot \left\{ \boldsymbol{E} \otimes \boldsymbol{E} + \boldsymbol{B} \otimes \boldsymbol{B} - \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2)\,\mathbf{1} \right\}.$$

It is readily shown that the latter can also be rewritten as

$$\boldsymbol{T}^F = J_F \mathbb{C}^{-1} \cdot \bar{\mathfrak{E}} \otimes \boldsymbol{E} + \mathfrak{B} \otimes \boldsymbol{B} - \mathscr{H}^F \boldsymbol{F}^{-1}. \tag{4.12}$$

The reader will find in Chapter 8 of [4] the proof that Eqs. (4.10) and (4.3.3), (4.3.4) follow simultaneously from the direct-motion and potential (ϕ and $\mathfrak{A}$) variations of the Lagrangian density

$$\mathscr{L} = \mathscr{L}^F + \frac{1}{2} \varrho_0(\boldsymbol{X})\, \boldsymbol{v}^2 - \hat{W}(\boldsymbol{F}, \bar{\mathfrak{E}}, \mathfrak{B}; \boldsymbol{X}) \tag{4.13}$$

with the complementary constitutive equations for (material) polarization and magnetization given by

$$\mathbb{M} = -\partial \hat{W}/\partial \mathfrak{B}, \quad \Pi = -\partial \hat{W}/\partial \bar{\mathfrak{E}}, \tag{4.14}$$

so that

$$\mathfrak{D} = J_F \mathbb{C}^{-1} \cdot \bar{\mathfrak{E}} + \Pi, \quad \mathfrak{H} = J_F^{-1} \mathbb{C} \cdot \bar{\mathfrak{B}} - \mathbb{M} \tag{4.15}$$

with Π and $\mathbb{M}$ related to the usual (Eulerian) fields $\boldsymbol{P}$ and $\boldsymbol{M}$ by

$$\Pi = J_F \boldsymbol{F}^{-1} \cdot \boldsymbol{P}, \quad \mathbb{M} = \left(\boldsymbol{M} + \frac{1}{C} \boldsymbol{v} \times \boldsymbol{P} \right) \cdot \boldsymbol{F} = \mathscr{M} \cdot \boldsymbol{F} \tag{4.16}$$

if (Lorentz-Heaviside EM units are used throughout)

$$\boldsymbol{D} = \boldsymbol{E} + \boldsymbol{P}, \quad \boldsymbol{H} = \boldsymbol{B} - \boldsymbol{M}. \tag{4.17}$$

Alternately, (4.10) can be written as

$$\frac{\partial}{\partial t} (\boldsymbol{p}_R + \boldsymbol{p}_R{}^F)\bigg|_X - \operatorname{div}_R (\boldsymbol{T} + \boldsymbol{T}^{\text{INT}} + \boldsymbol{T}^F) = 0, \tag{4.18}$$

wherein $\boldsymbol{T}$ is the first Piola-Kirchhoff stress associated with the Cauchy stress in accord with Eq. (2.2.1) and we have set

$$\boldsymbol{T}^{\text{INT}} = \Pi \otimes \boldsymbol{E} - \mathfrak{B} \otimes \mathscr{M} + (\mathbb{M} \cdot \mathfrak{B})\, \boldsymbol{F}^{-1}, \tag{4.19}$$

which emphasizes the contribution of polarization and magnetization as representative of interactions between free fields $\boldsymbol{E}$ and $\boldsymbol{B}$ and matter. The two-point tensor defined by the sum $\boldsymbol{T}^{\text{INT}} + \boldsymbol{T}^F$ is none other than the first Piola-Kirchhoff stress associated with the "Maugin-Collet" electromagnetic stress [19] (coinage by Ilyushin [22, p. 272]).

5 Inhomogeneities and fracture in dynamical electromagneto-elasticity

In writing down the equations of Section 4 we have assumed the presence of smooth inertial and electromagneto-elastic material inhomogeneities. This applies to a regular simply connected material domain. Like in the pure mechanical case recalled in Section 2, this can be materialized by the computation of a material inhomogeneity force f^{inh} in the same manner as in Eq. (2.9), i.e., formally

$$f^{\text{inh}} = \frac{\partial}{\partial t} \mathscr{P}^t \bigg|_X - \text{div}_R\, \boldsymbol{b}^t, \tag{5.1}$$

where $\mathscr{P}^t$ and $\boldsymbol{b}^t$ are total pseudomomentum and Eshelby stress acounting for electromagnetic effects. However, as remarked originally by Maugin and Epstein [23] in quasi-statics and Maugin, Epstein and Trimarco [24]–[26] in Galilean electrodynamics, the expression (4.6) shows that $\mathscr{L}^F$ cannot depend explicitly on X, so that we have the identity (compare to Eq. (2.17))

$$\boldsymbol{O} \equiv \left(\frac{\partial \mathscr{L}^F}{\partial \boldsymbol{X}}\right)_{\text{expl}} = \nabla_R \mathscr{L}^F - \text{div}_R \left(\boldsymbol{F}^T \frac{\partial \mathscr{L}^F}{\partial \boldsymbol{F}}\right) + \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}^F}{\partial \boldsymbol{V}}\right)\bigg|_X \tag{5.2}$$

at any regular point X in matter (and obviously outside). This result is tantamount to saying that a completely material formulation such as in (5.1) filters out the "pure" free field contributions or that the material manifold $\mathscr{M}^3$ is mechanically transparent to free electromagnetic fields. As a consequence, as shown in [24], [25], in the local equation (5.1) $\mathscr{P}^t$ and $\boldsymbol{b}^t$ are given by the expressions[4]:

$$\mathscr{P}^t = \varrho_0(X)\, \mathbb{C} \cdot \boldsymbol{V} + \frac{1}{C}\, \Pi \times \mathfrak{B}, \tag{5.3}$$

$$\boldsymbol{b}^t = \left(W - \frac{1}{2}\varrho_0 v^2\right) \mathbf{1}_R - \mathbb{C} \cdot \mathbb{S}^E + \Pi \otimes \bar{\mathfrak{E}} - \mathfrak{B} \otimes \mathbb{M}, \tag{5.4}$$

if, after imposing objectivity,

$$W = \hat{W}(\mathbb{E}, \bar{\mathfrak{E}}, \mathfrak{B}; X), \qquad \mathbb{S}^E = \partial \hat{W}/\partial \mathbb{E}, \qquad \Pi = -\partial \hat{W}/\partial \bar{\mathfrak{E}}, \qquad \mathbb{M} = -\partial \hat{W}/\partial \mathfrak{B}. \tag{5.5.1–4}$$

Clearly, the expressions (5.3) and (5.4) contain the free fields only insofar as they combine with polarization and magnetization, the latter two being true (thermodynamically extensive) material entities.

But the above reasoning holds true only at regular points X for the good reason that the right-hand side of Eq. (5.2) may not be integrable (over volume) in the vicinity of a singularity such as the tip of a crack (i.e., the limit of the sum of the r. h. s. of (5.2) over a region about a singular point is not zero as we shrink that region to the singular point). That is, we may say that if a smooth material manifold is indeed transparent to the free electromagnetic fields, singularities of this manifold may capture the singularities in the free electromagnetic fields. This is the case of what happens in electromagneto-elastic fracture as unambiguously demonstrated by Maugin and Dascalu [27]; this allows one to reconciliate the remark of Maugin and Epstein [23] with the quasi-static expressions proposed by Pak and Herrmann for electroelastic fracture

[4] The quantity $-(\mathbb{M} \cdot \mathfrak{B})$ – dipole energy – coming from the last contribution in (4.19) has been absorbed in $\hat{W}$ without loss in generality on account of (5.5.4).

[28], i.e., free fields in general remain involved in the J-integral of electroelastic fracture. What happens then in dynamical fracture? Had free fields not been involved, the question whether the Lagrangian or the Hamiltonian of free fields appears in dynamical fracture would be irrelevant. But this is not the case! Therefore, we must examine the question again in the larger framework as it is clear from a global-dissipation analysis in Cherepanov's style (such as proposed by Farat [29]) that it indeed is the Hamiltonian which does appear in the relevant J-integral. To envision this more general case we must thus consider the full expressions (which include the vanishing one of Eq. (5.2))

$$\mathscr{P}^t = \varrho_0(X)\,\mathbf{C}\cdot \boldsymbol{V} - \frac{1}{C}\,\boldsymbol{F}^T\cdot(\boldsymbol{E}\times\boldsymbol{B}) \tag{5.5.1}$$

and

$$\boldsymbol{b}^t = -\mathscr{L}\mathbf{1}_R + \boldsymbol{F}^T\cdot\frac{\partial\mathscr{L}^F}{\partial\boldsymbol{F}} - \mathbf{C}\cdot\mathbb{S}^E + \Pi\otimes\bar{\mathfrak{E}} - \mathfrak{B}\otimes\mathbb{M}, \tag{5.6.1}$$

or

$$\mathscr{P}^t = \varrho_0(X)\,\mathbf{C}\cdot\boldsymbol{V} + \frac{1}{C}\,J_F(\mathbf{C}^{-1}\cdot\bar{\mathfrak{E}})\times\mathfrak{B}, \tag{5.5.2}$$

and

$$\boldsymbol{b}^t = -\mathscr{L}\mathbf{1}_R - \mathbf{C}\cdot\mathbb{S}^E + \mathfrak{D}\otimes\bar{\mathfrak{E}} + \mathfrak{B}\otimes\mathfrak{H}, \tag{5.6.2}$$

where $\mathscr{L}$ is given by (4.13) with W depending now on $\mathbb{E}$ rather than on $\boldsymbol{F}$. We can now proceed to the analysis of the material force acting on the tip of a straight through crack in dynamical electromagneto-elasticity. It suffices to remark, following the analysis of Section 3, that $\mathscr{F}_1$ in the direction of $\boldsymbol{e}_1$ will now be given by

$$\mathscr{F}_1(\partial\Sigma) = \lim_{\Gamma\to 0}\int_\Gamma \boldsymbol{e}_1\cdot(\boldsymbol{b}^t - \mathscr{P}^t\otimes\mathscr{U})\cdot\boldsymbol{N}\,d\Gamma. \tag{5.7}$$

As the other contributions now offer no difficulty on account of the results of Section 3, we need concentrate only on the contribution due to electromagnetic pseudomomentum. We thus evaluate the quantity

$$\frac{1}{C}\,\boldsymbol{e}_1\cdot[J_F(\mathbf{C}^{-1}\cdot\bar{\mathfrak{E}})\times\mathfrak{B}]\,(\mathscr{U}\cdot\boldsymbol{N}) \tag{5.8}$$

on account of (4.4) and the fact that $\bar{\mathfrak{E}}$ and $\mathfrak{B}$ may present singularities of the order of $1/\sqrt{r}$ at the tip of the crack. We notice that

$$\begin{aligned}
&\mathscr{U}\cdot\boldsymbol{N} = -\dot{l}\boldsymbol{e}_1\cdot\boldsymbol{N},\\
&(\boldsymbol{e}_1\cdot\nabla_R)\,\mathfrak{A} = \dot{l}^{-1}\partial\mathfrak{A}/\partial t \cong C\dot{l}^{-1}\bar{\mathfrak{E}},\\
&\boldsymbol{e}_1\times(\nabla_R\times\mathfrak{A}) \cong -(\boldsymbol{e}_1\cdot\nabla)\,\mathfrak{A},
\end{aligned} \tag{5.9}$$

so that, in the neighborhood of the crack tip ($\dot{l} \neq 0$)

$$\frac{1}{C} e_1 \cdot [J_F(\mathfrak{C}^{-1} \cdot \bar{\mathfrak{E}}) \cdot \mathfrak{B}] (\mathscr{U} \cdot N) \cong -J_F(\bar{\mathfrak{E}} \cdot \mathfrak{C}^{-1} \cdot \bar{\mathfrak{E}}) (e_1 \cdot N). \tag{5.10}$$

Thus

$$\begin{aligned} &-\mathscr{L}(e_1 \cdot N) - (e_1 \cdot \mathscr{P}^t) (\mathscr{U} \cdot N) = -\mathscr{L}(e_1 \cdot N) + \varrho_0 \dot{u}^2 (e_1 \cdot N) + J_F(\bar{\mathfrak{E}} \cdot \mathfrak{C}^{-1} \cdot \bar{\mathfrak{E}}) (e_1 \cdot N) \\ &= \left\{ W - \frac{1}{2} \varrho_0 v^2 + \varrho_0 \dot{u}^2 + \frac{1}{2} J_F(B^2 + E^2) \right\} (e_1 \cdot N) \qquad (5.11) \\ &= \left\{ W + \frac{1}{2} \varrho_0 \dot{u}^2 + \frac{1}{2} J_F(\bar{\mathfrak{E}} \cdot \mathfrak{C}^{-1} \cdot \bar{\mathfrak{E}}) + \frac{1}{2} J_F^{-1}(\mathfrak{B} \cdot \mathfrak{C} \cdot \mathfrak{B}) \right\} (e_1 \cdot N) = \mathscr{H}^t(e_1 \cdot N), \end{aligned}$$

where $\mathscr{H}^t$ is the total (matter plus fields and interactions) Hamiltonian per unit volume in $\mathscr{K}_R$. Hence (5.7) reads

$$\mathscr{F}_1(\partial \Sigma) = \lim_{\Gamma \to 0} J_\Gamma^{\text{en}}, \tag{5.12}$$

wherein ($N_1 = e \cdot N$)

$$J_\Gamma{}^{\text{em}} = \int_\Gamma \left\{ \mathscr{H}^t N_1 - (e_1 \cdot \mathfrak{C}) \cdot (\mathbb{S}^E \cdot N) + (e_1 \cdot \bar{\mathfrak{E}}) (\mathfrak{D} \cdot N) + (e_1 \cdot \mathfrak{H}) (\mathfrak{B} \cdot N) \right\} d\Gamma, \tag{5.13}$$

which is the desired result. This can also be rewritten as

$$\begin{aligned} J_\Gamma{}^{\text{em}} = \int_\Gamma \Bigg\{ & \mathscr{H}^t N_1 - 2(e \cdot \mathbb{E}) \cdot (\mathbb{S}^E \cdot N) \\ & - \left[(e_1 \cdot \nabla_R) \phi + \frac{1}{C} \frac{\partial}{\partial t} (e_1 \cdot \mathfrak{A}) \right] (\mathfrak{D} \cdot N) + (e_1 \cdot \mathfrak{H}) (\mathfrak{B} \cdot N) \Bigg\} d\Gamma. \end{aligned} \tag{5.14}$$

In particular, with $e_1 \cdot \nabla_R = \partial/\partial X$ for quasi-statics, one obtains the reduction

$$J_\Gamma{}^{\text{em}} = \int_\Gamma \left\{ \left[W + \frac{1}{2} J_F(E^2 + B^2) \right] N_1 - T^\Gamma \cdot \frac{\partial u}{\partial X} - Q^\Gamma \frac{\partial \phi}{\partial X} + \mathfrak{H}_1(\mathfrak{B} \cdot N) \right\} d\Gamma, \tag{5.15}$$

wherein $T^\Gamma = \mathbb{S}^E \cdot N$ and $Q^\Gamma = \mathfrak{D} \cdot N$. In the electroelastic case we have the further reduction

$$J_\Gamma{}^E = \int_\Gamma \left\{ \left[W(\mathbb{E}, \mathfrak{E}) + \frac{1}{2} J_F \mathfrak{E} \cdot \mathfrak{C}^{-1} \cdot \mathfrak{E} \right] N_1 - T^\Gamma \cdot \frac{\partial u}{\partial X} - Q^\Gamma \frac{\partial \phi}{\partial X} \right\} d\Gamma \tag{5.16}$$

in agreement with Pak and Herrmann [28] or Parton and Kudryavtsev [30] — although the latter authors clearly have — see their Eq. (33.1) in p. 313 — in their derivation an essentially linear view of mechanical and electric behaviors. In the above Eqs. (5.13)–(5.16), the material may be anisotropic and nonlinear (allowing thus for piezoelectricity and higher order electroelastic couplings [21]), in particular from the electric point of view as may happen in ferroelectric crystals or ceramics.

6 Concluding remarks

The above-given derivations clearly exhibit the crucial role played by pseudomomentum in obtaining J-integrals of dynamic fracture in agreement with those of a global-dissipation analysis. As a matter of fact, pseudomomentum plays a fundamental part in all dynamical processes expressed on the material manifold. This is particularly true of linear and nonlinear wave phenomena including electromagnetic optics. For this subject matter that extends outside the scope of the present paper we refer the reader to original works [31]–[33]. As a matter of fact, Eq. (2.9) is the balance or unbalance of pseudomomentum while Eshelby's stress and the inhomogeneity force are but the flux and source terms associated with it!

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