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Jump Conditions and Boundary Conditions for a Multi-Continuum Theory for Composite Elastic Materials

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With 3 Figures

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Summary — Zusammenfassung

Jump Conditions and Boundary Conditions for a Multi-Continuum Theory for Composite Elastic Materials. The jump conditions for mass, momentum and energy across a propagating singularity surface are derived for a theory of mixtures developed for application to bonded composite materials. The jump conditions provide equations necessary for the study of shock and acceleration waves, and are also used to deduce the boundary conditions for the theory.

Sprung- und Randbedingungen einer Kontinuumstheorie elastischer Verbundwerkstoffe. Die Sprungbedingungen für Masse, Impuls und Energie an einer sich ausbreitenden Singularitätenfläche einer, zur Anwendung auf Verbundwerkstoffen entwickelten Theorie von Gemischen werden abgeleitet. Die Sprungbedingungen liefern zur Untersuchung von Stoß und Beschleunigungswellen benötigte Gleichungen, und sie werden zur Herleitung der Randbedingungen verwendet.

1. Introduction

In reference [1] a continuum theory of mixtures has been developed for application to bonded composite materials. Field equations were developed and a simple constitutive theory for elastic transversely isotropic materials was proposed.

In this note, the field equations of [1] are rederived in a way which also yields the jump conditions for mass, momentum and energy across a propagating singularity surface. These jump conditions are used to deduce the boundary conditions appropriate for the theory and also provide an extension to the mixture theory of composite materials of the equations necessary for the analysis of shock and acceleration wave propagation. In obtaining the latter equations, an interesting result is that the theory leads to a distinct shock surface for each composite constituent. Fundamental assumptions from [1] are reviewed, only Cartesian tensor notation is used, and results required from field theory are presented in an appendix.

2. Preliminaries

Consider a mixture of materials modeled by N superimposed continua. If these superimposed continua are to represent a bonded composite material, the stress-free, equilibrium temperature state of the mixture is of special importance, and will be called the reference configuration. Points in the reference configuration can be specified by vectors $\boldsymbol{X}$ having components X_K in a Cartesian reference coordinate system. If the superimposed continua are permitted to have individual motions, the subsequent motion of the composite material will carry the material particles originally located at $\boldsymbol{X}$ into various spatial positions specified by vectors $\boldsymbol{x}_{(\xi)}$, $\xi = 1, 2, \ldots, N$, having components $x_{(\xi)k}$ in a Cartesian spatial coordinate system. For simplicity, the two coordinate systems can be taken as superimposed. The individual motions can be expressed functionally by

$$x_{(\xi)k} = \chi_{(\xi)k}(X_K, t) \tag{2.1}$$

where the vector valued function $\boldsymbol{\chi}_{(\xi)}$ is called the motion. The velocity of the ξth constituent is defined by

$$v_{(\xi)k} = \left.\frac{\partial \chi_{(\xi)k}\ (X_K, t)}{\partial t}\right|_{X_K}. \tag{2.2}$$

The set of material particles, one for each constituent, located at $\boldsymbol{X}$ in the reference configuration are called congruent particles. A closely coupled mixture is then defined as one in which mechanical and thermal interactions occur only between congruent particles. As discussed in [1], this definition is motivated by the nature of the interactions in bonded composite materials.

The method in this note has been widely applied to single continua and can be applied to a broad class of continuum theories. The results are obtained by applying the balance and conservation postulates for mass, momentum and energy to a material volume containing a discontinuity surface. This procedure yields the usual field equations and also the jump conditions at a discontinuity surface, thus extending the shock and acceleration wave equations to a closely coupled mixture. Further, by identifying the discontinuity surface with a boundary, the boundary conditions for the mixture are obtained.

The principal tools required are the generalized Green-Gauss theorem and the rate of change of a tensor quantity in a material volume containing a discontinuity surface. These classical results [2], [3] are presented in the appendix.

3. Analysis

Consider a volume V in the reference configuration which is intersected by a surface Σ. The subsequent motions of the continua will carry the volume V into material volumes $v_{(\xi)}$, $\xi = 1, 2, \ldots, N$, and will carry the surface Σ into surfaces $\sigma_{(\xi)}$, $\xi = 1, 2, \ldots, N$, as shown in Fig. 1. In the analysis to follow, the surfaces $\sigma_{(\xi)}$ will be assumed to be discontinuity surfaces.

This introduction of multiple discontinuity surfaces may at first seem puzzling. It will be found in the derivations to follow that this concept arises naturally. However, it is also well motivated physically. Because of the way they are defined, the surfaces $\sigma_{(\xi)}$ are imbedded in congruent particles. Since these are the particles between which interactions occur in a closely coupled mixture, constituent interactions resulting from a discontinuity surface naturally occur between the surfaces $\sigma_{(\xi)}$. Finally, there is some experimental evidence that a shock wave in a

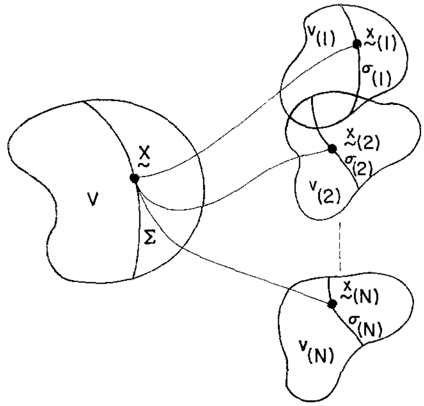

Fig. 1. A set of material volumes and surfaces

composite material propagates in the form of discrete, but interacting, waves in the individual constituents [4].

Conservation of Mass. If it is assumed that mass transfer between constituents does not occur, then the total mass of each constituent material volume $v_{(\xi)}$ must be constant. If $\varrho_{(\xi)}$ is the partial density of the ξth constituent,

$$\frac{d}{dt}\int\limits_{v_{(\xi)}} \varrho_{(\xi)}\, dv_{(\xi)} = 0. \tag{3.1}$$

Using Eq. (A. 3), this is

$$\int\limits_{v^-_{(\xi)}+v^+_{(\xi)}} \left[\frac{\partial \varrho_{(\xi)}}{\partial t} + (\varrho_{(\xi)} v_{(\xi)k})_{,k}\right] dv_{(\xi)} + \int\limits_{\sigma_{(\xi)}} [\![\varrho_{(\xi)}(v_{(\xi)k} - v^{\sigma}_{(\xi)k})]\!]\, n^{\sigma}_{(\xi)k}\, ds_{(\xi)} = 0. \tag{3.2}$$

New notation introduced in (3.2) is described in the appendix. Since $v_{(\xi)}$ can be chosen arbitrarily, (3.2) gives

$$\frac{\partial \varrho_{(\xi)}}{\partial t} + (\varrho_{(\xi)} v_{(\xi)k})_{,k} = 0. \tag{3.3}$$

Then choosing $\sigma_{(\xi)}$ arbitrarily leads to

$$[\![\varrho_{(\xi)}(v_{(\xi)k} - v^{\sigma}_{(\xi)k})]\!]\, n^{\sigma}_{(\xi)k} = 0. \tag{3.4}$$

These equations are identical to the usual result for a single continuum, and apply to each value of $\xi = 1, 2, \ldots, N$. Eq. (3.4) is the jump condition that must be satisfied at a propagating discontinuity such as a shock or acceleration wave. If the discontinuity $\sigma_{(\xi)}$ is a boundary, the constituent velocity $v_{(\xi)k}$ and the discontinuity velocity $v^{\sigma}_{(\xi)k}$ will be identical and (3.4) is satisfied identically. Eqs. (3.3) and (3.4) apply both to closely coupled mixtures and to ordinary mixtures.

Balance of Linear Momentum. Since interactions between constituents occur between congruent particles, the linear momentum postulate for a closely coupled mixture is that the rate of change of the total linear momentum of a set of material volumes $v_{(\xi)}$, $\xi = 1, 2, \ldots, N$, consisting of congruent particles of the mixture (see Fig. 1) is equal to the force exerted on the set by the partial stresses $t_{(\xi)kj}$ of the constituents and by the constituent external body forces $f_{(\xi)k}$

$$\frac{d}{dt}\sum_{\xi}\int_{v_{(\xi)}} \varrho_{(\xi)} v_{(\xi)k}\, dv_{(\xi)} = \sum_{\xi}\int_{s_{(\xi)}} t_{(\xi)kj} n_{(\xi)j}\, ds_{(\xi)} + \sum_{\xi}\int_{v_{(\xi)}} \varrho_{(\xi)} f_{(\xi)k}\, dv_{(\xi)}. \tag{3.5}$$

Using (A. 3) on the first term in (3.5) and (A. 2) on the second term, this becomes

$$\begin{aligned}
&\sum_{\xi}\int_{v_{(\xi)}^{-}+v_{(\xi)}^{+}} \left[\frac{\partial}{\partial t}(\varrho_{(\xi)} v_{(\xi)k}) + (\varrho_{(\xi)} v_{(\xi)k} v_{(\xi)j})_{,j}\right] dv_{(\xi)} \\
&\quad + \sum_{\xi}\int_{\sigma_{(\xi)}} [\![\varrho_{(\xi)} v_{(\xi)k}(v_{(\xi)j} - v_{(\xi)j}^{\sigma})]\!]\, n_{(\xi)j}^{\sigma}\, ds_{(\xi)} \\
&\qquad = \sum_{\xi}\int_{v_{(\xi)}^{-}+v_{(\xi)}^{+}} t_{(\xi)kj,j}\, dv_{(\xi)} + \sum_{\xi}\int_{\sigma_{(\xi)}} [\![t_{(\xi)kj}]\!]\, n_{(\xi)j}^{\sigma}\, ds_{(\xi)} \\
&\qquad + \sum_{\xi}\int_{v_{(\xi)}^{-}+v_{(\xi)}^{+}} \varrho_{(\xi)} f_{(\xi)k}\, dv_{(\xi)}.
\end{aligned} \tag{3.6}$$

Expanding the first term and using (3.3), (3.6) can be written

$$\begin{aligned}
&\sum_{\xi}\int_{v_{(\xi)}^{-}+v_{(\xi)}^{+}} [\varrho_{(\xi)} a_{(\xi)k} - t_{(\xi)kj,j} - \varrho_{(\xi)} f_{(\xi)k}]\, dv_{(\xi)} \\
&+ \sum_{\xi}\int_{\sigma_{(\xi)}} [\![\varrho_{(\xi)} v_{(\xi)k}(v_{(\xi)j} - v_{(\xi)j}^{\sigma}) - t_{(\xi)kj}]\!]\, n_{(\xi)j}^{\sigma}\, ds_{(\xi)} = 0
\end{aligned} \tag{3.7}$$

where $$a_{(\xi)k} = \frac{\partial v_{(\xi)k}}{\partial t} + v_{(\xi)j} v_{(\xi)k,j}.$$

The usual next step in derivations of this kind is to conclude that the integrands vanish by noting that the volume of integration is arbitrary. However, in (3.7) the volumes $v_{(\xi)}$ are *not* arbitrary since they are a set of material volumes made up of congruent particles and thus cannot be chosen independently. This difficulty was resolved [1], [5] for the first term in (3.7) by changing the integrations over $v_{(\xi)}$ into integrations over the single material volume V in the reference configuration by the transformation

$$dv_{(\xi)} = J_{(\xi)}\, dV \tag{3.8}$$

where $J_{(\xi)} = \det \dfrac{\partial x_{(\xi)k}}{\partial X_K}$ [6]. The material volume V can then be chosen arbitrarily. For the same reason, the second term in (3.7) must be transformed into integrals over a single surface Σ in the reference configuration. Thus, the jump conditions at a singularity in a closely coupled mixture can only be derived for surfaces $\sigma_{(\xi)}$ imbedded in congruent particles.

The transformation of the area integrals can be carried out by writing $n^{\sigma}_{(\xi)j}\, ds_{(\xi)} = da^{\sigma}_{(\xi)j}$, where $da^{\sigma}_{(\xi)j}$ is the constituent area vector in the current configuration, and then transforming the constituent area vector to the reference configuration by the usual transformation [6]

$$da^{\sigma}_{(\xi)j} = J_{(\xi)} X_{(\xi)K,j}\, dA_K \tag{3.9}$$

where $X_{(\xi)K,j} = \dfrac{\partial X_K}{\partial x_{(\xi)j}}$ and dA_K is the area vector in the reference configuration. Completing the transformation, we write $dA_K = N_K{}^{\Sigma}\, dS$, where $N_K{}^{\Sigma}$ is the unit normal to the surface Σ. With these transformations, (3.7) becomes

$$\begin{aligned} \int_V \sum_\xi \,&[\varrho_{(\xi)} a_{(\xi)k} - t_{(\xi)kj,j} - \varrho_{(\xi)} f_{(\xi)k}]\, J_{(\xi)}\, dV \\ &+ \int_\Sigma \sum_\xi [\![\varrho_{(\xi)} v_{(\xi)k} (v_{(\xi)j} - v^{\sigma}_{(\xi)j}) - t_{(\xi)kj}]\!]\, J_{(\xi)} \tilde{n}^{\sigma}_{(\xi)j}\, dS = 0 \end{aligned} \tag{3.10}$$

where $\tilde{n}^{\sigma}_{(\xi)j} = X_{(\xi)K,j} N_K{}^{\Sigma}$. Note that $\tilde{n}^{\sigma}_{(\xi)j}$ is normal to $\sigma_{(\xi)}$, although it will not in general be a unit vector. Also note in (3.10) that $J_{(\xi)} \tilde{n}^{\sigma}_{(\xi)j}$ can be evaluated on either side of the discontinuity if the material exists on both sides.

Now since V can be chosen arbitrarily it can be concluded from (3.10) that

$$\sum_\xi [\varrho_{(\xi)} a_{(\xi)k} - t_{(\xi)kj,j} - \varrho_{(\xi)} f_{(\xi)k}]\, J_{(\xi)} = 0 \tag{3.11}$$

and then since Σ can also be chosen arbitrarily, (3.10) gives

$$\sum_\xi [\![\varrho_{(\xi)} v_{(\xi)k} (v_{(\xi)j} - v^{\sigma}_{(\xi)j}) - t_{(\xi)kj}]\!]\, J_{(\xi)}\, \tilde{n}^{\sigma}_{(\xi)j} = 0. \tag{3.12}$$

Eqs. (3.11) and (3.12) can be written

$$[\varrho_{(\xi)} a_{(\xi)k} - t_{(\xi)kj,j} - \varrho_{(\xi)} f_{(\xi)k}]\, J_{(\xi)} = p'_{(\xi)k} \tag{3.13}$$

where $\sum_\xi p'_{(\xi)k} = 0$, and

$$[\![\varrho_{(\xi)} v_{(\xi)k} (v_{(\xi)j} - v^{\sigma}_{(\xi)j}) - t_{(\xi)kj}]\!]\, J_{(\xi)} \tilde{n}^{\sigma}_{(\xi)j} = b_{(\xi)k} \tag{3.14}$$

where $\sum_\xi b_{(\xi)k} = 0$.

The momentum balance Eq. (3.13) is equivalent to the one derived in [1] where $p'_{(\xi)k}$ is the constituent momentum transfer vector. Here the primary interest is in the jump condition (3.14). The vector $b_{(\xi)k}$ can be interpreted as a surface momentum transfer term. It represents the force interaction between constituents at a propagating singularity such as a shock or acceleration wave.

For the special case of a shock propagating into the undeformed medium, $J_{(\xi)} \tilde{n}^{\sigma}_{(\xi)j}$ can be evaluated ahead of the shock. Since there is no deformation ahead of the shock, $J_{(\xi)} = 1$ and $\tilde{n}^{\sigma}_{(\xi)j} = X_{(\xi)K,j}\, N_K{}^{\Sigma} = \delta_{Kj}\, N_K{}^{\Sigma} = n^{\sigma}_{(\xi)j}$, so that (3.14) has the simple form

$$[\![\varrho_{(\xi)} v_{(\xi)k} (v_{(\xi)j} - v^{\sigma}_{(\xi)j}) - t_{(\xi)kj}]\!]\, n^{\sigma}_{(\xi)j} = b_{(\xi)k}. \tag{3.15}$$

To obtain the boundary condition at a material interface or at a free surface, the discontinuity surface can be identified with the material interface or free surface so that the discontinuity velocity is equal to the particle velocity, $v^{\sigma}_{(\xi)j} = v_{(\xi)j}$, and (3.12) becomes the stress boundary condition for a closely coupled mixture.

$$\sum_{\xi} [\![t_{(\xi)kj}]\!] \, J_{(\xi)} \tilde{n}^{\sigma}_{(\xi)j} = 0. \tag{3.16}$$

If the boundary is a free surface, (3.16) reduces to

$$\sum_{\xi} t_{(\xi)kj} \, J_{(\xi)} \tilde{n}^{\sigma}_{(\xi)j} = 0. \tag{3.17}$$

For a linearized theory, let $\tilde{t}_{(\xi)kj}$ represent a linear constitutive functional which vanishes in the reference configuration, i.e. $\tilde{t}_{(\xi)kj}$ has no zeroth order term. To zeroth order, $J_{(\xi)} \tilde{n}^{\sigma}_{(\xi)j} = n^{\sigma}_{(\xi)j}$, so that the linearized forms of (3.16) and (3.17) lead to the familiar boundary conditions

$$\sum_{\xi} [\![\tilde{t}_{(\xi)kj}]\!] n^{\sigma}_{(\xi)j} = 0 \tag{3.18}$$

and, at a free surface,

$$\sum_{\xi} \tilde{t}_{(\xi)kj} n^{\sigma}_{(\xi)j} = 0. \tag{3.19}$$

Conservation of Energy. The conservation of energy postulate for a closely coupled mixture is that the rate of change of the total energy of a collection of material volumes consisting of congruent particles is equal to the rate of work done by partial stresses, body forces and interaction forces plus the rate of energy transfer by the constituent heat transfer vectors $q_{(\xi)k}$ and the constituent external heat supplies $h_{(\xi)}$.

$$\begin{aligned}
&\frac{d}{dt} \sum_{\xi} \int_{v_{(\xi)}} \left[\varrho_{(\xi)} \varepsilon_{(\xi)} + \frac{1}{2} \varrho_{(\xi)} v_{(\xi)k} v_{(\xi)k} \right] dv_{(\xi)} \\
&\qquad = \sum_{\xi} \int_{s_{(\xi)}} t_{(\xi)kj} \, n_{(\xi)j} \, v_{(\xi)k} \, ds_{(\xi)} + \sum_{\xi} \int_{v_{(\xi)}} \varrho_{(\xi)} f_{(\xi)k} \, v_{(\xi)k} \, dv_{(\xi)} \\
&\qquad\quad + \sum_{\xi} \int_{v_{(\xi)}} \frac{p_{(\xi)k}}{J_{(\xi)}} \, dv_{(\xi)} - \sum_{\xi} \int_{s_{(\xi)}} q_{(\xi)k} n_{(\xi)k} \, ds_{(\xi)} + \sum_{\xi} \int_{v_{(\xi)}} \varrho_{(\xi)} h_{(\xi)} \, dv_{(\xi)}
\end{aligned} \tag{3.20}$$

where $\varepsilon_{(\xi)}$ is the constituent specific internal energy. Applying (A. 2) and (A. 3) and using (3.3) and (3.13), this becomes

$$\begin{aligned}
&\sum_{\xi} \int_{v^{-}_{(\xi)} + v^{+}_{(\xi)}} \left[\varrho_{(\xi)} \frac{D\varepsilon_{(\xi)}}{Dt} - t_{(\xi)kj} \, v_{(\xi)k,j} + q_{(\xi)k,k} - \varrho_{(\xi)} h_{(\xi)} \right] dv_{(\xi)} \\
&+ \sum_{\xi} \int_{\sigma_{(\xi)}} \Big[\!\Big[\varrho_{(\xi)} \varepsilon_{(\xi)} (v_{(\xi)j} - v^{\sigma}_{(\xi)j}) + \frac{1}{2} \varrho_{(\xi)} v_{(\xi)k} \, v_{(\xi)k} (v_{(\xi)j} - v^{\sigma}_{(\xi)j}) \\
&\qquad - t_{(\xi)kj} \, v_{(\xi)k} + q_{(\xi)j} \Big]\!\Big] \, n^{\sigma}_{(\xi)j} \, ds_{(\xi)} = 0
\end{aligned} \tag{3.21}$$

where $\frac{D\varepsilon_{(\xi)}}{Dt} = \frac{\partial \varepsilon_{(\xi)}}{\partial t} + v_{(\xi)k}\,\varepsilon_{(\xi),k}$. Transforming the integrals to integrals on V and Σ as before, it is concluded that

$$\sum_\xi \left[\varrho_{(\xi)} \frac{D\varepsilon_{(\xi)}}{Dt} - t_{(\xi)kj}\, v_{(\xi)k,j} + q_{(\xi)k,k} - \varrho_{(\xi)} h_{(\xi)}\right] J_{(\xi)} = 0\,, \tag{3.22}$$

$$\begin{aligned}\sum_\xi \Big[\!\!\Big[\varrho_{(\xi)}\varepsilon_{(\xi)}(v_{(\xi)j} - v^\sigma_{(\xi)j}) + \frac{1}{2}\varrho_{(\xi)} v_{(\xi)k}\, v_{(\xi)k}(v_{(\xi)j} - v^\sigma_{(\xi)j}) \\ - t_{(\xi)kj}\, v_{(\xi)k} + q_{(\xi)j} \Big]\!\!\Big] J_{(\xi)}\, \tilde{n}^\sigma_{(\xi)j} = 0\end{aligned} \tag{3.23}$$

and these can be written

$$\left[\varrho_{(\xi)} \frac{D\varepsilon_{(\xi)}}{Dt} - t_{(\xi)kj} v_{(\xi)k,j} + q_{(\xi)k,k} - \varrho_{(\xi)} h_{(\xi)}\right] J_{(\xi)} = e'_{(\xi)} \tag{3.24}$$

where $\sum_\xi e'_{(\xi)} = 0$, and

$$\begin{aligned}\Big[\!\!\Big[\varrho_{(\xi)}\varepsilon_{(\xi)}(v_{(\xi)j} - v^\sigma_{(\xi)j}) + \frac{1}{2}\varrho_{(\xi)} v^2_{(\xi)}(v_{(\xi)j} - v^\sigma_{(\xi)j}) \\ - t_{(\xi)kj}\, v_{(\xi)k} + q_{(\xi)j} \Big]\!\!\Big] J_{(\xi)} \tilde{n}^\sigma_{(\xi)j} = c_{(\xi)}\end{aligned} \tag{3.25}$$

where $\sum_\xi c_{(\xi)} = 0$.

Again, the Eq. (3.24) is equivalent to the energy conservation equation derived in [1] where $e'_{(\xi)}$ is the constituent energy transfer term. In (3.25), $c_{(\xi)}$ is a surface energy transfer term. It represents the energy transfer between constituents at a propagating singularity.

For a shock propagating into undisturbed material, (3.25) becomes

$$\begin{aligned}\Big[\!\!\Big[\varrho_{(\xi)}\varepsilon_{(\xi)}(v_{(\xi)j} - v^\sigma_{(\xi)j}) + \frac{1}{2}\varrho_{(\xi)} v^2_{(\xi)}(v_{(\xi)j} - v^\sigma_{(\xi)j}) \\ - t_{(\xi)kj}\, v_{(\xi)k} + q_{(\xi)j} \Big]\!\!\Big] n^\sigma_{(\xi)j} = c_{(\xi)}\,.\end{aligned} \tag{3.26}$$

The boundary condition at a surface, where $v_{(\xi)j} = v^\sigma_{(\xi)j}$, is

$$\sum_\xi \Big[\!\!\Big[q_{(\xi)j} - t_{(\xi)kj} v_{(\xi)k} \Big]\!\!\Big] J_{(\xi)} \tilde{n}^\sigma_{(\xi)j} = 0 \tag{3.27}$$

and for a linearized theory, assuming $\tilde{q}_{(\xi)j}$ and $\tilde{t}_{(\xi)kj}$ to be constitutive functionals of first order and assuming $v_{(\xi)k}$ small, the boundary condition becomes

$$\sum_\xi \Big[\!\!\Big[q_{(\xi)j} \Big]\!\!\Big]\, n^\sigma_{(\xi)j} = 0 \tag{3.28}$$

and, at a free surface,

$$\sum_\xi \tilde{q}_{(\xi)j}\, n^\sigma_{(\xi)j} = 0\,. \tag{3.29}$$

In summary, Eqs. (3.16) and (3.27) constitute the boundary conditions for a nonlinear theory of closely coupled mixtures, while (3.18) and (3.28) are the boundary conditions for a linear theory. Eqs. (3.4), (3.14) and (3.25) are equivalent to the Rankine-Hugoniot equations for a propagating shock wave in a closely coupled mixture.

As a final example, the jump conditions will be written for a one-dimensional, plane shock wave propagating into an undisturbed region of a composite material in which the only motion behind the shock occurs in the direction of propagation. This would correspond, for example, to motion resulting from plane excitation in a direction parallel to the reinforcing of a layered or fibrous material. For this case, Eqs. (3.4), (3.15) and (3.26) become, in scalar form,

$$[\![\varrho_{(\xi)}(v_{(\xi)} - v^\sigma_{(\xi)})]\!] = 0, \tag{3.30}$$

$$[\![\varrho_{(\xi)}v_{(\xi)}(v_{(\xi)} - v^\sigma_{(\xi)}) + P_{(\xi)}]\!] = b_{(\xi)}, \tag{3.31}$$

$$\left[\!\!\left[(\varepsilon_{(\xi)} + \frac{1}{2} v^2_{(\xi)})\, \varrho_{(\xi)}(v_{(\xi)} - v^\sigma_{(\xi)}) + P_{(\xi)}v_{(\xi)} + q_{(\xi)}\right]\!\!\right] = c_{(\xi)} \tag{3.32}$$

where the partial pressure $P_{(\xi)} = -t_{(\xi)}$ has been introduced. With the exception of the transfer terms, the similarity of Eqs. (3.30)—(3.32) to the classical Rankine-Hugoniot equations is apparent.

Appendix

Generalized Green-Gauss Theorem and the Rate of Change of a Tensor Quantity in a Material Volume

Consider a volume v, with smooth surface s and outward norman $\boldsymbol{n}$, which is intersected by a smooth surface σ as shown in Fig. 2. Then consider the integral

$$\int_s \alpha_k n_k \, ds \tag{A. 1}$$

where α_k is a vector point function which may be discontinuous across σ. Let $\boldsymbol{n}^\sigma$ be a unit vector normal to σ and let s^-, v^- and s^+, v^+ denote the area and volume

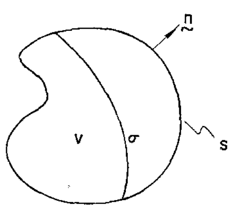

Fig. 2. A volume intersected by a discontinuity surface

of v behind and ahead of σ. ("Ahead" of σ is denoted by the arbitrary positive direction of $\boldsymbol{n}^\sigma$.) The integral (A. 1) can be written

$$\int_s \alpha_k n_k \, ds = \int_{v^- + v^+} \alpha_{k,k} \, dv + \int_\sigma [\![\alpha_k]\!] \, n_k^\sigma \, ds \tag{A. 2}$$

where $\alpha_{k,k} = \dfrac{\partial \alpha_k}{\partial x_k}$ and $[\![\alpha_k]\!] = \alpha_k^+ - \alpha_k^-$ is the jump in value of α_k across σ. Eq. (A. 2) is the generalized Green-Gauss theorem.

Now let v denote a material volume of a continuum and consider the rate of change of a tensor point function ψ integrated over v: $\frac{d}{dt}\int_v \psi\, dv$. The volume v at a time t and at a time $t + dt$ is shown in Fig. 3, where $\boldsymbol{v}$ is the continuum velocity and $\boldsymbol{v}^\sigma$ is the velocity of the surface σ. The rate of change of the integral $\int_v \psi\, dv$ is

$$\frac{d}{dt}\int_v \psi\, dv = \int_{v^- + v^+} \left[\frac{\partial \psi}{\partial t} + (\psi\, v_k)_{,k}\right] dv + \int_\sigma [\![\psi\,(v_k - v_k^\sigma)]\!]\, n_k^\sigma\, ds. \qquad \text{(A. 3)}$$

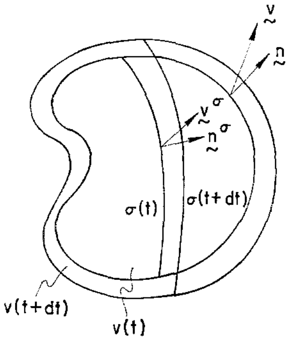

Fig. 3. A material volume intersected by a propagating discontinuity surface

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