Acta Mechanica 148, 187–197 (2001)

ACTA MECHANICA

Penny shaped crack in a three-dimensional piezoelectric strip under in-plane normal loadings

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(Received January 12, 2000)

Summary. The singular mechanical and electric fields in a three-dimensional piezoelectric ceramic strip containing a penny shaped crack under in-plane normal mechanical and electrical loadings based on the continuous electric boundary conditions on the crack surface are considered here. The potential theory and Hankel transforms are used to obtain a system of dual integral equations, which is then expressed as a Fredholm integral equation. All sorts of field intensity factors of Mode I are given, and numerical values for PZT-6B piezoelectric ceramic are graphically shown.

1 Introduction

Piezoelectric materials have been widely used in the devices such as sensors, transducers, and actuators for the reason of its interesting electro-mechanical nature. If piezoelectric materials have micro defects, the initiation and the propagation of cracks due to the stress concentration caused by mechanical and electrical loads may lead to the failure of these materials. Therefore, to prevent failure during service and to predict the service lifetime of piezoelectric components, the fracture mechanics of piezoelectric materials has been paid more attention to, and a lot of significant works are given. Pak [1] obtained the closed form solutions for an infinite piezoelectric medium under anti-plane loading by using a complex variable method. Park and Sun [2] obtained the closed form solutions for all three modes of fracture for an infinite piezoelectric medium containing a center crack subjected to combined mechanical and electrical loadings. Shindo et al. [3], [4] obtained the solutions for the infinite strip parallel or perpendicular to the crack under anti-plane loading using an integral transform method. Kwon and Lee [5] obtained the solutions for piezoelectric rectangular media with a center crack under anti-plane shear loading using an integral transform method.

In the piezoelectric fracture problems, how to impose the electrical boundary conditions on the crack surface is controversial. The impermeable boundary condition along the crack surface has been widely used in the previous works such as Pak [1], Park and Sun [2], Sosa [6]. As was pointed out by Zhang and Tong [7], Gao and Fan [8], Chen and Shioya [9], the results under impermeable conditions show a non-physical singularity around the crack and disagree with experimental results. Recently, Gao and Fan [10] suggested that the normal components of electric displacement and the tangential component of the electric field should be continuous across the crack surface because real cracks in piezoelectric media are filled with vacuum or air.

Because a three-dimensional cracks, such as a penny-shaped crack and an elliptical crack, exists in real media frequently, the fracture analyses for a three-dimensional crack in the piezoelectric material has been done recently. Wang [11] got the solution for an elliptical crack in infinite piezoelectric media using Fourier transform method. Wang et al. [12]–[14] suggested the general solutions to be expressed by the potential functions of three dimensional piezoelectric media. Kogan and Hui [15] gave the closed form solutions for a spheroidal piezoelectric inclusion in an infinite medium. Zhao et al. [16], [17] obtained the fundamental solutions for the unit concentrated displacement and electric potential discontinuity and the stress intensity factor for a circular crack in a piezoelectric solid. But all previous works were treated for unbounded media.

In this paper, we consider the penny shaped crack in a three dimensional piezoelectric ceramic strip under both in-plane mechanical and electrical loads. The continuous electric boundary condition on the crack surface proposed by Gao and Fan [10] is adopted. The potential theory [18], and Hankel transforms [19] are used to obtain a system of dual integral equations, which are then expressed by a Fredholm integral equation of the second kind. Numerical results for the various field intensity factors are given for PZT-6B piezoelectric ceramic.

2 Problem statement

Consider a piezoelectric strip of thickness $2h$ containing a center penny-shaped crack of diameter $2a$ subjected to the combined mechanical and electrical loads as shown in Fig. 1. The cylindrical coordinates (r, θ, z) are set at the center of the crack. The piezoelectric layer is transversely isotropic with hexagonal symmetry, and the z-axis is oriented in the poling direction. The strip is subjected to a constant normal stress σ_0 or a constant normal strain ε_0 at the edges, and the electrical boundary condition of a uniform electric displacement or a uniform electric field for the piezoelectric layer is considered [1]. Because of the symmetry in geometry and loading, it is possible to consider the problem for $0 \leq r \leq \infty$, $0 \leq z \leq h$.

In the axisymmetric problem, displacements and electric fields are independent of θ such that

$$u_r = u_r(r, z)\,, \qquad u_z = u_z(r, z)\,, \qquad u_\theta = 0\,, \tag{1}$$

$$E_r = E_r(r, z)\,, \qquad E_z = E_z(r, z)\,, \qquad E_\theta = 0\,, \tag{2}$$

where u_k and E_k $(k = r, \theta, z)$ are displacements and electric fields, respectively.

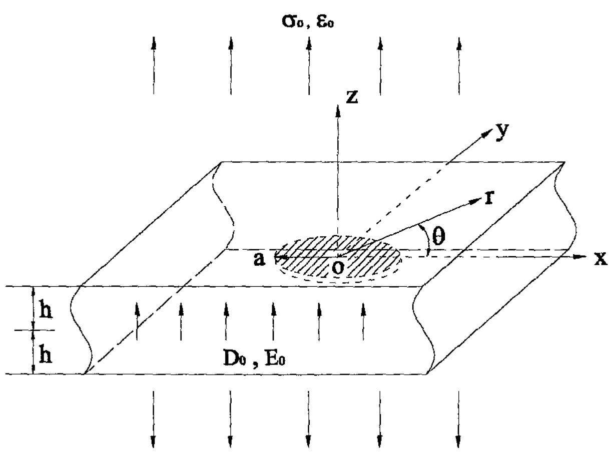

Fig. 1. Infinite piezoelectric strip with a penny shaped crack subject to the combined in-plane mechanical and electrical loadings

Define

$$E_r \equiv -\frac{\partial \phi}{\partial r}\,, \qquad E_z \equiv -\frac{\partial \phi}{\partial z}\,, \tag{3}$$

where ϕ is the electric potential.

The constitutive equations become

$$\begin{aligned}
\sigma_r &= c_{11}\varepsilon_r + c_{12}\varepsilon_\theta + c_{13}\varepsilon_z - e_{31}E_z\,,\\
\sigma_\theta &= c_{12}\varepsilon_r + c_{11}\varepsilon_\theta + c_{13}\varepsilon_z - e_{31}E_z\,,\\
\sigma_z &= c_{13}\varepsilon_r + c_{13}\varepsilon_\theta + c_{33}\varepsilon_z - e_{33}E_z\,,\\
\sigma_{rz} &= 2c_{44}\varepsilon_{rz} - e_{15}E_r\,,\\
D_r &= 2e_{15}\varepsilon_{rz} + d_{11}E_r\,,\\
D_z &= e_{31}\varepsilon_r + e_{31}\varepsilon_\theta + e_{33}\varepsilon_z + d_{33}E_z\,,
\end{aligned} \tag{4}$$

where $\sigma_k\,(k = r, \theta, z)$, σ_{rz} are normal and shear stresses, $D_k\,(k = r, z)$ are electric displacements, $(c_{11}, c_{12}, c_{13}, c_{33}, c_{44})$ are the elastic moduli measured in a constant electric field, (d_{11}, d_{33}) are the dielectric permittivities measured at a constant strain, and (e_{15}, e_{31}, e_{33}) are the piezoelectric constants.

The equilibrium equations are,

$$\begin{aligned}
&\frac{\partial \sigma_r}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0\,,\\
&\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\sigma_{rz}}{r} = 0\,,
\end{aligned} \tag{5}$$

and the equation of electrostatics is

$$\frac{\partial D_r}{\partial r} + \frac{\partial D_z}{\partial z} + \frac{D_r}{r} = 0\,. \tag{6}$$

To get the solutions which satisfy Eqs. (3)–(6), we define the potential in the forms [18]:

$$u_r = \frac{\partial \Phi}{\partial r}\,, \qquad u_z = k_1\,\frac{\partial \Phi}{\partial z}\,, \qquad \phi = -k_2\,\frac{\partial \Phi}{\partial z}\,, \tag{7}$$

where $\Phi(r, z)$ is the potential function,and k_1, k_2 are unknown constants.

Putting Eq. (7) into Eqs. (5) and (6), we can get the governing equation in the form

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + n\,\frac{\partial^2 \Phi}{\partial z^2} = 0\,, \tag{8}$$

where

$$\begin{aligned}
n &= \frac{c_{44} + (c_{13} + c_{44})\,k_1 - (e_{31} + e_{51})\,k_2}{c_{11}}\\
&= \frac{c_{33}k_1 - e_{33}k_2}{c_{44}k_1 + c_{13} + c_{44} - e_{15}k_2} = \frac{e_{33}k_1 + d_{33}k_2}{e_{15}k_1 + e_{15} + e_{31} + d_{11}k_2}\,.
\end{aligned} \tag{9}$$

From Eq. (9), we obtain

$$An^3 + Bn^2 + Cn + D = 0\,, \tag{10}$$

where

$$
\begin{aligned}
A &= c_{44}d_{11} + e_{15}^2 \,, \\
B &= (d_{11}c_{13}^2 - c_{11}c_{33}d_{11} + 2c_{13}c_{44}d_{11} - c_{11}c_{44}d_{33} + 2c_{13}e_{15}^2 + 2c_{13}e_{15}e_{31} \\
&\quad - c_{44}e_{31}^2 - 2c_{11}e_{15}e_{33})/c_{11} \,, \\
C &= (c_{33}c_{44}d_{11} - c_{13}^2 d_{33} + c_{11}c_{33}d_{33} - 2c_{13}c_{44}d_{33} + c_{33}e_{15}^2 + 2c_{33}e_{15}e_{31} \\
&\quad + c_{33}e_{31}^2 - 2c_{13}e_{15}e_{33} - 2c_{13}e_{31}e_{33} - 2c_{44}e_{31}e_{33} + c_{11}e_{33}^2)/c_{11} \,, \\
D &= -c_{44}(c_{33}d_{33} + e_{33}^2)/c_{11} \,.
\end{aligned}
\tag{11}
$$

According to Eq. (10), the governing Eq. (8) becomes

$$
\frac{\partial^2 \Phi_i}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi_i}{\partial r} + \frac{\partial^2 \Phi_i}{\partial {z_i}^2} = 0 \,, \quad (i = 1, 2, 3) \,, \tag{12}
$$

where

$$
z_i = \frac{z}{\sqrt{n_i}} = s_i z \,, \quad (i = 1, 2, 3) \,. \tag{13}
$$

$n_i \, (i = 1, 2, 3)$ are the roots of Eq. (10) and $\Phi_i \, (r, z) \, (i = 1, 2, 3)$ is the potential function corresponding to $n_i \, (i = 1, 2, 3)$.

According to superposition, the displacement and the electric potential equations become

$$
u_r = \sum_{i=1}^{3} \frac{\partial \Phi_i}{\partial r} \,, \qquad u_z = \sum_{i=1}^{3} k_{1i} \frac{\partial \Phi_i}{\partial z} \,, \qquad \phi = -\sum_{i=1}^{3} k_{2i} \frac{\partial \Phi_i}{\partial z} \,, \tag{14}
$$

where k_{1i} and $k_{2i} \, (i = 1, 2, 3)$ are determined from Eq. (9).

According to the suggestions of Gao and Fan [10] we set up the following boundary conditions:

$$
\begin{aligned}
&\sigma_z (r, 0) = 0 \quad (0 \le r < a) \,, \\
&u_z (r, 0) = 0 \quad (a \le r < \infty) \,,
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&D_z (r, 0^+) = D_z (r, 0^-) \quad (0 \le r < a) \,, \\
&E_r (r, 0^+) = E_r (r, 0^-) \quad (0 \le r < a) \,, \\
&\phi (r, 0) = 0 \qquad\qquad\quad (a \le r < \infty) \,,
\end{aligned}
\tag{16}
$$

$$
\sigma_{rz} (r, 0) = 0 \,, \tag{17}
$$

$$
\sigma_{rz} (r, h) = 0 \,. \tag{18}
$$

There may be four possible cases of combined mechanical and electrical loadings in the edge as follows:

$$
\text{(Case 1)} \quad \sigma_z (r, h) = \sigma_0 \,, \quad D_z (r, h) = D_0 \,, \tag{19}
$$

$$
\text{(Case 2)} \quad \varepsilon_z (r, h) = \varepsilon_0 \,, \quad E_z (r, h) = E_0 \,, \tag{20}
$$

$$
\text{(Case 3)} \quad \sigma_z (r, h) = \sigma_0 \,, \quad E_z (r, h) = E_0 \,, \tag{21}
$$

$$
\text{(Case 4)} \quad \varepsilon_z (r, h) = \varepsilon_0 \,, \quad D_z (r, h) = D_0 \,, \tag{22}
$$

where $\sigma_0, \varepsilon_0, D_0$ and E_0 are the uniform applied stress, strain, electric displacement and electric field, respectively.

3 Solution procedure

Applying Hankel transform of order 0 to Eq. (12), we can get the solution

$$\Phi_i(r, z_i) = \int_0^\infty \frac{1}{\xi} \left[A_i(\xi)\cosh(\xi z_i) + B_i(\xi)\sinh(\xi z_i)\right] J_0(\xi r)\, d\xi\,, \tag{23}$$

where $A_i(\xi)$ and $B_i(\xi)$ $(i = 1, 2, 3)$ are the unknown functions to be determined by boundary conditions.

The field equations are obtained in the forms:

$$\begin{aligned}
u_r &= -\sum_{i=1}^{3} \int_0^\infty \left[A_i(\xi)\cosh(\xi z_i) + B_i(\xi)\sinh(\xi z_i)\right] J_1(\xi r)\, d\xi\,,\\
u_z &= \sum_{i=1}^{3} k_{1i} s_i \int_0^\infty \left[A_i(\xi)\sinh(\xi z_i) + B_i(\xi)\cosh(\xi z_i)\right] J_0(\xi r)\, d\xi + a_0 z\,,\\
\phi &= -\sum_{i=1}^{3} k_{2i} s_i \int_0^\infty \left[A_i(\xi)\sinh(\xi z_i) + B_i(\xi)\cosh(\xi z_i)\right] J_0(\xi r)\, d\xi - b_0 z\,,\\
\varepsilon_z &= \sum_{i=1}^{3} k_{1i} s_i^2 \int_0^\infty \xi\left[A_i(\xi)\cosh(\xi z_i) + B_i(\xi)\sinh(\xi z_i)\right] J_0(\xi r)\, d\xi + a_0\,,\\
E_r &= -\sum_{i=1}^{3} k_{2i} s_i \int_0^\infty \xi\left[A_i(\xi)\sinh(\xi z_i) + B_i(\xi)\cosh(\xi z_i)\right] J_1(\xi r)\, d\xi\,,\\
E_z &= \sum_{i=1}^{3} k_{2i} s_i^2 \int_0^\infty \xi\left[A_i(\xi)\cosh(\xi z_i) + B_i(\xi)\sinh(\xi z_i)\right] J_0(\xi r)\, d\xi + b_0\,,\\
\sigma_z &= \sum_{i=1}^{3} F_{1i} \int_0^\infty \xi\left[A_i(\xi)\cosh(\xi z_i) + B_i(\xi)\sinh(\xi z_i)\right] J_0(\xi r)\, d\xi + c_0\,,\\
\sigma_{rz} &= -\sum_{i=1}^{3} F_{3i} \int_0^\infty \xi\left[A_i(\xi)\sinh(\xi z_i) + B_i(\xi)\cosh(\xi z_i)\right] J_1(\xi r)\, d\xi\,,\\
D_z &= \sum_{i=1}^{3} F_{2i} \int_0^\infty \xi\left[A_i(\xi)\cosh(\xi z_i) + B_i(\xi)\sinh(\xi z_i)\right] J_0(\xi r)\, d\xi + d_0\,,
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
F_{1i} &= (c_{33}k_{1i} - e_{33}k_{2i})\, s_i^2 - c_{13}\,,\\
F_{2i} &= (e_{33}k_{1i} + d_{33}k_{2i})\, s_i^2 - e_{31}\,, \quad (i = 1, 2, 3)\\
F_{3i} &= \left[c_{44}(1 + k_{1i}) - e_{15}k_{2i}\right] s_i\,,
\end{aligned} \tag{25}$$

and $a_0, b_0, c_0 = c_{33}a_0 - e_{33}b_0$ and $d_0 = e_{33}a_0 + d_{33}b_0$ are unknown constants to be determined from edge loading conditions.

By applying edge loading conditions, Eqs. (19)–(22), the following equations are obtained:

$$\sum_{i=1}^{3} G_{1i}\left[A_i(\xi)\cosh(\xi s_i h) + B_i(\xi)\sinh(\xi s_i h)\right] = 0\,, \tag{26}$$

$$\sum_{i=1}^{3} G_{2i}\left[A_i(\xi)\cosh(\xi s_i h) + B_i(\xi)\sinh(\xi s_i h)\right] = 0\,, \tag{27}$$

where

$$G_{1i} = (c_{33}k_{1i} - e_{33}k_{2i})\, s_i^{\,2} - c_{13}\,, \quad (\text{Case } 1,\, 3)\,, \qquad (i = 1,2,3)\,,$$
$$\phantom{G_{1i}} = k_{1i}s_i^{\,2}\,, \quad (\text{Case } 2,\, 4)\,, \tag{28}$$

$$G_{2i} = (e_{33}k_{1i} + d_{33}k_{2i})\, s_i^{\,2} - e_{31}\,, \quad (\text{Case } 1,\, 4)\,, \qquad (i = 1,2,3)\,,$$
$$\phantom{G_{2i}} = k_{2i}s_i^{\,2}\,, \quad (\text{Case } 2,\, 3)\,, \tag{29}$$

and a_0, b_0 are evaluated as follows:

$$(\text{Case } 1) \quad a_0 = \frac{d_{33}\sigma_0 + e_{33}D_0}{c_{33}d_{33} + e_{33}^2}\,, \quad b_0 = \frac{c_{33}D_0 - e_{33}\sigma_0}{c_{33}d_{33} + e_{33}^2}\,, \quad c_0 = \sigma_0\,, \quad d_0 = D_0\,, \tag{30}$$

$$(\text{Case } 2) \quad a_0 = \varepsilon_0\,, \quad b_0 = E_0\,, \quad c_0 = c_{33}\varepsilon_0 - e_{33}E_0\,, \quad d_0 = e_{33}\varepsilon_0 + d_{33}E_0\,, \tag{31}$$

$$(\text{Case } 3) \quad a_0 = \frac{\sigma_0 + e_{33}E_0}{c_{33}}\,, \quad b_0 = E_0\,, \quad c_0 = \sigma_0\,, \quad d_0 = \frac{e_{33}\sigma_0 + (c_{33}d_{33} + e_{33}^2)\, E_0}{c_{33}}\,, \tag{32}$$

$$(\text{Case } 4) \quad a_0 = \varepsilon_0\,, \quad b_0 = \frac{D_0 - e_{33}\varepsilon_0}{d_{33}}\,, \quad c_0 = \frac{(c_{33}d_{33} + e_{33}^2)\,\varepsilon_0 - e_{33}D_0}{d_{33}}\,, \quad d_0 = D_0\,. \tag{33}$$

From the conditions of Eqs. (16)–(18), the following equations are obtained:

$$\sum_{i=1}^{3} k_{2i}s_iB_i(\xi) = 0\,, \tag{34}$$

$$\sum_{i=1}^{3} F_{3i}B_i(\xi) = 0\,, \tag{35}$$

$$\sum_{i=1}^{3} F_{3i}\left[A_i(\xi)\sinh(\xi s_i h) + B_i(\xi)\cosh(\xi s_i h)\right] = 0\,. \tag{36}$$

From Eqs. (34) and (35), the relations between the coefficients, $B_i(\xi)$ $(i = 1,2,3)$, are obtained in the forms

$$B_2(\xi) = M_1B_1(\xi)\,,$$
$$B_3(\xi) = M_2B_1(\xi)\,, \tag{37}$$

where

$$M_1 = \frac{F_{31}k_{23}s_3 - F_{33}k_{21}s_1}{F_{33}k_{22}s_2 - F_{32}k_{23}s_3}\,, \qquad M_2 = \frac{F_{32}k_{21}s_1 - F_{31}k_{22}s_2}{F_{33}k_{22}s_2 - F_{32}k_{23}s_3}\,. \tag{38}$$

From Eqs. (26), (27) and (36) using Eqs. (37) and (38), the relations between the coefficients are obtained as following:

$$A_1(\xi) = M_{41}(\xi)B_1(\xi)\,,$$
$$A_2(\xi) = M_{42}(\xi)B_1(\xi)\,,$$
$$A_3(\xi) = M_{43}(\xi)B_1(\xi)\,, \tag{39}$$

where

$$M_{41}(\xi) = \frac{\begin{array}{l}\{[M_{32}(\xi)G_{12}G_{33} - M_{31}(\xi)G_{22}G_{33}]\cosh(\xi s_2 h)\sinh(\xi s_3 h)\\ +[M_{31}(\xi)G_{23}G_{32} - M_{32}(\xi)G_{13}G_{32}]\cosh(\xi s_3 h)\sinh(\xi s_2 h)\\ +[M_{33}(\xi)G_{13}G_{22} - M_{33}(\xi)G_{12}G_{23}]\cosh(\xi s_2 h)\cosh(\xi s_3 h)\}\end{array}}{\Delta(\xi)},$$

$$M_{42}(\xi) = \frac{\begin{array}{l}\{[M_{31}(\xi)G_{21}G_{33} - M_{32}(\xi)G_{11}G_{33}]\cosh(\xi s_1 h)\sinh(\xi s_3 h)\\ +[M_{32}(\xi)G_{13}G_{31} - M_{31}(\xi)G_{23}G_{31}]\cosh(\xi s_3 h)\sinh(\xi s_1 h)\\ +[M_{33}(\xi)G_{11}G_{23} - M_{33}(\xi)G_{13}G_{21}]\cosh(\xi s_1 h)\cosh(\xi s_3 h)\}\end{array}}{\Delta(\xi)}, \tag{40}$$

$$M_{43}(\xi) = \frac{\begin{array}{l}\{[M_{32}(\xi)G_{11}G_{32} - M_{31}(\xi)G_{21}G_{32}]\cosh(\xi s_1 h)\sinh(\xi s_2 h)\\ +[M_{31}(\xi)G_{22}G_{31} - M_{32}(\xi)G_{12}G_{31}]\cosh(\xi s_2 h)\sinh(\xi s_1 h)\\ +[M_{33}(\xi)G_{12}G_{21} - M_{33}(\xi)G_{11}G_{22}]\cosh(\xi s_1 h)\cosh(\xi s_2 h)\}\end{array}}{\Delta(\xi)},$$

$$\begin{aligned}\Delta(\xi) = {} & (G_{11}G_{22}G_{33} - G_{12}G_{21}G_{33})\cosh(\xi s_1 h)\cosh(\xi s_2 h)\sinh(\xi s_3 h)\\ & + (G_{13}G_{21}G_{32} - G_{11}G_{23}G_{32})\cosh(\xi s_1 h)\sinh(\xi s_2 h)\cosh(\xi s_3 h)\\ & + (G_{12}G_{23}G_{31} - G_{13}G_{22}G_{31})\sinh(\xi s_1 h)\cosh(\xi s_2 h)\cosh(\xi s_3 h),\end{aligned}$$

$$M_{31}(\xi) = G_{11}\sinh(\xi s_1 h) + M_1 G_{12}\sinh(\xi s_2 h) + M_2 G_{13}\sinh(\xi s_3 h),$$

$$M_{32}(\xi) = G_{21}\sinh(\xi s_1 h) + M_1 G_{22}\sinh(\xi s_2 h) + M_2 G_{23}\sinh(\xi s_3 h),$$

$$M_{33}(\xi) = G_{31}\cosh(\xi s_1 h) + M_1 G_{32}\cosh(\xi s_2 h) + M_2 G_{33}\cosh(\xi s_3 h),$$

$$G_{3i} = F_{3i}$$

From Eqs. (15), (37) and (39), a system of dual integral equations is obtained as follows:

$$\begin{aligned}&\int_0^\infty N(\xi)B_1(\xi)\xi J_0(\xi r)\,d\xi = -\frac{c_0}{n_0}, \quad (0 \le r < a),\\ &\int_0^\infty B_1(\xi)J_0(\xi r)\,d\xi = 0, \quad (a \le r < \infty),\end{aligned} \tag{41}$$

where

$$N(\xi) = [F_{11}M_{41}(\xi) + F_{12}M_{42}(\xi) + F_{13}M_{43}(\xi)]/n_0, \tag{42}$$

$$n_0 = -[F_{11} + F_{12}M_1 + F_{13}M_2], \tag{43}$$

and $J_0(\,)$ is the zero-order Bessel function of the first kind.

Equations (41) may be solved by using the new function $\psi(\alpha)$ defined by

$$B_1(\xi) = \int_0^a \psi(\alpha)\sin(\xi\alpha)\,d\alpha. \tag{44}$$

Inserting Eq. (44) into Eq. (41), we obtain a Fredholm integral equation of the second kind in the form

$$\Psi(\Xi) + \int_0^1 \Psi(H)L(\Xi, H)\,dH = \Xi, \tag{45}$$

where

$$L(\Xi, H) = \frac{2}{\pi} \int_0^\infty \left[N\left(\frac{S}{a}\right) - 1 \right] \sin(\Xi S) \sin(HS)\, dS\,, \tag{46}$$

$$N\left(\frac{S}{a}\right) = -\frac{F_{11} M_{41}\left(\frac{S}{a}\right) + F_{12} M_{42}\left(\frac{S}{a}\right) + F_{13} M_{43}\left(\frac{S}{a}\right)}{F_{11} + F_{12} M_1 + F_{13} M_2}\,, \tag{47}$$

$$S = \xi a\,, \quad \Xi = \frac{\alpha}{a}\,, \quad H = \frac{\beta}{a}\,, \quad \Psi(\Xi) = -\frac{\pi}{2a} \frac{n_0}{c_0}\, \psi(\alpha)\,, \quad \Psi(H) = -\frac{\pi}{2a} \frac{n_0}{c_0}\, \psi(\beta)\,. \tag{48}$$

4 Field intensity factors

Each kind of intensity factor is obtained in the form

$$K^\sigma = K_I = \lim_{r \to a^+} \sqrt{2\pi(r-a)}\, \sigma_z(r, 0) = \frac{2}{\pi} \sqrt{\pi a}\, c_0 \Psi(1)\,, \tag{49}$$

$$K^D = \lim_{r \to a^+} \sqrt{2\pi(r-a)}\, D_z(r, 0) = \frac{2}{\pi} \sqrt{\pi a}\, \frac{F_{21} + F_{22} M_1 + F_{23} M_2}{F_{11} + F_{12} M_1 + F_{13} M_2}\, c_0 \Psi(1)\,, \tag{50}$$

$$K^\varepsilon = \lim_{r \to a^+} \sqrt{2\pi(r-a)}\, \varepsilon_z(r, 0) = \frac{2}{\pi} \sqrt{\pi a}\, \frac{k_{11} {s_1}^2 + k_{12} {s_2}^2 M_1 + k_{13} {s_3}^2 M_2}{F_{11} + F_{12} M_1 + F_{13} M_2}\, c_0 \Psi(1)\,, \tag{51}$$

$$K^E = \lim_{r \to a^+} \sqrt{2\pi(r-a)}\, E_z(r, 0) = \frac{2}{\pi} \sqrt{\pi a}\, \frac{k_{21} {s_1}^2 + k_{22} {s_2}^2 M_1 + k_{23} {s_3}^2 M_2}{F_{11} + F_{12} M_1 + F_{13} M_2}\, c_0 \Psi(1)\,, \tag{52}$$

where $K^\sigma, K^D, K^\varepsilon$ and K^E are the stress intensity, electric displacement intensity, strain intensity, and electric field intensity factor, respectively.

If the thickness of the strip, h, reaches infinity, $\Psi(1)$ approaches 1, in the case that the stress intensity factor is identical with that of Kogan and Hui [15] for an infinite body in Case 1.

We should recognize that the $\Psi(1)$ in each case is different due to the difference of $N\left(\frac{S}{a}\right)$ in Eq. (47). In Case 1, the stress intensity factor is dependent on the mechanical load, and the electric displacement intensity factor depends on the material properties and the mechanical load, but not on the electrical load. These tendencies are consistent with those of Gao and Fan [8] in the two-dimensional mixed mode problem. Also field intensity factors are independent of the electrical loading under constant stress loading, but dependent on it under constant strain loading, and these results agree with those of Shindo et al. [3], [4], and Zhang and Hack [20] in two-dimensional mode III problem.

5 Numerical results

To investigate the changes of field intensity factors according to the dimensions, Eq. (45) was evaluated by Gaussian quadrature formula. The material properties of PZT-6B ceramic considered here are as follows:

Elastic constants (10^{10} N/m^2): $c_{11} = 16.8,\ c_{12} = 6.0,\ c_{13} = 6.0,\ c_{33} = 16.3,\ c_{44} = 2.71\,,$
Piezoelectric constants (C/m^2): $e_{15} = 4.6,\ e_{31} = -0.9,\ e_{33} = 7.1\,,$
Dielectric permittivity (10^{-10} F/m): $d_{11} = 36,\ d_{33} = 34\,.$

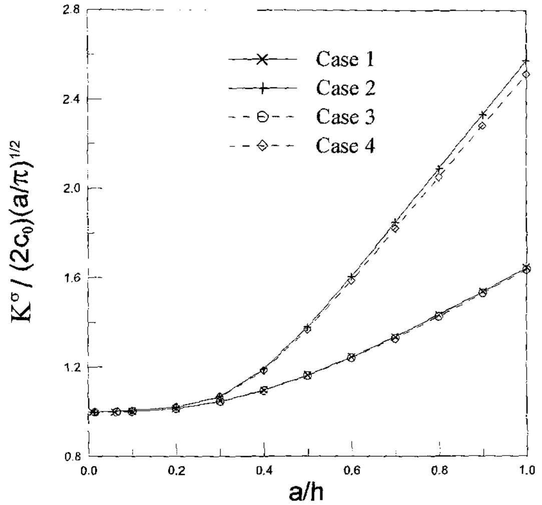

Fig. 2. Change of the stress intensity factor with the ratio between crack radius to layer thickness a/h for PZT-6B ceramic

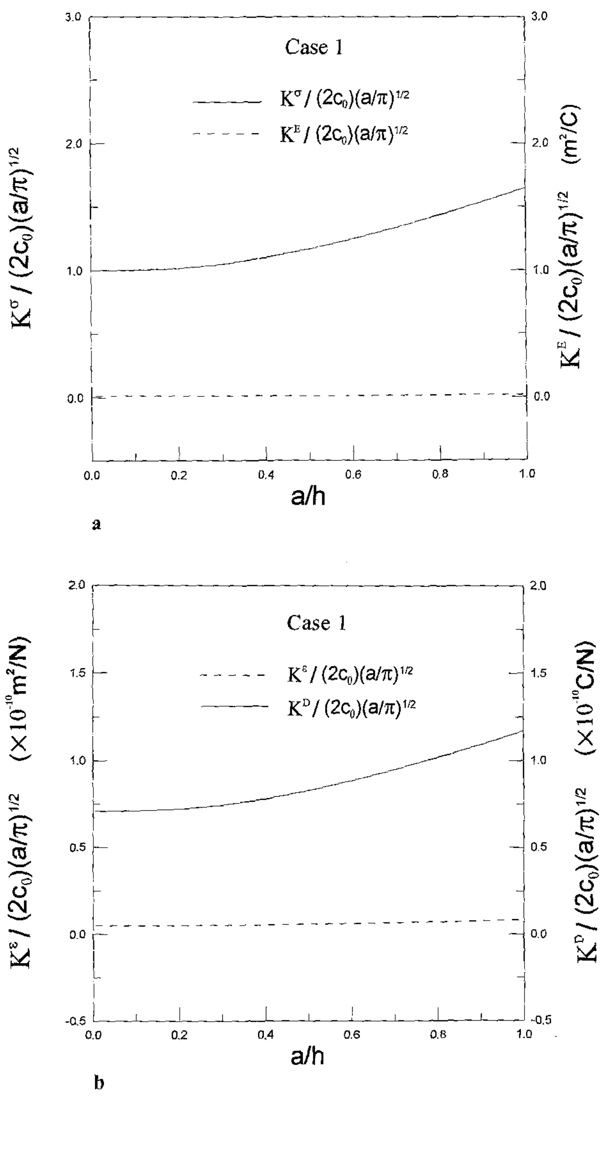

Fig. 3. Change of field intensity factors with the variation of the ratio between crack radius to layer thickness a/h for PZT-6B ceramic in Case 1

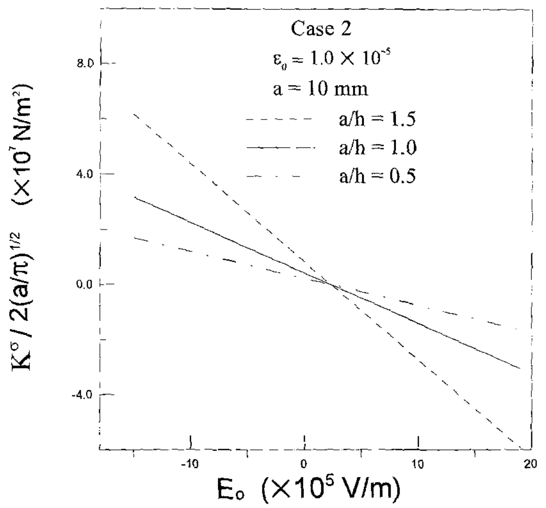

Fig. 4. Change of normalized stress intensity factor $K^\sigma/2(a/\pi)^{1/2}$ with the electrical field E_0 for PZT-6B ceramic in Case 2

The normalized stress intensity factors, $K^\sigma/(2c_0)\,(a/\pi)^{1/2}$, in every case are shown in Fig. 2. It shows that the magnitude and trends of normalized stress intensity factors in Cases 1 and 3 are similar and those in Cases 2 and 4 are similar. Therefore, we conclude that the normalized stress intensity factor is affected more by the conditions of mechanical loadings than by the conditions of electrical loadings.

The variations of the normalized intensity factors according to the ratio a/h are shown in Fig. 3 (Case 1). It is shown that the normalized intensity factors increase with the increase of the ratio a/h, and that the stress intensity factor and the electric field intensity factor are much larger than the strain intensity factor and the electric displacement intensity factor.

Figure 4 shows the variation of $K^\sigma/2(a/\pi)^{1/2}$ according to the applied electric field E_0 with various a/h values for a PZT-6B ceramic in case of a crack diameter $2a = 20$ mm and $\varepsilon_0 = 1.0 \times 10^{-5}$. It is concluded from Fig. 4 that the stress intensity factor may have negative values according to the direction of the electric field. In Case 4, the tendency of the variation of the stress intensity factor with the electric displacement is similar to that of the electric field in Case 2.

6 Conclusions

The field equations and intensity factors for a penny shaped crack in a transversely isotropic piezoelectric ceramic strip under in-plane mechanical and electrical loadings are obtained by the potential theory and the integral transform method. The continuous electric boundary conditions are used on the crack surface. Various field intensity factors are obtained from the solution of a Fredholm integral equation of the second kind. The normalized intensity factors increase with the increase of the ratio of crack radius to the strip thickness. For the case of constant stress loading, the electric field intensity factor and the electric displacement intensity factor depend on the material constants and the applied mechanical load, but not on the applied electrical load. For the case of constant strain loading, the field intensity factors depend on the applied mechanical and electrical loads.

Acknowledgements

This work was supported by the Brain Korea 21 project.

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