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A New Family of Routh Approximants*

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Abstract. Based on the combinatorial Routh $\alpha-\beta$ and $\gamma-\delta$ expansions of a stable transfer function, a new energy decomposition tree for linear systems is developed. The pertinent properties to the energy decomposition tree are investigated, and an algorithm is derived for synthesizing transfer functions from the tree. The synthesis process naturally leads to a new family of Routh approximants to the system. It is indicated that the selection of Routh approximants based on the values of impulse-response energy is often inadequate because there may be a number of different Routh approximants with the same order and the same impulse-response energy. In such cases, an additional performance criterion, such as the integral of squared error of impulse response or unit-step response, has to be used to select a suitable Routh approximant.

1. Introduction

The Routh approximation method uses the Routh stability-test algorithm to generate reduced-order models for linear, time-invariant, continuous-time systems. Since it was originally proposed by Hutton and Friedland [5], the Routh approximation method has received considerable attention [6], [14], [15], [17], [23], [24], [26]–[28], [30], [31], [33]. It has also been recognized as one of the most significant and popular stable model-reduction methods. The significance and popularity of the Routh approximation method are mainly due to the low computational burden for obtaining a set of stable models with different orders and the ability to guarantee a stable reduced-order model that preserves the first time moments and/or Markov parameters of the original stable system. Another useful capability of the Routh approximation method is that it produces in the reduction process the val-

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ues of impulse-response energy of the Routh approximants, which often serve as a criterion for selecting the order of a reduced model.

The advantages of Routh approximation have encouraged several authors to extend the method to obtain stable reduced-order models for other types of systems. Bistritz [1], Hsieh and Hwang [4], Hwang and Hsieh [11], and Hwang and Shih [12] have applied the Routh approximation method along with bilinear transformation to derive stable reduced-order models for linear discrete-time systems. By using the matrix Routh algorithm, Hwang and Guo [9] and Ramakrishnan et al. [22] have developed the matrix Routh approximant technique for obtaining reduced-degree matrix-fraction descriptions for linear multi-input multi-output systems. Recently, Guo et al. [2] have extended the Routh approximation method to the model reduction of two-dimensional (2-D) separable-denominator discrete systems.

In parallel with the study of extending the Routh approximation method to a broader class of systems, considerable effort is devoted to improve the quality of Routh approximants. Because the Routh approximants derived from the Routh $\gamma - \delta$ expansion [27] fit only the initial time moments of the original systems, they may not produce a good approximation in the portion of transient response. Besides, as indicated by Rao [24], the pole zero cancellations in a high-order original system can affect the approximation quality of Routh approximants. To overcome these potential disadvantages, several modified Routh approximation methods have been proposed [3], [7], [8], [10], [13], [16], [18], [32], [34]. Basically, these modifications combine the use of a Routh stability-test array to generate the denominator polynomial with other criteria, such as the minimization of the integral of squared error of time responses [7], [8], [10], [13], the matching of more time moments [18], [34], and the fitting of both time moments and Markov parameters [3], [29], [32], to yield the numerator polynomial of a reduced-order model. Recently, Hsieh and Hwang [3] have proposed a modified Routh approximation method that combines, in a balanced fashion, both the Routh $\alpha - \beta$ and $\gamma - \delta$ expansions for obtaining stable reduced models. Because a Routh $\alpha - \beta$ approximant retains the high-frequency or transient characteristics and a Routh $\gamma - \delta$ approximant retains the low-frequency or steady-state characteristics of the system, this modified Routh approximation method can, in general, give accurate and satisfactory reduced-order models. However, the advantage of associating the impulse-response energy of a modified Routh approximant with the combined Routh $\alpha - \beta$ and $\gamma - \delta$ parameters has not been explored yet.

In this paper, the impulse-response energy approximation technique [19]–[21], [25] is generalized to obtain Routh approximants that retain various combinations of the $\alpha - \beta$ and $\gamma - \delta$ energy parameters of the system. The generalization is based on combinatorially using the Routh $\alpha - \beta$ and $\gamma - \delta$ expansions to peel off the energy components of a linear system described by its transfer function. This decomposition procedure allows one to build a energy decomposition tree for the system. Some interesting properties associated with the tree are identified. In addition, a synthesis procedure is presented to transform the energy decomposition tree to a tree of Routh approximants. It is also remarked that in a Routh approximant tree there are several Routh approximants with the same order and the same impulse-response energy. This fact indicates that the selection of a Routh approximant to the system based on the value of impulse-response energy is inadequate. Therefore, an additional performance criterion of model reduction, such as the integral of squared error of impulse or unit-step response, should be also evaluated for selecting a proper Routh approximant.

2. Impulse-response energy decomposition

Consider a stable system described by the nth-order transfer function

$$G_n(s) = \frac{b_0 + b_1 s + \cdots + b_{n-1}s^{n-1}}{a_0 + a_1 s + \cdots + a_n s^n} \triangleq \frac{B(s)}{A(s)} . \tag{1}$$

Denote the impulse response of the system by

$$g_n(t) = \mathcal{L}^{-1}\{G_n(s)\} . \tag{2}$$

The impulse-response energy of the system, $G_n(s)$, is defined by

$$I_n \triangleq \int_0^\infty g_n^2(t)\,dt = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} G_n(s)G_n(-s)\,ds \tag{3}$$

where $j = \sqrt{-1}$. In the following subsections, the Routh $\alpha - \beta$ and $\gamma - \delta$ expansions are applied to decompose the transfer function $G_n(s)$ and its impulse-response energy I_n.

2.1. *Routh $\gamma - \delta$ expansion.*

The denominator polynomial, $A(s)$, of the transfer function $G_n(s)$ may be decomposed as

$$A(s) = A_1(s^2) + sA_2(s^2) \tag{4}$$

where

$$A_1(s^2) \triangleq \sum_{k=0}^{n_1} a_{1,k}s^{2k} = a_0 + a_2 s^2 + \cdots + a_{2n_1}s^{2n_1} \tag{5a}$$

$$A_2(s^2) \triangleq \sum_{k=0}^{n_2} a_{2,k}s^{2k} = a_1 + a_3 s^2 + \cdots + a_{2n_2+1}s^{2n_2} \tag{5b}$$

and n_i is the integer part of $(n+1-i)/2$. Substituting the even-odd decomposition (4) for $A(s)$ and letting $B_1(s) \triangleq b_{1,0} + b_{1,1}s + \cdots + b_{1,n-1}s^{n-1} = B(s)$, we can expand the transfer function $G_n(s)$ as

$$G_n(s) = \frac{\frac{B_1(s)}{sA_2(s^2)}}{1 + \frac{A_1(s^2)}{sA_2(s^2)}} = \frac{\frac{\delta_1}{s} + \frac{B_2(s)}{A_2(s^2)}}{1 + \frac{\gamma_1}{s} + \frac{sA_3(s^2)}{A_2(s^2)}} \tag{6}$$

where

$$\gamma_1 = \frac{A_1(0)}{A_2(0)} = \frac{a_{1,0}}{a_{2,0}} \tag{7a}$$

$$A_3(s^2) \triangleq \sum_{k=0}^{n_3} a_{3,k} s^{2k} = \frac{1}{s^2}(A_1(s^2) - \gamma_1 A_2(s^2)) \tag{7b}$$

$$\delta_1 = \frac{B_1(0)}{A_2(0)} = \frac{b_{1,0}}{a_{2,0}} \tag{7c}$$

$$B_2(s) \triangleq \sum_{k=0}^{n-2} b_{2,k} s^k = \frac{1}{s}(B_1(s) - \delta_1 A_2(s^2)) \ . \tag{7d}$$

By iteratively performing the expansions

$$\frac{A_i(s^2)}{sA_{i+1}(s^2)} = \frac{1}{s}\gamma_i + \frac{sA_{i+2}(s^2)}{A_{i+1}(s^2)} \tag{8a}$$

$$\gamma_i = \frac{A_i(0)}{A_{i+1}(0)} = \frac{a_{i,0}}{a_{i+1,0}} \tag{8b}$$

$$A_{i+2}(s^2) \triangleq \sum_{k=0}^{n_{i+2}} a_{i+2,k} s^{2k} = \frac{1}{s^2}(A_i(s^2) - \gamma_i A_{i+1}(s^2)) \tag{8c}$$

and

$$\frac{B_i(s)}{sA_{i+1}(s^2)} = \frac{1}{s}\delta_i + \frac{B_{i+1}(s)}{A_{i+1}(s^2)} \tag{9a}$$

$$\delta_i = \frac{B_i(0)}{A_{i+1}(0)} = \frac{b_{i,0}}{a_{i+1,0}} \tag{9b}$$

$$B_{i+1}(s^2) \triangleq \sum_{i=0}^{n-i-1} b_{i+1,k} s^k = \frac{1}{s}(B_i(s) - \delta_i A_{i+1}(s^2)) \tag{9c}$$

for $i = 1, 2, \ldots, n$, we can finally expand $G_n(s)$ into the following Routh $\gamma - \delta$ form [27]:

$$G_n(s) = \sum_{i=1}^{n} \frac{\delta_i}{s} \prod_{k=1}^{i} W_{n,k}(s) \tag{10}$$

where

$$W_{n,k}(s) = \cfrac{1}{\cfrac{\gamma_k}{s} + \cfrac{1}{\cfrac{\gamma_{k+1}}{s} + \cfrac{1}{\ddots \, \cfrac{\gamma_n}{s}}}} \tag{11}$$

for $k = 2, 3, \ldots, n$. As for $W_{n,1}(s)$, the first term in the preceding expansion is $1 + \frac{\gamma_1}{s}$. According to the Routh stability criterion, the transfer function $G_n(s)$ is asymptotically stable if and only if $\gamma_i 0$ for $i = 1, 2, \ldots, n$. Given the coefficients

γ_i and δ_i for $i = 1, 2, \ldots, n$, the impulse-response energy of the system $G_n(s)$ can be evaluated by

$$I_n = \sum_{i=1}^{n} \frac{\delta_i^2}{2\gamma_i} . \tag{12}$$

2.2. Routh $\alpha - \beta$ expansion.

Alternatively, the denominator polynomial, $A(s)$, of the transfer function $G_n(s)$ can be split into

$$\begin{aligned} A(s) &= (a_n s^n + a_{n-2}s^{n-2} + \cdots) + (a_{n-1}s^{n-1} + a_{n-3}s^{n-3} + \cdots) \\ &\triangleq \hat{A}_1(s) + \hat{A}_2(s) \end{aligned} \tag{13}$$

where

$$\begin{aligned} \hat{A}_1(s) &\triangleq \sum_{k=0}^{n_1} \hat{a}_{1,k} s^{n-2k} \\ &= a_n s^n + a_{n-2}s^{n-2} + \cdots + \begin{cases} a_1 s & \text{for } n \text{ odd} \\ a_0 & \text{for } n \text{ even} \end{cases} \end{aligned} \tag{14a}$$

$$\begin{aligned} \hat{A}_2(s) &\triangleq \sum_{k=0}^{n_2} \hat{a}_{2,k} s^{n-1-2k} \\ &= a_{n-1}s^{n-1} + a_{n-3}s^{n-3} + \cdots + \begin{cases} a_0 & \text{for } n \text{ odd} \\ a_1 s & \text{for } n \text{ even.} \end{cases} \end{aligned} \tag{14b}$$

Let $B(s)$ be written as

$$B(s) = \hat{B}_1(s) \triangleq \sum_{k=0}^{n-1} \hat{b}_{1,k} s^{n-1-k} . \tag{15}$$

Then the transfer function $G_n(s)$ can be expressed as

$$G_n(s) = \frac{\hat{B}_1(s)}{\hat{A}_1(s) + \hat{A}_2(s)} = \frac{\frac{\hat{B}_1(s)}{\hat{A}_2(s)}}{1 + \frac{\hat{A}_1(s)}{\hat{A}_2(s)}} . \tag{16}$$

Performing the long divisions

$$\frac{\hat{A}_i(s)}{\hat{A}_{i+1}(s)} = \alpha_i s + \frac{\hat{A}_{i+2}(s)}{\hat{A}_{i+1}(s)} \tag{17a}$$

$$\alpha_i = \lim_{s\to\infty} \frac{\hat{A}_i(s)}{s\hat{A}_{i+1}(s)} = \frac{\hat{a}_{i,0}}{\hat{a}_{i+1,0}} \tag{17b}$$

$$\hat{A}_{i+2}(s) = \sum_{k=0}^{n_{i+2}} \hat{a}_{i+2,k} s^{n-i-1-2k} = \hat{A}_i(s) - \alpha_i s \hat{A}_{i+1}(s) \tag{17c}$$

and

$$\frac{\hat{B}_i(s)}{\hat{A}_{i+1}(s)} = \beta_i + \frac{\hat{B}_{i+1}(s)}{\hat{A}_{i+1}(s)} \tag{18a}$$

$$\beta_i = \lim_{s\to\infty} \frac{\hat{B}_i(s)}{\hat{A}_{i+1}(s)} = \frac{\hat{b}_{i,0}}{\hat{a}_{i+1,0}} \tag{18b}$$

$$\hat{B}_{i+1}(s) = \sum_{k=0}^{n-i-1} \hat{b}_{i+1,k} s^{n-i-1-k} = \hat{B}_i(s) - \beta_i \hat{A}_{i+1}(s) \tag{18c}$$

for $i = 1, 2, \ldots, n$, we can expand $G_n(s)$ into the following Routh $\alpha - \beta$ form [5]:

$$G_n(s) = \sum_{i=1}^{n} \beta_i \prod_{k=1}^{i} V_{n,k}(s) \tag{19}$$

where

$$V_{n,k}(s) = \cfrac{1}{\alpha_k s + \cfrac{1}{\alpha_{k+1} s + \cfrac{1}{\ddots \cfrac{1}{\alpha_n s}}}} \tag{20}$$

for $k = 2, 3, \ldots, n$. For $V_{n,1}(s)$, the first term in the preceding expansion is $1+\alpha_1 s$ rather than $\alpha_1 s$. Because the transfer function $G_n(s)$ is stable, the coefficients α_i, $i = 1, 2, , \ldots, n$, of the Routh $\alpha - \beta$ expansion are all positive. Besides, the impulse-response energy of the system $G_n(s)$ can be evaluated alternatively from the α and β coefficients as follows [5]:

$$I_n = \sum_{i=1}^{n} \frac{\beta_i^2}{2\alpha_i} . \tag{21}$$

2.3. System and energy decompositions.

Let $G_{n-1,0}(s)$ be the $(n-1)$th-order decomposed transfer function formed from the polynomials $\hat{B}_2(s)$, $\hat{A}_2(s)$, and $\hat{A}_3(s)$ of the first-level Routh $\alpha - \beta$ expansion of $G_n(s)$ as follows:

$$G_{n-1,0}(s) = \frac{\hat{B}_2(s)}{\hat{A}_2(s) + \hat{A}_3(s)} . \tag{22}$$

It is obvious that $G_{n-1,0}(s)$ is stable and its impulse-response energy, denoted by $I_{n-1,0}$, is given by

$$\begin{aligned} I_{n-1,0} &\triangleq \int_0^\infty g_{n-1,0}^2(t)\, dt = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} G_{n-1,0}(s) G_{n-1,0}(-s)\, ds \\ &= \sum_{i=2}^{n} \frac{\beta_i^2}{2\alpha_i} \end{aligned} \tag{23}$$

where $g_{n-1,0}(t) = \mathcal{L}^{-1}\{G_{n-1,0}(s)\}$. Comparing (21) and (23), we have

$$I_n = \frac{\beta_1^2}{2\alpha_1} + I_{n-1,0} \,. \tag{24}$$

Similarly, we can form another $(n-1)$th-order decomposed stable transfer function $G_{n-1,1}(s)$ from the polynomials $B_2(s)$, $A_2(s^2)$, and $A_3(s^2)$ of the first-level Routh $\gamma - \delta$ expansion of $G_n(s)$ as follows:

$$G_{n-1,1}(s) = \frac{B_2(s)}{A_2(s^2) + sA_3(s^2)} \,. \tag{25}$$

The impulse-response energy of $G_{n-1,1}(s)$, denoted by $I_{n-1,1}$, is given by

$$I_{n-1,1} \triangleq \int_0^\infty g_{n-1,1}^2(t)\,dt = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} G_{n-1,1}(s)G_{n-1,1}(-s)\,ds = \sum_{i=2}^{n} \frac{\delta_i^2}{2\gamma_i} \tag{26}$$

where $g_{n-1,1}(t) = \mathcal{L}^{-1}\{G_{n-1,1}(s)\}$. Comparing (26) and (12), we have

$$I_n = \frac{\delta_1^2}{2\gamma_1} + I_{n-1,1} \,. \tag{27}$$

According to the results in (24) and (27), we can regard the Routh $\alpha - \beta$ and $\gamma - \delta$ expansions as impulse-response energy decomposition schemes. Hence, by performing combinatorially first-level Routh $\alpha - \beta$ and $\gamma - \delta$ expansions on $G_n(s)$ and the decomposed transfer functions, we can develop an energy decomposition tree as shown in Figure 1. In this tree, the entry $G_{n-i+1,k}(s)$ is an $(n-i+1)$th-order transfer function that can be generally expressed as

$$G_{n-i+1,k}(s) = \frac{\hat{B}_{i,k}(s)}{\hat{A}_{i,k}(s) + \hat{A}_{i+1,k}(s)} \tag{28}$$

for the purpose of performing Routh $\alpha - \beta$ expansion, and as

$$G_{n-i+1,k}(s) = \frac{B_{i,k}(s)}{A_{i,k}(s^2) + sA_{i+1,k}(s^2)} \tag{29}$$

for the purpose of performing Routh $\gamma - \delta$ expansion. Let the polynomials $\hat{A}_{i,k}(s)$, $\hat{B}_{i,k}(s)$, $A_{i,k}(s^2)$, and $B_{i,k}(s)$ in (28) and (29) be denoted by

$$\hat{A}_{i,k}(s) = \sum_{l=0}^{n_i} \hat{a}_{i,k,l}s^{n+1-i-2l} \tag{30a}$$

$$\hat{B}_{i,k}(s) = \sum_{l=0}^{n-i} \hat{b}_{i,k,l}s^{n-i-l} \tag{30b}$$

$$B_{i,k}(s) = \sum_{l=0}^{n-i} b_{i,k,l}s^{l} \tag{30c}$$

$$A_{i,k}(s^2) = \sum_{l=0}^{n_i} a_{i,k,l}s^{2l} \,. \tag{30d}$$

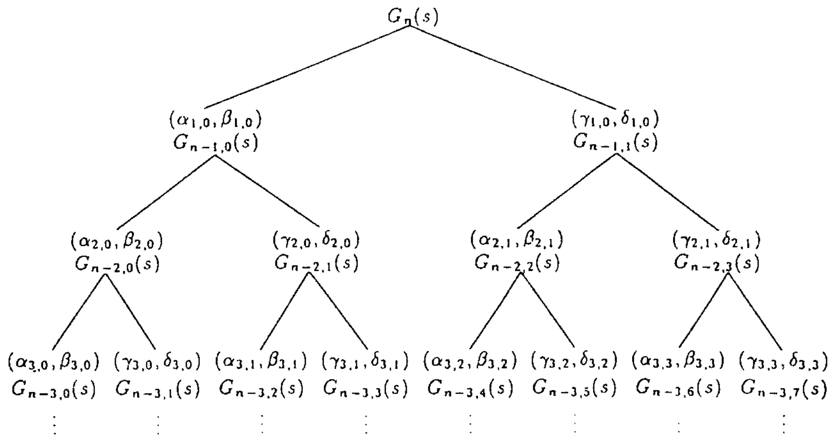

Figure 1. Energy decomposition tree.

Then it is obvious that they satisfy the following relations:

$$B_{i,k}(s) = \hat{B}_{i,k}(s) \tag{31a}$$

$$A_{i,k}(s^2) = \begin{cases} \hat{A}_{i,k}(s) & \text{for } n-i+1 \text{ even} \\ \hat{A}_{i+1,k}(s) & \text{for } n-i+1 \text{ odd} \end{cases} \tag{31b}$$

$$A_{i+1,k}(s^2) = \begin{cases} \frac{1}{s}\hat{A}_{i+1,k}(s) & \text{for } n-i+1 \text{ even} \\ \frac{1}{s}\hat{A}_{i,k}(s) & \text{for } n-i+1 \text{ odd.} \end{cases} \tag{31c}$$

Now, return to (28) and perform the Routh $\alpha - \beta$ expansion to obtain

$$G_{n-i+1,k}(s) = \frac{\beta_{i,k} + \frac{\hat{B}_{i+1,k}(s)}{\hat{A}_{i+1,k}(s)}}{1 + \alpha_{i,k}s + \frac{\hat{A}_{i+2,k}(s)}{\hat{A}_{i+1,k}(s)}} \tag{32}$$

where

$$\alpha_{i,k} = \lim_{s\to\infty} \frac{\hat{A}_{i,k}(s)}{s\hat{A}_{i+1,k}(s)} = \frac{\hat{a}_{i,k,0}}{\hat{a}_{i+1,k,0}} \tag{33a}$$

$$\hat{A}_{i+2,k}(s) = \hat{A}_{i,k}(s) - \alpha_{i,k}s\hat{A}_{i+1,k}(s) \tag{33b}$$

$$\beta_{i,k} = \lim_{s\to\infty} \frac{\hat{B}_{i,k}(s)}{\hat{A}_{i+1,k}(s)} = \frac{\hat{b}_{i,k,0}}{\hat{a}_{i+1,k,0}} \tag{33c}$$

$$\hat{B}_{i+1,k}(s) = \hat{B}_{i,k}(s) - \beta_{i,k}\hat{A}_{i+1,k}(s)\ . \tag{33d}$$

From the Routh $\alpha-\beta$ expansion of $G_{n-i+1,k}(s)$ in (32), we can obtain the $(n-i)$th-order decomposed transfer function

$$G_{n-i,2k}(s) = \frac{\hat{B}_{i+1,k}(s)}{\hat{A}_{i+1,k}(s) + \hat{A}_{i+2,k}(s)}\ . \tag{34}$$

Hence, the polynomials in the general representations

$$G_{n-i,2k}(s) = \begin{cases} \dfrac{\hat{B}_{i+1,2k}(s)}{\hat{A}_{i+1,2k}(s) + \hat{A}_{i+2,2k}(s)} & \text{for } \alpha - \beta \text{ expansion} \\ \dfrac{B_{i+1,2k-1}(s)}{A_{i+1,2k}(s^2) + sA_{i+2,2k}(s^2)} & \text{for } \gamma - \delta \text{ expansion} \end{cases} \tag{35}$$

are given by

$$\hat{B}_{i+1,2k}(s) = \hat{B}_{i+1,k}(s) \tag{36a}$$

$$\hat{A}_{i+1,2k}(s) = \hat{A}_{i+1,k}(s) \tag{36b}$$

$$\hat{A}_{i+2,2k}(s) = \hat{A}_{i+2,k}(s) \tag{36c}$$

and

$$B_{i+1,2k}(s) = \hat{B}_{i+1,k}(s) \tag{37a}$$

$$A_{i+1,2k}(s^2) = \begin{cases} \hat{A}_{i+1,k}(s) & \text{for } n-i \text{ even} \\ \hat{A}_{i+2,k}(s) & \text{for } n-i \text{ odd.} \end{cases} \tag{37b}$$

$$A_{i+2,2k}(s^2) = \begin{cases} \frac{1}{s}\hat{A}_{i+2,k}(s) & \text{for } n-i \text{ even} \\ \frac{1}{s}\hat{A}_{i+1,k}(s) & \text{for } n-i \text{ odd.} \end{cases} \tag{37c}$$

As shown in the preceding subsections, the impulse-response energy of $G_{n-i+1,k}(s)$, denoted by $I_{n-i+1,k}$, can be expressed in terms of that of $G_{n-i,2k}(s)$, denoted by $I_{n-i,2k}$, and the coefficients $\alpha_{i,k}$ and $\beta_{i,k}$ as

$$I_{n-i+1,k} = I_{n-i,2k} + \frac{\beta_{i,k}^2}{2\alpha_{i,k}} . \tag{38}$$

Alternatively, we can apply the Routh $\gamma - \delta$ expansion to decompose the transfer function $G_{n-i+1,k}(s)$ as

$$G_{n-i+1,k}(s) = \frac{\frac{\delta_{i,k}}{s} + \frac{B_{i+1,k}(s)}{A_{i+1,k}(s^2)}}{1 + \frac{\gamma_{i,k}}{s} + \frac{sA_{i+2,k}(s^2)}{A_{i+1,k}(s^2)}} \tag{39}$$

where

$$\gamma_{i,k} = \lim_{s\to 0} \frac{A_{i,k}(s^2)}{A_{i+1,k}(s^2)} = \frac{a_{i,k,0}}{a_{i+1,k,0}} \tag{40a}$$

$$A_{i+2,k}(s^2) = \frac{1}{s^2}(A_{i,k}(s^2) - \gamma_{i,k}A_{i+1,k}(s^2)) \tag{40b}$$

$$\delta_{i,k} = \lim_{s\to 0} \frac{B_{i,k}(s)}{A_{i+1,k}(s^2)} = \frac{b_{i,k,0}}{a_{i+1,k,0}} \tag{40c}$$

$$B_{i+1,k}(s) = \frac{1}{s}(B_{i,k}(s) - \delta_{i,k}A_{i+1,k}(s^2)) . \tag{40d}$$

Let the $(n-i)$th-order decomposed transfer function associated with the expansion (39) be

$$G_{n-i,2k+1}(s) = \frac{B_{i+1,k}(s)}{A_{i+1,k}(s^2) + sA_{i+2,k}(s^2)} \,. \tag{41}$$

Then the polynomials in the general representations of $G_{n-i,2k+1}(s)$,

$$G_{n-i,2k+1}(s) = \begin{cases} \dfrac{\hat{B}_{i+1,2k+1}(s)}{\hat{A}_{i+1,2k+1}(s) + \hat{A}_{i+2,2k+1}(s)} & \text{for } \alpha - \beta \text{ expansion} \\ \dfrac{B_{i+1,2k}(s)}{A_{i+1,2k+1}(s) + sA_{i+2,2k+1}(s^2)} & \text{for } \gamma - \delta \text{ expansion} \end{cases} \tag{42}$$

are given by

$$\hat{B}_{i+1,2k+1}(s) = B_{i+1,k}(s) \tag{43a}$$

$$\hat{A}_{i+1,2k+1}(s) = \begin{cases} A_{i+1,k}(s^2) & \text{for } n-i \text{ even} \\ sA_{i+2,k}(s^2) & \text{for } n-i \text{ odd} \end{cases} \tag{43b}$$

$$\hat{A}_{i+2,2k+1}(s) = \begin{cases} sA_{i+2,k}(s^2) & \text{for } n-i \text{ even} \\ A_{i+1,k}(s^2) & \text{for } n-i \text{ odd} \end{cases} \tag{43c}$$

and

$$B_{i+1,2k+1}(s) = B_{i+1,k}(s) \tag{44a}$$

$$A_{i+1,2k+1}(s^2) = A_{i+1,k}(s^2) \tag{44b}$$

$$A_{i+2,2k+1}(s^2) = A_{i+2,k}(s^2) \,. \tag{44c}$$

Similarly, the impulse-response energy of $G_{n-i+1,k}(s)$ can be evaluated from that of $G_{n-i,2k+1}(s)$, denoted by $I_{n-1,2k+1}$, and the coefficients $\gamma_{i,k}$ and $\delta_{i,k}$ as follows:

$$I_{n-i+1,k} = I_{n-i,2k+1} + \frac{\delta_{i,k}^2}{2\gamma_{i,k}} \,. \tag{45}$$

Before ending this section, we note that the energy parameter pairs and the decomposed transfer functions of $G_{n-i}(s)$ have the following properties:

(i) For $n-i$ even,

$$(\alpha_{i+1,k}, \beta_{i+1,k}) = (\alpha_{i+2,2k+1}, \beta_{i+2,2k+1}) \tag{46a}$$

$$(\gamma_{i+1,k}, \delta_{i+1k}) = (\gamma_{i+2,2k}, \delta_{i+2,2k}) \tag{46b}$$

$$G_{n-i-2,4k+1}(s) = G_{n-i-2,4k+2}(s) \,. \tag{46c}$$

(ii) For $n-i$ odd,

$$\frac{\beta_{i+1,k}^2}{\alpha_{i+1,k}} + \frac{\delta_{i+2,2k}^2}{\gamma_{i+2,k}} = \frac{\delta_{i+1,k}^2}{\gamma_{i+1,k}} + \frac{\beta_{i+2,2k+1}^2}{\alpha_{i+2,2k+1}} \,. \tag{47}$$

A proof of these properties for $i = 0$ is given in the Appendix.

3. Derivation of a new family of Routh approximants

By using the energy parameter pairs of the energy decomposition tree in Figure 1, a tree of Routh approximants to $G_n(s)$ can be obtained. The Routh-approximant tree constructed from the $\alpha - \beta$ and $\gamma - \delta$ energy parameter pairs of the energy decomposition tree is shown in Figure 2. In this figure, the entry $\hat{G}_{m,k}(s)$ is the mth-order transfer function, which is constructed such that it has the sequence of energy parameter pairs (μ_i, ν_i), $i = 1, 2, \ldots, m$, where

$$(\mu_i, \nu_i) = \begin{cases} (\alpha_{i,k(i)}, \beta_{i,k(i)}) & \text{if } d_{i-1} = 0 \\ (\gamma_{i,k(i)}, \delta_{i,k(i)}) & \text{if } d_{i-1} = 1 \,. \end{cases} \tag{48}$$

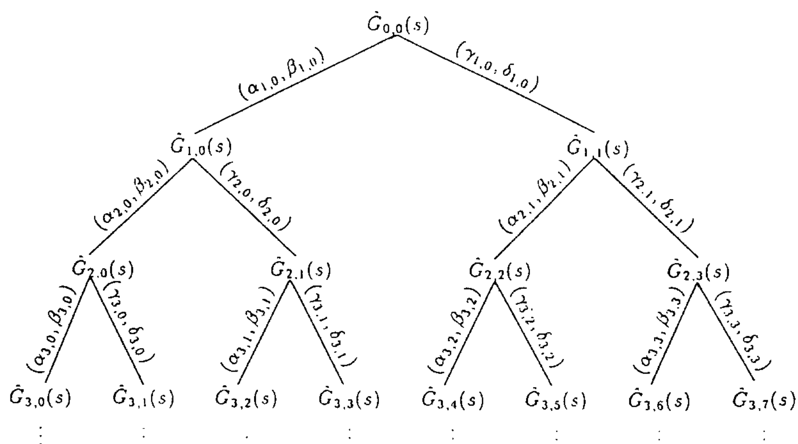

Figure 2. Routh-approximant tree.

In equation (48), d_i, $i = 0, 1, \ldots, m-1$, are the digits of the m-bit binary representation $d_{m-1}d_{m-2}\cdots d_1 d_0$ of the number k, and $k(i)$ is the integer part of $k/2^{m+1-i}$. As an illustration, we consider the case of $m = 4$ and $k = 5$. The 4-bit binary representation of $k = 5$ is 0101, so we have $d_3 = 0$, $d_2 = 0$, $d_1 = 0$, $d_0 = 0$, and $k(1) = 0$, $k(2) = 0$, $k(3) = 1$, $k(4) = 2$. Hence, the transfer function $\hat{G}_{4,5}(s)$ has the sequence of energy parameter pairs $(\alpha_{1,0}, \beta_{1,0})$, $(\gamma_{2,0}, \delta_{2,0})$, $(\alpha_{3,1}, \beta_{3,1})$, and $(\gamma_{4,2}, \delta_{4,2})$.

In what follows, we derive an algorithm for synthesizing transfer function $\hat{G}_{m,k}(s)$ from the sequence of energy parameter pairs (μ_i, ν_i), $i = 1, 2, \ldots, m$. Let

$$\begin{aligned} g_i(s) &= \frac{q_{i,0} + q_{i,1}s + \cdots + q_{i,i-1}s^{i-1}}{p_{i,0} + p_{i,1}s + \cdots + p_{i,i}s^i} \\ &\triangleq \frac{q_i(s)}{p_i(s)} \end{aligned} \tag{49}$$

be the ith-order transfer function that has the sequence of energy parameter pairs (μ_l, ν_l), $l = m - i + 1, m - i + 2, \ldots, m$. Then, if $(\mu_{m-i}, \nu_{m-i}) = (\alpha_{m-i,k(m-i)}, \beta_{m-i,k(m-i)})$, $g_i(s)$ and $g_{i+1}(s)$ are related by

$$g_{i+1}(s) = \frac{q_{i+1}(s)}{p_{i+1}(s)} = \begin{cases} \dfrac{\nu_{m-i} + \dfrac{q_i(s)}{(p_i(s) + p_i(-s))/2}}{1 + \mu_{m-i}s + \dfrac{(p_i(s) - p_i(-s))/2}{(p_i(s) + p_i(-s))/2}} & \text{for } i \text{ even} \\ \dfrac{\nu_{m-i} + \dfrac{q_i(s)}{(p_i(s) - p(-s))/2}}{1 + \mu_{m-i}s + \dfrac{(p_i(s) - p(-s)/2}{(p_i(s) - p_i(s))/2}} & \text{for } i \text{ odd.} \end{cases} \tag{50}$$

Comparing the last two equations gives

$$q_{i+1}(s) = \begin{cases} 1/2\nu_{m-i}(p_i(s) + p_i(-s)) + q_i(s) & \text{for } i \text{ even} \\ 1/2\nu_{m-i}(p_i(s) - p_i(-s)) + q_i(s) & \text{for } i \text{ odd} \end{cases} \tag{51a}$$

and

$$p_{i+1}(s) = \begin{cases} 1/2\mu_{m-i}s(p_i(s) + p_i(-s)) + p_i(s) & \text{for } i \text{ even} \\ 1/2\mu_{m-i}s(p_i(s) - p_i(-s)) + p_i(s) & \text{for } i \text{ odd.} \end{cases} \tag{51b}$$

Also, if $(\mu_{m-i}, \nu_{m-i}) = (\gamma_{m-i,k(m-i)}, \delta_{m-i,k(m-i)})$, the numerator and the denominator polynomials of the transfer functions $g_{i+1}(s)$ and $g_i(s)$ are related by

$$\begin{aligned} g_{i+1}(s) &= \frac{q_{i+1}(s)}{p_{i+1}(s)} \\ &= \frac{\dfrac{\nu_{m-i}}{s} + \dfrac{q_i(s)}{(p_i(s) + p(-s))/2}}{1 + \dfrac{\mu_{m-i}}{s} + \dfrac{(p_i(s) - p_i(-s))/2}{(p_i(s) + p_i(-s))/2}} . \end{aligned} \tag{52}$$

Comparing the last two equations, we have

$$q_{i+1}(s) = \nu_{m-i}(p_i(s) + p_i(-s))/2 + sq_i(s) \tag{53a}$$

$$p_{i+1}(s) = \mu_{m-i}(p_i(s) + p_i(-s))/2 + sp_i(s) . \tag{53b}$$

By using the initial conditions

$$g_1(s) = \frac{q_1(s)}{p_1(s)} = \begin{cases} \frac{\alpha_{1,0}\beta_{1,0}}{\alpha_{1,0}s+1} & \text{for } d_{m-1} = 0 \\ \frac{\gamma_{1,0}\delta_{1,0}}{s+\gamma_{1,0}} & \text{for } d_{m-1} = 1 \end{cases} \tag{54}$$

and using the recursive relations (51) and (53), we can obtain $\hat{G}_{m,k}(s) = g_m(s)$ from the sequence of energy parameter pairs in (48).

We have completed the derivation of synthesis formulas for constructing a tree of Routh approximants to $G_n(s)$ from its energy parameter pairs in the energy decomposition tree. Some interesting properties of the Routh approximant tree in Figure 2 are now outlined as follows:

(i) The impulse-response energy of the mth Routh approximant $\hat{G}_{m,k}(s)$ is given by

$$\hat{I}_{m,k} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \hat{G}_{m,k}(s)\hat{G}_{m,k}(-s)\,ds = \sum_{i=1}^{m} \frac{\nu_i^2}{2\mu_i} . \tag{55}$$

(ii) The impulse-response energy $\hat{I}_{m,k}$ satisfies the following relations:

$$\hat{I}_{m+1,2k} = \hat{I}_{m,k} + \frac{\beta_{m,k}^2}{2\alpha_{m,k}} \tag{56a}$$

$$\hat{I}_{m+1,2k+1} = \hat{I}_{m,k} + \frac{\delta_{m,k}^2}{2\gamma_{m,k}} . \tag{56b}$$

(iii) If the number of 1's in the m-bit binary representation of k_1, denoted by $\rho(m, k_1)$, is equal to that of k_2, $\rho(m, k_2)$, then

$$\hat{I}_{m,k_1} = \hat{I}_{m,k_2} . \tag{57}$$

Hence, there are only $m+1$ different values of impulse-response energy for the family of 2^m mth-order Routh approximants $\hat{G}_{m,k}(s)$, $k = 0, 1, \ldots, 2^m - 1$.

(iv) If $n - m$ is even and $\rho(m, k_1) = \rho(m, k_2)$, then $\hat{G}_{m,k_1}(s) = \hat{G}_{m,k_2}(s)$. In this case, there are only $m + 1$ different mth-order Routh approximants.

(v) The mth-order Routh approximant $\hat{G}_{m,k}(s)$ fits $G_n(s)$ at $s = 0$ by:

$$\frac{d^{i-1}}{ds^{i-1}}\hat{G}_{m,k}(0) = \frac{d^{i-1}}{ds^{i-1}}\hat{G}(0) , \quad i = 1, 2, \ldots, \begin{cases} \rho(m,k) & \text{for } n, m \text{ even} \\ \tau_1(m,k) & \text{for } n, m \text{ odd} \end{cases} \tag{58}$$

where $\tau_1(m, k)$ (resp. $\tau_0(m, k)$) denotes the maximum number of leading 1's (resp. 0's) in the m-bit binary representation of k. Moreover, $\hat{G}_{m,k}(s)$ fits $G_n(s)$ at $s = \infty$ by

$$\hat{M}_{m,k,i} = M_{n,i} , \quad i = 1, 2, \ldots, \begin{cases} m - \rho(m,k) & \text{for } n, m \text{ even} \\ \tau_0(m,k) & \text{for } n, m \text{ odd} \end{cases} \tag{59}$$

where $\hat{M}_{m,k,i}$ and $M_{n,i}$ are, respectively, the coefficients of the negative power series of $\hat{G}_{m,k}(s)$ and $G_n(s)$; i.e.,

$$\hat{G}_{m,k}(s) = \sum_{i=1}^{\infty} \hat{M}_{m,k,i} s^{-i} \tag{60a}$$

$$G_n(s) = \sum_{i=1}^{\infty} M_{n,i} s^{-i} . \tag{60b}$$

It is noted that properties (i) and (ii) follow obviously from the results derived in Section 2. Properties (iii) and (iv) can be proved by using the relations (46) and (47) of the energy decomposition tree. Moreover, the Padé fitting properties in (58) and (59) can be verified by a direct calculation.

Finally, it is remarked that if $n - m$ is odd, then all the 2^m mth-order Routh approximants, $\hat{G}_{m,k}(s)$, $k = 0, 1, \ldots, 2^m - 1$, are distinct. However, there are only $m + 1$ different values of impulse-response energy among these 2^m m-th order Routh approximants. This fact indicates that the selection of Routh approximants based on the impulse-response energy is inadequate. Hence, another criterion has to be used to select the optimal prespecified-order Routh approximant.

4. Examples

To illustrate properties of the new family of Routh approximants, we provide two examples in this section.

Example 1. Consider the stable fourth-order transfer function

$$G_4(s) = \frac{81.6103s^3 + 506.6497s^2 + 99.8432s + 5}{s^4 + 105.2s^3 + 521.01s^2 + 101.05s + 5} \; .$$

The first four terms of the negative and positive power series of $G_4(s)$ are given by

$$G_4(s) = 1 - 0.24136s + 2.00583s^2 - 20.1055s^3 + \cdots$$
$$G_4(s) = 81.6103s^{-1} - 8078.75s^{-2} + 807464.9s^{-3} - 80744445s^{-4} + \cdots \; .$$

Using the energy decomposition algorithm of Section 2, the energy parameter pairs and decomposed subsystems of the energy decomposition tree are computed and given as follows:

(1) $i = 1$

$$(\alpha_{1,0}, \beta_{1,0}) = (0.009506, 0.775763)$$
$$G_{3,0}(s) = \frac{506.65s^2 + 21.4523s + 5}{105.2s^2 + 502.049s^2 + 101.05s + 5}$$
$$(\gamma_{1,0}, \delta_{1,0}) = (0.049480, 0.049480)$$
$$G_{3,1}(s) = \frac{81.6103s^2 + 501.44s + 99.8432}{s^3 + 105.2s^2 + 515.805s + 101.05} \; .$$

(2) $i = 2$

$$(\alpha_{2,0}, \beta_{2,0}) = (0.202288, 0.974233)$$
$$G_{2,0}(s) = \frac{21.4523s + 0.128832}{502.049s^2 + 100.039s + 5}$$
$$(\gamma_{2,0}, \delta_{2,0}) = (0.049480, 0.049480)$$
$$G_{2,1}(s) = \frac{501.444s + 21.4523}{105.2s^2 + 514.844s + 101.05}$$
$$(\alpha_{2,1}, \beta_{2,1}) = (0.009506, 0.775763)$$

$$G_{2,2}(s) = G_{2,1}(s)$$
$$(\gamma_{2,1}, \delta_{2,1}) = (0.195908, 0.193568)$$
$$G_{2,3}(s) = \frac{81.4167s + 501.444}{s^2 + 105.004s + 515.805} .$$

(3) $i = 3$

$$(\alpha_{3,0}, \beta_{3,0}) = (5.198490, 0.314440)$$
$$G_{3,0}(s) = \frac{0.128832}{100.039s + 5}$$
$$(\gamma_{3,0}, \delta_{3,0}) = (0.049981, 0.001288)$$
$$G_{3,1}(s) = \frac{21.4523}{520.049s + 100.039}$$
$$(\alpha_{3,1}, \beta_{3,1}) = (\alpha_{3,2}, \beta_{3,2}) = (0.204334, 0.973973)$$
$$G_{1,2}(s) = G_{1,4}(s) = \frac{21.4523}{514.844s + 101.05}$$
$$(\gamma_{3,1}, \delta_{3,1}) = (\gamma_{3,2}, \delta_{3,2}) = (0.196273, 0.041668)$$
$$G_{1,3}(s) = G_{1,5}(s) = \frac{501.444}{105.2s + 514.844}$$
$$(\alpha_{3,3}, \beta_{3,3}) = (0.009523, 0.775367)$$
$$G_{1,6}(s) = \frac{501.444}{105.004s + 515.808}$$
$$(\gamma_{3,3}, \delta_{3,3}) = (4.912234, 4.775474)$$
$$G_{1,7}(s) = \frac{81.4167}{s + 105.004} .$$

(4) $i = 4$

$$(\alpha_{4,0}, \beta_{4,0}) = (20.00771, 0.02577)$$
$$(\gamma_{4,0}, \delta_{4,0}) = (0.049980, 0.001288)$$
$$(\alpha_{4,1}, \beta_{4,1}) = (5.198490, 0.214440)$$
$$(\gamma_{4,1}, \delta_{4,1}) = (0.192363, 0.041250)$$
$$(\alpha_{4,2}, \beta_{4,2}) = (\alpha_{4,4}, \beta_{4,4}) = (5.094944, 0.212294)$$
$$(\gamma_{4,2}, \delta_{4,2}) = (\gamma_{4,4}, \delta_{4,4}) = (0.196273, 0.041668)$$
$$(\alpha_{4,3}, \beta_{4,3}) = (\alpha_{4,5}, \beta_{4,5}) = (0.204334, 0.973973)$$
$$(\gamma_{4,3}, \delta_{4,3}) = (\gamma_{4,5}, \delta_{4,5}) = (4.893955, 4.766581)$$
$$(\alpha_{4,6}, \beta_{4,6}) = (0.203573, 0.972159)$$
$$(\gamma_{4,6}, \delta_{4,6}) = (4.912233, 4.775474)$$
$$(\alpha_{4,7}, \beta_{4,7}) = (0.009523, 0.775367)$$
$$(\gamma_{4,7}, \delta_{4,7}) = (105.0041, 81.416732) .$$

From the preceding results, it can be verified that the properties in (46) and (47) are:

$$
\begin{aligned}
(\gamma_{2,0}, \delta_{2,0}) &= (\gamma_{1,0}, \delta_{1,0}) \\
(\alpha_{2,1}, \beta_{2,1}) &= (\alpha_{1,0}, \beta_{1,0}) \\
(\gamma_{4,0}, \delta_{4,0}) &= (\gamma_{3,0}, \delta_{3,0}) \\
(\alpha_{4,1}, \beta_{4,1}) &= (\alpha_{3,0}, \beta_{3,0}) \\
(\gamma_{4,2}, \delta_{4,2}) &= (\gamma_{3,1}, \delta_{3,1}) \\
(\alpha_{4,3}, \beta_{4,3}) &= (\alpha_{3,1}, \beta_{3,1}) \\
(\gamma_{4,4}, \delta_{4,4}) &= (\gamma_{3,2}, \delta_{3,2}) \\
(\alpha_{4,5}, \beta_{4,5}) &= (\alpha_{3,2}, \beta_{3,2}) \\
(\gamma_{4,6}, \delta_{4,6}) &= (\gamma_{3,3}, \delta_{3,3}) \\
(\alpha_{4,7}, \beta_{4,7}) &= (\alpha_{3,3}, \beta_{3,3}) \ , .
\end{aligned}
$$

From the energy parameter pairs of the energy decomposition tree, we construct the Routh approximant tree. In the following, we list the whole family of Routh approximants and their impulse-response energy, the integral of squared error of impulse response, denoted by J_0, and the integral of squared error of unit-step response, denoted by J_1. Also, the negative and positive power series of each Routh approximant $\hat{G}_{m,k}(s)$ are included to demonstrate the partial Padé fitting properties (58) and (59).

(1) $m = 1$

$$
\begin{aligned}
\hat{G}_{1,0}(s) &= \frac{81.6103}{105.2 + s} \\
&= 0.775763 - 0.007374s + \cdots \\
&= 81.6103s^{-1} - 8585.404s^{-2} + \cdots \\
\hat{I}_{1,0} &= \frac{\beta_{1,0}^2}{2\alpha_{1,0}} = 31.65514 \\
J_0 &= 0.152136
\end{aligned}
$$

$$
\begin{aligned}
\hat{G}_{1,1}(s) &= \frac{0.0494805}{0.0494805 + s} \\
&= 1 - 20.21s + \cdots \\
&= 0.4948s^{-1} - 0.002448s^{-2} + \cdots \\
\hat{I}_{1,1} &= \frac{\delta_{1,0}}{2\gamma_{1,0}} = 0.024740 \\
J_0 &= 33.93220 \\
J_1 &= 9.76757 \ .
\end{aligned}
$$

(2) $m = 2$

$$\begin{aligned}
\hat{G}_{2,0}(s) &= \frac{506.6497 + 81.6103s}{520.0494 + 105.2s + s^2} \\
&= 0.974234 - 0.040148s + 0.006248s^2 + \cdots \\
&= 81.6103s^{-1} - 8078.75s^{-2} + 807443.5s^{-3} + \cdots \\
\hat{I}_{2,0} &= \hat{I}_{1,0} + \frac{\beta_{2,0}^2}{2\alpha_{2,0}} = 34.0011 \\
J_0 &= 8.848 \times 10^{-5}
\end{aligned}$$

$$\begin{aligned}
\hat{G}_{2,1}(s) = \hat{G}_{2,2}(s) &= \frac{5.205344 + 81.6103s}{5.205344 + 105.2s + s^2} \\
&= 1 - 4.53182s + 91.39604s^2 + \cdots \\
&= 81.6103s^{-1} - 8580.198s^{-2} + 902212.0s^{-3} + \cdots \\
\hat{I}_{2,1} &= \hat{I}_{1,0} + \frac{\delta_{2,0}^2}{2\gamma_{2,0}} \\
&= \hat{I}_{1,0} + \frac{\beta_{2,1}^2}{2\alpha_{2,1}} = \hat{I}_{2,2} = 31.6799 \\
J_0 &= 0.147721 \\
J_1 &= 0.438170 \\
\hat{G}_{2,3}(s) &= \frac{0.009694 + 0.193568s}{0.009694 + 0.195908s + s^2} \\
&= 1 - 0.24136s - 98.2830s^2 + \cdots \\
&= 0.193568s^{-1} - 0.028228s^{-2} + 0.003654s^{-3} + \cdots \\
\hat{I}_{2,3} &= \hat{I}_{1,1} + \frac{\delta_{2,1}^2}{2\gamma_{2,1}} = 0.120368 \\
J_0 &= 33.7459 \\
J_1 &= 2.38060
\end{aligned}$$

(3) $m = 3$

$$\begin{aligned}
\hat{G}_{3,0}(s) &= \frac{99.05856 + 506.6497s + 81.6103s^2}{100.0386 + 521.000s + 105.2s^2 + s^3} \\
&= 0.990204 - 0.092433s + 0.255886s^2 - 1.24535s^3 + \cdots \\
&= 81.6103s^{-1} - 8078.754s^{-2} + 807465s^{-3} - 80744444s^{-4} + \cdots \\
\hat{I}_{3,0} &= \hat{I}_{2,0} + \frac{\beta_{3,0}^2}{2\alpha_{3,0}} = 34.0056 \\
J_0 &= 3.19 \times 10^{-6}
\end{aligned}$$

$$\begin{aligned}
\hat{G}_{3,1}(s) &= \frac{20.8337 + 506.650s + 81.6130s^2}{25.9925 + 520.297s + 105.2s^2 + s^3} \\
&= 0.80153 + 3.44780s - 69.1195s^2 + 1369.60s^3 + \cdots \\
&= 81.6103s^{-1} - 8078.75s^{-2} + 807444s^{-3} - 80741902s^{-4} + \cdots \\
\hat{I}_{3,1} &= \hat{I}_{2,0} + \frac{\delta_{3,0}^2}{2\gamma_{3,0}} = 34.0011 \\
J_0 &= 1.106 \times 10^{-3}
\end{aligned}$$

$$\begin{aligned}
\hat{G}_{3,2}(s) &= \frac{45.2370 + 501.444s + 81.6103s^2}{25.4747 + 515.086s + 105.2s^2 + s^3} \\
&= 1.77576 - 16.2210s + 323.852s^2 - 6481.21s^3 + \cdots \\
&= 81.6103s^{-1} - 8083.96s^{-2} + 808441s^{-3} - 80886178s^{-4} + \cdots \\
\hat{I}_{3,2} &= \hat{I}_{2,1} + \frac{\beta_{3,1}^2}{2\alpha_{3,1}} = 34.0011 \\
J_0 &= 1.465 \times 10^{-2}
\end{aligned}$$

$$\begin{aligned}
\hat{G}_{3,3}(s) &= \frac{1.81424 + 4.38343s + 81.6130s^2}{1.02167 + 20.6576s + 105.2s^2 + s^3} \\
&= 1.77576 - 31.6146s + 536.263s^2 - 7589.39s^3 + \cdots \\
&= 81.6103s^{-1} - 8581.02s^{-2} + 901039s^{-3} - 94612150s^{-4} + \cdots \\
\hat{I}_{3,3} &= \hat{I}_{2,1} + \frac{\delta_{3,1}^2}{2\gamma_{3,1}} = 31.6843 \\
J_0 &= 2.438 \times 10^{-1}
\end{aligned}$$

$$\begin{aligned}
\hat{G}_{3,4}(s) &= \frac{25.4747 + 501.444s + 81.6598s^2}{25.4747 + 514.844s + 105.249s^2 + s^3} \\
&= 1 - 0.526s + 9.70450s^2 - 193.994s^3 + \cdots \\
&= 81.6598s^{-1} - 8093.02s^{-2} + 809789s^{-3} - 81065218s^{-4} + \cdots \\
\hat{I}_{3,4} &= \hat{I}_{2,2} + \frac{\beta_{3,2}^2}{2\alpha_{3,2}} = 34.0011 \\
J_0 &= 4.685 \times 10^{-5}
\end{aligned}$$

$$\begin{aligned}
\hat{G}_{3,5}(s) &= \frac{1.02167 + 4.38343s + 81.6598s^2}{20.6479s + 105.249s^2 + s^3} \\
&= 1 - 15.9195s + 298.644s^2 - 4396.60s^3 + \cdots \\
&= 81.6598s^{-1} - 8590.26s^{-2} + 902435s^{-3} - 94803628s^{-4} + \cdots
\end{aligned}$$

$$\hat{I}_{3,5} = \hat{I}_{2,2} + \frac{\delta_{3,2}^2}{2\gamma_{3,2}} = 31.6843$$
$$J_0 = 2.214 \times 10^{-1}$$
$$J_1 = 6.203$$

$$\hat{G}_{3,6}(s) = \frac{1.01787 + 20.3254s + 81.4662s^2}{1.01787 + 20.5711s + 105.05s^2 + s^3}$$
$$= 1 - 0.241360s - 18.2854s^2 - 393.679s^3 + \cdots$$
$$= 81.4662s^{-1} - 8537.88s^{-2} + 895272s^{-3} - 93875933s^{-4} + \cdots$$
$$\hat{I}_{3,6} = \hat{I}_{2,3} + \frac{\beta_{3,3}^2}{2\alpha_{3,3}} = 31.6843$$
$$J_0 = 1.376 \times 10^{-1}$$
$$J_1 = 9.530 \times 10^{-2}$$
$$\hat{G}_{3,7}(s) = \frac{0.047617 + 0.950850s + 4.82495s^2}{0.047617 + 0.962343s + 4.96171s^2 + s^3} + \cdots$$
$$= 1 - 0.241360s - 2.00582s^2 - 36.3888s^3 + \cdots$$
$$= 4.82496s^{-1} - 22.9892s^{-2} + 109.470s^{-3} - 521.266s^{-4} + \cdots$$
$$\hat{I}_{3,7} = \hat{I}_{2,3} + \frac{\delta_{3,3}^2}{2\gamma_{3,3}} = 2.44163$$
$$J_0 = 28.1523$$
$$J_1 = 5.778 \times 10^{-2}\,.$$

Example 2. Consider the fifth-order transfer function

$$G_5(s) = \frac{10s^4 + 262s^3 + 1148s^2 + 2100s + 900}{s^5 + 20s^4 + 147s^3 + 435s^2 + 570s + 300}$$

whose impulse-response energy is 29.04249. The set of $2^3 = 8$ third-order Routh approximants $\hat{G}_{3,k}(s)$, $k = 0, 1, \ldots, 7$, and the performance measures are computed to be:

$$\hat{G}_{3,0}(s) = \frac{10s^2 + 262s + 1103.69}{s^3 + 20s^2 + 142.569s + 346.377}$$
$$\hat{I}_{3,0} = 26.1801$$
$$J_0 = 0.792569$$

$$\hat{G}_{3,1}(s) = \hat{G}_{3,2}(s) = \hat{G}_{3,4}(s)$$
$$= \frac{10s^2 + 262s + 203.108}{s^3 + 20s^2 + 128.635s + 67.7027}$$

$$
\begin{aligned}
\hat{I}_{3,1} &= 17.8906 \\
J_0 &= 3.347263 \\
J_1 &= 2.904820 \\
\hat{G}_{3,3}(s) &= \hat{G}_{3,5}(s) = \hat{G}_{3,6}(s) \\
&= \frac{10s^2 + 114.349s + 49.0066}{s^3 + 20s^2 + 31.0375s + 16.3355} \\
\hat{I}_{3,3} &= 15.0060 \\
J_0 &= 5.536489 \\
J_1 &= 0.902978
\end{aligned}
$$

$$
\begin{aligned}
\hat{G}_{3,7}(s) &= \frac{9.47715s^2 + 18.1093s + 7.76113}{s^3 + 3.61034s^2 + 4.91538s + 2.58704} \\
\hat{I}_{3,0} &= 23.2988 \\
J_0 &= 2.200249 \\
J_1 &= 0.143134\,.
\end{aligned}
$$

There are only four different third-order Routh approximants. It is seen that in this set of third-order Routh approximants, $\hat{G}_{3,0}(s)$ has the largest value of impulse-response energy and the least value of integral of squared error of impulse response, whereas $\hat{G}_{3,7}(s)$ has the least value of integral of squared error of unit-step response.

Because $n - m = 5 - 4 = 1$ is odd, there are 16 different fourth-order Routh approximants. The values of impulse-response energy, integral of squared error of impulse response, integral of squared error of unit-step response, and steady-state response for the unit-step input for each of these approximants are listed in Table 1. There are only five different values of impulse-response energy for this set of Routh approximants. Hence, the values of integral of squared error of impulse or unit-step response are useful for selecting a proper reduced-order model.

Model $G_{4,k}(s)$	$\hat{I}_{4,k}$	J_0	J_1	$G_{4,k}(\infty)$
$k = 0$	29.03082	0.4363579	∞	4.194840
$k = 1$	26.19181	1.565106	∞	1.905439
$k = 2$	26.19181	1.155872	∞	4.591816
$k = 3$	20.74127	4.098899	∞	4.591816
$k = 4$	26.19181	0.6390586	∞	2.5
$k = 5$	20.74127	3.388695	∞	2.5

Table 1. The performance evaluations of 16 fourth-order Routh approximants for Examples 2.

Model $G_{4,k}(s)$	$\hat{I}_{4,k}$	J_0	J_1	$G_{4,k}(\infty)$
$k = 6$	20.74127	2.670035	∞	2.5
$k = 7$	23.30719	1.863767	∞	2.5
$k = 8$	26.19181	0.6563492	0.3487138	3
$k = 9$	20.74127	3.765633	3.066499	3
$k = 10$	20.74127	2.393686	0.5566352	3
$k = 11$	23.30719	1.945676	0.3972896	3
$k = 12$	20.74127	2.774710	0.2930052	3
$k = 13$	23.30719	1.906522	0.1041621	3
$k = 14$	23.30719	3.044335	0.1761163	3
$k = 15$	29.03415	0.6527974	0.0039807	3
$G_5(s)$	29.04249	0	0	3

Table 1 continued.

5. Conclusions

In this paper, both Routh $\alpha - \beta$ and $\gamma - \delta$ expansions are regarded as energy decomposition schemes for stable linear systems described by their transfer functions. By combinatorially applying these two energy decomposition schemes to a stable transfer function and the decomposed subsystems, an energy decomposition tree for the system is constructed. The leaves of the tree are just the Routh $\alpha - \beta$ and $\gamma - \delta$ expansion coefficient pairs. A synthesis algorithm has been developed to derive a new family of Routh approximants from the energy decomposition tree. The elegant properties of the Routh-approximant family have also been explored and have been demonstrated by two examples. It is worth noting that, for a given system, there may exist different Routh approximants that have the same order and the same value of impulse-response energy. Hence, to judge the goodness of a Routh approxmant by the impulse-response energy may not suffice.

Appendix: Proof of properties (46) and (47)

Without loss of generality, we prove properties (46) and (47) for $i = 0$. In other words, we shall prove

$$(\alpha_{1,0}, \beta_{1,0}) = (\alpha_{2,1}, \beta_{2,1}) \tag{A1.a}$$

$$(\gamma_{1,0}, \delta_{1,0}) = (\gamma_{2,0}, \delta_{2,0}) \tag{A1.b}$$

$$G_{n-2,1}(s) = G_{n-2,2}(s) \tag{A1.c}$$

for even n, and

$$\frac{\beta_{1,0}^2}{2\alpha_{1,0}} + \frac{\delta_{2,0}^2}{2\gamma_{2,0}} = \frac{\beta_{2,1}}{\alpha_{2,1}} + \frac{\delta_{1,0}^2}{2\gamma_{1,0}} \tag{A2}$$

for odd n. To this end, we apply the Routh $\alpha - \beta$ and $\gamma - \delta$ expansion algorithms developed in Section 2 to obtain the following expressions for the energy parameter coefficients $\alpha_{1,0}, \beta_{1,0}, \gamma_{1,0}, \delta_{1,0}, \gamma_{2,0}, \delta_{2,0}, \alpha_{2,1}, \beta_{2,1}$ and the decomposed subsystems $G_{n-2,1}(s)$ and $G_{n-2,2}(s)$ in terms of the coefficients a_i's and b_i's of $G_n(s)$.

(i) For odd n,

$$\alpha_{1,0} = \frac{a_n}{a_{n-1}}, \quad \beta_{1,0} = \frac{b_{n-1}}{a_{n-1}} \tag{A3.a}$$

$$\gamma_{1,0} = \frac{a_0}{a_1}, \quad \delta_{1,0} = \frac{b_0}{a_1} \tag{A3.b}$$

$$\gamma_{2,0} = \frac{a_0}{a_1 - \alpha_{1,0}a_0}, \quad \delta_{2,0} = \frac{b_0 - \beta_{1,0}a_0}{a_1 - \alpha_{1,0}a_0} \tag{A3.c}$$

$$\alpha_{2,1} = \frac{a_n}{a_{n-1} - \gamma_{1,0}a_n}, \quad \beta_{2,1} = \frac{b_{n-1} - \delta_{1,0}a_n}{a_{n-1} - \gamma_{1,0}a_n} \tag{A3.d}$$

$$G_{n-1,1}(s) = \frac{\begin{array}{c} b_{n-2}s^{n-3} + [(b_{n-3} - \beta_{1,0}a_{n-3}) - \delta_{2,0}(a_{n-2} - \alpha_{1,0}a_{n-3})]s^{n-4} \\ + \cdots + [(b_2 - \beta_{1,0}a_2) - \delta_{2,0}(a_3 - \alpha_{1,0}a_2)]s + b_1 \end{array}}{\begin{array}{c} a_{n-1}s^{n-2} + (a_{n-2} - \alpha_{1,0}a_{n-3})s^{n-3} + [a_{n-3} - \gamma_{2,0}(a_{n-2} - \alpha_{1,0}a_{n-3})]s^{n-4} \\ + (a_{n-4} - \alpha_{1,0}a_{n-5})s^{n-5} + \cdots + [a_2 - \gamma_{2,0}(a_3 - \alpha_{1,0}a_2)]s + (a_1 - \alpha_{1,0}a_0) \end{array}} \tag{A3.e}$$

$$G_{n-1,2}(s) = \frac{\begin{array}{c} b_{n-2}s^{n-3} + [(b_{n-3} - \delta_{1,0}a_{n-2}) - \beta_{2,1}(a_{n-3} - \gamma_{1,0}a_{n-2})]s^{n-4} \\ + b_{n-4}s^{n-5} + \cdots + [(b_2 - \delta_{1,0}a_3) - \beta_{2,1}(a_2 - \gamma_{1,0}a_3)]s + b_1 \end{array}}{\begin{array}{c} (a_{n-1} - \gamma_{1,0}a_n)s^{n-2} + [a_{n-2} - \alpha_{2,1}(a_{n-3} - \gamma_{1,0}a_{n-2})]s^{n-3} \\ + \cdots + (a_4 - \gamma_{1,0}a_5)s^3 + [a_3 - \alpha_{2,1}(a_2 - \gamma_{1,0}a_3)]s^2 + (a_2 - \gamma_{1,0}a_3)s + a_1 \end{array}} \tag{A3.f}$$

(ii) For even n,

$$\alpha_{1,0} = \frac{a_n}{a_{n-1}}, \quad \beta_{1,0} = \frac{b_{n-1}}{a_{n-1}} \tag{A4.a}$$

$$\gamma_{1,0} = \frac{a_0}{a_1}, \quad \delta_{1,0} = \frac{b_0}{a_1} \tag{A4.b}$$

$$\gamma_{2,0} = \frac{a_0}{a_1}, \quad \delta_{2,0} = \frac{b_0}{a_1} \tag{A4.c}$$

$$\alpha_{2,1} = \frac{a_n}{a_{n-1}}, \quad \beta_{2,1} = \frac{b_{n-1}}{a_{n-1}} \tag{A4.d}$$

$$G_{n-1,1}(s) = \frac{\begin{array}{c}(b_{n-2} - \delta_{2,0}a_{n-1})s^{n-3} + (b_{n-3} - \beta_{1,0}a_{n-3})s^{n-4} \\ + \cdots + (b_2 - \delta_{2,0}a_3)s + (b_1 - \beta_{1,0}a_1)\end{array}}{\begin{array}{c}a_{n-1}s^{n-2} + (a_{n-2} - \alpha_{1,0}a_{n-3} - \gamma_{2,0}a_{n-1})s^{n-3} + a_{n-3}s^{n-4} \\ +(a_{n-1} - \alpha_{1,0}a_{n-5} - \gamma_{2,0}a_{n-3})s^{n-5} + \cdots + (a_2 - \alpha_{1,0}a_1 - \gamma_{2,0}a_3)s + a_1\end{array}} \tag{A4.e}$$

$$G_{n-1,2}(s) = \frac{\begin{array}{c}(b_{n-2} - \delta_{1,0}a_{n-1})s^{n-3} + (b_{n-3} - \beta_{2,1}a_{n-3})s^{n-4} \\ + \cdots + (b_2 - \delta_{1,0}a_3)s + (b_1 - \beta_{2,1}a_1)\end{array}}{\begin{array}{c}a_{n-1}s^{n-2} + (a_{n-2} - \gamma_{1,0}a_{n-1} - \alpha_{2,1}a_{n-3})s^{n-3} \\ + \cdots + a_3s^2 + (a_2 - \gamma_{1,0}a_3 - \alpha_{2,1}a_1)s + a_1\end{array}}\,. \tag{A4.f}$$

It is obvious that (A1) follows (A4) immediately. Besides, we have from (A3) that

$$\begin{aligned}\frac{\beta_{1,0}^2}{2\alpha_{1,0}} + \frac{\delta_{2,0}^2}{2\gamma_{2,0}} &= \frac{\beta_{1,0}^2}{2\alpha_{1,0}} + \frac{(b_0 - \beta_{1,0}a_0)^2}{2a_0(a_1 - \alpha_{1,0}a_0)} \\ &= \frac{\beta_{1,0}^2}{2\alpha_{1,0}} + \frac{(\delta_{1,0} - \beta_{1,0}\gamma_{1,0})^2}{2\gamma_{1,0}(1 - \alpha_{1,0}\gamma_{1,0})} \\ &= \frac{\alpha_{1,0}\delta_{1,0}^2 + \gamma_{1,0}\beta_{1,0}^2 - 2\alpha_{1,0}\beta_{1,0}\gamma_{1,0}\delta_{1,0}}{2\alpha_{1,0}\gamma_{1,0}(1 - \alpha_{1,0}\gamma_{1,0})}\end{aligned}$$

and

$$\begin{aligned}\frac{\delta_{1,0}^2}{2\gamma_{1,0}} + \frac{\beta_{2,1}^2}{2\alpha_{2,1}} &= \frac{\delta_{1,0}^2}{2\gamma_{1,0}} + \frac{(b_{n-1} - \delta_{1,0}a_n)^2}{2a_n(a_{n-1} - \gamma_{1,0}a_n)} \\ &= \frac{\delta_{1,0}^2}{2\gamma_{1,0}} + \frac{(\beta_{1,0} - \alpha_{1,0}\delta_{1,0})^2}{2\delta_{1,0}(1 - \alpha_{1,0}\gamma_{1,0})} \\ &= \frac{\alpha_{1,0}\delta_{1,0}^2 + \gamma_{1,0}\beta_{1,0}^2 - 2\alpha_{1,0}\beta_{1,0}\gamma_{1,0}\delta_{1,0}}{2\alpha_{1,0}\gamma_{1,0}(1 - \alpha_{1,0}\gamma_{1,0})}\,.\end{aligned}$$

Hence, (A2) follows. The proof is now completed. □

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