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# Optimal Harvesting-Coefficient Control of Steady-State Prey–Predator Diffusive Volterra–Lotka Systems

A. W. Leung

Department of Mathematical Sciences, University of Cincinnati,
Cincinnati, OH 45221-0025, USA

**Abstract.** This article considers the optimal control of the harvesting of a prey–predator system in an environment. The species are assumed to be in steady state under diffusion and Voterra–Lotka type of interaction. They are harvested for economic profit, leading to reduction of growth rates; and the problem is to control the spatial distributions of harvests so as to optimize the return. Precise conditions are found so that the optimal control can be rigorously characterized as the solution of an optimality system of nonlinear elliptic partial differential equations. Moreover, a constructive approximation scheme for optimal control is given.

**Key Words.** Elliptic systems, Optimal harvest, Prey–predator.

**AMS Classification.** 49K20, 90A30, 35J60, 92D40, 49M99.

## 1. Introduction

This paper considers the optimal harvesting control of two interacting populations. The species concentrations satisfy a prey–predator Volterra–Lotka system under diffusion. They are in a steady-state situation with no-flux boundary conditions:

$$
\begin{aligned}
&\Delta u + u\left[(a_1(x) - f_1(x)) - b_1 u - c_1 v\right] = 0 \qquad \text{in } \Omega,\\
&\Delta v + v\left[(a_2(x) - f_2(x)) + c_2 u - b_2 v\right] = 0 \qquad \text{in } \Omega, \qquad (1.1)\\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega.
\end{aligned}
$$

The functions $u(x)$, $v(x)$ respectively describe prey and predator population concentrations with intrinsic growth rates $a_1(x)$, $a_2(x)$. The functions $f_1(x)$, $f_2(x)$ denote the distribution of control harvesting on the biological species. Such problem arises naturally in ecosystems, e.g., fisheries and agriculture, when various species are harvested for economic return. The parameters $b_1$, $c_i$, $i = 1, 2$, designate crowding and interaction effects which are assumed constant for simplicity. The optimal control criteria is to maximize profit which is the difference between economic revenue and cost. This is expressed by the payoff functional

$$J(f_1, f_2) = \int_\Omega \{K_1 u f_1 + K_2 v f_2 - M_1 f_1^2 - M_2 f_2^2\}\, dx, \tag{1.2}$$

where $K_1$, $K_2$ are constants describing the price of the prey and predator species, and $M_1$, $M_2$ are constants describing the costs of the controls $f_1, f_2$. Here $\int_\Omega u f_1\, dx$ and $\int_\Omega v f_2\, dx$ represent the total harvest of respectively $u, v$ which depends on $f_i$ through (1.1). Analogous models appear in, e.g., [4] and [13].

For the case of only one species with a single equation under control simplier analogous nonlinear and linear problems were studied in [10] and [14]. However, there are two species here; and they are under a prey–predator type of interaction which is usually more difficult to analyze than the competing or cooperative case, because the relation between the species is not symmetric. It requires painstaking effort to find explicit conditions for the rigorous characterization of the optimal control and for justification of the existence of the solution of the resulting nonlinear system of four equations. The conditions on the various coefficients are much more elaborate than those given in [10], and some of them seem incompatible with each other. However, Example 4.1 shows that they can all be simultaneously satisfied. Our results provide a framework for further investigation to consider whether some of the hypotheses can be successively relaxed for more practical applications. In Section 4 the optimality system is solved by an iterative scheme. The system does not satisfy the conditions in [8], because the nonlinear terms are not really monotonic in each component. Consequently, it requires special treatment to find a particular scheme so that an oscillatory sequence is obtained for approximating each component. We have assumed that the cost in the payoff functional depends quadratically on the control in the form $M_i f_i^2$ in a customary way in (1.2). The condition can certainly be modified to obtain a new payoff functional for $J(f_1, f_2)$.

We assume $\Omega$ is a bounded domain in $R^n$ with $\delta\Omega \in C^2$; $\Delta$ and $\partial/\partial v$ denote respectively the Laplacian and outward normal derivative. $K_i$, $M_i$, $b_i$, and $c_i$, $i = 1, 2$, are positive constants. We make the following assumptions and notations:

$$\begin{aligned} &a_i(x) \geq 0, \quad f_i(x) \geq 0 \qquad \text{a.e. in } \Omega,\\ &a_i \in L^\infty(\Omega), \quad f_i \in L^\infty(\Omega), \qquad i = 1, 2; \end{aligned} \tag{1.3}$$

$$L_+^\infty(\Omega) = \{f \mid f \in L^\infty(\Omega),\ f \geq 0 \text{ a.e. in } \Omega\}, \tag{1.4}$$

and

$$\mathscr{C}(\delta_1, \delta_2) = \{(f_1, f_2) \mid 0 \leq f_1 \leq \delta_i \text{ a.e. in } \Omega,\ i = 1, 2\} \tag{1.5}$$

for $\delta_i > 0$, $i = 1, 2$. Finally, we denote an optimal control (if it exists) to be an

$(f_1^*, f_2^*) \in \mathscr{C}(\delta_1, \delta_2)$ such that

$$J(f_1^*, f_2^*) = \sup\{J(f_1, f_2) | (f_1, f_2) \in \mathscr{C}(\delta_1, \delta_2)\}. \tag{1.6}$$

In Section 2 we discuss the existence and uniqueness of positive solutions (1.1). Then we show the existence of an optimal control for our problem (1.1), (1.2), (1.6). In Section 3 we find stronger conditions which enable the characterization of an optimal control in terms of a solution of an elliptic optimality system of four equations. Several theorems and corollaries are given with increasingly more stringent hypothesis, consequently giving rise to increasingly less-elaborate optimality systems and results. In Section 4 we construct monotone sequences closing in to all appropriate solutions of an optimality system for the last case in Section 3. If the monotone increasing and decreasing sequences converge to the same function, then the optimal control is unique. An example satisfying all the hypotheses is given at the end of Section 4. For convenience, we denote $\bar{a}_i = \text{ess sup}_{x \in \Omega} a_i(x)$, $\tilde{a}_i = \text{ess inf}_{x \in \Omega} a_i(x)$, for $i = 1, 2$.

## 2. Existence and Uniqueness of Positive Solutions, Existence of Optimal Control

We first establish the existence of a positive solution of (1.1) for an arbitrary fixed control $(f_1, f_2) \in \mathscr{C}(\delta_1, \delta_2)$ in Theorem 2.1, under hypotheses (H1) and (H2). Here, the solution may not be unique. Under further hypothesis (H3), Theorem 2.2 shows the uniqueness of the solution in the appropriate range, for each given control. Theorem 2.3 shows the existence of an optimal control, when solutions are uniquely defined for each fixed given control.

**Theorem 2.1.** *Suppose that $a_i(x)$, $b_i$, $c_i$, and $\delta_i$ satisfy the hypotheses*:

(H1) $\tilde{a}_1 - (c_1/b_2)(\bar{a}_2 + c_2\bar{a}_1/b_1) > \delta_1 > 0.$
(H2) $\tilde{a}_2 > \delta_2 > 0.$

*Then for each pair $(f_1, f_2) \in \mathscr{C}(\delta_1, \delta_2)$, problem* (1.1) *has a strictly positive solution $(u, v) = (u(f_1, f_2), v(f_1, f_2))$, i.e., $u, v > 0$ in $\bar{\Omega}$, and with each component in $W^{2,p}(\Omega)$ for any $p \in (n, \infty)$. Moreover, the estimate*

$$\|u(f_1, f_2)\|_{2,p}, \|v(f_1, f_2)\|_{2,p} \leq \text{constant} \tag{2.1}$$

*is valid uniformly for all $(f_1, f_2) \in \mathbb{C}(\delta_1, \delta_2)$.*

*Proof.* Define constant functions:

$$\psi_1(x) \equiv \frac{\bar{a}_1}{b_1}, \qquad \psi_2(x) \equiv \frac{1}{b_2}\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right),$$

$$\phi_1(x) \equiv \frac{1}{b_1}\left[\tilde{a}_1 - \frac{c_1}{b_2}\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right) - \delta_1\right],$$

and

$$\phi_2(x) \equiv \frac{1}{b_2}\left[\tilde{a}_2 - \delta_2 + \frac{c_2}{b_1}\left\{\tilde{a}_1 - \frac{c_1}{b_2}\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right) - \delta_1\right\}\right] \tag{2.2}$$

for all $x$ in $\bar{\Omega}$. It is clear from (H1) and (H2) that $\psi_i$, $\phi_i$, $i = 1, 2$, are strictly positive in $\bar{\Omega}$. It can be readily seen that

$$\begin{aligned}\Delta\psi_1 &+ \psi_1[(a_1(x) - f_1(x)) - b_1\psi_1 - c_1 v] \\ &= \frac{\bar{a}_1}{b_1}[(a_1(x) - f_1(x)) - \bar{a}_1 - c_1 v] \leq 0 \end{aligned} \tag{2.3}$$

for all $\phi_2 \leq v \leq \psi_2$, $x \in \Omega$. Moreover, we have

$$\begin{aligned}\Delta\phi_1 &+ \phi_1[(a_1(x) - f_1(x)) - b_1\phi_1 - c_1 v] \\ &= \phi_1(a_1(x) - \tilde{a}_1) + (\delta_1 - f_1(x)) + c_1 b_2^{-1}(\bar{a}_2 + c_2\bar{a}_1 b_1^{-1}) - c_1 v \geq 0 \end{aligned} \tag{2.4}$$

for all $\phi_2 \leq v \leq \psi_2$, $0 \leq f_1 \leq \delta_1$, $x \in \Omega$. Similarly, we obtain

$$\Delta\psi_2 + \psi_2[(a_2(x) - f_2(x)) - b_2\psi_2 + c_2 u] \leq 0, \tag{2.5}$$

$$\Delta\phi_2 + \phi_2[(a_2(x) - f_2(x)) - b_2\phi_2 + c_2 u] \geq 0, \tag{2.6}$$

for all $\phi_1 \leq u \leq \psi_1$, $0 \leq f_2 \leq \delta_2$, $x \in \Omega$.

Let $X_i = \{w \in C(\bar{\Omega}), \phi_i \leq w \leq \psi_i\}$, $i = 1, 2$. Define the map $T: X_1 \times X_2 \to X_1 \times X_2$ as $T(y_1, y_2) = (z_1, z_2)$ for $(y_1, y_2) \in X_1 \times X_2$, where $z_1, z_2 \in W^{2,p}(\Omega)$, $p > n$, and $(z_1, z_2)$ is determined uniquely as the solution of the following:

$$\begin{aligned} &\Delta z_1 - Qz_1 + y_1[a_1(x) - f_1(x) - b_1 y_1 - c_1 y_2] + Qy_1 = 0 \qquad \text{in } \Omega, \\ &\Delta z_2 - Qz_2 + y_2[a_2(x) - f_2(x) + c_2 y_1 - b_2 y_2] + Qy_2 = 0 \qquad \text{in } \Omega, \\ &\frac{\partial z_1}{\partial \nu} = \frac{\partial z_2}{\partial \nu} = 0 \qquad \text{on } \delta\Omega. \end{aligned}$$

Here $Q > 0$ is a constant. Using (2.3)–(2.6) and the maximum principle for the $W^{2,p}(\Omega)$ solution with Neumann boundary condition we can show as in Theorem 3.1 in [9] that $(z_1, z_2) \in X_1 \times X_2$. Using Theorem 15.1 in [1], we can obtain a uniform bound for the $W^{2,p}(\Omega)$ norm of $z_1, z_2$. Following the proof in [9], we can then use such a bound to show $T$ is compact and eventually obtain a fixed point. Such a fixed point is a solution of (1.1) in $X_1 \times X_2$, and the uniform bound for the $W^{2,p}(\Omega)$ norm gives precisely (2.1). For more details, see [9]. □

**Theorem 2.2.** *Assume hypotheses* (H1) *and* (H2). *Let*

$$\begin{aligned} S \overset{\text{def}}{=} \min\Bigg\{ &b_2 b_1^{-1}\left[\tilde{a}_1 - \frac{c_1}{b_2}\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right) - \delta_1\right]\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right)^{-1}, \\ &(\tilde{a}_2 - \delta_2)b_1(\bar{a}_1 b_2)^{-1}\Bigg\}. \end{aligned} \tag{2.7}$$

*Suppose further that*

(H3) $c_1 c_2 (b_1 b_2)^{-1} < S^4$

*is satisfied. Then, for each pair* $(f_1, f_2) \in \mathscr{C}(\delta_1, \delta_2)$, *problem* (1.1) *has a unique solution*

*$(u, v)$, $u, v \in W^{2,p}(\Omega)$ for any $p \in (n, \infty)$, with the property that*

$$\phi_1 \leq u(x) \leq \psi_1, \quad \phi_2 \leq v(x) \leq \psi_2 \qquad \text{in } \bar{\Omega}. \tag{2.8}$$

*Here $\phi_i$, $\psi_i$, $i = 1, 2$, are given in* (2.2).

**Remark 2.1.** Hypotheses (H1) and (H2) imply that $S$ is positive. Hypothesis (H3) is readily satisfied if $c_1$ or $c_2$ is reduced to being sufficiently small.

*Proof.* For convenience, let

$$G_1 = b_1 b_2^{-1}\left[\tilde{a}_1 - \frac{c_1}{b_2}\left(\bar{a}_2 + \frac{c_2 \bar{a}_1}{b_1}\right) - \delta_1\right]^{-1}\left(\bar{a}_2 + \frac{c_2 \bar{a}_1}{b_1}\right)$$

and

$$G_2 = (\tilde{a}_2 - \delta_2)^{-1} b_1^{-1} \bar{a}_1 b_2.$$

We now define $\hat{U}(x)$, $\hat{V}(x)$, $\tilde{U}(x)$, and $\tilde{V}(x)$ to be respectively solutions of the scalar problems:

$$\Delta \hat{U} + \hat{U}[(a_1(x) - f_1(x)) - b_1 \hat{U}] = 0 \quad \text{in } \Omega, \qquad \frac{\partial \hat{U}}{\partial v} = 0 \quad \text{on } \partial\Omega, \tag{2.9}$$

$$\Delta \hat{V} + \hat{V}\left[(a_2(x) - f_2(x)) + \frac{c_2 \bar{a}_1}{b_1} - b_2 \hat{V}\right] = 0 \quad \text{in } \Omega, \qquad \frac{\partial \hat{V}}{\partial v} = 0 \quad \text{on } \partial\Omega, \tag{2.10}$$

$$\Delta \tilde{U} + \tilde{U}[(a_1(x) - f_1(x)) - b_1 \tilde{U} - c_1 \hat{V}(x)] = 0 \quad \text{in } \Omega, \qquad \frac{\partial \tilde{U}}{\partial v} = 0 \quad \text{on } \partial\Omega, \tag{2.11}$$

$$\Delta \tilde{V} + \tilde{V}[(a_2(x) - f_2(x)) - b_2 \tilde{V}] = 0 \quad \text{in } \Omega, \qquad \frac{\partial \tilde{V}}{\partial v} = 0 \quad \text{on } \partial\Omega, \tag{2.12}$$

Using the constant functions $\bar{a}_1/b_1$ and $(1/b_1)[\tilde{a}_1 - \delta_1]$ as upper and lower solutions for (2.9), we can readily obtain, by means of monotone iterations from the upper solution as in [7] and [8], a unique solution $\hat{U}$ of (2.9) in $W^{2,p}(\Omega)$ for any $p \in (n, \infty)$; and $(1/b_1)(\tilde{a}_1 - \delta_1) \leq \hat{U}(x) \leq \bar{a}_1/b_1$ for all $x \in \bar{\Omega}$. Similarly, we obtain the unique positive solutions in $W^{2,p}(\Omega)$,

$$b_2^{-1}(\tilde{a}_2 - \delta_2) \leq \hat{V}(x) \leq b_2^{-1}\left(\bar{a}_2 + \frac{c_2 \bar{a}_1}{b_1}\right),$$

$$b_1^{-1}\left[\tilde{a}_1 - \delta_1 - \frac{c_1}{b_2}\left(\bar{a}_2 + \frac{c_2 \bar{a}_1}{b_1}\right)\right] \leq \tilde{U}(x) \leq b_1^{-1}(\tilde{a}_1 - \delta_1),$$

$$b_2^{-1}(\tilde{a}_2 - \delta_2) \leq \tilde{V}(x) \leq b_2^{-1}\bar{a}_2,$$

respectively for (2.10), (2.11), and (2.12). We thus have the comparison,

$$\hat{U}(x) \leq G_2 \tilde{V}(x), \qquad \hat{V}(x) \leq G_1 \tilde{U}(x) \qquad \text{in } \bar{\Omega}. \tag{2.13}$$

We next inductively define $u_i, v_i$ to be strictly positive functions in $W^{2,p}(\Omega)$, starting with $u = \hat{U}$, $v_1$ satisfying

$$\Delta v_1 + v_1[(a_2(x) - f_2(x)) + c_2\hat{U}(x) - b_2 v_1] = 0 \quad \text{in } \Omega, \qquad \frac{\partial v_1}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

and $u_i, v_i$, $i = 2, 3, \ldots$, satisfying

$$\begin{aligned} &\Delta u_i + u_i[(a_1(x) - f_1(x)) - b_1 u_i - c_1 v_{i-1}] = 0 \quad \text{in } \Omega, \qquad \frac{\partial u_i}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\ &\Delta v_i + v_i[(a_2(x) - f_2(x)) + c_2 u_i - b_2 v_i] = 0 \quad \text{in } \Omega, \qquad \frac{\partial v_i}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.14}$$

Using the maximum principle, we can deduce as in Sections 5.2 and 5.3 in [8], that we have, for all $x \in \bar{\Omega}$,

$$\begin{aligned} &\tilde{U} \leq u_2 \leq u_4 \leq u_6 \cdots \leq u_5 \leq u_3 \leq u_1 \leq \hat{U}, \\ &\tilde{V} \leq v_2 \leq v_4 \leq v_6 \cdots \leq v_5 \leq v_3 \leq v_1 \leq \hat{V}. \end{aligned} \tag{2.15}$$

Using Green's identity and (2.14), we obtain, for $i \geq 1$,

$$\begin{aligned} 0 &= \int (u_{2i+2}\Delta u_{2i+1} - u_{2i+1}\Delta u_{2i+2})\, dx \\ &= -\int_\Omega u_{2i+1}u_{2i+2}[b_1(u_{2i+2} - u_{2i+1}) + c_1(v_{2i+1} - v_{2i})]\, dx, \end{aligned} \tag{2.16}$$

$$0 = \int_\Omega v_{2i}v_{2i+1}[c_2(u_{2i} - u_{2i+1}) + b_2(v_{2i+1} - v_{2i})]\, dx, \tag{2.17}$$

$$0 = \int_\Omega u_{2i}u_{2i+1}[b_1(u_{2i} - u_{2i+1}) + c_1(v_{2i-1} - v_{2i})]\, dx, \tag{2.18}$$

$$0 = \int_\Omega v_{2i-1}v_{2i}[c_2(u_{2i} - u_{2i-1}) + b_2(v_{2i-1} - v_{2i})]\, dx. \tag{2.19}$$

Using (2.16), (2.17) and (2.13), (2.15) we deduce that

$$\begin{aligned} \int_\Omega (u_{2i+1} - u_{2i+2})u_{2i+1}u_{2i+2}\, dx &= \frac{c_1}{b_1}\int_\Omega (v_{2i+1} - v_{2i})u_{2i+1}u_{2i+2}\, dx \\ &\leq \frac{c_1}{b_1}\int_\Omega G_2^2(v_{2i+1} - v_{2i})v_{2i}v_{2i+1}\, dx \\ &= G_2^2\frac{c_1}{b_1}\frac{c_2}{b_2}\int_\Omega (u_{2i+1} - u_{2i})v_{2i}v_{2i+1}\, dx. \end{aligned} \tag{2.20}$$

Then we use (2.18), (2.19) and (2.13), (2.15) again to obtain

$$\begin{aligned}\int_\Omega (u_{2i-1} - u_{2i})u_{2i}u_{2i+1}\,dx &= \frac{c_1}{b_1}\int_\Omega (v_{2i-1} - v_{2i})u_{2i}u_{2i+1}\,dx \\ &\le \frac{c_1}{b_1}\int_\Omega G_2^2(v_{2i-1} - v_{2i})v_{2i-1}v_{2i}\,dx \\ &= G_2^2\frac{c_1}{b_1}\frac{c_2}{b_2}\int_\Omega (u_{2i-1} - u_{2i})v_{2i-1}v_{2i}\,dx. \qquad (2.21)\end{aligned}$$

Combining (2.20), (2.21) and using (2.13), (2.15) we obtain, for $i \ge 1$,

$$\begin{aligned}&\int_\Omega (u_{2i+1} - u_{2i+2})u_{2i+1}u_{2i+2}\,dx \\ &\quad\le G_2^4G_1^4\left(\frac{c_1}{b_1}\right)^2\left(\frac{c_2}{b_2}\right)^2\int_\Omega (u_{2i-1} - u_{2i})u_{2i-1}u_{2i}\,dx. \qquad (2.22)\end{aligned}$$

By means of (2.22), we conclude that if (H3) is satisfied, then $\lim_{i\to\infty}\int_\Omega(u_{2i+1} - u_{2i+2})u_{2i+1}u_{2i+2}\,dx = 0$. By (2.15), the limits

$$\lim_{i\to\infty} u_{2i+1} \overset{(\text{def})}{=} u^* > 0 \quad\text{and}\quad \lim_{i\to\infty} u_{2i+2} \overset{(\text{def})}{=} u^* > 0$$

must exist. The argument above shows that $u^* = u_*$ a.e. in $\Omega$. Further, using the maximum principle described above and the subsequently modified comparison theorem as in Theorem 5.2-1 in [8], we can show as in Section 5.2 in [8] that any solution $(u, v)$ of (1.1) with $\tilde{U} \le u \le \hat{U}$, $\tilde{V} \le v \le \hat{V}$ in $\bar{\Omega}$, $u, v \in W^{2,p}(\Omega)$, must satisfy

$$u_* \le u \le u^*, \qquad \lim_{i\to\infty} v_{2i} \overset{(\text{def})}{=} v_* \le v \le v^* \overset{(\text{def})}{=} \lim_{i\to\infty} v_{2i+1}, \qquad x \in \bar{\Omega}.$$

(For more details, see Theorem 5.2-4 in [9] and Theorem 2.1 in [14].) Since $u^* = u_*$, we can show that $v^*$ and $v_*$ satisfy the same equation and again use the comparison as above to conclude that $v^* = v_*$ (see Theorem 5.2-3 in [8]). Comparing $\phi_i$, $\psi_i$, $i = 1, 2$, with the estimates for $\tilde{U}$, $\hat{U}$, $\tilde{V}$, $\hat{V}$, we conclude that any solution $(u, v)$ of (1.1) satisfying (2.8) must have $u = u^* = u_*$, $v = v^* = v_*$ in $\bar{\Omega}$. The existence part follows from Theorem 2.1. □

**Remark 2.2.** In Theorem 2.2 uniform $\|\ \|_{2,p}$ bound for $u, v$ can be obtained for all $(f_1, f_2) \in \mathscr{C}(\delta_1, \delta_2)$, $p > n$, as in Theorem 2.1.

**Remark 2.3.** Under the hypotheses of Theorem 2.2, the functional $J(f_1, f_2)$ is uniquely defined if $(u, v)$ is chosen as the one solution satisfying (2.8).

**Theorem 2.3.** *Assume hypotheses* (H1) *and* (H2) *and that* $(u(f_1, f_2), v(f_1, f_2))$ *is defined uniquely so that* (2.8) *and* (2.1) *are satisfied uniformly for all* $(f_1, f_2) \in \mathscr{C}(\delta_1, \delta_2)$. *Then* $(f_1^*, f_2^*) \in \mathscr{C}(\delta_1, \delta_2)$ *exists such that* $J(f_1^*, f_2^*)$ *is the optimal control for all* $(f_1, f_2) \in \mathscr{C}(\delta_1, \delta_2)$.

**Remark 2.4.** By Theorem 2.2 and Remark 2.3, the addition of hypothesis (H3) to (H1) and (H2) ensures that $(u(f_1, f_2), v(f_1, f_2))$ can be chosen uniquely in the way described in Theorem 2.3. Hence, under hypotheses (H1)–(H3), an optimal control does exist.

*Proof.* The uniform boundedness of $(u(f_1, f_2), v(f_1, f_2))$ for all $(f_1, f_2) \in \mathscr{C}(\delta_1, \delta_2)$ implies that $\sup\{J(f_1, f_2) | (f_1, f_2) \in \mathscr{C}(\delta_1, \delta_2)\} < \infty$. Let $(f_{1n}, f_{2n}) \in \mathscr{C}(\delta_1, \delta_2)$ be a maximizing sequence. Then a subsequence, again denoted as $(f_{1n}, f_{2n})$ for convenience, exists so that

$$f_{in} \to f_i^* \qquad \text{weakly in } L^2(\Omega), \text{ with } \quad (f_1^*, f_2^*) \in \mathscr{C}(\delta_1, \delta_2),$$

and

$$\left.\begin{aligned} u_n &\equiv u(f_{1n}, f_{2n}) \to \bar{u}^* \\ v_n &\equiv v(f_{1n}, f_{2n}) \to \bar{v}^* \end{aligned}\right\} \quad \text{strongly in } W^{1,2}(\Omega)$$

(by using (2.1)). Passing to the limit as $n \to \infty$ in

$$\int_\Omega (\nabla u_n \nabla\varphi - (a_1 - f_{1n}) u_n \varphi + b_1 u_n^2 \varphi + c_1 u_n v_n \varphi)\, dx = 0$$

and

$$\int_\Omega (\nabla v_n \nabla\varphi - (a_2 - f_{2n}) v_n \varphi - c_2 v_n u_n \varphi + b_2 v_n^2 \varphi)\, dx = 0,$$

for all $\varphi \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, and noting that, for example,

$$\int_\Omega f_{1n} u_n \varphi\, dx \to \int_\Omega f_1^* \bar{u}^* \varphi\, dx \qquad \text{for all} \quad \varphi \in L^\infty(\Omega),$$

we conclude that $(\bar{u}^*, \bar{v}^*)$ is a solution of (1.1) with $(f_1, f_2)$ replaced by $(f_1^*, f_2^*)$. Since $(u_n, v_n)$ are uniquely defined in a certain range of values, hence its limit $(\bar{u}^*, \bar{v}^*)$ is within the same bounds. Consequently, (1.1) implies that $\|u\|_{2,p}, \|v\|_{2,p}$ is bounded by the same constant as in (2.1). By assumption, $u(f_1^*, f_2^*)$ is uniquely defined so that such properties are satisfied. We thus conclude that $(\bar{u}^*, \bar{v}^*) = (u(f_1^*, f_2^*), v(f_1^*, f_2^*))$. Finally, the conclusion follows from the semicontinuity of $J$; that is, we have $J(f_1^*, f_2^*) = \sup\{J(f_1, f_2) | (f_1, f_2) \in \mathscr{C}(\delta_1, \delta_2)\}$. □

## 3. Derivation of the Optimality System

In this section we need stronger assumptions on the intrinsic growth rate functions $a_i(x)$, $i = 1, 2$. When (H1) and (H2) are respectively strengthened to (H1*) and (H2*) and additional assumptions are made on the interaction rates between the species, Lemma 3.1 shows the differentiability of $u(f_1, f_2)$ and $v(f_1, f_2)$ with respect to $(f_1, f_2)$. The additional assumptions are satisfied, for instance, when the interspecies interactions are small compared with the intraspecies interactions. Theorem 3.1

gives a characterization of an optimal control in terms of solutions of an elliptic system of four equations. The optimal control is related to the solution of the systems in terms of various inequalities. Corollary 3.1 shows that under further assumptions on the cost and price parameters $M_i$, $K_i$, $i = 1, 2$, the optimal control can be exactly characterized by a solution of the optimality system of four equations.

**Lemma 3.1.** *Assume that* $\delta_1, \delta_2$ *exist such that*

(H1*) $0 < \delta_1 \le \frac{1}{3}\{2\tilde{a}_1 - \bar{a}_1 - (2c_1/b_2)(\bar{a}_2 + c_2\bar{a}_1/b_1)\}$,
(H2*) $0 < \delta_2 \le \frac{1}{3}\{2\tilde{a}_2 - \bar{a}_2 - c_2\bar{a}_1/b_1\}$,

*and that* $u(f_1, f_2)$, $v(f_1, f_2)$ *is uniquely defined for all* $(f_1, f_2) \in \mathscr{C}(\delta_1, \delta_2)$ *in the sense described in Theorem 2.3. Further suppose*

(H4) $c_1\bar{a}_1/b_1 + (c_2/b_2)(\bar{a}_2 + c_2\bar{a}_1/b_1) < 2\min\{\delta_1, \delta_2, 1\}$.

*Then the mappings* $\mathscr{C}(\delta_1, \delta_2) \ni (f_1, f_2) \mapsto u(f_1, f_2), v(f_1, f_2) \in W^{1,2}(\Omega)$ *are differentiable in the following sense:*

$$\begin{gathered}\left(\frac{u(f_1 + \beta_i\bar{f}_1, f_2) - u(f_1, f_2)}{\beta_i}, \frac{v(f_1 + \beta_i\bar{f}_1, f_2) - v(f_1, f_2)}{\beta_i}\right) \to (\xi, \eta),\\ \left(\frac{u(f_1, f_2 + \beta_i\bar{f}_2) - u(f_1, f_2)}{\beta_i}, \frac{v(f_1, f_2 + \beta_i\bar{f}_2) - v(f_1, f_2)}{\beta_i}\right) \to (\tilde{\xi}, \tilde{\eta})\end{gathered} \tag{3.1}$$

*componentwise weakly in* $W^{1,2}(\Omega)$ *for some* $\beta_i \to 0$, *for any given* $(f_1, f_2) \in \mathscr{C}(\delta_1, \delta_2)$ *and* $\bar{f}_1, \bar{f}_2 \in L^\infty(\Omega)$ *such that* $(f_1 + \beta_i\bar{f}_1, f_2 + \beta_i\bar{f}_2) \in \mathscr{C}(\delta_1, \delta_2)$. *Further,* $(\xi, \eta)$ *is a solution of*

$$\begin{aligned}&\Delta\xi + [(a_1 - f_1) - 2b_1u(f_1, f_2) - c_1v(f_1, f_2)]\xi - c_1u(f_1, f_2)\eta\\ &\qquad = u(f_1, f_2)\bar{f}_1 \qquad \text{in } \Omega,\\ &\Delta\eta + c_2v(f_1, f_2)\xi + [(a_2 - f_2) - 2b_2v(f_1, f_2) + c_2u(f_1, f_2)]\eta = 0 \qquad \text{in } \Omega,\\ &\frac{\partial\xi}{\partial\nu} = \frac{\partial\eta}{\partial\nu} = 0 \qquad \text{on } \partial\Omega:\end{aligned} \tag{3.2}$$

*and* $(\tilde{\xi}, \tilde{\eta})$ *is a solution of*

$$\begin{aligned}&\Delta\tilde{\xi} + [(a_1 - f_1) - 2b_1u(f_1, f_2) - c_1v(f_1, f_2)]\tilde{\xi} - c_1u(f_1, f_2)\tilde{\eta} = 0 \qquad \text{in } \Omega,\\ &\Delta\tilde{\eta} + c_2v(f_1, f_2)\tilde{\xi} + [(a_2 - f_2) - 2b_2v(f_1, f_2) + c_2u(f_1, f_2)]\tilde{\eta}\\ &\qquad = v(f_1, f_2)\bar{f}_2 \qquad \text{in } \Omega,\\ &\frac{\partial\tilde{\xi}}{\partial\nu} = \frac{\partial\tilde{\eta}}{\partial\nu} = 0 \qquad \text{on } \partial\Omega.\end{aligned} \tag{3.3}$$

*Here* $\xi, \eta, \tilde{\xi}, \tilde{\eta}$ *are in* $W^{2,2}(\Omega)$.

*Proof.* From (1.1), we deduce that

$$\xi_\beta = \frac{u(f_1 + \beta \bar{f}_1, f_2) - u(f_1, f_2)}{\beta}, \qquad \eta_\beta = \frac{v(f_1 + \beta \bar{f}_1, f_2) - v(f_1, f_2)}{\beta} \tag{3.4}$$

satisfy

$$\begin{aligned} &\Delta \xi_\beta + [(a_1 - f_1) - b_1 u(f_1 + \beta \bar{f}_1, f_2) - b_1 u(f_1, f_2) - c_1 v(f_1, f_2)]\xi_\beta \\ &\qquad - c_1 u(f_1 + \beta \bar{f}_1, f_2)\eta_\beta = u(f_1 + \beta \bar{f}_1, f_2)\bar{f}_1 \qquad \text{in } \Omega, \\ &\Delta \eta_\beta + c_2 v(f_1 + \beta \bar{f}_1, f_2)\xi_\beta \\ &\qquad + [(a_2 - f_2) - b_2 v(f_1 + \beta \bar{f}_1, f_2) - b_2 v(f_1, f_2) + c_2 u(f_1, f_2)]\eta_\beta \\ &\qquad = 0 \qquad \text{in } \Omega, \\ &\frac{\partial \xi_\beta}{\partial \nu} = \frac{\partial \eta_\beta}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{3.5}$$

if $(f_1, f_2)$ and $(f_1 + \beta \bar{f}_1, f_2) \in \mathscr{C}(\delta_1, \delta_2)$. The lower bounds in (2.8) imply that

$$\begin{aligned} &b_1[u(f_1 + \beta \bar{f}_1, f_2) + u(f_1, f_2)] - a_1 + f_1 + c_1 v(f_1, f_2) \\ &\qquad \geq 2\tilde{a}_1 - \frac{2c_1}{b_2}\left(\bar{a}_2 + \frac{c_2 \bar{a}_1}{b_1}\right) - 2\delta_1 - a_1 + c_1 \phi_2 \geq \delta_1 \qquad \text{in } \bar{\Omega}, \end{aligned} \tag{3.6}$$

where the last inequality follows from (H1*). The first equation in (3.5), and (3.6) give

$$\min\{\delta_1, 1\} \|\xi_\beta\|_{1,2}^2 \leq \|u(f_1 + \beta \bar{f}_1, f_2)\|_\infty \|\xi_\beta\|_2 [c_1 \|\eta_\beta\|_2 + \|\bar{f}_1\|_2]. \tag{3.7}$$

The bounds in (2.8) and (H2*) also imply that

$$b_2[v(f_1 + \beta \bar{f}_1, f_2) + v(f_1, f_2)] - a_2 + f_2 - c_2 u(f_1, f_2) \geq \delta_2 \qquad \text{in } \bar{\Omega}.$$

The last inequality and the second equation in (3.5) give

$$\min\{\delta_2, 1\} \|\eta_\beta\|_{1,2}^2 \leq c_2 \|v(f_1 + \beta \bar{f}_1, f_2)\|_\infty \|\xi_\beta\|_2 \|\eta_\beta\|_2. \tag{3.8}$$

Since $\|u(f_1 + \beta \bar{f}_1, f_2)\|_\infty \leq \bar{a}_1/b_1$ and $\|v(f_1 + \beta \bar{f}_1, f_2)\|_\infty \leq (1/b_2)(\bar{a}_2 + c_2 \bar{a}_1/b_1)$, inequalities (3.7) and (3.8) and (H4) yield

$$\begin{aligned} &\min\{\delta_1, 1\} \|\xi_\beta\|_{1,2}^2 + \min\{\delta_2, 1\} \|\eta_\beta\|_{1,2}^2 \\ &\qquad \leq k \min\{\delta_1, \delta_2, 1\} \|\xi_\beta\|_2 \|\eta_\beta\|_2 + \text{const} \|\xi_\beta\|_2 \end{aligned} \tag{3.9}$$

for some $k \in (0, 2)$. Inequality (3.9) hence leads to

$$\hat{k}(\|\xi_\beta\|_{1,2}^2 + \|\eta_\beta\|_{1,2}^2) \leq \text{const} \|\xi_\beta\|_{1,2} \tag{3.10}$$

for some $\hat{k} > 0$. This gives a uniform bound for $\|\xi_\beta\|_{1,2}$ and $\|\eta_\beta\|_{1,2}$ for all $(f_1, f_2)$, $(f_1 + \beta \bar{f}_1, f_2) \in \mathscr{C}(\delta_1, \delta_2)$ with $\bar{f}_1$ fixed. We can thus choose a sequence $\beta_i \to 0$ such that we have a weakly convergent sequence as described in (3.1).

Since $c_1 u(f_1, f_2)\eta + u(f_1, f_2)\bar{f}_1$ is a function in $L^2(\Omega)$, the first equation in (3.2) and the results in [1] imply that $\xi \in W^{2,2}(\Omega)$. Similarly, the second equation in (3.2) implies that $\eta \in W^{2,2}(\Omega)$.

Analogously, we obtain the result involving the second part of (3.1) and the solution $(\tilde{\xi}, \tilde{\eta})$ of (3.3). □

For convenience, we denote constants

$$E_1 = E_1(\bar{a}_1, \tilde{a}_1, \bar{a}_2, b_1, b_2, c_1, c_2) = 2\tilde{a}_1 - \bar{a}_1 - \frac{2c_1}{b_2}\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right), \tag{3.11}$$

$$E_2 = E_2(\bar{a}_2, \tilde{a}_2, \bar{a}_1, b_1, c_2) = 2\tilde{a}_2 - \bar{a}_2 - \frac{c_2\bar{a}_1}{b_1}. \tag{3.12}$$

In order to obtain a better characterization of the optimal control of the problem, we need to strengthen hypotheses (H1*) and (H2*) to

$$\text{(H1**)}\ \ 0 < \delta_1 \le \min\left\{\tfrac{1}{4}E_1, \tfrac{1}{3}\left[E_1 + \frac{c_1}{b_2}\frac{\bar{a}_2}{2} - \frac{c_2}{b_2}\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right)\frac{K_2}{2K_1}\right]\right\},$$
$$\text{(H2**)}\ \ 0 < \delta_2 \le \tfrac{1}{4}\left[E_2 - \frac{8c_1^2c_2K_1}{b_1^2b_2K_2}\,\bar{a}_1^2\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right)E_1^{-1}E_2^{-1}\right].$$

Note that the right-hand sides of (H1*) and (H2*) are respectively $\frac{1}{3}E_1$ and $\frac{1}{3}E_2$. We are not looking for the best possible sufficient condition for the characterization; hypotheses (H1**) and (H2**) are used because they can be readily satisfied if $c_1$ and/or $c_2$ are sufficiently small.

**Theorem 3.1.** *Assume hypotheses* (H1**), (H2**) *and* (H4) *and that* $(u(f_1, f_2), v(f_1, f_2))$ *is uniquely defined for all* $(f_1, f_2) \in \mathscr{C}(\delta_1, \delta_2)$ *in the sense described in Theorem* 2.3. *Suppose* $(f_1^*, f_2^*) \in \mathscr{C}(\delta_1, \delta_2)$ *is an optimal control. Then let* $(u, v, p_1, p_2)$ *be any solution of*

$$\left.\begin{aligned}
&\Delta u + (a_1 - f_1^*)u - b_1u^2 - c_1uv = 0 \quad \text{in } \Omega,\\
&\Delta v + (a_2 - f_2^*)v + c_2uv - b_2v^2 = 0 \quad \text{in } \Omega,\\
&\Delta p_1 + (a_1 - f_1^*)p_1 - (2b_1u + c_1v)p_1 + c_2vp_2 = -K_1f_1^* \quad \text{in } \Omega,\\
&\Delta p_2 + (a_2 - f_2^*)p_2 - (2b_2v - c_2u)p_2 - c_1up_1 = -K_2f_2^* \quad \text{in } \Omega,\\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial p_1}{\partial \nu} = \frac{\partial p_2}{\partial \nu} = 0 \quad \text{on } \Omega,
\end{aligned}\right\} \tag{3.13}$$

*with* $u, v, p_1, p_2 \in H^{2,2}(\Omega)$, *satisfying*

$$\begin{aligned}
&\phi_1 \le u \le \psi_1, \quad \phi_2 \le v \le \psi_2 \quad \text{in } \Omega,\\
&-4\bar{a}_1c_1c_2K_1\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right)(b_1b_2E_1E_2)^{-1} \le p_1 \le K_1 \quad \text{in } \Omega,\\
&-2\bar{a}c_1K_1(b_1E_2)^{-1} \le p_2 \le \frac{K_2}{2} \quad \text{in } \Omega
\end{aligned} \tag{3.14}$$

*then the control* $(f_1^*, f_2^*)$ *must satisfy, for* $i = 1, 2$,

$$f_i^*(x) \geq \frac{(K_i - p_i(x))u_i(x)}{2M_i} \quad \text{in } \{x \in \Omega \mid f_i^*(x) < \delta_i\} \text{ a.e.,} \tag{3.15a}$$

$$f_i^*(x) \leq \frac{(K_i - p_i(x))u_i(x)}{2M_i} \quad \text{in } \{x \in \Omega \mid f_i^*(x) > 0\} \text{ a.e.,} \tag{3.15b}$$

$$f_i^*(x) = \frac{(K_i - p_i(x))u_i(x)}{2M_i} \quad \text{in } \{x \in \Omega \mid 0 < f_i^*(x) < \delta_i\} \text{ a.e.,} \tag{3.15c}$$

*Here, we denote* $(u, v) = (u_1, u_2)$ *for convenience. (Recall* $\phi_i, \psi_i$, $i = 1, 2$, *are defined in* (2.2).)

*Proof.* Since hypotheses (H1**) and (H2**) imply (H1) and (H2), Theorem 2.3 ensures the existence of an optimal control $(f_1^*, f_2^*) \in \mathscr{C}(\delta_1, \delta_2)$.

For $\bar{f}_1 \in L_+^\infty(\Omega)$, $\varepsilon > 0$, define

$$\bar{f}_1^\varepsilon = \begin{cases} \bar{f}_1 & \text{if } \; f_1^* \leq \delta_1 - \varepsilon \| \bar{f}_1 \|_\infty, \\ 0 & \text{elsewhere.} \end{cases} \tag{3.16}$$

Then, for $\beta > 0$ small enough, we have $J(f_1^*, f_2^*) \geq J(f_1^* + \beta \bar{f}_1^\varepsilon, \bar{f}_2^*)$. Dividing by $\beta$, and letting $\beta$ tend to zero appropriately as in Lemma 3.1, we obtain

$$\int_\Omega K_1 f_1^* \xi + K_1 u(f_1^*, f_2^*) \bar{f}_1^\varepsilon + K_2 f_2^* \eta - 2M_1 f_1^* \bar{f}_1^\varepsilon \, dx \leq 0, \tag{3.17}$$

where $(\xi, \eta)$ is a solution of (3.2) with $f_1, f_2, \bar{f}_1$ respectively replaced by $f_1^*, f_2^*, \bar{f}_1^\varepsilon$. Let $(p_1, p_2)$ be any solution of

$$\left.\begin{aligned} &-\Delta p_1 - (a_1 - f_1^*)p_1 + (2b_1 u(f_1^*, f_2^*) + c_1 v(f_1^*, f_2^*))p_1 \\ &\qquad - c_2 v(f_1^*, f_2^*)p_2 = K_1 f_1^* \qquad \text{in } \Omega, \\ &-\Delta p_2 - (a_2 - f_2^*)p_2 + (2b_2 v(f_1^*, f_2^*) - c_2 u(f_1^*, f_2^*))p_2 \\ &\qquad + c_1 u(f_1^*, f_2^*)p_1 = K_2 f_2^* \qquad \text{in } \Omega, \\ &\frac{\partial p_1}{\partial \nu} = \frac{\partial p_2}{\partial \nu} = 0 \qquad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.18}$$

Replacing $K_1 f_1^*$ and $K_2 f_2^*$ in (3.17) by the left-hand side of (3.18) and integrating by parts, we obtain by means of the equation for $(\xi, \eta)$ that

$$\int_\Omega \bar{f}_1^\varepsilon [p_1 u(f_1^*, \; f_2^*) - K_1 u(f_1^*, f_2^*) + 2M_1 f_1^*] \, dx \geq 0. \tag{3.19}$$

Letting $\varepsilon \to 0^+$, (3.19) leads to

$$f_1^*(x) \geq \frac{(K_1 - p_1(x))u(f_1^*, f_2^*)}{2M_1} \quad \text{in } \{x \in \Omega \mid f_1^*(x) < \delta_1\}. \tag{3.20}$$

This proves (3.15a) for $i = 1$. The rest of the proof for (3.15) for $i = 1, 2$ is analogous to that of Theorem 3.1 in [10], the details are thus omitted here. Comparing (3.18) with (3.13) and noting the definition of $u(f_1^*, f_2^*)$, $v(f_1^*, f_2^*)$, we see that it remains to show that $(u, v, p_1, p_2)$ as described in the statement of the theorem actually exists.

The proof of the existence of the solution with the range of values as described in (3.14) is carried out as in Theorem 2.1 by using upper and lower solutions for the system. For convenience, we denote

$$\begin{aligned}\psi_3(x) &= K_1, \qquad \phi_3(x) = -4\bar{a}_1 c_1 c_2 K_1\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right)(b_1 b_2 E_1 E_2)^{-1} \\ \psi_4(x) &= \frac{K_2}{2}, \qquad \phi_4(x) = -2\bar{a}_1 c_1 K_1 (b_1 E_2)^{-1}\end{aligned} \tag{3.21}$$

for $x \in \bar{\Omega}$. Consider, for all $\phi_1 \le u \le \psi_1$, $\phi_2 \le v \le \psi_2$, $\phi_4 \le p_2 \le \psi_4$, that the expression

$$\begin{aligned}&\Delta\psi_3 + (a_1 - f_1^*)\psi_3 - (2b_1 u + c_1 v)\psi_3 + c_2 v p_2 + K_1 f_1^* \\ &\le \bar{a}_1 K_1 - 2\left[\tilde{a}_1 - \frac{c_1}{b_2}\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right) - \delta_1\right]K_1 - c_1\frac{1}{b_2}(\tilde{a}_2 - \delta_2)K_1 \\ &\quad + \frac{c_2}{b_2}\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right)\frac{K_2}{2} + K_1\delta_1 \\ &\le K_1\left[\bar{a}_1 - 2\tilde{a}_1 + \frac{2c_1}{b_2}\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right)\right] - \frac{c_1}{b_2}\frac{\bar{a}_2}{2}K_1 \\ &\quad + \frac{c_2}{b_2}\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right)\frac{K_2}{2} + 3K_1\delta_1\end{aligned}$$

is true in $\Omega$, since $\delta_2 \le \frac{1}{2}E_2$ implies $\tilde{a}_2 - \delta_2 \ge \bar{a}_2/2$. Thus we have, for such situations,

$$\Delta\psi_3 + (a_1 - f_1^*)\psi_3 - (2b_1 u + c_1 v)\psi_3 + c_2 v p_2 + K_1 f_1^* \le 0 \tag{3.22}$$

provided

$$\delta_1 \le \tfrac{1}{3}\left[E_1 + \frac{c_1}{b_2}\frac{\bar{a}_2}{2} - \frac{c_2}{b_2}\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right)\frac{K_2}{2K_1}\right],$$

which is assumed in (H1**).

For all $\phi_1 \le u \le \psi_1$, $\phi_2 \le v \le \psi_2$, $\phi_3 \le p_1 \le \psi_3$, $x \in \Omega$, consider the expression

$$\begin{aligned}&\Delta\psi_4 + (a_2 - f_2^*)\psi_4 - (2b_2 v - c_2 u)\psi_4 - c_1 u p_1 + K_2 f_2^* \\ &\le \frac{\bar{a}_2 K_2}{2} - 2(\tilde{a}_2 - \delta_2)\frac{K_2}{2} \\ &\quad + c_2\frac{a_1}{b_1}\frac{K_2}{2} + c_1\frac{\bar{a}_1}{b_1}4\bar{a}_1 c_1 c_2 K_1\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right)(b_1 b_2 E_1 E_2)^{-1} + K_2\delta_2.\end{aligned}$$

The above expression is $\leq 0$ provided that

$$\left[\frac{\bar{a}_2}{2} - \tilde{a}_2 + \frac{c_2\bar{a}_1}{2b_1} + 2\delta_2\right]K_2 + K_1 4\bar{a}_1^2 c_1^2 c_2\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right)(b_1^2 b_2 E_1 E_2)^{-1} \leq 0.$$

Consequently, hypothesis (H2**) implies that

$$\Delta\psi_4 + (a_2 - f_2^*)\psi_4 - (2b_2 v - c_2 u)\psi_4 - c_1 u p_1 + K_2 f_2^* \leq 0 \tag{3.23}$$

in $\Omega$ for the appropriate $u$, $v$, $p_1$ described above.

For $\phi_1 \leq u \leq \psi_1$, $\phi_2 \leq v \leq \psi_2$, $\phi_4 \leq p_2 \leq \psi_4$, $x \in \Omega$, we have

$$\begin{aligned}
&\Delta\phi_3 + (a_1 - f_1^*)\phi_3 - (2b_1 u + c_1 v)\phi_3 + c_2 v p_2 + K_1 f_1^* \\
&\quad \geq -4\bar{a}_1 c_1 c_2 K_1\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right)(b_1 b_2 E_1 E_2)^{-1} \\
&\quad \times \left\{\bar{a}_1 - 2\left[\tilde{a}_1 - \frac{c_1}{b_2}\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right) - \delta_1\right] - \frac{c_1\bar{a}_2}{2b_2}\right\} \\
&\quad - c_2\frac{1}{b_2}\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right)\cdot 2\bar{a}_1 c_1 K_1 (b_1 E_2)^{-1}
\end{aligned}$$

is valid because $\delta_2 \leq \frac{1}{2}E_2$ implies $v \geq (\tilde{a}_2 - \delta_2)/b_2 \geq \bar{a}_2/2b_2$. Consequently,

$$\Delta\phi_3 + (a_1 - f_1^*)\phi_3 - (2b_1 u + c_1 v)\phi_3 + c_2 v p_2 + K_1 f_1^* \geq 0 \tag{3.24}$$

in the described region provided that

$$2\delta_1 \leq 2\tilde{a}_1 - \bar{a}_1 - \frac{2c_1}{b_2}\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right) + \frac{\bar{a}_2 c_1}{2b_2} - \frac{E_1}{2} = \frac{E_1}{2} + \frac{\bar{a}_2 c_1}{2b_2},$$

which is clearly true due to (H1**).

For $\phi_1 \leq u \leq \psi_1$, $\phi_2 \leq v \leq \psi_2$, $\phi_3 \leq p_1 \leq \psi_3$, $x \in \Omega$, we have

$$\begin{aligned}
&\Delta\phi_4 + (a_2 - f_2^*)\phi_4 - (2b_2 v - c_2 u)\phi_4 - c_1 u p_1 + K_2 f_2^* \\
&\quad \geq 2\bar{a}_1 c_1 K_1 (b_1 E_2)^{-1}\left\{-\bar{a}_2 + 2[\tilde{a}_2 - \delta_2] - \frac{c_2\bar{a}_1}{b_1}\right\} - c_1\frac{\bar{a}_1}{b_1}K_1.
\end{aligned}$$

Hypothesis (H2**) implies that $\delta_2 < E_2/4$, and thus

$$-\bar{a}_2 + 2[\tilde{a}_2 - \delta_2] - \frac{c_2\bar{a}_1}{b_1} > \frac{1}{2}\left[2\tilde{a}_2 - \bar{a}_2 - \frac{c_2\bar{a}_1}{b_1}\right] = \frac{1}{2}E_2.$$

Consequently, we have

$$\begin{aligned}
&\Delta\phi_4 + (a_2 - f_2^*)\phi_4 - (2b_2 v - c_2 u)\phi_4 - c_1 u p_1 + K_2 f_2^* \\
&\quad > 2\bar{a}_1 c_1 K_1 (b_1 E_2)^{-1}\cdot\frac{1}{2}E_2 - \frac{c_1 a_1}{b_1}K_1 = 0
\end{aligned} \tag{3.25}$$

in the region described.

Since the first two equations of (3.13) are independent of $p_1, p_2$, we can show $\Delta\psi_1 + (a_1 - f_1^*)\psi_1 - b_1\psi_1^2 - c_1 v\psi_1 \geq 0, \Delta\phi_1 + (a_1 - f_1^*)\phi_1 - b_1\phi_1^2 - c_1 v\phi_1 \leq 0$ for all $\phi_2 \leq v \leq \psi_2, \phi_3 \leq p_1 \leq \psi_3, \phi_4 \leq p_2 \leq \psi_4, x \in \Omega$ in exactly in the same way as in Theorem 2.1. Similarly, we show $\Delta\psi_2 + (a_2 - f_2^*)\psi_2 + c_2 u\psi_2 - b_2\psi_2^2 \leq 0$, $\Delta\phi_2 + (a_2 - f_2^*)\phi_2 + c_2 u\phi_2 - b_2\phi_2^2 \geq 0$, for all $\phi_1 \leq v \leq \psi_1$, $\phi_3 \leq p_1 \leq \psi_3$, $\phi_4 \leq p_2 \leq \psi_4$, $x \in \Omega$. Then we follow the same method in the last part of the proof of Theorem 2.1 to construct a natural mapping $T$ from $X_1 \times X_2 \times X_3 \times X_4$ into itself, where $X_i = \{w \in C(\bar{\Omega}); \phi_i \leq w \leq \psi_i\}$, $i = 1, 2, 3, 4$. Following the same arguments, the fixed point of $T$ gives a solution of (3.13) with each component in $H^{2,2}(\Omega)$. □

**Corollary 3.1.** *Assume all the hypotheses of Theorem* 3.1; *and moreover*

$$M_1 > \frac{K_1 \bar{a}_1}{2b_1\delta_1}\left[1 + 4\bar{a}_1 c_1 c_2\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right)(b_1 b_2 E_1 E_2)^{-1}\right], \tag{3.26}$$

$$M_2 > \frac{1}{2b_2\delta_2}\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right)[K_2 + 2\bar{a}_1 c_1 K_1 (b_1 E_2)^{-1}], \tag{3.27}$$

*where*

$$E_1 = 2\tilde{a}_1 - \bar{a}_1 - \frac{2c_1}{b_2}\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right), \qquad E_2 = 2\tilde{a}_2 - \bar{a}_2 - \frac{c_2\bar{a}_1}{b_1}.$$

*Let* $(f_1^*, f_2^*) \in \mathscr{C}(\delta_1, \delta_2)$ *be an optimal control. Then*

$$f_1^* = \frac{(K_1 - p_1)u}{2M_1}, \qquad f_2^* = \frac{(K_2 - p_2)v}{2M_2}, \tag{3.28}$$

*where* $(u, v, p_1, p_2)$ *is a solution of the optimality system*:

$$\left.\begin{aligned}
&\Delta u + \left[\left(a_1 - \frac{(K_1 - p_1)u}{2M_1}\right) - b_1 u - c_1 v\right]u = 0 \quad \text{in } \Omega,\\
&\Delta v + \left[\left(a_2 - \frac{(K_2 - p_2)v}{2M_2}\right) + c_2 u - b_2 v\right]v = 0 \quad \text{in } \Omega,\\
&\Delta p_1 + a_1 p_1 + \frac{(K_1 - p_1)^2 u}{2M_1} - (2b_1 u + c_1 v)p_1 + c_2 v p_2 = 0 \quad \text{in } \Omega,\\
&\Delta p_2 + a_2 p_2 + \frac{(K_2 - p_2)^2 v}{2M_2} - (2b_2 v - c_2 u)p_2 - c_1 u p_1 = 0 \quad \text{in } \Omega,\\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial p_1}{\partial \nu} = \frac{\partial p_2}{\partial \nu} = 0 \quad \text{on } \partial\Omega
\end{aligned}\right\} \tag{3.29}$$

*with* $u, v, p_1, p_2 \in H^{2,2}(\Omega)$ *satisfying* (3.14).

*Proof.* By Theorem 3.1, $(f_1^*, f_2^*)$ must satisfy (3.15a–c), $i = 1, 2$, where $(u, v, p_1, p_2)$ is a solution of (3.13) with conditions (3.14).

From (3.15a), in the set $\{x \in \Omega \mid f_1^*(x) = 0\}$ we have

$$0 = f_1^*(x) \geq \frac{(K_1 - p_1(x))u(x)}{2M_1} \geq 0. \tag{3.30}$$

Thus the first equality in (3.28) must hold in this set. From (3.15b), in the set $\{x \in \Omega \mid f_1^*(x) = \delta_1\}$, we have

$$\begin{aligned}\delta_1 = f_1^*(x) &\leq \frac{(K_1 - p_1(x))}{2M_1}\, u(x) \\ &\leq \frac{\bar{a}_1}{2M_1 b_1}\left[K_1 + 4\bar{a}_1 c_1 c_2 K_1\left(\bar{a}_2 + \frac{c_2 \bar{a}_1}{b_1}\right)(b_1 b_2 E_1 E_2)^{-1}\right] < \delta_1\end{aligned} \tag{3.31}$$

due to (3.14) and (3.26). Thus there are no $x \in \Omega$ where $f_1^*(x) = \delta_1$. From (3.15c) and (3.30), the first equality in (3.28) holds for all $x \in \Omega$.

From (3.15a), in the set $\{x \in \Omega \mid f_2^*(x) = 0\}$, we have

$$0 = f_2^*(x) \geq \frac{(K_2 - p_2(x)v(x)}{2M_2} \geq 0. \tag{3.32}$$

Thus the second equality in (3.28) must hold in this set. From (3.15b), in the set $\{x \in \Omega \mid f_2^*(x) = \delta\}$, we have

$$\begin{aligned}\delta_2 = f_2^*(x) &\leq \frac{(K_2 - p_2(x))}{2M_2}\, v(x) \\ &\leq \frac{1}{2M_2 b_2}\left(\bar{a}_2 + \frac{c_2 \bar{a}_1}{b_1}\right)[K_2 + 2\bar{a}_1 c_1 K_1 (b_1 E_2)^{-1}] < \delta_2\end{aligned} \tag{3.33}$$

due to (3.14) and (3.27). Thus there are no $x \in \Omega$ where $f_2^*(x) = \delta_2$. From (3.15c) and (3.32), the second inequality in (3.28) holds for all $x \in \Omega$. Combining (3.13) and (3.28), we conclude that $(u, v, p_1, p_2)$ satisfies (3.29). This completes the proof. □

If some additional conditions are made, we can show that the solution $(u, v, p_1, p_2)$ of (3.29), which characterizes the optimal control, is actually positive.

**Corollary 3.2.** *Assume all the hypotheses of Corollary* 3.1 *and further*

(H5) $c_1 \leq \dfrac{K_2^2}{K_1}\dfrac{b_1}{b_2}\dfrac{\bar{a}_2}{\bar{a}_1}\dfrac{1}{8M_2}$.

*Let* $(f_1^*, f_2^*) \in \mathscr{C}(\delta_1, \delta_2)$ *be an optimal control. Then* $(f_1^*, f_2^*)$ *satisfies* (3.28), *where* $(u, v, p_1, p_2)$ *is a solution of the optimality system* (3.29) *with each component in* $H^{2,2}(\Omega)$ *satisfying*

$$\begin{aligned}&\phi_1 \leq u \leq \psi_1, \quad \phi_2 \leq v \leq \psi_2 \qquad \text{in } \Omega, \\ &0 \leq p_1 \leq K_1, \quad 0 \leq p_2 \leq \frac{K_2}{2} \qquad \text{in } \Omega.\end{aligned} \tag{3.34}$$

*Proof.* In Corollary 3.1 we show by means of (3.14), (3.27), and (3.33) that there are no $x \in \Omega$ where $f_2^*(x) = \delta_2$. Applying (3.15a) at any characterization solution in Theorem 3.1, we conclude that

$$f_2^*(x) \geq \frac{K_2}{4M_2}\,\phi_2(x) \qquad \text{for all} \quad x \in \Omega. \tag{3.35}$$

Since (H1**) implies $\tilde{a}_1 - (c_1/b_2)(\bar{a}_2 + c_2\bar{a}_1/b_1) - \delta_1 > 0$ and (H2**) implies that $\tilde{a}_2 - \delta_2 > \bar{a}_2/2$, hypothesis (H5) leads to

$$c_1 < \frac{K_2^2 b_1}{4K_1\bar{a}_1 M_2}\,\phi_2. \tag{3.36}$$

Let $\tilde{\phi}_3(x) = \tilde{\phi}_4(x) \equiv 0$ in $\Omega$ and $\psi_i(x)$, $i = 3, 4$, be the same as in Theorem 3.1. For all $\phi_1 \leq u \leq \psi_1$, $\phi_2 \leq v \leq \psi_2$, $\tilde{\phi}_3 \leq p_1 \leq \psi_3$, $x \in \Omega$, we can show that

$$\Delta\tilde{\phi}_4 + (a_2 - f_2^*)\tilde{\phi}_4 - (2b_2 v - c_2 u)\tilde{\phi}_4 - c_1 u p_1 + K_2 f_2^* \geq 0 \tag{3.37}$$

by (3.36) and (3.25).

For all $\phi_1 \leq u \leq \psi_1$, $\phi_2 \leq v \leq \psi_2$, $\tilde{\phi}_3 \leq p_1 \leq \psi_3$, $x \in \Omega$, we can show by means of hypothesis (H2**) that

$$\Delta\psi_4 + (a_2 - f_2^*)\psi_4 - (2b_2 v - c_2 u)\psi_4 - c_1 u p_1 + K_2 f_2^* \leq 0. \tag{3.38}$$

For all $\phi_1 \leq u \leq \psi_1$, $\phi_2 \leq v \leq \psi_2$, $\tilde{\phi}_4 \leq p_2 \leq \psi_4$, $x \in \Omega$, we have

$$\Delta\tilde{\phi}_3 + (a_1 - f_1^*)\tilde{\phi}_3 - (2b_1 u + c_1 v)\tilde{\phi}_3 + c_2 v p_2 + K_1 f_1^* \geq 0; \tag{3.39}$$

and since $\delta_2 \leq \frac{1}{2}E_2$ implies that $\tilde{a}_2 - \delta_2 \geq \bar{a}_2/2$, we can use hypothesis (H1**) to obtain

$$\Delta\psi_3 + (a_1 - f_1^*)\psi_3 - (2b_1 u + c_1 v)\psi_3 + c_2 v p_2 + K_1 f_1^* \leq 0. \tag{3.40}$$

We can then follow the same argument as in the final part of the proof of Theorem 3.1 to conclude that (3.13) has a solution $(u, v, p_1, p_2)$ satisfying (3.34) as well as (3.14). Moreover, (3.15a–c) are all satisfied.

Furthermore, in Corollary 3.1, we show that any $x \in \Omega$ where $f_1^*(x) = \delta_1$ or $f_2^*(x) = \delta_2$ cannot exist. At the points where $f_1^*(x) = 0$ or $f_2^*(x) = 0$, we show that equality must hold in (3.30) and (3.32). Thus for the $(u, v, p_1, p_2)$ mentioned above, (3.28) must hold, and the system (3.29) is satisfied as well as (3.13). This complete the proof. □

## 4. Solution of the Optimality System by a Monotone Scheme

For simplicity, we always assume all the hypotheses of Corollary 3.2 in this section, so that a positive solution of the optimality system (3.29) can be found with the property (3.34). We further deduce a constructive method of approximating or computing the positive solution $(u, v, p_1, p_2)$ of the optimality system. We construct monotone sequences converging from above and below to give upper and lower estimates for $(u, v, p_1, p_2)$. In the case where the limits of the upper and lower integrates agree, then the optimal control problem is completely solved.

Choose a large constant $R > 0$ so that the following four expressions:

$$-\left[a_1(x) - \frac{(K_1 - p_1)u}{2M_1} - b_1 u - c_1 v\right]u - Ru,$$

$$-\left[a_2(x) - \frac{(K_2 - p_2)v}{2M_2} + c_2 u - b_2 v\right]v - Rv,$$

$$-\left[a_1(x)p_1 + \frac{(K_1 - p_1)^2 u}{2M_1} - (2b_1 u + c_1 v)p_1 + c_2 v p_2\right] - Rp_1,$$

and

$$-\left[a_2(x)p_2 + \frac{(K_2 - p_2)^2 v}{2M_2} - (2b_2 v - c_2 u)p_2 - c_1 v p_1\right] - Rp_2$$

are decreasing respectively in the four corresponding variables $u$, $v$, $p_1$, $p_2$ for all $x \in \Omega$, $\phi_1 \le u \le \psi_1$, $\phi_2 \le v \le \psi_2$, $0 \le p_1 \le K_1$, $0 \le p_2 \le K_2/2$, when the other three variables are fixed. (Recall the definitions of $\phi_i$, $\psi_i$, $i = 1, 2$, in (2.2).) For convenience, let

$$u_0 \equiv \phi_1, \quad u_{-1} \equiv \psi_1; \qquad v_0 \equiv \phi_2, \quad v_{-1} \equiv \psi_2;$$
$$p_{1,0} \equiv 0, \quad p_{1,-1} \equiv K_1; \qquad p_{2,0} \equiv 0, \quad p_{2,-1} \equiv \frac{K_2}{2}. \tag{4.1}$$

We can readily verify that these constant functions satisfy

$$\Delta u_{-1} - Ru_{-1} \le -u_{-1}\left[a_1(x) - \frac{(K_1 - p_{1,-1})u_{-1}}{2M_1} - b_1 u_{-1} - c_1 v_0\right] - Ru_{-1} \quad \text{in } \Omega \tag{4.2}$$

and

$$\Delta u_0 - Ru_0 \ge -u_0\left[a_1(x) - \frac{(K_1 - p_1)u_0}{2M_1} - b_1 u_0 - c_1 v\right] - Ru_0 \quad \text{in } \Omega \tag{4.3}$$

for each $v, p_1$ respectively in the intervals $[v_0, v_{-1}]$, $[p_{1,0}, p_{1,-1}]$. The last inequality is true because

$$a_1(x) - \frac{(K_1 - p_1)u_0}{2M_1} - b_1 u_0 - c_1 v \ge a_1 - \frac{K_1}{2M_1}\phi_1 - b_1\phi_1 - c_1\psi_2$$
$$\ge a_1 - \frac{\delta_1}{\bar{a}_1}\tilde{a}_1 - \tilde{a}_1 + \delta_1 > 0$$

by using (3.26). Moreover, we have

$$\Delta v_{-1} - Rv_{-1} \ge -v_{-1}\left[a_2(x) - \frac{(K_2 - p_2)v_{-1}}{2M_2} + c_2 u - b_2 v_{-1}\right] - Rv_{-1} \quad \text{in } \Omega \tag{4.4}$$

for each $u, p_2$ respectively in the intervals $[u_0, u_{-1}]$, $[p_{2,0}, p_{2,-1}]$, since

$$a_2(x) - \frac{(K_2 - p_2)v_{-1}}{2M_2} + c_2 u - b_2 v_{-1} \le a_2(x) + c_2\psi_1 - b_2\psi_2$$
$$= a_2(x) - \bar{a}_2 \le 0.$$

On the other hand,

$$\Delta v_0 - Rv_0 \ge -v_0\left[a_2(x) - \frac{(K_2 - p_2)v_0}{2M_2} + c_2 u - b_2 v_0\right] - Rv_0 \qquad \text{in } \Omega \qquad (4.5)$$

for each $u \in [u_0, u_{-1}]$, $p_2 \in [p_{2,0}, p_{2,-1}]$, since

$$a_2(x) - \frac{(K_2 - p_2)v_0}{2M_2} + c_2 u - b_2 v_0 \ge a_2(x) - \frac{K_2}{2M_2}\phi_2 - b_2\phi_2 + c_2\phi_1$$
$$> a_2(x) - \frac{K_2}{2M_2}\frac{1}{b_2}\left(\bar{a}_2 + \frac{c_1\bar{a}_1}{b_1}\right) - \tilde{a}_2 + \delta_2$$
$$> a_2(x) - \delta_2 - \tilde{a}_2 + \delta_2 \ge 0.$$

(Here, we have used the hypothesis (3.27).) Further, we verify

$$\Delta p_{1,-1} - Rp_{1,-1}$$
$$\le -\left[a_1(x)p_{1,-1} + \frac{(K_1 - p_{1,-1})^2 u}{2M_1} - (2b_1u_0 + c_1v_0)p_{1,-1} + c_2 v p_2\right]$$
$$- Rp_{1,-1} \qquad \text{in } \Omega \qquad (4.6)$$

for each $u \in [u_0, u_{-1}]$, $v \in [v_0, v_{-1}]$, $p_2 \in [p_{2,0}, p_{2,-1}]$, since

$$a_1(x)p_{1,-1} + \frac{(K_1 - p_{1,-1})^2 u}{2M_1} - (2b_1u_0 + c_1v_0)p_{1,-1} + c_2 v p_2$$
$$\le K_1\left[\bar{a}_1 - 2\tilde{a}_1 + 2\delta_1 + \frac{2c_1}{b_2}\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right) - \frac{\bar{a}_2}{2}\frac{c_1}{b_2}\right]$$
$$+ \frac{c_2}{b_2}\left(\bar{a}_2 + \frac{c_2\bar{a}_1}{b_1}\right)\frac{K_2}{2} \le 0.$$

The last inequality is true due to hypothesis (H1**). We also have

$$\Delta p_{1,0} - Rp_{1,0} \ge -\left[a_1(x)p_{1,0} + \frac{(K_1 - p_{1,0})^2}{2M_1}u - (2b_1u + c_1v)p_{1,0} + c_2 v p_2\right]$$
$$- Rp_{1,0} \qquad \text{in } \Omega \qquad (4.7)$$

for each $u \in [u_0, u_{-1}]$, $v \in [v_0, v_{-1}]$, $p_2 \in [p_{2,0}, p_{2,-1}]$, since

$$a_1(x)p_{1,0} + \frac{(K_1 - p_{1,0})^2 u}{2M_1} - (2b_1u_0 + c_1v_0)p_{1,0} + c_2 v p_2 \ge \frac{K_1^2 u_0}{2M_1} > 0 \qquad \text{in } \Omega.$$

For the last component, we have

$$\begin{aligned}\Delta p_{2,-1} - Rp_{2,-1} &\\ \leq -\Bigg[ a_2(x)p_{2,-1} + \frac{(K_2 - p_{2,-1})^2}{2M_2} v - (2b_2 v_0 - c_2 u)p_{2,-1} - c_1 u_0 p_{1,0}\Bigg] &\\ - Rp_{2,-1} \qquad \text{in } \Omega & \qquad (4.8)\end{aligned}$$

for each $u \in [u_0, u_{-1}]$, $v \in [v_0, v_{-1}]$, since

$$\begin{aligned}&a_2(x)p_{2,-1} + \frac{(K_2 - p_{2,-1})^2 v}{2M_2} - (2b_2 v_0 - c_2 u)p_{2,-1} - c_1 u_0 p_{1,0}\\ &\leq \frac{\bar{a}_2 K_2}{2} + \frac{K_2}{2}\cdot\frac{\delta_2}{2} - K_2\tilde{a}_2 + K_2\delta_2 + c_2\frac{\bar{a}_1}{b_1}\frac{K_2}{2} < 0.\end{aligned}$$

In the last line, we use hypotheses (3.27) and (H2**). Moreover, we have

$$\begin{aligned}\Delta p_{2,0} - Rp_{2,0} &\\ \geq -\Bigg[ a_2(x)p_{2,0} + \frac{(K_2 - p_{2,0})^2 v}{2M_2} - (2b_2 v - c_2 u)p_{2,0} - c_1 u p_1\Bigg] &\\ - Rp_{2,0} \qquad \text{in } \Omega & \qquad (4.9)\end{aligned}$$

for each $u \in [u_0, u_{-1}]$, $v \in [v_0, v_{-1}]$, $p_1 \in [p_{1,0}, p_{1,-1}]$, since

$$\begin{aligned}&a_2(x)p_{2,0} + \frac{(K_2 - p_{2,0})^2 v}{2M_2} - (2b_2 v - c_2 u)p_{2,0} - c_1 u p_1\\ &\geq \frac{K_2^2}{2M_2}\phi_2 - c_1\psi_1 K_1 > \frac{2K_1\bar{a}_1 c_1}{b_1\phi_2}\phi_2 - c_1\psi_1 K_1 = \frac{K_1\bar{a}_1}{b_1}c_1 > 0 \qquad \text{in } \Omega.\end{aligned}$$

(Here, we use (H1**), (H2**), and (H5), see (3.36).)

We now inductively define sequences of functions $u_k(x)$, $v_k(x)$, $p_{1,k}(x)$, $p_{2,k}(x)$ in $\Omega$, $k = 1, 2, \ldots$, as solutions of scalar problems as follows:

$$\begin{aligned}\Delta u_k - Ru_k = -u_{k-2}\Bigg[ a_1 - \frac{(K_1 - p_{1,k-2})u_{k-2}}{2M_1} - b_1 u_{k-2} - c_1 v_{k-1}\Bigg] &\\ - Ru_{k-2} \text{ in } \Omega, & \qquad (4.10)\end{aligned}$$

$$\begin{aligned}\Delta v_k - Rv_k = -v_{k-2}\Bigg[ a_2 - \frac{(K_2 - p_{2,k-2})v_{k-2}}{2M_2} + c_2 u_k - b_2 v_{k-2}\Bigg] &\\ - Rv_{k-2} \text{ in } \Omega, & \qquad (4.11)\end{aligned}$$

$$\begin{aligned}\Delta p_{1,k} - Rp_{1,k} &\\ = -\Bigg[ a_1 p_{1,k-2} + \frac{(K_1 - p_{1,k-2})^2 u_k}{2M_1} - (2b_1 u_{k-1} + c_1 v_{k-1})p_{1,k-2} &\\ + c_2 v_k p_{2,k-2}\Bigg] - Rp_{1,k-2} \qquad \text{in } \Omega, & \qquad (4.12)\end{aligned}$$

$$\begin{aligned}\Delta p_{2,k} - Rp_{2,k} \\ = -\Bigg[ a_2 p_{2,k-2} + \frac{(K_2 - p_{2,k-2})^2 v_k}{2M_2} - (2b_2 v_{k-1} + c_2 u_k) p_{2,k-2} \\ - c_1 u_{k-1} p_{1,k-1} \Bigg] - Rp_{2,k-2} \quad \text{in } \Omega,\end{aligned} \tag{4.13}$$

where

$$\frac{\partial u_k}{\partial \nu} = \frac{\partial v_k}{\partial \nu} = \frac{\partial p_{1,k}}{\partial \nu} = \frac{\partial p_{2,k}}{\partial \nu} = 0 \qquad \text{on } \partial\Omega. \tag{4.14}$$

**Theorem 4.1.** *Assume all the hypotheses of Corollary 3.2. The sequences of functions $u_k(x)$, $v_k(x)$, $p_{1,k}(x)$, $p_{2,k}(x)$ defined above, satisfy the relations*

$$u_0 \le u_2 \le \cdots \le u_{2r} \le \cdots \le u_{2r-1} \cdots \le u_1 \le u_{-1}, \tag{4.15}$$

$$v_0 \le v_2 \le \cdots \le v_{2r} \cdots \le v_{2r-1} \le \cdots \le v_1 \le v_{-1}, \tag{4.16}$$

$$p_{i,0} \le p_{i,2} \le \cdots \le p_{i,2r} \cdots \le p_{i,2r-1} \le \cdots \le p_{i,1} \le p_{i,-1}, \qquad i = 1, 2, \tag{4.17}$$

*for all $x \in \Omega$. Moreover, any solution $(u, v, p_1, p_2)$ of problem (3.29) with the property*

$$u_0 \le u \le u_{-1}, \qquad v_0 \le v \le v_{-1}, \qquad p_{i,0} \le p_i \le p_{i,-1}, \qquad i = 1, 2, \quad \text{in } \Omega, \tag{4.18}$$

*must satisfy*

$$u_{2r} \le u \le u_{2r-1}, \qquad v_{2r} \le v \le v_{2r-1}, \qquad p_{i,2r} \le p_i \le p_{i,2r-1}, \qquad i = 1, 2, \quad \text{in } \Omega, \tag{4.19}$$

*for all positive integers r.*

*Proof.* Using the equation satisfied by $u_1$ and inequality (4.2), we obtain $\Delta(u_{-1} - u_1) - R(u_{-1} - u_1) \le 0$ in $\Omega$, $(\partial/\partial\nu)(u_{-1} - u_1) = 0$ on $\partial\Omega$. Hence $u_1 \le u_{-1}$ in $\Omega$. Similarly, using (4.3) and the choice of $R$, we deduce that $\Delta(u_0 - u_1) - R(u_0 - u_1) \ge 0$ in $\Omega$, $(\partial/\partial\nu)(u_0 - u_1) = 0$ on $\partial\Omega$. Thus, we have

$$u_0 \le u_1 \le u_{-1} \qquad \text{in } \Omega. \tag{4.20}$$

Using the equation for $v_1$, inequalities (4.4), (4.5), (4.20), and the choice of $R$, we obtain

$$\begin{aligned}\Delta(v_{-1} - v_1) - R(v_{-1} - v_1 \le 0 \quad \text{in } \Omega, \qquad \frac{\partial}{\partial \nu}(v_{-1} - v_1) = 0 \quad \text{on } \partial\Omega, \\ \Delta(v_0 - v_1) - R(v_0 - v_1) \ge 0 \quad \text{in } \Omega, \qquad \frac{\partial}{\partial \nu}(v_0 - v_1) = 0 \quad \text{on } \partial\Omega,\end{aligned} \tag{4.21}$$

and

$$v_0 \le v_1 \le v_{-1} \quad \text{in } \Omega. \tag{4.22}$$

Again, using the equations for $p_1$ and $p_2$, the inequalities above, and the choice of $R$, we deduce similarly that

$$p_{i,0} \le p_{i,1} \le p_{i,-1}, \qquad i = 1, 2, \qquad \text{in } \Omega. \tag{4.23}$$

Next, we show

$$\Delta(u_1 - u_2) - R(u_1 - u_2) \le 0 \quad \text{in } \Omega, \qquad \frac{\partial}{\partial \nu}(u_1 - u_2) = 0 \quad \text{on } \partial\Omega,$$

$$\Delta(u_0 - u_2) - R(u_0 - u_2) \ge 0 \quad \text{in } \Omega, \qquad \frac{\partial}{\partial \nu}(u_0 - u_2) = 0 \quad \text{on } \partial\Omega.$$

(Here, we use (4.3) at $v = v_1$, $p_1 = p_{1,0}$.) Thus we obtain

$$u_0 \le u_2 \le u_1 \qquad \text{in } \Omega. \tag{4.24}$$

Similarly, comparing the related equations and using the inequalities above, we deduce successively that

$$v_0 \le v_2 \le v_1 \qquad \text{in } \Omega, \tag{4.25}$$

$$p_{1,0} \le p_{1,2} \le p_{1,1} \quad \text{and} \quad p_{2,0} \le p_{2,2} \le p_{2,1} \qquad \text{in } \Omega. \tag{4.26}$$

From the above inequalities we have the validity of the inequalities

$$\begin{aligned} &u_{2n} \le u_{2n+2} \le u_{2n+1} \le u_{2n-1}, \quad v_{2n} \le v_{2n+2} \le v_{2n+1} \le v_{2n-1} \qquad \text{in } \Omega, \\ &p_{1,2n} \le p_{1,2n+2} \le p_{1,2n+1} \le p_{1,2n-1}, \\ &\quad p_{2,2n} \le p_{2,2n+2} \le p_{2,2n+1} \le p_{2,2n-1} \qquad \text{in } \Omega, \end{aligned} \tag{4.27}$$

for $n = 0$. We then use the comparison method as above to prove the validity of (4.27) for any positive integer $n$ by induction. This proves (4.15)–(4.17).

To prove the second part of the theorem, we use the comparison method on the appropriate equations as above, and proceed by induction on $r$ in proving inequality (4.19). For more details of analogous procedures, see Theorem 5.5-1 in [8]. □

**Example 4.1.** Let $\Omega$ be any domain on the $(x, y)$ plane with $C^2$ boundary, with $\{(x, y) | 0 \le x \le 1,\ 0 \le y \le 1\} \subset \Omega$. Consider the system (1.1), with $a_i(x, y) = 16 + 2 \sin xy$, $\bar{a}_i = 18$, $\tilde{a}_i = 14$ for $i = 1, 2$, $b_1 = 12$, $b_2 = 7.8$, $c_1 = 0.35$, $c_2 = 0.5$. The problem is to maximize (1.2) with $K_1 = 0.5$, $K_2 = 1$, $M_1 = 0.5$, $M_2 = 1$, for all $f_i \in L^\infty(\Omega)$ in (1.5), where $\delta_1 = 2.079$, $\delta_2 = 2.308$.

We can verify that the hypotheses (H1**), (H2**), (H3), and (H4) are all satisfied, thus Theorem 3.1 applies. Moreover, conditions (3.26), (3.27), and (H5) are also valid; hence Corollary 3.1, 3.2, and Theorem 4.1 are all applicable to this example. The optimal solution can be characterized by positive solutions of the elliptic system of four equations (3.29). Moreover, the approximation Theorem 4.1 applies.

## References


1. S. Agmon, A. Douglis, and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, Comm. Pure Appl. Math., 12, 623–727, 1959.
2. H. T. Banks, Modeling and Control in Biomedial Sciences, Lecture Notes in Biomathematics, Vol. 6, Springer-Verlag, New York, 1975.
3. V. Barbu, Optimal Control of Variational Inequalities, Research Notes in Mathematics, Vol. 100, Pitman, London, 1984.
4. C. Clark, Mathematical Bioeconomics: The Optimal Management of Renewable Resources, Wiley, New York, 1976.
5. P. Fife, Mathematical Aspects of Reacting and Diffusing Systems, Lecture Notes in Biomathematics, Vol. 28, Springer-Verlag, New York, 1979.
6. G. Ladde, V. Lakshmikantham, and V. Vatsala, Monotone Iterative Techniques for Nonlinear Differential Equations, Pitman, Boston, 1985.
7. A. Leung, Monotone schemes for semilinear elliptic systems related to ecology, Math. Methods Appl. Sci., 4, 272–285, 1982.
8. A. Leung, Systems of Nonlinear Partial Differential Equations, Applications to Biology and Engineering, Kluwer, Dordrecht, 1989.
9. A. Leung and G. Fan, Existence of positive solutions for elliptic systems—degenerate and nondegenerate ecological models, J. Math. Anal. Appl., 151, 512–531, 1990.
10. A. Leung and S. Stojanovic, Optimal control for elliptic Volterra–Lotka equations, J. Math. Anal. Appl., 173, 603–619, 1993.
11. L. Li, Coexistence theorems of steady states for predator–prey interacting systems, Trans. Amer. Math. Soc., 305, 143–166, 1988.
12. J. L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, Berlin, 1971.
13. A. Okubo, Diffusion and Ecological problems: Mathematical Models, Springer-Verlag, Berlin, 1980.
14. S. Stojanovic, Optimal damping control and nonlinear elliptic systems, SIAM J. Control Optim., 29, 594–608, 1991.


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