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Optimal Control of Semilinear Elliptic Equations with Pointwise Constraints on the Gradient of the State*

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Abstract. In this paper we are concerned with optimal control problems governed by an elliptic semilinear equation, the control being distributed in Ω. The existence of constraints on the control as well as pointwise constraints on the gradient of the state is assumed. A convenient choice of the control space permits us to derive the optimality conditions and study the adjoint state equation, which has derivatives of measures as data. In order to carry out this study, we prove a trace theorem and state Green's formula by using the transposition method.

Key Words. Optimal control, State constraints, Semilinear elliptic equations, Optimality conditions, Lagrange multipliers.

AMS Classification. 49K20, 49J20.

1. Introduction

In the last 10 years several papers dealing with optimal control problems with pointwise state constraints have appeared: [1], [6], [7], [15], and [16] for linear equations and convex control problems, [8] and [9] for linear equations and

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control in the coefficients, and [3]–[5] for semilinear equations. Bermúdez and Martínez [2] and Luneville [14] have considered this type of problem in connection with some realistic problems. However, only two of the previous papers have included the study of optimal control problems with pointwise constraints on the gradient of the state: Mackenroth [16] derived the optimality conditions for a control problem governed by a linear elliptic equation assuming the constraint on the gradient only in a compact subset of Ω and taking controls in the Sobolev spaces $H^1(\Omega)$ in the case of a distributed control or $H^{5/2}(\Gamma)$ in the case of a boundary control; Bonnans and Casas [3] considered a semilinear elliptic equation and constraints on the gradient in $\bar{\Omega}$, but the adjoint state equation was not studied nor properly formulated.

This paper deals with optimal control problems of semilinear elliptic equations subject to control constraints and pointwise constraints on the gradient of the state. The control is distributed and it is assumed to belong to $L^r(\Omega)$ for some $r > n$. Thus we avoid taking $H^1(\Omega)$ or other Sobolev spaces as the control space, which is not very realistic.

We begin studying a control problem of a system governed by a Dirichlet problem in Section 2. The optimality conditions (enounced in Section 3 and proved in Section 5) ameliorate those obtained in the previous papers in two aspects: more general state constraints are treated and a simpler Lagrange multiplier (associated with the state constraints) than in [3] and [16] is achieved. Also, we deduce some regularity results on the optimal control. A detailed study of the adjoint state equation is carried out in Section 4. Finally, in the last section, we show briefly the results corresponding to the Neumann case.

2. Distributed Control for the Dirichlet Problem

Let Ω be an open bounded subset of R^n ($n \geq 2$) with $C^{1,1}$ boundary Γ (see, for example, [12]). Let us consider the following boundary-value problem:

$$\begin{cases} Ay + \varphi(y) = u & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma, \end{cases} \tag{2.1}$$

with $u \in L^r(\Omega)$, for $r > n$, and

$$Ay = -\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(x)\, \partial_{x_i} y(x)) + a_0(x)y(x),$$

$$\begin{cases} a_{ij} \in C^{0,1}(\bar{\Omega}) \quad \text{and} \quad a_0 \in L^\infty(\Omega), \\ \exists m > 0 \quad \text{such that} \quad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq m|\xi|^2, \quad \forall \xi \in R^n, \quad \forall x \in \Omega, \\ a_0(x) \geq 0 \quad \text{a.e.} \quad x \in \Omega, \end{cases} \tag{2.2}$$

$$\varphi\colon R \to R \quad \text{is an increasing function of class } C^1 \text{ with } \quad \varphi(0) = 0. \tag{2.3}$$

We consider the following control problem:

$$\text{(P)} \quad \begin{cases} \text{Minimize } J(u) \\ u \in K \quad \text{and} \quad \nabla y_u \in C, \end{cases}$$

where $J\colon U \to R$ is defined by

$$J(u) = \tfrac{1}{2} \int_\Omega |y_u(x) - y_d(x)|^2 \, dx + \frac{N}{\sigma} \int_\Omega |u(x)|^\sigma \, dx,$$

y_u being the solution of (2.1), y_d given in $L^2(\Omega)$, $\sigma \in (1, +\infty)$, $N \geq 0$, and $U = L^\tau(\Omega)$, with $\tau = \max\{\sigma, r\}$; K is a nonempty convex closed subset of U; C is a closed convex subset of $C(X)^n$ with nonempty interior, X being a compact subset of $\bar{\Omega}$. Furthermore we assume that one of the following hypotheses is satisfied:

H1. *K is bounded in $L^r(\Omega)$, $r > n$, and $\sigma \leq r$.*

H2. *$N > 0$ and $\sigma \geq r > n$.*

Let us give some examples about control constraints that fall into our formulation:

$$K = \{u \in L^r(\Omega)\colon \|u\|_{L^r(\Omega)} \leq 1\},$$

$$K = \{u \in L^\infty(\Omega)\colon a \leq u(x) \leq b \text{ a.e. } x \in \Omega\}$$

with $-\infty < a < b < +\infty$, and $\sigma = 2$, which satisfy H1,

$$K = L^r(\Omega) \quad \text{or} \quad K = \{u \in L^r(\Omega)\colon u(x) \geq 0 \text{ a.e. } x \in \Omega\}$$

and $\sigma = r$, that satisfy H2.

Typical state constraints that we can consider are the following:

$$X = \bar{\Omega} \quad \text{and} \quad C = \{\mathbf{z} \in C(\bar{\Omega})^n\colon |\mathbf{z}_j(x)| \leq \delta_j,\ \forall x \in \bar{\Omega},\ 1 \leq j \leq n\},$$

$$X = \Gamma \quad \text{and} \quad C = \{\mathbf{z} \in C(\Gamma)^n\colon \mathbf{z}(x) \cdot \mathbf{v}(x) \geq \delta,\ \forall x \in \Gamma\},$$

or

$$X \subset \Omega \quad \text{and} \quad C = \{\mathbf{z} \in C(X)^n\colon |\mathbf{z}(x)| \leq \delta,\ \forall x \in X\},$$

where δ and δ_j are given constants and $\mathbf{v}(x)$ denotes the unit outward normal vector to Γ at the point x.

In the rest of this section we will see that (P) is a well-posed problem in the sense that there exists a unique solution $y_u \in C^1(\bar{\Omega})$ for each $u \in L^r(\Omega)$; the functional $u \to y_u$ is of class C^1 and there exists at least one solution of (P), under the assumption of existence of feasible controls (i.e., $u \in K$ such that $\nabla y_u \in C$).

Theorem 1. *For each $u \in L^r(\Omega)$, with $r > n$, there exists a unique element $y_u \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ solution of (2.1). Moreover, there exists a constant C_1 independent of u such that*

$$\|y_u\|_{W^{2,r}(\Omega)} \leq C_1 \|u\|_{L^r(\Omega)}. \tag{2.4}$$

Finally, if $\{u_k\} \subset L^r(\Omega)$ *converges to* u *weakly (or weak* in the case* $r = +\infty$*) in* $L^r(\Omega)$*, then* $\{y_{u_k}\}$ *converges to* y_u *weakly in* $W^{2,r}(\Omega)$.

Proof. For every positive integer k let us consider the functions $\varphi_k: R \to R$ defined by

$$\varphi_k(\theta) = \begin{cases} \varphi(\theta) & \text{if } |\theta| \le k, \\ \varphi(+k) & \text{if } \theta \ge +k, \\ \varphi(-k) & \text{if } \theta \le -k. \end{cases}$$

Let $\mathscr{A}_k: H_0^1(\Omega) \to H^{-1}(\Omega)$ be the operator defined by

$$\langle \mathscr{A}_k y, z\rangle = a(y, z) + \int_\Omega \varphi_k(y(x))z(x)\,dx,$$

where $a: H_0^1(\Omega) \times H_0^1(\Omega) \to R$ is the bilinear form associated with the operator A:

$$a(y, z) = \sum_{i,j=1}^n \int_\Omega a_{ij}(x)\,\partial_{x_i} y(x)\,\partial_{x_j} z(x)\,dx + \int_\Omega a_0(x)y(x)z(x)\,dx.$$

It is easy to verify that $\mathscr{A}_k$ is a bounded, hemicontinuous, strictly monotone, and coercive operator, therefore $\mathscr{A}_k$ is bijective; see [13].

On the other hand, since $r > n$, the linear form $F: H_0^1(\Omega) \to R$ given by

$$\langle F, z\rangle = \int_\Omega u(x)z(x)\,dx$$

is continuous. Therefore there exists a unique element $y_k \in H_0^1(\Omega)$ such that $\mathscr{A}_k y_k = F$, i.e., y_k is the unique solution of the problem

$$\begin{cases} Ay + \varphi_k(y) = u & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma. \end{cases}$$

Now let us take

$$\psi_k(\theta) = |\varphi_k(\theta)|^{r-2}\varphi_k(\theta) \quad \text{and} \quad z_k(x) = \psi_k(y_k(x)).$$

Since $r > n \ge 2$ and ψ_k is a Lipschitz function, with $\psi_k(0) = 0$, we have that $z_k \in H_0^1(\Omega)$; see, for instance, [19]. Moreover,

$$\psi_k'(\theta) = (r-1)|\varphi_k(\theta)|^{r-2}\varphi_k'(\theta) \ge 0,$$

hence

$$\begin{aligned} \|\varphi_k(y_k)\|_{L^r(\Omega)}^r &\le \sum_{i,j=1}^n \int_\Omega \psi_k'(y_k(x))a_{ij}(x)\,\partial_{x_i} y_k(x)\,\partial_{x_j} y_k(x)\,dx \\ &\quad + \int_\Omega a_0(x)y_k(x)z_k(x)\,dx + \int_\Omega \varphi_k(y_k(x))z_k(x)\,dx \\ &= \langle \mathscr{A}_k y_k, z_k\rangle = \langle F, z_k\rangle = \int_\Omega u(x)z_k(x)\,dx \\ &\le \|u\|_{L^r(\Omega)}\|z_k\|_{L^{r'}(\Omega)} = \|u\|_{L^r(\Omega)}\|\varphi_k(y_k)\|_{L^r(\Omega)}^{r-1}, \end{aligned}$$

where $1/r + 1/s = 1$. From the previous inequality it follows that

$$\|\varphi_k(y_k)\|_{L^r(\Omega)} \leq \|u\|_{L^r(\Omega)}. \tag{2.5}$$

Taking $f_k = u - \varphi_k(y_k)$, it is obvious that y_k is the unique solution of the Dirichlet problem

$$\begin{cases} Ay = f_k & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma. \end{cases}$$

From the regularity results for boundary-value problems (see [12]) and using (2.5), we obtain that $y_k \in W^{2,r}(\Omega)$ and the following estimate is satisfied:

$$\|y_k\|_{W^{2,r}(\Omega)} \leq c_1 \|f_k\|_{L^r(\Omega)} \leq 2c_1 \|u\|_{L^r(\Omega)}, \tag{2.6}$$

with c_1 independent of k. Since $W^{2,r}(\Omega)$ is included in $C(\bar{\Omega})$, the inclusion being continuous, we deduce from (2.6) that there exists a constant $c_2 > 0$ independent of k such that

$$\|y_k\|_{C(\bar{\Omega})} \leq c_2.$$

Hence, $\varphi_k(y_k) = \varphi(y_k)$ for every $k \geq c_2$, which means that y_k is a solution of the Dirichlet problem (2.1). The uniqueness of the solution of (2.1) in

$$W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$$

is an immediate consequence of the properties of A and φ. Therefore every y_k, with $k \geq c_2$, is equal to this unique solution and (2.6) implies (2.4).

Finally, the continuous dependence of y_u with respect to u follows easily from inequality (2.4) and the compact inclusion $W^{2,r}(\Omega) \subset C(\bar{\Omega})$ (see [17]), which allow us to pass to the limit in the state equation. □

The differentiability of the relation between the control and the state can be readily deduced from the implicit function theorem:

Theorem 2. *The mapping $G: L^r(\Omega) \to W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ defined by $G(u) = y_u$ is of class C^1 and for every $u, v \in L^r(\Omega)$ the element $z = DG(u) \cdot v$ is the unique solution of the Dirichlet problem*

$$\begin{cases} Az + \varphi'(y_u)z = v & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma. \end{cases} \tag{2.7}$$

Taking a minimizing sequence and arguing in the standard way, we obtain the existence of a solution for the optimal control problem (P):

Theorem 3. *Under the hypotheses assumed in Section 2 and assuming the existence of a feasible control (i.e., a control $u \in K$ such that $\nabla y_u \in C$), then problem* (P) *has at least one solution. Moreover, if φ is linear, then the solution is unique.*

3. Optimality Conditions

Hereafter we use the following notations: $M(X)$ denotes the space of real regular Borel measures in X, that is, the dual space of $C(X)$ (remember that X is a compact subset of $\bar{\Omega}$). The norm in $M(X)$ is given by

$$\|\mu\|_{M(X)} = |\mu|(X) = \sup\left\{\int_X y\, d\mu\colon y \in C(X) \text{ and } \|y\|_\infty \leq 1\right\}, \tag{3.1}$$

where $|\mu|$ is the total variation measure; see [18]. Hence, $M(X)^n$ is the dual space of $C(X)^n$ too. Obviously every element $\boldsymbol{\mu} \in M(X)^n$ can be decomposed as a sum of two measures $\boldsymbol{\mu} = \boldsymbol{\mu}_\Omega + \boldsymbol{\mu}_\Gamma$, $\boldsymbol{\mu}_\Omega$ and $\boldsymbol{\mu}_\Gamma$ being regular real Borel measures in $\bar{\Omega}$, concentrated in $\Omega \cap X$ and $\Gamma \cap X$, respectively.

The next theorem establishes the optimality conditions for (P).

Theorem 4. *Let $\bar{u}$ be a solution of problem* (P), *then there exist a real number $\bar{\lambda} \geq 0$ and elements $\bar{y} \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$, $\bar{p} \in L^t(\Omega)$ for all $t < n/(n-1)$, and $\boldsymbol{\mu} \in M(X)^n$ satisfying*

$$\bar{\lambda} + \|\boldsymbol{\mu}\|_{M(X)^n} > 0, \tag{3.2}$$

$$\begin{cases} A\bar{y} + \varphi(\bar{y}) = \bar{u} & \text{in } \Omega, \\ \bar{y} = 0 & \text{on } \Gamma, \end{cases} \tag{3.3}$$

$$\begin{cases} A^*\bar{p} + \varphi'(\bar{y})\bar{p} = \bar{\lambda}(\bar{y} - y_d) - \operatorname{div} \boldsymbol{\mu}_\Omega & \text{in } \Omega, \\ \bar{p} = \boldsymbol{\alpha}_A \boldsymbol{\mu}_\Gamma & \text{on } \Gamma, \end{cases} \tag{3.4}$$

$$\int_X (\mathbf{z} - \nabla\bar{y})\, d\boldsymbol{\mu} \leq \mathbf{0}, \qquad \forall \mathbf{z} \in C, \tag{3.5}$$

$$\int_\Omega (\bar{p} + \bar{\lambda} N |\bar{u}|^{\sigma-2}\bar{u})(u - \bar{u})\, dx \geq 0, \qquad \forall u \in K, \tag{3.6}$$

where A^* is the adjoint operator of A,

$$\boldsymbol{\alpha}_A(x) = \frac{-\mathbf{v}(x)}{\mathbf{v}_A(x)\cdot\mathbf{v}(x)} \quad \text{and} \quad \mathbf{v}_A(x) = (a_{ij}(x))\mathbf{v}(x).$$

Moreover, if the following Slater condition is verified

$$\exists (u_0, z_0) \in K \times (W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega))/(\nabla\bar{y} + \nabla z_0) \in \mathring{C}, \tag{3.7}$$

where z_0 is the solution of the following Dirichlet problem

$$\begin{cases} Az_0 + \varphi'(\bar{y})z_0 = u_0 - \bar{u} & \text{in } \Omega, \\ z_0 = 0 & \text{on } \Gamma, \end{cases}$$

then the system (3.3)–(3.6) *is satisfied with $\bar{\lambda} = 1$.*

This theorem is proved in Section 5. Now let us see whether the previous optimality conditions can be simplified in some particular cases.

Remark 1. In the case $X \subset \Omega$, since $\boldsymbol{\mu}$ is concentrated in X, then $\boldsymbol{\mu}_\Gamma = 0$ and the adjoint state verifies $\bar{p} = 0$ on Γ. Consequently, we have that $\bar{p}$ is an element of $W^{2,r}_{\text{loc}}(\Omega \backslash X) \subset C^1(\bar{\Omega} \backslash X)$.

Analogously, in case $X \subset \Gamma$ we have that $\boldsymbol{\mu}_\Omega = 0$ and therefore $\bar{p}$ belongs to $W^{2,r}_{\text{loc}}(\Omega) \subset C^1(\Omega)$.

Remark 2. If φ is a linear function, then the Slater condition (3.7) becomes

$$\exists u_0 \in K \qquad \text{such that} \quad \nabla y_{u_0} \in \mathring{C}.$$

Thus, assuming φ is linear and

$$C = \{\mathbf{z} \in C(X)^n \colon |\mathbf{z}(x)| \leq \delta, \ \forall x \in X\}, \tag{3.8}$$

the Slater condition is satisfied for every $\delta > \delta_0$, where δ_0 is the first value for which (P) possesses a feasible control.

This condition allows us to obtain the optimality system (3.3)–(3.6) with $\bar{\lambda} = 1$. Using Clarke's terminology [11], the problem is "normal" in the sense that the functional J to be minimized appears in the optimality system. Sometimes it is possible to deduce normality of (P) without proving the Slater condition. For instance, if C is defined as above, then the optimality system is verified with $\bar{\lambda} = 1$ for almost all $\delta \in [\delta_0, +\infty)$. The proof of this fact can be done using Clarke's result [11]; see [3].

When there are no control constraints ($K = L^r(\Omega)$), then (P) is always normal: if $\bar{\lambda} = 0$ it follows from (3.6) that $\bar{p} = 0$ and so (3.4) and (3.5) imply that $\boldsymbol{\mu} = 0$, which contradicts (3.2). Therefore it is sufficient to replace $\bar{p}$ and $\boldsymbol{\mu}$ by $\bar{p}/\bar{\lambda}$ and $\boldsymbol{\mu}/\bar{\lambda}$, respectively.

On the other hand, normality has repercussions on optimal control regularity; see Remark 3.

The following corollary improves the results of Bonnans and Casas [3] and Mackenroth [16] by establishing that the Lagrange multiplier $\boldsymbol{\mu}$ associated with the state constraints can be reduced to one measure in X when the restriction $|\nabla y(x)| \leq \delta$ is considered.

Corollary 1. *Let $\bar{u}$ be a solution of problem* (P), *with*

$$C = \{\mathbf{z} \in C(X)^n \colon |\mathbf{z}(x)| \leq \delta, \ \forall x \in X\},$$

where $|\cdot|$ denotes the euclidean norm in R^n. Then there exist a real number $\bar{\lambda} \geq 0$ and elements $\bar{y} \in W^{2,r}(\Omega) \cap W^{1,r}_0(\Omega)$, $\bar{p} \in L^t(\Omega)$ for all $t < n/(n-1)$, and $\bar{\mu} \in M(X)$ satisfying

$$\bar{\lambda} + \|\bar{\mu}\|_{M(X)} > 0, \tag{3.9}$$

$$\begin{cases} A\bar{y} + \varphi(\bar{y}) = \bar{u} & \text{in } \Omega, \\ \bar{y} = 0 & \text{on } \Gamma, \end{cases} \tag{3.10}$$

$$\begin{cases} A^*\bar{p} + \varphi'(\bar{y})\bar{p} = \bar{\lambda}(\bar{y} - y_d) - \dfrac{1}{\delta}\operatorname{div}(\nabla\bar{y} \cdot \bar{\mu}_\Omega) & \text{in } \Omega, \\ \bar{p} = \beta_A \bar{\mu}_\Gamma & \text{on } \Gamma, \end{cases} \tag{3.11}$$

$$\int_X (z(x) - |\nabla \bar{y}(x)|)\, d\bar{\mu} \le 0, \qquad \forall z \in \bar{B}_\delta(0), \tag{3.12}$$

$$\int_\Omega (\bar{p} + \bar{\lambda} N |\bar{u}|^{\sigma-2}\bar{u})(u - \bar{u})\, dx \ge 0, \qquad \forall u \in K, \tag{3.13}$$

where $\bar{B}_\delta(0)$ is the closed ball of $C(X)$ with radius δ and center 0 and

$$\beta_A(x) = \frac{-\partial_\nu \bar{y}(x)}{\delta \mathbf{v}_A(x) \cdot \mathbf{v}(x)}.$$

Furthermore, $\bar{\mu}$ is a positive measure concentrated in the set

$$X^+ = \{x \in X : |\nabla \bar{y}(x)| = \delta\}.$$

In particular, if the equality $|\nabla \bar{y}(x)| = \delta$ is satisfied at a finite set of points $\{x_j\}_{j=1}^m \subset X$, then we have

$$\bar{\mu} = \sum_{j=1}^m \lambda_j \delta_{x_j},$$

where $\lambda_j \ge 0$ and δ_{x_j} is the Dirac measure concentrated at x_j.

Proof. Let $\boldsymbol{\mu} \in M(X)^n$ as in Theorem 4 and let us consider the measure $\bar{\mu} \in M(X)$ defined by

$$\bar{\mu} = \frac{1}{\delta} \nabla \bar{y} \cdot \boldsymbol{\mu},$$

i.e.,

$$\int_X z\, d\bar{\mu} = \frac{1}{\delta} \sum_{j=1}^n \int_X z\, \partial_{x_j} \bar{y}\, d\boldsymbol{\mu}^j, \qquad \forall z \in C(X).$$

From Lemma 1, we deduce that $\bar{\mu}$ is a positive measure concentrated in X^+ and $\boldsymbol{\mu} = (1/\delta)\nabla \bar{y} \cdot \bar{\mu}$, therefore (3.9) and (3.11) follow from (3.2) and (3.4), respectively. To complete the optimality conditions, it remains to prove (3.12). Let $z \in \bar{B}_\delta(0)$ be arbitrary and let us take $\mathbf{z} = (1/\delta) z \nabla \bar{y}$, which obviously belongs to C. Therefore, using (3.5), Lemma 1, and remarking that the support of $\bar{\mu}$ is equal to that of $\boldsymbol{\mu}$, we get

$$0 \ge \int_X (\mathbf{z} - \nabla \bar{y})\, d\boldsymbol{\mu} = \int_{X^+} (z - |\nabla \bar{y}|) \frac{1}{\delta} \nabla \bar{y}\, d\boldsymbol{\mu}$$

$$= \int_{X^+} (z - |\nabla \bar{y}|)\, d\bar{\mu} = \int_X (z - |\nabla \bar{y}|)\, d\bar{\mu}. \qquad \square$$

Lemma 1. *Let $\boldsymbol{\mu} \in M(X)^n$ be a measure satisfying* (3.5), *with C given by* (3.8), *then $\bar{\mu} = (1/\delta)\nabla \bar{y} \cdot \boldsymbol{\mu}$ is a positive measure concentrated in the set*

$$X^+ = \{x \in X : |\nabla \bar{y}(x)| = \delta\}.$$

Moreover, $\boldsymbol{\mu} = (1/\delta)\nabla \bar{y} \cdot \bar{\mu}$.

Proof. We divide the proof into several steps.

Step 1: $\bar{\mu}$ is a positive measure. Let $B \subset X$ be a Borel set and let us consider the function

$$\psi(x) = (1 - \chi_B(x))\nabla\bar{y}(x),$$

where χ_B is the characteristic function of B. Let us denote by σ the positive measure in X:

$$\sigma = \sum_{j=1}^{n} |\mu^j|,$$

where $|\mu^j|$ is the total variation measure of μ^j, $1 \leq j \leq n$. Applying Lusin's theorem [18] we deduce the existence of a sequence $\{\varphi_k\}_{k=1}^{\infty} \subset C(X)^n$ such that

$$\sigma(\{x \in X: \varphi_k(x) \neq \psi(x)\}) < \frac{1}{k}.$$

Now we define

$$\psi_k(x) = \begin{cases} \varphi_k(x) & \text{if } |\varphi_k(x)| \leq \delta, \\ \dfrac{\delta}{|\varphi_k(x)|}\varphi_k(x) & \text{if } |\varphi_k(x)| > \delta. \end{cases}$$

Then it is obvious that $\{\psi_k\}_{k=1}^{\infty} \subset C$ and also

$$\sigma(\{x \in X: \psi_k(x) \neq \psi(x)\}) < \frac{1}{k}.$$

Using the dominated convergence theorem and (3.5) we obtain

$$\begin{aligned} \bar{\mu}(B) &= \frac{1}{\delta}\sum_{j=1}^{n}\int_B \partial_{x_j}\bar{y}(x)\, d\mu^j(x) \\ &= \frac{1}{\delta}\sum_{j=1}^{n}\left(\int_X \partial_{x_j}\bar{y}(x)\, d\mu^j(x) - \int_X (1 - \chi_B(x))\, \partial_{x_j}\bar{y}(x)\, d\mu^j(x)\right) \\ &= \frac{1}{\delta}\lim_{k\to\infty}\sum_{j=1}^{n}\left(\int_X \partial_{x_j}\bar{y}(x)\, d\mu^j(x) - \int_X \psi_k^j(x)\, d\mu^j(x)\right) \\ &= \frac{1}{\delta}\lim_{k\to\infty}\langle \mu, \nabla\bar{y} - \psi_k\rangle \geq 0, \end{aligned}$$

which proves that $\bar{\mu}$ is a positive measure.

Step 2: μ^j is absolutely continuous with respect to $\bar{\mu}$, $1 \leq j \leq n$. For each $j = 1, \ldots, n$, we can take a Borel function h_j: $X \to R$, with $|h_j(x)| = 1$, $\forall x \in X$, and $d|\mu^j| = h_j\, d\mu^j$; see, for example, [18].

Now given a Borel set $B \subset X$ such that $\bar{\mu}(B) = 0$, let us consider the function

$$\psi(x) = \frac{\delta}{\sqrt{n}}\chi_B(x)\sum_{j=1}^{n} h_j(x)\mathbf{e}_j + (1 - \chi_B(x))\nabla\bar{y}(x),$$

where $\mathbf{e}_j$ is the jth canonical vector of R^n. It is obvious that $|\boldsymbol{\psi}(x)| \le \delta$, $\forall x \in X$. Then we can take a sequence $\{\boldsymbol{\psi}_k\}_{k=1}^{\infty} \subset C$ as in step 1 and obtain

$$\begin{aligned}\sum_{j=1}^{n} |\boldsymbol{\mu}^j|(B) &= \sum_{j=1}^{n} \int_B h_j(x)\, d\boldsymbol{\mu}^j(x) \\ &= \frac{\sqrt{n}}{\delta} \left(\sum_{j=1}^{n} \int_X [\boldsymbol{\psi}^j(x) - \partial_{x_j} \bar{y}(x)]\, d\boldsymbol{\mu}^j(x) + \int_B \partial_{x_j} \bar{y}(x)\, d\boldsymbol{\mu}^j(x) \right) \\ &= \frac{\sqrt{n}}{\delta} \lim_{k \to \infty} \langle \boldsymbol{\mu}, \boldsymbol{\psi}_k - \nabla \bar{y} \rangle + \sqrt{n}\, \bar{\mu}(B) \le 0,\end{aligned}$$

from where $\boldsymbol{\mu}^j(B) = 0$ for every $j = 1, \dots, n$ and therefore $\boldsymbol{\mu}^j \ll \bar{\mu}$.

Step 3: $\boldsymbol{\mu} = (1/\delta)\nabla \bar{y} \cdot \bar{\mu}$ *and* $\sup(\bar{\mu}) \subset X^+$. Since $\boldsymbol{\mu}^j \ll \bar{\mu}$, we can apply the Radon–Nikodym theorem to deduce the existence of a function $f_j \in L^1(\bar{\mu})$ such that $d\boldsymbol{\mu}^j = f_j\, d\bar{\mu}$, $1 \le j \le n$. Let us denote $\mathbf{f} = (f_j)_{j=1}^n$. Now (3.5) is equivalent to

$$\int_X \mathbf{f}(x)\mathbf{z}(x)\, d\bar{\mu}(x) \le \frac{1}{\delta} \int_X \mathbf{f}(x) \nabla \bar{y}(x)\, d\bar{\mu}(x), \qquad \forall \mathbf{z} \in \bar{B}_1(0), \tag{3.14}$$

$\bar{B}_1(0)$ being the closed unit ball of $C(X)^n$.

From the definition of $\bar{\mu}$ we deduce that, for every Borel set $B \subset X$,

$$\begin{aligned}\int_B d\bar{\mu} = \bar{\mu}(B) &= \frac{1}{\delta} \sum_{j=1}^{n} \int_B \partial_{x_j} \bar{y}(x)\, d\boldsymbol{\mu}^j(x) \\ &= \frac{1}{\delta} \sum_{j=1}^{n} \int_B \partial_{x_j} \bar{y}(x) f_j(x)\, d\bar{\mu} \\ &= \frac{1}{\delta} \int_B \nabla \bar{y}(x) \mathbf{f}(x)\, d\bar{\mu}(x),\end{aligned}$$

which implies

$$\frac{1}{\delta} \nabla \bar{y}(x) \mathbf{f}(x) = 1 \qquad \text{a.e.} \quad [\bar{\mu}] x \in X. \tag{3.15}$$

From here we obtain with the aid of relation $\nabla \bar{y}(x) \in C$, $\forall x \in X$, that

$$1 = \frac{1}{\delta} \nabla \bar{y}(x) \mathbf{f}(x) \le \frac{1}{\delta} |\nabla \bar{y}(x)|\, |\mathbf{f}(x)| \le |\mathbf{f}(x)| \qquad \text{a.e.} \quad [\bar{\mu}] x \in X. \tag{3.16}$$

Let us take $\mathbf{g}(x) = \mathbf{f}(x)/|\mathbf{f}(x)|$ and again apply the Lusin theorem to get a sequence $\{\boldsymbol{\psi}_k\}_{k=1}^{n} \subset C(X)^n$ such that

$$|\boldsymbol{\psi}_k(x)| \le 1 \quad \text{and} \quad \bar{\mu}(\{x \in X \colon \boldsymbol{\psi}_k(x) \ne \mathbf{g}(x)\}) < \frac{1}{k}.$$

Then from (3.14) and (3.15) it follows that

$$\int_X |\mathbf{f}(x)|\, d\bar{\mu}(x) = \lim_{k\to\infty} \int_X \boldsymbol{\psi}_k(x)\cdot \mathbf{f}(x)\, d\bar{\mu}(x)$$
$$\leq \frac{1}{\delta}\int_X \mathbf{f}(x)\nabla\bar{y}(x)\, d\bar{\mu}(x) = \int_X d\bar{\mu}(x),$$

which together with (3.16) implies that

$$|\mathbf{f}(x)| = 1 \qquad \text{a.e.} \quad [\bar{\mu}]x\in X. \tag{3.17}$$

Therefore

$$1 = \frac{1}{\delta}\nabla\bar{y}(x)\mathbf{f}(x) \leq \frac{1}{\delta}|\nabla\bar{y}(x)|\,|\mathbf{f}(x)| \leq \frac{1}{\delta}|\nabla\bar{y}(x)| \leq 1 \qquad \text{a.e.} \quad [\bar{\mu}]x\in X, \tag{3.18}$$

which means that $\bar{\mu}$ is concentrated in X^+. Moreover, since the balls in R^n for the euclidean metric are stictly convex, (3.15), (3.17), and (3.18) imply that $\mathbf{f}(x) = (1/\delta)\nabla\bar{y}(x)$ a.e. $[\bar{\mu}]x\in X$ and hence $\boldsymbol{\mu} = (1/\delta)\nabla\bar{y}\cdot\bar{\mu}$. □

Corollary 2. *If $X \subset \Gamma$ and the state constraint set C is given by*

$$C = \{\mathbf{z}\in C(X)^n\colon \mathbf{z}(x)\cdot\mathbf{v}(x) \geq \delta,\ \forall x\in X\},$$

then the optimality conditions (3.9)–(3.13) *are verified with* (3.11) *and* (3.12) *replaced by*

$$\begin{cases} A^*\bar{p} + \varphi'(\bar{y})\bar{p} = \bar{\lambda}(\bar{y} - y_d) & \text{in}\ \Omega, \\ \bar{p} = \dfrac{-\bar{\mu}_\Gamma}{\mathbf{v}_A\cdot\mathbf{v}} & \text{on}\ \Gamma, \end{cases} \tag{3.19}$$

and

$$\int_X (z - \partial_\nu\bar{y})\, d\bar{\mu}_\Gamma \leq 0, \qquad \forall z\in\tilde{C}, \tag{3.20}$$

where $\tilde{C}$ is given by

$$\tilde{C} = \{z\in C(X)\colon z(x) \geq \delta,\ \forall x\in X\}.$$

Moreover, $\bar{\mu}_\Gamma$ is a negative measure concentrated in

$$X^- = \{x\in X\colon \partial_\nu\bar{y}(x) = \delta\}.$$

Proof. It is enough to apply Theorem 4 by taking $\bar{\mu}_\Gamma = \mathbf{v}\cdot\boldsymbol{\mu}_\Gamma$, defined by

$$\int_\Gamma z\, d\bar{\mu}_\Gamma = \sum_{j=1}^{n}\int_X z\nu_j\, d\boldsymbol{\mu}_\Gamma^j.$$

To prove this let us remark that we can decompose each canonical vector $\mathbf{e}_j$ in the way

$$\mathbf{e}_j = \nu_j(x)\cdot\mathbf{v}(x) + \mathbf{t}_j(x), \qquad \forall x\in\Gamma,$$

where $\mathbf{t}_j(x)$ is a tangent vector to Γ at x and hence

$$\partial_{x_j}\bar{y}(x) = \nabla\bar{y}(x)\cdot\mathbf{e}_j = v_j(x)\,\partial_\nu\bar{y}(x) + \partial_{t_j}\bar{y}(x) = v_j(x)\,\partial_\nu\bar{y}(x),$$

the last equality being a consequence of the fact that $\bar{y} = 0$ on Γ.

Now taking $\mathbf{z} = z\cdot\mathbf{v}$, with $z \in \tilde{C}$, we have that $\mathbf{z} \in C$ and furthermore using (3.5) we deduce

$$\int_X (z - \partial_\nu\bar{y})\,d\bar{\mu}_\Gamma = \sum_{j=1}^n \int_X (z - \partial_\nu\bar{y})v_j\,d\boldsymbol{\mu}_\Gamma^j = \int_X (\mathbf{z} - \nabla\bar{y})\,d\boldsymbol{\mu}_\Gamma \le 0.$$

Arguing as in Theorem 3 of [7], we deduce from (3.20) that $\bar{\mu}_\Gamma$ is a negative measure concentrated in X^-. □

Remark 3. From (3.6) we can deduce some qualitative properties of optimal control $\bar{u}$. For example, if $N = 0$ or $\bar{\lambda} = 0$ and

$$K = \{u \in L^\infty(\Omega)\colon a \le u(x) \le b \text{ a.e. } x \in \Omega\}, \tag{3.21}$$

then $\bar{u}$ has the behavior of Bang-Bang type. More precisely, $\bar{u}(x) = a$ if $\bar{p}(x) > 0$ and $\bar{u}(x) = b$ if $\bar{p}(x) < 0$.

However, if $N\bar{\lambda} \neq 0$, then $\bar{u}$ can possess additional regularity properties. Thus if there is not any constraint on the control, then from (3.6) it follows that

$$\bar{u}(x) = \frac{-1}{(N\bar{\lambda})^{1/(\sigma-1)}}\,|\bar{p}(x)|^{(2-\sigma)/(\sigma-1)}\bar{p}(x),$$

where σ is greater than or equal to $r > n$, by virtue of (H2). Hence $\bar{u}$ is continuous in the points where $\bar{p}$ is continuous. If C is given by (3.8) and we suppose $y_d \in L^r(\Omega)$, then (3.11) and Lemma 1 imply that $\bar{p}$ belongs to $W^{2,r}_{\mathrm{loc}}(X_0) \subset C^1(X_0)$, where

$$X_0 = \{x \in X\colon |\nabla\bar{y}(x)| < \delta\}.$$

Therefore, $\bar{u}$ is continuous in X_0 and it is of class C^1 at the points x belonging to $X_0\backslash Z(\bar{p})$, where

$$Z(\bar{p}) = \{x \in \bar{\Omega}\colon \bar{p}(x) = 0\}.$$

If K is the set of positive controls, the situation is very similar, because in this case

$$\bar{u}(x) = \max\left\{0, \frac{-1}{(N\bar{\lambda})^{1/(\sigma-1)}}\,|\bar{p}(x)|^{(2-\sigma)/(\sigma-1)}\bar{p}(x)\right\}.$$

When K is given by (3.21), then σ can be taken equal to 2 and we have

$$\bar{u}(x) = \mathrm{Proj}_{[a,b]}\left(\frac{-1}{N\bar{\lambda}}\,\bar{p}(x)\right).$$

Thus we can again deduce continuity of $\bar{u}$ at the points where the state constraint is not active.

Obviously, additional regularity can be obtained for $\bar{u}$ when $X \subset \Gamma$. For instance, $\bar{u} \in C(\bar{\Omega}) \cap C^1(\bar{\Omega}\backslash Z(\bar{p}))$ if there is not any constraint on the control. If K is defined by (3.21) and $\sigma = 2$, then $\bar{u}$ is a locally Lipschitz function in $\bar{\Omega}$.

4. Study of the Adjoint State Equation

In this section we use the transposition method to prove that (3.4) has a unique solution in $L^t(\Omega)$ for every $1 \le t < n/(n-1)$. First we prove a trace theorem which gives sense to the boundary condition of (3.4). We begin by introducing some notation: for every $1 \le t < n/(n-1)$, $V^t(\Omega)$ denotes the space

$$V^t(\Omega) = \{p \in L^t(\Omega): A^*p \in C_0^1(\Omega)'\},$$

endowed with the norm

$$\|p\|_{V^t(\Omega)} = \|p\|_{L^t(\Omega)} + \|A^*p\|_{C_0^1(\Omega)'},$$

that turns into a Banach space. Here $C_0^1(\Omega)$ is the space of the C^1 functions in $\bar{\Omega}$ which, together with all their partial derivatives of first order, vanish on Γ. In $C_0^1(\Omega)$ we consider the usual norm

$$\|z\|_{C_0^1(\Omega)} = \max_{x \in \Omega}|z(x)| + \sum_{j=1}^{n} \max_{x \in \Omega}|\partial_{x_j} z(x)|.$$

$C_0(\Omega)$ denotes the Banach space formed by the continuous functions on $\bar{\Omega}$ which vanish on Γ, endowed with the supremum norm.

Using the identity $M(\Omega) = C_0(\Omega)'$, it is possible to characterize the dual space of $C_0^1(\Omega)$:

Lemma 2. *The dual space $C_0^1(\Omega)'$ is isometrically isomorphic to the Banach space consisting of those distributions $T \in D'(\Omega)$ satisfying*

$$T = \mu_0 - \sum_{j=1}^{n} \partial_{x_j}\mu_j \qquad \textit{for some} \quad \mu_j \in M(\Omega), \quad j = 0, 1, \ldots, n, \tag{4.1}$$

normed by

$$\|T\|_{C_0^1(\Omega)'} = \inf\left\{\sum_{j=0}^{n} \|\mu_j\|_{M(\Omega)}: (\mu_0, \ldots, \mu_n) \textit{ satisfies } (4.1)\right\}. \tag{4.2}$$

Now let us prove a trace theorem and Green's formula.

Theorem 5. *There exists a unique linear and continuous mapping*

$$\gamma: V^t(\Omega) \to W^{-1/t,t}(\Gamma)$$

satisfying

$$\gamma(p) = p|_\Gamma, \qquad \forall p \in D(\bar{\Omega}), \tag{4.3}$$

and

$$\langle \gamma(p), \partial_{\nu_A} z \rangle = \langle A^*p, z \rangle_\Omega - \int_\Omega pAz\, dx, \tag{4.4}$$

for every $z \in W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega)$, where $1/t + 1/t' = 1$,

$$\partial_{\nu_A} z(x) = \nabla z(x) \cdot \mathbf{v}_A(x)$$

and

$$\langle A^*p, z\rangle_\Omega = \int_\Omega z\, d\mu_0 + \sum_{j=1}^n \int_\Omega \partial_{x_j} z\, d\mu_j,$$

assuming that

$$A^*p = \mu_0 - \sum_{j=1}^n \partial_{x_j}\mu_j;$$

see Lemma 2.

Proof. We begin by remarking that the linear continuous functional

$$\partial_{\nu_A}\colon W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega) \to W^{1/t,t'}(\Gamma)$$

is surjective. The proof of this fact, given in [10], utilizes in an essential way the Lipschitz continuity of coefficients a_{ij} and the $C^{1,1}$ regularity of Γ; see [12] for the case $\mathbf{v}_A = \mathbf{v}$. Using the open mapping theorem, we deduce the existence of a constant c_1 depending only on A, t', and Ω such that

$$\inf\{\|z\|_{W^{2,t'}(\Omega)}\colon \partial_{\nu_A} z = g\} \le c_1 \|g\|_{W^{1/t,t'}(\Gamma)}. \tag{4.5}$$

Now, given $g \in W^{1/t,t'}(\Gamma)$, let us take $z \in W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega)$ verifying that $\partial_{\nu_A} z = g$ and define

$$\langle \gamma(p), g\rangle = \langle \gamma(p), \partial_{\nu_A} z\rangle = \langle A^*p, z\rangle_\Omega - \int_\Omega pAz\, dx.$$

Prove that $\gamma(p)$ is well defined as an element of $W^{-1/t,t}(\Gamma) = (W^{1/t,t'}(\Gamma))'$. First, from the inequality $t < n/(n-1)$ it follows that $t' > n$ and therefore

$$W^{2,t'}(\Omega) \subset C^1(\bar\Omega),$$

thus the term on the right-hand side of (4.4) makes sense. Moreover, if z_1, $z_2 \in W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega)$ and $\partial_{\nu_A} z_1 = \partial_{\nu_A} z_2 = g$, then we must prove that

$$\langle A^*p, z_1\rangle_\Omega - \int_\Omega pAz_1\, dx = \langle A^*p, z_2\rangle_\Omega - \int_\Omega pAz_2\, dx.$$

To do so, let us consider $z = z_1 - z_2 \in W_0^{2,t'}(\Omega)$ and a sequence $\{z_k\} \subset D(\Omega)$ converging to z in $W_0^{2,t'}(\Omega)$. We thus obtain

$$\langle A^*p, z\rangle_\Omega - \int_\Omega pAz\, dx = \lim_{k\to\infty}\left\{\langle A^*p, z_k\rangle_\Omega - \int_\Omega pAz_k\, dx\right\} = 0,$$

the last equality being a consequence of the derivative definition in the distribution sense.

The continuity of $\gamma(p)$ and γ is deduced as follows:

$$\begin{aligned}|\langle \gamma(p), g\rangle| &\le \|\mu_0\|_{M(\Omega)}\|z\|_{C_0(\Omega)} + \sum_{j=1}^n \|\mu_j\|_{M(\Omega)}\|\partial_{x_j} z\|_{C(\bar\Omega)} + \|p\|_{L^t(\Omega)}\|Az\|_{L^{t'}(\Omega)}\\ &\le c_2\left(\|\mu_0\|_{M(\Omega)} + \sum_{j=1}^n \|\mu_j\|_{M(\Omega)} + \|p\|_{L^t(\Omega)}\right)\|z\|_{W^{2,t'}(\Omega)}.\end{aligned}$$

From (4.2) and (4.5), taking the infimum in $(\mu_0, \ldots, \mu_n)$ and z, respectively, we derive

$$|\langle \gamma(p), g\rangle| \leq c_1 c_2 \|p\|_{V^t(\Omega)} \|g\|_{W^{1/t,t'}(\Gamma)}.$$

From the definition of γ and using Green's formula for regular functions, it is possible to prove that (4.3) is satisfied. The uniqueness follows from (4.4) and the surjectivity of the mapping ∂_{ν_A} onto $W^{1/t,t'}(\Gamma)$. □

Theorem 6. *Given $\boldsymbol{\mu} \in M(\bar{\Omega})^n$, $y \in L^\infty(\Omega)$, and $t \in (1, n/(n-1))$, the problems*

$$\text{(PD1)} \quad \begin{cases} \text{Find} \quad p \in V^t(\Omega) \quad \text{such that} \\ A^*p + \varphi'(y)p = -\operatorname{div} \boldsymbol{\mu}_\Omega \qquad \text{in } \Omega, \\ p = \boldsymbol{\alpha}_A \boldsymbol{\mu}_\Gamma \qquad \text{on } \Gamma, \end{cases}$$

with $\boldsymbol{\alpha}_A$ defined as in Theorem 4, and

$$\text{(PD2)} \quad \begin{cases} \text{Find} \quad p \in L^t(\Omega) \quad \text{such that} \\ \displaystyle\int_\Omega p(Az + \varphi'(y)z)\, dx = \int_{\bar{\Omega}} \nabla z \, d\boldsymbol{\mu}, \qquad \forall z \in W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega), \end{cases}$$

are equivalent and have a unique solution.

Proof. First let us prove that problem (PD2) has a unique solution. Following [12] we know that the linear operator

$$\mathscr{A}: W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega) \to L^{t'}(\Omega)$$

defined by $\mathscr{A}(z) = Az + \varphi'(y)z$ is an isomorphism thanks to the hypotheses on A and Γ. Hence the adjoint operator

$$\mathscr{A}^*: L^t(\Omega) \to (W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega))'$$

is an isomorphism too. Since.

$$z \to \int_{\bar{\Omega}} \nabla z \, d\boldsymbol{\mu}$$

defines an element of $(W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega))'$, we deduce the existence and uniqueness of a function $p \in L^t(\Omega)$ verifying

$$\int_\Omega p(Az + \varphi'(y)z)\, dx = \int_\Omega p\mathscr{A}z \, dx = \langle \mathscr{A}^*(p), z\rangle = \int_{\bar{\Omega}} \nabla z \, d\boldsymbol{\mu}$$

for all $z \in W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega)$.

It remains to prove the equivalence between (PD1) and (PD2). If p is a solution of (PD2), taking $z \in D(\Omega)$ as arbitrary in the previous equality we get

$$A^*p + \varphi'(y)p = -\operatorname{div} \boldsymbol{\mu}_\Omega \qquad \text{in } \Omega, \tag{4.6}$$

which implies that p belongs to $V^t(\Omega)$ and the boundary condition is well defined by Theorem 5. From Green's formula (4.4) and (4.6) we obtain

$$\langle \gamma(p), \partial_{\nu_A} z\rangle = \langle A^*p, z\rangle_\Omega - \int_\Omega pAz\, dx$$
$$= \int_\Omega \nabla z\, d\boldsymbol{\mu}_\Omega - \int_\Omega p(Az + \varphi'(y)z)\, dx = -\int_\Gamma \nabla z\, d\boldsymbol{\mu}_\Gamma. \tag{4.7}$$

Arguing analogously to the proof of Corollary 2, we can decompose each canonical vector $\mathbf{e}_j$ in the form

$$\mathbf{e}_j = \frac{\nu_j(x)}{\boldsymbol{\nu}_A(x)\cdot\boldsymbol{\nu}(x)}\boldsymbol{\nu}_A(x) + \mathbf{t}_j(x), \qquad \forall x \in \Gamma, \tag{4.8}$$

where $\mathbf{t}_j(x)$ is a tangent vector to Γ at x and hence for every $z \in W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega)$ we have

$$\partial_{x_j} z(x) = \frac{\nu_j(x)}{\boldsymbol{\nu}_A(x)\cdot\boldsymbol{\nu}(x)}\partial_{\nu_A} z(x) + \partial_{t_j} z(x) = -\boldsymbol{\alpha}_A^j(x)\,\partial_{\nu_A} z(x), \tag{4.9}$$

the last equality being a consequence of the fact that $z = 0$ on Γ. Let us remark that (2.2) implies

$$\boldsymbol{\nu}_A(x)\cdot\boldsymbol{\nu}(x) \geq m > 0, \qquad \forall x \in \Gamma,$$

and so $\boldsymbol{\alpha}_A^j \in C^{0,1}(\Gamma)$ thanks to the fact that a_{ij} are Lipschitz functions and Γ is of class $C^{1,1}$. Combining (4.7) and (4.9) it follows that

$$\langle \gamma(p), \partial_{\nu_A} z\rangle = \int_\Gamma \partial_{\nu_A} z\boldsymbol{\alpha}_A\, d\boldsymbol{\mu}_\Gamma, \qquad \forall z \in W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega),$$

which, together with the surjectivity of ∂_{ν_A}, provides $\gamma(p) = \boldsymbol{\alpha}_A\boldsymbol{\mu}_\Gamma$ and therefore p is a solution of (PD1).

Finally, let p be a solution of (PD1), then using Green's formula (4.4), the fact that p is a solution of (PD1) and (4.9), we deduce

$$\int_\Omega p(Az + \varphi'(y)z)\, dx = \langle A^*p, z\rangle_\Omega - \langle \gamma(p), \partial_{\nu_A} z\rangle + \int_\Omega p\varphi'(y)z\, dx$$
$$= \int_\Omega \nabla z\, d\boldsymbol{\mu}_\Omega + \int_\Gamma \nabla z\, d\boldsymbol{\mu}_\Gamma = \int_{\bar\Omega} \nabla z\, d\boldsymbol{\mu},$$

which proves that p is a solution of (PD2). □

Corollary 3. *Let $\bar\lambda \geq 0$, $\bar y \in W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega)$, and $\boldsymbol{\mu} \in M(X)^n$ be as in Theorem 4, then there exists a unique solution of* (3.4), *$\bar p \in V^t(\Omega)$, for every $t \in [1, n/(n-1))$.*

Proof. It is enough to decompose (3.4) as a sum of the problems (PD1) and

$$\begin{cases} A^*p + \varphi'(\bar y)p = \bar\lambda(\bar y - y_d) & \text{in } \Omega, \\ p = 0 \quad \text{on } \Gamma, \end{cases}$$

and remark that every measure $\mu \in M(X)$ can be considered as a measure in $\bar{\Omega}$ with support in X. □

5. Proof of Theorem 4

Before proving Theorem 4 we need the next abstract result of the existence of the Lagrange multiplier, whose proof can be found in [10]; see also [3].

Theorem 7. *Let U and Z be two Banach spaces and let $K \subset U$ and $C \subset Z$ be two convex subsets, C having a nonempty interior. Let $\bar{u} \in K$ be a solution of the optimization problem:*

$$(\mathrm{P}) \quad \begin{cases} \mathrm{Min}\ J(u) \\ u \in K \quad \text{and} \quad G(u) \in C, \end{cases}$$

where $J\colon U \to (-\infty, +\infty]$ and $G\colon U \to Z$ are two Gâteaux differentiable mappings at $\bar{u}$. Then there exist a real number $\bar{\lambda} \geq 0$ and an element $\bar{\mu} \in Z'$ such that

$$\bar{\lambda} + \|\bar{\mu}\|_{Z'} > 0, \tag{5.1}$$

$$\langle \bar{\mu}, z - G(\bar{u}) \rangle \leq 0, \qquad \forall z \in C, \tag{5.2}$$

$$\langle \bar{\lambda} J'(\bar{u}) + [DG(\bar{u})]^* \bar{\mu}, u - \bar{u} \rangle \geq 0, \qquad \forall u \in K. \tag{5.3}$$

Moreover, $\bar{\lambda}$ can be taken equal to 1 if the following condition of Slater type is satisfied:

$$\exists u_0 \in K \qquad \text{such that} \quad G(\bar{u}) + DG(\bar{u}) \cdot (u_0 - \bar{u}) \in \mathring{C}. \tag{5.4}$$

Applying this theorem with U as the control space, $Z = C(X)^n$, J the functional to minimize, G the mapping that associates to each control the gradient of the corresponding state, which is differentiable (Theorem 2), K the convex subset of U, and C the convex subset of $Z = C(X)^n$ with nonempty interior, we deduce the existence of $\bar{\lambda}$ and $\boldsymbol{\mu}$ satisfying (3.2) and (3.5). Now let us take $\bar{y} = y_{\bar{u}}$ and $\bar{p} \in V^t(\Omega)$, the unique solution of (3.4); see Corollary 3. Then it remains to prove inequality (3.6), which is done by using the corresponding inequality (5.3). For this it is enough to establish the identity

$$\bar{\lambda} J'(\bar{u}) \cdot v + \langle [DG(\bar{u})]^* \boldsymbol{\mu}, v \rangle = \int_\Omega (\bar{p} + \bar{\lambda} N |\bar{u}|^{\sigma - 2} \bar{u}) v \, dx, \qquad \forall v \in U.$$

Let us take $z \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ as a solution of

$$\begin{cases} Az + \varphi'(\bar{y}) z = v & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma. \end{cases}$$

Then, using Theorem 2, we get

$$\begin{aligned}\bar{\lambda}J'(\bar{u})\cdot v &+ \langle[DG(\bar{u})]^*\boldsymbol{\mu}, v\rangle \\ &= \bar{\lambda}\int_\Omega(\bar{y}-y_d)z\,dx + \bar{\lambda}N\int_\Omega|\bar{u}|^{\sigma-2}\bar{u}v\,dx + \int_\Omega \nabla z\,d\boldsymbol{\mu} \\ &= \int_\Omega \bar{p}(Az+\varphi'(\bar{y})z)\,dx + \bar{\lambda}N\int_\Omega|\bar{u}|^{\sigma-2}\bar{u}v\,dx = \int_\Omega(\bar{p}+\bar{\lambda}N|\bar{u}|^{\sigma-2}\bar{u})v\,dx.\end{aligned}$$

6. Distributed Control for the Neumann Problem

In this section we consider the optimal control problem (P) associated with the following Neumann problem:

$$\begin{cases} Ay + \varphi(y) = u & \text{in } \Omega, \\ \partial_{\nu_A} y = g & \text{on } \Gamma, \end{cases} \tag{6.1}$$

where $u \in L^r(\Omega)$, $g \in W^{1/s,r}(\Gamma)$, $r > n$, and $1/r + 1/s = 1$, while Ω, Γ, A, and φ satisfy the hypotheses of Section 2 with the following additional assumption:

$$a_0 \not\equiv 0. \tag{6.2}$$

For the sake of brevity we résumé in the following remark the analogous results of Section 2 corresponding to this case:

Remark 4. 1. Problem (6.1) has a unique solution $y_u \in W^{2,r}(\Omega)$ and there exists a constant C_2 independent of u and g verifying

$$\|y_u\|_{W^{2,r}(\Omega)} \le C_2(\|u\|_{L^r(\Omega)} + \|g\|_{W^{1/s,r}(\Gamma)}). \tag{6.3}$$

It can be obtained, for instance, from Theorem 1 of [10] and the regularity results of [12].

2. The mapping $G\colon L^r(\Omega) \to W^{2,r}(\Omega)$ given by $G(u) = y_u$ is of class C^1 and for every $u, v \in L^r(\Omega)$ the element $z = DG(u)\cdot v$ is the unique solution of the Neumann problem

$$\begin{cases} Az + \varphi'(y_u)z = v & \text{in } \Omega, \\ \partial_{\nu_A} z = 0 & \text{on } \Gamma. \end{cases} \tag{6.4}$$

3. Assume the existence of a feasible control, then problem (P) has at least one solution. Let us remark that, depending on the state constraints, the existence of a feasible control can impose certain limitations on g. For example, if C is given by (3.8) and $\Gamma \subset X$, then g must verify the inequality

$$|g(x)| = |\partial_{\nu_A} y_u(x)| = |\nabla y_u(x)\cdot \mathbf{v}_A(x)| \le \delta\|(a_{ij}(x))\|, \qquad \forall x \in \Gamma.$$

The optimality conditions for (P) are stated in the next theorem.

Theorem 8. *Let $\bar{u}$ be a solution of problem* (P), *then there exist a real number $\bar{\lambda} \geq 0$ and elements $\bar{y} \in W^{2,r}(\Omega)$, $\bar{p} \in L^t(\Omega)$ for every $t < n/(n-1)$, and $\mathbf{\mu} \in M(X)^n$ satisfying*

$$\bar{\lambda} + \|\mathbf{\mu}\|_{M(X)^n} > 0, \tag{6.5}$$

$$\begin{cases} A\bar{y} + \varphi(\bar{y}) = \bar{u} & \text{in } \Omega, \\ \partial_{\nu_A}\bar{y} = g & \text{on } \Gamma, \end{cases} \tag{6.6}$$

$$\begin{cases} A^*\bar{p} + \varphi'(\bar{y})\bar{p} = \bar{\lambda}(\bar{y} - y_d) - \operatorname{div} \mathbf{\mu}_\Omega & \text{in } \Omega, \\ \partial_{\nu_{A^*}}\bar{p} = -\sum_{j=1}^{n} \partial_t \mathbf{\mu}_\Gamma^j & \text{on } \Gamma, \end{cases} \tag{6.7}$$

$$\int_X (\mathbf{z} - \nabla\bar{y})\, d\mathbf{\mu} \leq \mathbf{0}, \qquad \forall \mathbf{z} \in C, \tag{6.8}$$

$$\int_\Omega (\bar{p} + \bar{\lambda}N|\bar{u}|^{\sigma-2}\bar{u})(u - \bar{u})\, dx \geq 0, \qquad \forall u \in K, \tag{6.9}$$

where A^* is the adjoint operator of A and each $\mathbf{t}_j(x)$ is the tangent vector to the manifold Γ at the point x given by

$$\mathbf{t}_j(x) = \mathbf{e}_j - \frac{\nu_j(x)}{\mathbf{\nu}_A(x)\cdot\mathbf{\nu}(x)}\,\mathbf{\nu}_A(x),$$

$\mathbf{e}_j$ being the jth *canonical vector of R^n.*

As in Section 3, it is possible to state some normality results for problem (P), simplifications of the optimality conditions in some particular cases (see Corollaries 1 and 2), and regularity of optimal control.

The proof of Theorem 8 follows in the same way as that of Theorem 4: it utilizes Theorem 7 and the results about the adjoint state equation. Thus the rest of the paper is devoted to the study of (6.7).

We begin by proving the existence of a normal derivative with respect to $\mathbf{\nu}_{A^*}$ in $W^{-1-1/t,t}(\Gamma) = (W^{1+1/t,t'}(\Gamma))'$ for the elements of $V^t(\Omega)$ with $t < n/(n-1)$:

Theorem 9. *There exists a unique linear and continuous mapping*

$$\partial_{\nu_{A^*}}: V^t(\Omega) \to W^{-1-1/t,t}(\Gamma)$$

satisfying

$$\partial_{\nu_{A^*}}(p) = \nabla p \cdot \mathbf{\nu}_{A^*}, \qquad \forall p \in D(\bar{\Omega}), \tag{6.10}$$

with $\mathbf{\nu}_{A^*}(x) = (a_{ji}(x))\mathbf{\nu}(x)$ and

$$\langle \partial_{\nu_{A^*}}(p), \gamma(z)\rangle = \langle \gamma(p), \partial_{\nu_A} z\rangle - \langle A^*p, z\rangle_\Omega + \int_\Omega pAz\, dx, \tag{6.11}$$

for every $z \in W^{2,t'}(\Omega)$, where $1/t + 1/t' = 1$ and $\gamma(p)$ is defined in Theorem 5.

Proof. Firstly, let us remark that the linear continuous functional

$$\gamma: W^{2,t'}(\Omega) \to W^{1+1/t,t'}(\Gamma)$$

is surjective, see [12]. Hence, given $h \in W^{1+1/t,t'}(\Gamma)$, let us take z in $W^{2,t'}(\Omega)$ verifying that $\gamma(z) = h$ and define

$$\langle \partial_{\nu_{A^*}}(p), h \rangle = \langle \gamma(p), \partial_{\nu_A} z \rangle - \langle A^* p, z \rangle_\Omega + \int_\Omega pAz\, dx. \tag{6.12}$$

Let us prove that $\partial_{\nu_{A^*}}(p)$ is well defined as an element of $W^{-1-1/t,t}(\Gamma)$. Thanks again to the inclusion $W^{2,t'}(\Omega) \subset C^1(\bar{\Omega})$, the term on the right-hand side of (6.11) makes sense. Moreover, if $z_1, z_2 \in W^{2,t'}(\Omega)$ and $\gamma(z_1) = \gamma(z_2) = h$, then $z = z_1 - z_2$ belongs to $W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega)$ and so, by virtue of (4.4), we obtain

$$\langle \gamma(p), \partial_{\nu_A} z \rangle - \langle A^* p, z \rangle_\Omega + \int_\Omega pAz\, dx = 0,$$

which proves that the definition (6.12) is independent of the choice of z.

Now, taking absolute values in (6.12), we deduce

$$\begin{aligned} |\langle \partial_{\nu_{A^*}}(p), h \rangle| &\le \|\gamma(p)\|_{W^{-1/t,t}(\Gamma)} \|\partial_{\nu_A} z\|_{W^{1/t,t'}(\Gamma)} + \|\mu_0\|_{M(\Omega)} \|z\|_{C(\bar{\Omega})} \\ &\quad + \sum_{j=1}^n \|\mu_j\|_{M(\Omega)} \|\partial_{x_j} z\|_{C(\bar{\Omega})} + \|p\|_{L^t(\Omega)} \|Az\|_{L^{t'}(\Omega)} \\ &\le c_1 \left(\|\gamma(p)\|_{W^{-1/t,t}(\Gamma)} + \sum_{j=0}^n \|\mu_j\|_{M(\Omega)} + \|p\|_{L^t(\Omega)} \right) \|z\|_{W^{2,t'}(\Omega)}. \end{aligned}$$

Taking the infimum in $(\mu_0, \ldots, \mu_n)$ and z, respectively, and using Theorem 5 and Lemma 2 we derive

$$|\langle \partial_{\nu_{A^*}}(p), h \rangle \le c_2 \|p\|_{V^t(\Omega)} \|h\|_{W^{1+1/t,t'}(\Gamma)}.$$

From the definition of $\partial_{\nu_{A^*}}$ and using Green's formula for regular functions, it is possible to prove that (6.10) is satisfied. Uniqueness follows from (6.11) and the surjectivity of the mapping γ onto $W^{1+1/t,t'}(\Gamma)$. □

Definition 1. Given a measure $\mu \in M(\Gamma)$ and a continuous vector field denoted by $\mathbf{t}: \Gamma \to R^n$, such that $\mathbf{t}(x)$ is a tangent vector to Γ at the point x, we define $\partial_{\mathbf{t}}\mu$ as an element of $W^{-1-1/t,t}(\Gamma)$ by the formula

$$\langle \partial_{\mathbf{t}}\mu, h \rangle = -\int_\Gamma \partial_{\mathbf{t}} h\, d\mu, \qquad \forall h \in W^{1+1/t,t'}(\Gamma).$$

Let us see that $\partial_{\mathbf{t}}\mu$ is a continuous linear form over $W^{1+1/t,t'}(\Gamma)$. Given $z \in W^{2,t'}(\Omega)$ with $\gamma(z) = h$, we have

$$\begin{aligned} \left| \int_\Gamma \partial_{\mathbf{t}} h\, d\mu \right| &= \left| \int_\Gamma \nabla z \cdot \mathbf{t}\, d\mu \right| = \left| \sum_{j=1}^n \int_\Gamma \partial_{x_j} z \mathbf{t}_j\, d\mu \right| \\ &\le \sum_{j=1}^n \left(\int_\Gamma |\partial_{x_j} z|\, d|\mu| \right) \|\mathbf{t}_j\|_{C(\Gamma)} \le \|\mu\|_{M(\Gamma)} \|\mathbf{t}\|_{C(\Gamma)} \|z\|_{C^1(\bar{\Omega})^n} \\ &\le c\|z\|_{W^{2,t'}(\Omega)}. \end{aligned}$$

Now taking the infimum among the elements $z \in W^{2,t'}(\Omega)$ with trace equal to h we get the search result

$$\left| \int_\Gamma \partial_t h \, d\mu \right| \le c \|h\|_{W^{1+1/t,t'}(\Gamma)}.$$

After this definition and Theorem 9, the boundary condition of (6.7) is correctly defined. Finally, we state the existence and uniqueness of the solution for problem (6.7).

Theorem 10. *Given $\boldsymbol{\mu} \in M(\bar{\Omega})^n$, $y \in L^\infty(\Omega)$, and $t \in (1, n/(n-1))$, the problems*

$$\text{(PN1)} \quad \begin{cases} \text{Find} \quad p \in V^t(\Omega) \quad \text{such that} \\ A^* p + \varphi'(y)p = -\operatorname{div} \boldsymbol{\mu}_\Omega \qquad \text{in } \Omega, \\ \partial_{\nu_{A^*}} p = -\sum_{j=1}^n \partial_{\mathbf{t}_j} \boldsymbol{\mu}_\Gamma^j \qquad \text{on } \Gamma, \end{cases}$$

with $\mathbf{t}_j$ defined as in Theorem 8, *and*

$$\text{(PN2)} \quad \begin{cases} \text{Find} \quad p \in V^t(\Omega) \quad \text{such that} \\ \displaystyle\int_\Omega p(Az + \varphi'(y)z)\, dx + \langle \gamma(p), \partial_{\nu_A} z \rangle \\ \displaystyle\qquad = \sum_{j=1}^n \left(\int_\Omega \partial_{x_j} z \, d\boldsymbol{\mu}_\Omega^j + \int_\Gamma \partial_{\mathbf{t}_j} z \, d\boldsymbol{\mu}_\Gamma^j \right), \qquad \forall z \in W^{2,t'}(\Omega), \end{cases}$$

are equivalent and have a unique solution.

Proof. To prove that (PN2) has a unique solution let us take the linear operator

$$\mathscr{A}: W^{2,t'}(\Omega) \to L^{t'}(\Omega) \times W^{1/t,t'}(\Gamma)$$

defined by

$$\mathscr{A}(z) = (Az + \varphi'(y)z, \partial_{\nu_A} z).$$

Once more, using the results of [12], we get that $\mathscr{A}$ is an isomorphism. Hence the adjoint operator

$$\mathscr{A}^*: L^t(\Omega) \times W^{-1/t,t}(\Gamma) \to (W^{2,t'}(\Omega))'$$

is an isomorphism too. Therefore there exists a unique element (p, q) in $L^t(\Omega) \times W^{-1/t,t}(\Gamma)$ satisfying

$$\int_\Omega p(Az + \varphi'(y)z)\, dx + \langle q, \partial_{\nu_A} z \rangle = \sum_{j=1}^n \left(\int_\Omega \partial_{x_j} z \, d\boldsymbol{\mu}_\Omega^j + \int_\Gamma \partial_{\mathbf{t}_j} z \, d\boldsymbol{\mu}_\Gamma^j \right) \tag{6.13}$$

for all $z \in W^{2,t'}(\Omega)$. Now taking $z \in D(\Omega)$ as arbitrary in the previous equality we get

$$A^* p + \varphi'(y)p = -\operatorname{div} \boldsymbol{\mu}_\Omega \qquad \text{in } \Omega, \tag{6.14}$$

which implies that p belongs to $V^t(\Omega)$. For every $z \in W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega)$, it follows from (4.4), (6.13), and (6.14) that

$$\langle \gamma(p), \partial_{\nu_A} z \rangle = \langle q, \partial_{\nu_A} z \rangle,$$

or, equivalently, $\gamma(p) = q$. This, together with (6.13), proves that p is the unique solution of (PN2).

It is sufficient to choose $z \in W^{2,t'}(\Omega)$, with $\partial_{\nu_A} z = 0$, in (6.11) and utilize (6.13) and Definition 1 to get that p verifies the boundary condition of (PN1). Therefore, remembering (6.14), we deduce that p is also a solution of (PN1).

Finally, if p is a solution of (PN1), it follows directly from (6.11) that p is a solution of (PN2). □

Corollary 4. *Let $\bar{\lambda} \geq 0$, $\bar{y} \in W^{2,t'}(\Omega)$, and $\boldsymbol{\mu} \in M(X)^n$ be as in Theorem 8, then there exists a unique $\bar{p}$ solution of (6.7) belonging to $V^t(\Omega)$ for every $t \in [1, n/(n-1))$.*

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