Acta Mechanica 93, 205–222 (1992)

ACTA MECHANICA

Alternative approach to shakedown as a solution of a min-max problem

S. Pycko and **Z. Mróz,** Warsaw, Poland

(Received May 21, 1991)

Summary. Melan's classical shakedown theorem for continuous media considered as a problem of mathematical programming with constraints is reformulated and reduced to a solution of a certain min-max problem. A similar approach is presented for the structural theory described in terms of generalized variables. A distinction is made between alternating plasticity and incremental collapse modes in the analysis of structures with nonsandwich cross-sections.

1 Introduction

In the last thirty years the classical shakedown theory, concerning elastic-plastic structures subjected to quasi static variable repeated loads has been well established [2], [7], [9], [11]. On the other hand, theoretical developments were not followed by a similar progress in numerical analysis. In recent years only few methods have been proposed, solving shakedown problems with nonlinear yield condition [10], [12], [14], [15].

In this contribution a new formulation of shakedown analysis is studied, which allows to investigate new numerical techniques for determination of the shakedown load multiplier. It transforms the classical problem based on the method of mathematical programming to the following one:

$$\lambda_{sh} = \min_{\varrho_{ij}} \max_{x,t} f\big(\sigma_{ij}^E(x, t) + \varrho_{ij}(x)\big)/k(x) \tag{1}$$

where

$k(x)$	plastic modulus,
$\mu_{sh} = 1/\lambda_{sh}$	shakedown load multiplier for the given load domain,
$f(.)$	arbitrary yield function of stress, homogeneous of degree one,
$\sigma_{ij}^E(x, t)$	elastic stress tensor for given time-variable load,
$\varrho_{ij}(x)$	residual stress resulting from plastic strain tensor.

It was Hill [1], who introduced a similar concept of extremal field, applicable in the limit analysis. Next Zwoliński and Bielawski [10] used this kind of formulation in a numerical procedure in order to obtain shakedown multiplier.

The objective of this paper is to provide a theoretical background of such an approach for both continuous media and structural theory described in terms of generalized variables.

2 Basic relations

Let the material be elastic-plastic and the total strain tensor ε_{ij} be additively decomposed within small strain theory into elastic ε_{ij}^e and plastic ε_{ij}^p parts

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p. \tag{2}$$

The elastic strain tensor is related to the stress tensor by Hooke's law via a positive definite elasticity tensor E_{ijkl}, with its usual symmetry properties

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl}^e. \tag{3}$$

The stress tensor is bounded in the stress space by a convex yield function homogeneous of degree one in stress

$$f(t\sigma_{ij}) = |t| f(-\sigma_{ij}), \quad \text{and} \quad f(\sigma_{ij}) = f(-\sigma_{ij}), \tag{4}$$

whereas the plastic strain rate tensor is given by the associated flow rule. Active and passive plastic loading processes are defined by

$$\dot{\varepsilon}_{ij} = \dot{\lambda}\frac{\delta f}{\delta \sigma_{ij}}, \quad f(\sigma_{ij}) \leqq k, \quad \dot{\lambda} \geqq 0, \quad \dot{\lambda}[f(\sigma_{ij}) - k] = 0, \quad \dot{\lambda}\dot{f} = 0. \tag{5}$$

The total stress tensor consists of elastic and residual portions

$$\sigma_{ij}(\mathbf{x}, t) = \sigma_{ij}^E(\mathbf{x}, t) + \varrho_{ij}(\mathbf{x}, t) \tag{6}$$

where the elastic stress tensor is obtained for purely elastic response of the structure subjected to external loads: body forces f_i and surface tractions t_i

$$\sigma_{ij,j}^E + f_i = 0 \quad \text{in } V, \tag{7.1}$$

$$\sigma_{ij}^E n_j = t_i \quad \text{on } S_T, \tag{7.2}$$

$$u_i^E = 0 \quad \text{on } S_U. \tag{7.3}$$

The residual stress tensor is in equilibrium with vanishing load

$$\begin{aligned} &\varrho_{ij,j} = 0 \quad \text{in } V, \\ &\varrho_{ij} n_j = 0 \quad \text{on } S_T, \end{aligned} \tag{8}$$

and results directly from the plastic strain tensor ε_{ij}^p

$$\begin{aligned} &\varrho_{ij} = E_{ijkl}(\varepsilon_{kl}^r - \varepsilon_{kl}^p), \\ &\varepsilon_{ij}^r = \frac{1}{2}(u_{i,j}^r + u_{j,i}^r), \\ &u_i^r = 0 \quad \text{on } S_U. \end{aligned} \tag{9}$$

Practical engineering problems usually involve loads depending on more than one parameter. Such a load may vary arbitrarily within some prescribed limits, which can be described by means

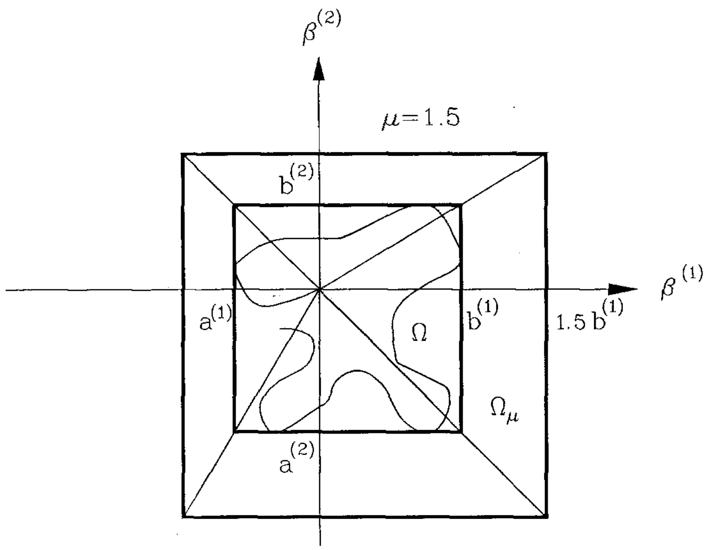

Fig. 1. Ranges of possible variation of loads f_i, t_i and f_i^μ, t_i^μ for $\mu = 1.5$

of a finite number of time-dependent load multipliers β

$$f_i = \sum_{l=1}^{r} \beta^{(l)}(t) f_i^{(l)}(\mathbf{x}), \qquad t_i = \sum_{l=1}^{r} \beta^{(l)}(t)\, t_i^{(l)}(\mathbf{x}),$$
$$a^{(l)} \leqq \beta^{(l)} \leqq b^{(l)}, \qquad l = 1, \ldots, r, \tag{10}$$

where $a^{(l)}$, $b^{(l)}$ are the specified lower and upper limits, r is the number of independent load systems.

The range of possible admissible variation of load multipliers is defined as a domain Ω in the r-dimensional space of parameters β. We can also consider the load, which is proportionally less or greater than the given one, Eq. (10)

$$f_i^\mu = \mu f_i, \qquad t_i^\mu = \mu t_i, \tag{11}$$

and for which the load multipliers are contained within load domain Ω_μ (Fig. 1). For this load the following static and kinematic shakedown theorems read:

Theorem 1: Melan's shakedown theorem
If for an elastic-plastic structure subjected to external agencies μf_i, and μt_i there exists a statically admissible time-independent residual stress field $\varrho_{ij}(\mathbf{x})$ satisfying Eq. (8) *such that for all possible load variations within prescribed limits, Eq.* (10), *the following condition holds:*

$$f\big(\mu\sigma_{ij}^E(\mathbf{x}, t) + \varrho_{ij}(\mathbf{x})\big) \leqq k(\mathbf{x}), \tag{12}$$

then the structure will shake down in any load program contained within these limits.

If we are interested in the maximal multiplier μ for which shakedown occurs it is necessary to solve the following optimization problem of mathematical programming:

$$\mu_{sh} = \max \mu,$$
$$f\big(\mu\sigma_{ij}^E(\mathbf{x}, t) + \varrho_{ij}(\mathbf{x})\big) \leqq k(\mathbf{x}),$$
$$\varrho_{ij,j} = 0 \quad \text{in } V, \tag{13}$$
$$\varrho_{ij} n_j = 0 \quad \text{on } S_T.$$

Depending on the form of the yield function, the problem can be regarded as a linear or nonlinear one.

Theorem 2: Koiter's shakedown theorem
The structure will shake down to the load μf_i, μt_i, if for all load paths $\beta_{(l)}(t)$ and for all kinematically admissible strain rate fields $\dot{\varepsilon}_{ij}^p$ (i.e. which result in compatible strain increments)

$$\triangle\bar{\varepsilon}_{ij}(\mathbf{x}) = \int_{t_1}^{t_2} \dot{\varepsilon}_{ij}^p(\mathbf{x}, t)\, dt = \frac{1}{2}\left(\bar{u}_{i,j}^r(\mathbf{x}) + \bar{u}_{j,i}^r(\mathbf{x})\right) \quad \text{in } V, \quad \bar{u}_i^r = 0 \quad \text{on } S_U, \tag{14}$$

the following inequality holds:

$$\int_{t_1}^{t_2} \int_V \mu\sigma_{ij}^E(\mathbf{x}, t)\, \dot{\varepsilon}_{ij}^p\, dV dt \leqq \int_{t_1}^{t_2} \int_V D(\dot{\varepsilon}_{ij}^p)\, dV dt. \tag{15}$$

Kinematical formulation of shakedown problems (Theorem 2) allows us to distinguish the following main phenomena of collapse:

(i) incremental collapse
(ii) alternating plasticity
(iii) mixed mode with both incremental and alternating plastic strain components.

Definition 1: *Incremental collapse occurs if the plastic strain increment components in each load cycle are of the same sign, and, after sufficient number of cycles, the total strains become so large that the structure departs markedly from its original form and becomes unserviceable.*

Definition 2: *Alternating plasticity occurs if the plastic strain increments change sign during each cycle, so that the total plastic increments after each cycle become equal to zero at every point in the structure.*

3 Alternative shakedown theorem

Let us assume a slightly different problem than the classical one, Eq. (13). Instead of enlarging the initial load domain, Eq. (10) proportionally to the multiplier μ, we keep it all time constant

$$\mu = 1 \tag{16}$$

and consider only the initial load domain described by f_i, t_i. We introduce an alternative multiplier λ which will be connected with changes in magnitude of the yield stresses. Considering this multiplier it will be possible to answer whether the shakedown takes place.

Definition 3: *For a given statically admissible time-independent residual stress field $\varrho_{ij}(\mathbf{x})$, the maximal multiplier λ obtained from the equality*

$$f\left(\sigma_{ij}^E(\mathbf{x}, t) + \varrho_{ij}(\mathbf{x})\right) = \lambda k(\mathbf{x}), \tag{17}$$

valid at every point $\mathbf{x} \in V$, is defined as the statical alternative multiplier λ_{st}^a

$$\lambda_{st}^a = \max_{x,t} \lambda(\mathbf{x}, t, \varrho_{ij}) = \max_{x,t} f\left(\sigma_{ij}^E(\mathbf{x}, t) + \varrho_{ij}(\mathbf{x})\right)/k(\mathbf{x}). \tag{18}$$

Theorem 3: Alternative static shakedown theorem
If from all statically admissible time-independent residual stress fields $\varrho_{ij}(\mathbf{x})$, we can find that one, which minimizes the alternative multiplier λ_{st}^a

$$\lambda_{st}^a = \min_{\varrho_{ij}(\mathbf{x})} \lambda_{st}^a(\varrho_{ij}(\mathbf{x})), \tag{19}$$

then the load obtained by multiplication of the initial load f_i, t_i by the inverse of alternative shakedown multiplier $(\lambda_{st}^a)^{-1}$ is the shakedown load

$$f_i^{sh} = \frac{f_i}{\lambda_{sh}^a}, \qquad t_i^{sh} = \frac{t_i}{\lambda_{sh}^a}. \tag{20}$$

Proof: Let shakedown alternative multiplier λ_{sh}^a correspond to the time-independent residual stress field $\bar{\varrho}_{ij}(\mathbf{x})$. According to Definition 3 the inequality

$$f(\sigma_{ij}^E(\mathbf{x}, t) + \bar{\varrho}_{ij}(\mathbf{x})) \leqq \lambda_{sh}^a k(\mathbf{x}), \tag{21}$$

holds at any point $\mathbf{x}$ of the structure and any time t.

Let us define a shakedown multiplier μ_{sh}

$$\mu_{sh} = 1/\lambda_{sh}^a. \tag{22}$$

Multiplying inequality (21) by μ_{sh} in view of homogeneity of degree one of the yield function, we arrive at the following relation:

$$f(\mu_{sh}\sigma_{ij}^E(\mathbf{x}, t) + \mu_{sh}\bar{\varrho}_{ij}(\mathbf{x})) \leqq k(\mathbf{x}). \tag{23}$$

Because the inequality is valid for the time-independent residual stress field $\mu_{sh}\bar{\varrho}_{ij}(\mathbf{x})$ at any time t and at any point $\mathbf{x} \in V$, then, according to the Melan's shakedown theorem, the structure will shake down, for a load being in equilibrium with elastic stresses

$$\begin{aligned} \bar{f}_i &= \mu_{sh} f_i, \qquad \mu_{sh}\sigma_{ij,j}^E(\mathbf{x}) + \bar{f}_i = 0, \\ \bar{t}_i &= \mu_{sh} t_i, \qquad \mu_{sh}\sigma_{ij}(\mathbf{x})^E\, n_j = \bar{t}_i. \end{aligned} \tag{24}$$

Similarly to the previous considerations, let us consider any other time-independent residual stress field $\hat{\varrho}_{ij}(\mathbf{x})$ and corresponding alternative static multiplier $\hat{\lambda}_{st}^a$. According to Eq. (19)

$$\lambda_{sh}^a \leqq \hat{\lambda}_{st}^a. \tag{25}$$

Denoting by $\hat{\mu}_{st} = 1/\hat{\lambda}_{st}^a$ we arrive at the inequality

$$f(\hat{\mu}_{st}\sigma_{ij}^E(\mathbf{x}, t) + \hat{\mu}_{st}\varrho_{ij}(\mathbf{x})) \leqq k(\mathbf{x}), \tag{26}$$

which is valid for any time $t > 0$. In this case shakedown takes place for loads $\hat{\mu}_{st} f_i$, $\hat{\mu}_{st} t_i$.

In view of Eqs. (25) and (22) the inequality

$$(\lambda_{sh}^a)^{-1} \geqq (\hat{\lambda}_{st}^a)^{-1} \tag{27}$$

is satisfied, so that the loads $(\lambda_{sh}^a)^{-1} f_i$, $(\lambda_{sh}^a)^{-1} t_i$ are the maximal loads for which shakedown occurs.

Remark. The residual stress field, which ensures the limit shakedown load $(\lambda^a_{sh})^{-1} f_i$, $(\lambda^a_{sh})^{-1} t_i$ is equal to

$$\varrho_{ij}(\mathbf{x}) = \frac{\bar{\varrho}_{ij}(\mathbf{x})}{\lambda^a_{sh}}. \tag{28}$$

Let us note that the relation (19) can be formulated in a more convenient form for numerical application:

$$\lambda^a_{sh} = \min_{\varrho_{ij}} \max_{x,t} f(\sigma^E_{ij}(\mathbf{x}, t) + \varrho_{ij}(\mathbf{x}))/k(\mathbf{x}). \tag{29}$$

A similar formulation (without any derivation) was used in [10] to derive a powerful numerical procedure allowing to solve shakedown problems with nonlinear yield condition.

4 Formulation in terms of generalized variables

4.1 Description

Let us introduce at the level of cross-section ξ generalized stresses Q_r, $r = 1, \ldots, n$ and generalized strains q_r [7], [9] in such a way that the virtual work principle employed for these generalized quantities is the same as in the case of continuous media

$$Q_r(\xi)\, q_r(\xi) = \int_\xi \sigma_{ij}(\mathbf{x})\, \varepsilon_{ij}(\mathbf{x})\, dx. \tag{30}$$

In order to pass from continuum to the structural theory described at the level of cross-section, it is usually assumed that the strain distribution within the given cross-section ξ is determined by means of a kinematic hypothesis

$$\varepsilon_{ij}(\mathbf{x}) = \bar{D}_{ij}(\mathbf{x}, q_r(\xi)) = D^r_{ij}(\mathbf{x})\, q_r(\xi). \tag{31}$$

In the classical shell and plate theory operator $\bar{D}_{ij}$ is usually linear according to the last equality in Eq. (31).

Taking into account Eqs. (31) and (30), we arrive at the definition of generalized stress

$$Q_r(\xi) = \int_\xi \sigma_{ij}(\mathbf{x})\, D^r_{ij}(\mathbf{x})\, dx = \phi_r(\sigma_{ij}), \tag{32}$$

where ϕ_r is also linear.

In the case of validity of Hooke's law, Eq. (3), the respective relation between the generalized stress and strain holds

$$Q_r = K_{rm} q_m, \tag{33}$$

where

$$K_{rm} = \int_\xi D^r_{ij}(\mathbf{x})\, E_{ijkl} D^m_{kl}(\mathbf{x})\, dx \tag{34}$$

is a symmetric positive definite stiffness matrix of the cross-section ξ.

From Hooke's law (3) and kinematic hypothesis (31) we obtain linear dependence between elastic stress σ^E_{ij} and generalized stress $Q_r{}^E$ defined at the level of cross-section ξ

$$\sigma^E_{ij}(\mathbf{x}) = h^r_{ij}(\mathbf{x})\, Q_r{}^E(\xi), \tag{35}$$

where

$$h^r_{ij} = E_{ijkl} D^m_{kl}(\mathbf{x})\, K^{-1}_{mr}(\xi). \tag{36}$$

In general, for elastic-plastic material there is no one-to-one relationship between field σ_{ij} and vector Q_r [9]. There exists some non-vanishing stress field $S_{ij}(\mathbf{x})$ called pseudoresidual (Fig. 2b) resulting in zero generalized stresses

$$Q_r = \phi_r(S_{ij}(\mathbf{x})) = 0. \tag{37}$$

In this way the total residual stress field in the structure can be divided into

$$\varrho_{ij}(\mathbf{x}) = \varrho_{ij}^{res}(\mathbf{x}) + S_{ij}(\mathbf{x}), \tag{38}$$

where

$$\phi_r(\varrho_{ij}^{res}) = Q_r^{res} \neq 0, \tag{39}$$

$$\phi_r(S_{ij}) = 0. \tag{40}$$

Then the total stress field in the body can be expressed as follows (Fig. 2b):

$$\sigma_{ij}(\mathbf{x}) = \sigma_{ij}^E + \varrho_{ij} = h_{ij}^r(\mathbf{x})\, Q_r(\xi) + S_{ij}(\mathbf{x}), \qquad Q_r = Q_r^E + Q_r^{res}, \tag{41}$$

where Q_r^E is the elastic generalized stress vector, Q_r^{res} denotes the difference between the total actual vector Q_r and the elastic vector Q_r^E, S_{ij} is the pseudoresidual stress field resulting in vanishing generalized stress vector, cf. Eqs. (39), (40). Let us introduce at the level of cross-section the limit locus in terms of the non-dimensional generalized stress (Fig. 2a)

$$F^L\left(\sum_r \frac{Q_r}{Q_{or}}\right) \leqq 0. \tag{42}$$

Let us note that F^L needs not to be a homogeneous function of Q_r/Q_{or} but follows from homogeneous yield function of stress of degree one. The equality sign in Eq. (42) corresponds to fully plastic state of the cross-section, and at every point $\mathbf{x} \in \xi$ the stress tensor is on the yield surface

$$f\left(\frac{\sigma_{ij}(\mathbf{x})}{k(\mathbf{x})}\right) - 1 = 0, \qquad \mathbf{x} \in \xi. \tag{43}$$

Similarly to the definition of the limit locus an elastic or initial locus is introduced in the space of Q_r/Q_{or} (Fig. 2a)

$$F^E\left(\sum_r \frac{Q_r}{Q_{or}}\right) \leqq 0 \tag{44}$$

which ensures purely elastic response of the virgin structure.

The equilibrium equations for a structural theory in generalized variables assume the form

$$L_k[Q_r(\xi, t)] + P_k(\xi, t) = 0, \qquad k = 1, 2, \ldots \tag{45}$$

where L_k denote linear differential operators, P_k are the external load integrals over the cross-section ξ, which vary in time according to Eq. (10),

$$P_k(\xi, t) = \sum_{l=1}^{r} \beta_{(l)}(t)\, P_k^{(l)}(\xi). \tag{46}$$

The shakedown analysis consists in determination of two load multipliers. One of them preserves the structure against alternating plasticity, and the other one against incremental

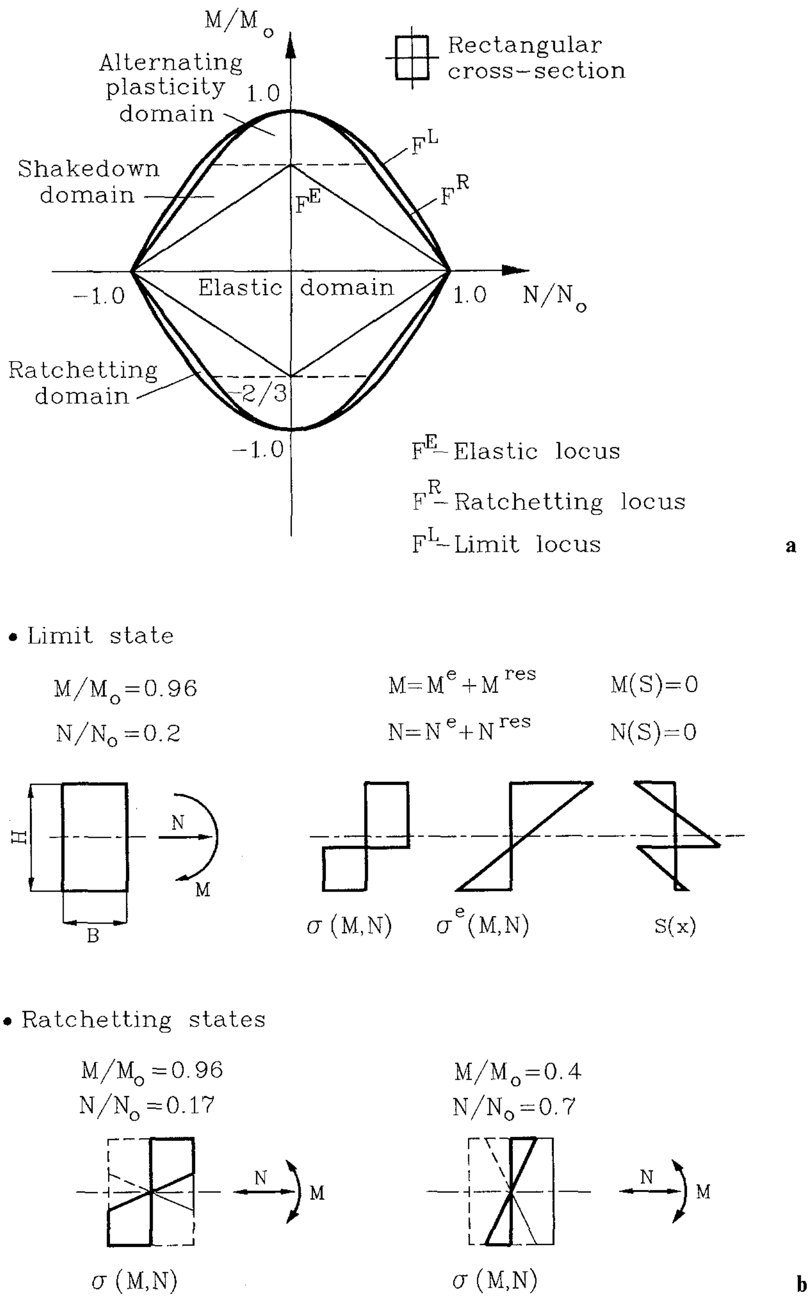

Fig. 2. a Limit, ratchetting and elastic loci for rectangular cross-section for plane beam **b** Distributions of statically admissible stress within cross-section for limit and ratchetting loci

collapse (ratchetting). The latter requires the ratchetting locus to be determined. First let us define a non-ratchetting domain $\mathscr{D}^{NR}$ in terms of generalized stresses. Namely

$$Q_r \in \mathscr{D}^{NR} \tag{47}$$

$$\underset{\mathbf{x}\in\xi}{\exists} \; \underset{Q_r\in\mathscr{D}^{NR}}{\forall} \; \underset{S_{ij}(\mathbf{x})}{\exists} \; f\big(h^r_{ij}(\mathbf{x})\, Q_r + S_{ij}(\mathbf{x})\big) < k, \qquad \mathbf{x} \in V_e. \tag{48}$$

In Eq. (48) we assume existence of an elastic fiber in the cross-section. In fact the ratchetting effect will not occur if there is an elastic domain V_e, no matter how small within the cross-sectional domain, for all stress states associated with the specified loading program. Thus a non-ratchetting domain corresponds to the elastic response for $\mathbf{x} \in V_e$, whereas the remaining cross-sectional portion undergoes plastic strains.

On the other hand the ratchetting locus (Fig. 2) is defined as

$$F^R = \partial \mathscr{D}^{NR}. \tag{49}$$

The specification of a ratchetting domain for a beam of rectangular cross-section subjected to cyclic bending moment and fixed axial force was discussed in [5], [8] for static and kinematic approach, respectively. However, it turns out that the ratchetting locus is very close to the limit locus. In the case of sandwich cross-section these loci coincide:

$$F^R = F^L. \tag{50}$$

For such a description in terms of generalized variables the separate safety criteria against inadaptation (nonshakedown) were derived:

Theorem 4:
A given structure subjected to variable loads $\mu P_k(\xi, t)$ is safe with respect to incremental collapse (Definition 1) if there exists a time-independent residual generalized stress vector $Q_r^{res}(\xi)$ such that for any instant t the following condition holds:

$$F\left(\sum_r \frac{\mu Q_r^E(\xi, t) + Q_r^{res}(\xi)}{Q_{or}}\right) \leqq 0. \tag{51}$$

If:

(i) *$F = F^R$ and $F^R \neq F^L$, the maximal multiplier μ satisfying Eq. (51) is a lower bound multiplier,*

(ii) *$F = F^L$ and $F^R \neq F^L$ (the case considered in [9]), the maximal multiplier μ is an upper bound multiplier,*

(iii) *$F^R = F^L$ (i.e. the case of sandwich cross-section) μ is an actual shakedown multiplier against incremental collapse (ratchetting).*

A similar theorem can be stated with respect to elastic shakedown or alternating plasticity [9].

Theorem 5:
A given structure subjected to variable loads $\mu P_k(\xi t)$ is safe with respect to the alternating plasticity (Definition 2) if, for every ξ, there exists such a pseudoresidual stress field $S_{ij}(\mathbf{x})$ for which the following is fulfilled:

$$f(\mu h_{ij}^r(\mathbf{x})\, Q_r^E(\xi, t) + S_{ij}(\mathbf{x})) \leqq k(\mathbf{x}) \tag{52}$$

for any time $t > 0$.

4.2 On accuracy of the generalized variables approach

Obviously, the generalized stress and strain formulation involves some approximations, namely

(i) imposing kinematical constraints (Eq. (31)), the pseudoresidual stress $S_{ij}(\mathbf{x})$ does not enter the equilibrium condition and constitutes an internal variable (similar to the back stress in the kinematic hardening rule). In the case when $F = F^L$, some forms of incremental collapse may be ruled out with respect to a continuum formulation [8].

In the case when $F = F^R$ the pseudoresidual stress $S_{ij}(\mathbf{x})$ has been taken into consideration at the level of cross-section. Let us recall that F^R was obtained for arbitrary variation $Q_r \in \mathscr{D}^{NR}$. But in the real structure a domain of variation of $Q_r(\xi, t) \in \mathscr{D}_\xi^{NR}$ depends on load variation and location of the cross-section ξ in the structure and may be much smaller than the considered one. Thus the ratchetting locus $F_\xi{}^R$ may be greater than F^R. For this reason the shakedown multiplier determined for $F = F^R$ may be a lower bound multiplier.

(ii) the static variables $Q_r{}^E$, $Q_r{}^o$ satisfy the equilibrium equations in the integral form (45). Hence the continuum equations (7.1), (7.2) and (8) are satisfied in an approximate manner.

Theorem 5 is fully derived from kinematical formulation for continuous media and directly transformed to the case of generalized variables. From this reason only the second error mentioned above may be meaningful.

4.3 Incremental collapse mode

Let us consider an alternative approach (Sect. 3) for the load multiplier $\mu = 1$ and external forces P_k (generalized variable formulation). It is assumed that the function F is expressed in terms of generalized variables, Eq. (51).

Definition 4: *For a given statically admissible time-independent generalized residual stress vector $Q_r{}^{res}(\xi)$, the maximal multiplier η obtained from the equality*

$$F\left(\sum_r \frac{Q_r{}^E(\xi, t) + Q_r{}^{res}(\xi)}{\eta Q_{or}(\xi)}\right) = 0, \tag{53}$$

is defined as the alternative multiplier η^a

$$\eta^a(Q_r{}^{res}) = \max_{\xi,t} \eta(\xi, t, Q_r{}^{res}). \tag{54}$$

Usually it is possible to solve Eq. (53) with respect to η

$$\eta(\xi, t, Q_r{}^{res}) = \bar{F}\left(\sum_r \frac{Q_r{}^E(\xi, t) + Q_r{}^{res}(\xi)}{Q_{or}(\xi)}\right), \tag{55}$$

and relation (54) can be rewritten as follows:

$$\eta^a(Q_r{}^{res}) = \max_{\xi,t} \bar{F}\left(\sum_r \frac{Q_r{}^E(\xi, t) + Q_r{}^{res}(\xi)}{Q_{or}(\xi)}\right). \tag{56}$$

For such a description it is possible to find the alternative shakedown multiplier η^{inc} with respect to the incremental collapse.

Theorem 6:

If from all statically admissible time-independent residual stress vectors $Q_r{}^{res}(\xi)$ we can find that one which minimizes the alternative multiplier η^a

$$\eta^{inc} = \min_{Q_r{}^{res}} \eta^a(Q_r{}^{res}) = \min_{Q_r{}^{res}} \max_{\xi,t} \bar{F}\left(\sum_r \frac{Q_r{}^E(\xi, t) + Q_r{}^{res}(\xi)}{Q_{or}}\right), \tag{57}$$

then the load obtained by multiplication of the initial load P_k by the inverse of the alternative shakedown multiplier $(\eta^{inc})^{-1}$ is the shakedown load with respect to the incremental collapse.

$$P_k^{inc} = \frac{P_k}{\eta^{inc}}. \tag{58}$$

Proof: Let the multiplier η^{inc} correspond to the time independent generalized residual stress vector $\bar{Q}_r^{res}(\xi)$. According to Definition 4 there is

$$\eta^{inc} = \bar{\eta}^a(Q_r^{res}) \geqq \eta(\xi, t, \bar{Q}^{res}), \tag{59}$$

and

$$\frac{1}{\bar{\eta}^a} < \frac{1}{\eta}. \tag{60}$$

It follows from Eq. (53) that for specified $\bar{Q}^{res}(\xi)$ the inequality

$$F\left(\sum_r \frac{1}{\bar{\eta}^a} \frac{Q_r^E(\xi, t) + \bar{Q}_r^{res}(\xi)}{Q_{or}(\xi)}\right) \leqq 0, \tag{61}$$

is satisfied for any ξ and $t > 0$.

According to Theorem 4, the structure shakes down with respect to the incremental collapse to the load

$$\bar{\mu}^{inc} P_k = \frac{1}{\bar{\eta}^a} P_k. \tag{62}$$

Assume another arbitrary residual vector $\hat{Q}_r^{res}(\xi)$ and the corresponding alternative multiplier $\hat{\eta}^a(\hat{Q}_r^{res})$.

Similarly to the previous considerations, the structure shakes down to the load

$$\hat{\mu} P_k = \frac{1}{\hat{\eta}^a} P_k. \tag{63}$$

In view of Eq. (57) there is

$$\eta^{inc} = \bar{\eta}^a \leqq \hat{\eta}^a, \tag{64}$$

and

$$\bar{\mu}^{inc} P_k \geqq \hat{\mu} P_k, \tag{65}$$

so the theorem is proved.

Let us notice that the proof of this theorem does not require any assumption concerning the homogeneity of the yield function F.

4.4 Alternating plasticity mode

Let us turn back to the continuum formulation and decompose the load multiplier $\beta_{(l)}(t)$ into the time independent and time dependent parts according to the relations (10):

$$\beta_{(l)}(t) = \beta_{(l)}^o + \beta_{(l)}^s(t), \quad \beta_{(l)}^o = \frac{a_{(l)} + b_{(l)}}{2}, \quad |\beta_{(l)}^s(t)| \leqq \frac{a_{(l)} - b_{(l)}}{2}. \tag{66}$$

Now the following theorem can be proved:

Theorem 7:
The constant load described by $\beta^o_{(l)}$ cannot affect the alternating plasticity condition, which is dependent only on the symmetrically varying loads specified by $\beta^s_{(l)}(t)$.

Proof: In fact, since for the alternating plasticity state, the total plastic strain increment over a cycle is equal to zero

$$\Delta\bar{\varepsilon}^p_{ij}(\mathbf{x}) = \int_{t_1}^{t_2} \dot{\varepsilon}_{ij}(\mathbf{x}, t)\, dt = 0, \qquad \mathbf{x} \in V, \tag{67}$$

for the constant multiplier $\beta^o_{(l)}$ the following integral vanishes:

$$\int_{t_1}^{t_2} \sum_{l=1}^{r} \beta^o_{(l)} \int_V \sigma^{E(l)}_{ij}(\mathbf{x})\, \dot{\varepsilon}^p_{ij}(\mathbf{x}, t)\, dt\, dV = \sum_{l=1}^{r} \beta^o_{(l)} \int_V \sigma^{E(l)}_{ij}(\mathbf{x}) \int_{t_1}^{t_2} \dot{\varepsilon}^p_{ij}(\mathbf{x}, t)\, dt\, dV = 0. \tag{68}$$

Thus, Koiter's inequality (15) is reduced to the form

$$\int_{t_1}^{t_2} \sum_{l=1}^{r} \beta_{(l)}(t) \int_V \sigma^{E(l)}_{ij}(\mathbf{x})\, \dot{\varepsilon}^p_{ij}(\mathbf{x}, t)\, dV dt$$

$$= \int_{t_1}^{t_2} \sum_{l=1}^{r} \beta^s_{(l)}(t) \int_V \sigma^{E(l)}_{ij}(\mathbf{x})\, \dot{\varepsilon}^p_{ij}(\mathbf{x}, t)\, dV dt \leqq \int_{t_1}^{t_2} \int_V D\big(\dot{\varepsilon}^p_{ij}(\mathbf{x}, t)\big)\, dV dt \tag{69}$$

according to the partition, Eq. (66), so the theorem is proved.

An alternative proof can be obtained from the partial shakedown theorem stated by Mróz [5]. It is assumed that the alternating plasticity response is associated with the plastic zones V_p and the remaining elastic portion V_e. If the static load is applied to the structure, so that the statically admissible stress state equilibrating this load superposed with the previous elastic one does not violate the yield condition beyond V_p and does not change the stress distribution within V_p, it can be deduced that the constant load does not affect the plastic behaviour due to the alternating plasticity.

Let us return to the generalized variable approach and consider only the symmetric loads specified by the symmetrically varying multipliers $\beta^s_{(l)}(t)$ (the constant load specified by $\beta^o_{(l)}$ does not affect the alternating plasticity condition)

$$\mu P_k{}^s(\xi, t) = \mu \sum_{l=1}^{r} \beta^s_{(l)}(t)\, P_k{}^{(l)}(\xi). \tag{70}$$

For this load there exist such instants of time $\hat{t}_1, \hat{t}_2$ for which at every point ξ the stress tensors are located on the opposite sides of the symmetric yield function

$$f\big(\mu h^r_{ij} Q_r{}^{Es}(\xi, \hat{t}_1)\big) = f\big(-\mu h^r_{ij} Q_r{}^{Es}(\xi, \hat{t}_2)\big), \qquad Q_r{}^{Es}(\xi, \hat{t}_1) = -Q_r{}^{Es}(\xi, \hat{t}_2) \tag{71}$$

where $Q_r{}^{Es}$ defines generalized stress obtained for a purely elastic structure subjected to symmetrical loads $P_k{}^s$. Let us notice that for this case the pseudoresidual stress field $S_{ij}(\mathbf{x})$ which ensures the highest multiplier μ, cf. Eq. (52), is equal to zero and Theorem 5 is reduced to the following one:

Theorem 8:
The shakedown multiplier μ^{alt} with respect to the alternating plasticity is obtained from the following optimization problem:

$$\mu^{alt} = \max_{\mathbf{x},t} \mu,$$

$$f\big(\mu h^r_{ij}(\mathbf{x})^r\, Q_r{}^{Es}(\xi, t)\big) \leqq k. \tag{72}$$

Let us notice that the problem (52) using the pseudoresidual stress field $S_{ij}(\mathbf{x})$ has been completely reduced to the pure elastic problem (72). Theorem 8 states that the load domain multiplier μ, for which first yielding occurs at any point $\mathbf{x}$ in the structure, is also the shakedown multiplier with respect to the alternating plasticity. For typical shell and plate structures and linear kinematical hypothesis (31) it is sufficient to check the external fibers of the cross-section ξ instead of each point $\mathbf{x} \in \xi$. In this way the problem is reduced to the elastic problem considered only at the external fibers of the cross-sections.

Introducing η^E into elastic locus (44) similarly as in the case of limit locus

$$F^E\left(\sum_r \frac{Q_r^E}{\eta^E Q_{or}}\right) = 0, \qquad \eta^E = \bar{F}^E\left(\sum_r \frac{Q_r^{Es}}{Q_{or}}\right), \tag{73}$$

the optimization problem (72) is equivalent to the following one:

$$1/\mu^{alt} = \eta^{alt} = \max_{\xi,t} \bar{F}^E\left(\sum_r \frac{Q_r^{Es}(\xi, t)}{Q_{or}}\right). \tag{74}$$

4.5 Shakedown multiplier

To determine the shakedown multiplier within generalized variable approach it is necessary:

(i) to find the shakedown alternative multiplier η^{inc} with respect to the incremental collapse (elasto-plastic analysis, Eq. (57))

$$\eta^{inc} = \min_{Q_r^{res}} \eta^a(Q_r^{res}) = \min_{Q_r^{res}} \max_{\xi,t} \bar{F}\left(\sum_r \frac{Q_r^E(\xi, t) + Q_r^{res}(\xi)}{Q_{or}}\right), \qquad \mu^{inc} = (\eta^{inc})^{-1} \tag{75}$$

(ii) to determine the alternative shakedown multiplier η^{alt} with respect to the alternating plasticity (elastic analysis, Eq. (74))

$$\eta^{alt} = \max_{\xi,t} \bar{F}^E\left(\sum_r \frac{Q_r^{Es}(\xi, t)}{Q_{or}}\right), \qquad \mu^{alt} = (\eta^{alt})^{-1} \tag{76}$$

(iii) to find the maximal alternative multiplier which will be responsible for the collapse of the structure

$$\eta^{sh} = \max(\eta^{inc}, \eta^{alt}) \tag{77}$$

and to determine the shakedown loads as

$$P_k^{sh} = \mu^{sh} P_k = P_k/\eta^{sh}, \qquad \mu^{sh} = 1/\eta^{sh}. \tag{78}$$

Possible inaccuracies are associated with the generalized stress formulation (cf. Sect. 4.2).

5 Examples

Example 1. Determination of alternative multiplier η with respect to incremental collapse for rectangular cross-section

Let us consider the cross-section of a plane rectangular beam with bending moment M and axial force N as the generalized stress variable, and yield function F^L written in the form

$$F^L\left(\frac{N}{N_o}, \frac{M}{M_o}\right) = \left(\frac{N}{N_o}\right)^2 + \left|\frac{M}{M_o}\right| - 1 = 0 \tag{79}$$

where N_o, M_o define yielding axial force and yielding bending moment, respectively. Let us remember (Theorem 4) that for $F = F^L$ we may obtain an upper bound of shakedown load.

Using the definition of the alternative multiplier η with respect to the incremental collapse

$$\left(\frac{N}{\eta N_o}\right)^2 + \left|\frac{M}{\eta M_o}\right| - 1 = 0 \tag{80}$$

and solving this equation with respect to η we obtain

$$\eta = \frac{1}{2}\left(\left|\frac{M}{M_o}\right| + \sqrt{\left(\frac{M}{M_o}\right)^2 + 4\left(\frac{N}{N_o}\right)^2}\right) \quad \text{for} \quad \eta > 0. \tag{81}$$

Then in the case of a plane frame with rectangular cross-section of the beam element, the min-max problem (57) can be formulated as

$$\eta^{inc} = \min_{\theta} \max_{\xi,t} \frac{1}{2}\left(\left|\frac{M^E(\xi,t) + M^{res}(\xi,\theta)}{M_o}\right| + \sqrt{\left(\frac{M^E(\xi,t) + M^{res}(\xi,\theta)}{M_o}\right)^2 + 4\left(\frac{N^E(\xi,t) + N^{res}(\xi,\theta)}{N_o}\right)^2}\right) \tag{82}$$

where

$M^E, M^{res}, N^E, N^{res}$	elastic and residual bending moments and axial forces, respectively,
ξ	specified cross-section,
θ	free parameters (i.e. plastic rotation and elongation in nodes of beam elements).

Example 2. Plane frame under two cyclic loads

Let us consider a plane frame with a rectangular cross-section (Fig. 3a), subjected to two cyclically varying loads P_1 and P_2 according to the program given in Table 1.

Let us first determine the shakedown alternative multiplier η^{inc} with respect to incremental collapse. Our aim is to solve the min-max problem formulated in Example 1, Eq. (82).

Table 1. Loading program

Loading program	Loading	
	P_1	P_2
I	P	0
II	P	P
III	0	P
IV	0	0

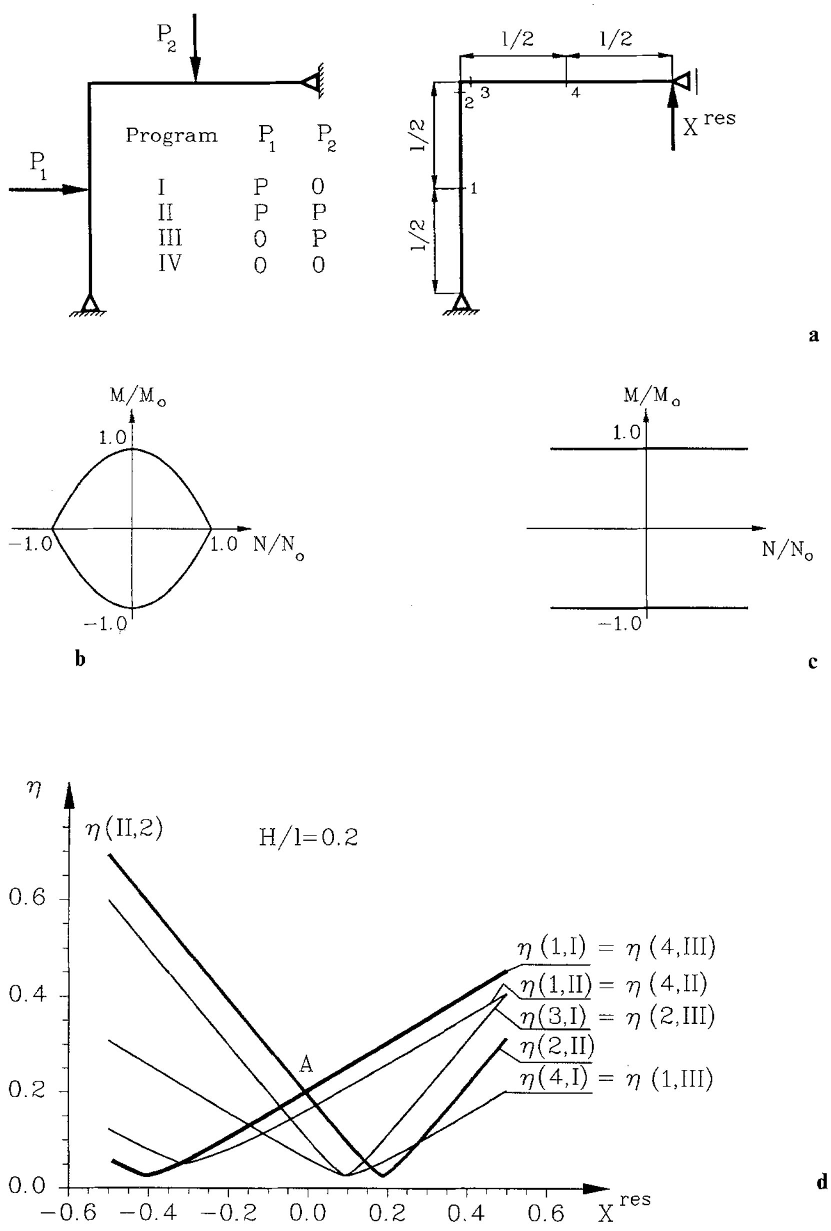

Fig. 3. a Plane frame under cyclic loading **b** Yield locus for rectangular cross-section **c** Yield locus for sandwich cross-section **d** Relation of alternative multiplier against incremental collapse η and residual force X^{res}

To this end, let us analyze the behaviour of η specified by the formula

$$\eta^{inc}(\xi_i, t_j, X^{res}) = \frac{1}{2}\left(\left|\frac{M^E(\xi_i, t_j) + M^{res}(\xi_i, X^{res})}{M_o}\right| + \sqrt{\left(\frac{M^E(\xi_i, t_j) + M^{res}(\xi_i, X^{res})}{M_o}\right)^2 + 4\left(\frac{N^E(\xi_i, t_j) + N^{res}(\xi_i, X^{res})}{N_o}\right)^2}\right) \quad (83)$$

Table 2. Elastic and residual bending moments and axial forces

Loading program	M^E/Pl at node			N^E/P at node	
	1	2, 3	4	1, 2	3, 4
I	13/64	−3/32	−3/64	−3/32	−19/32
II	10/64	−6/32	10/64	−22/32	−22/32
III	−3/64	−3/32	13/64	−19/32	−3/32
	M^{res}			N^{res}	
	$0.5X^{res}l$	$X^{res}l$	$0.5X^{res}l$	X^{res}	X^{res}

with respect to the vertical redundant force X^{res} (Fig. 3 d). Each curve η is determined for one set (ξ_i, t_j), where ξ_i denotes the nodal points in the structure $\xi_i = 1, 2, 3, 4$ and t_j denotes the time instant corresponding to one of the vertices I, II, III, IV of load polyhedron. According to the theorem stated in [6] for shakedown analysis it is sufficient to consider only the vertices of load polyhedron I, II, III, IV instead of any load variation in the interior of this polyhedron.

The respective elastic generalized forces for any load vertex different from origin and residual states are listed in Table 2.

A solution of the min-max problem is illustrated by a point A in Fig. 3 d and is obtained as an intersection of curves $\eta(2, II, X^{res})$ and $\eta(1, I, X^{res}) \equiv \eta(4, III, X^{res})$ plotted in the $\eta - X^{res}$ plane.

Taking into account the yield moduli for rectangular cross-section

$$M_o = \sigma_o H^2/4, \quad N_o = \sigma_o H, \quad M_o/N_o = H/4 \tag{84}$$

we arrive for the ratio $H/l = 0.2$ at the following results:

$$X^{res} = -0.006\,487, \quad \eta^{inc} = 0.200\,007 Pl/M_o, \quad \mu^{inc} P = \frac{P}{\eta^{inc}} = 4.999\,8 M_o/l. \tag{85}$$

For comparison, for the sandwich cross-section and the yield locus independent of axial forces (Fig. 3 c) (case equivalent to $H/l = 0$) we obtain

$$X^{res} = -0.010\,417, \quad \eta^{inc} = 0.197\,917 Pl/M_o, \quad \mu^{inc} P = \frac{P}{\eta^{inc}} = 5.052\,6 M_o/l. \tag{86}$$

It is seen that the difference between Eqs. (85) and (86) is very small.

To obtain the maximal multiplier with respect to alternating plasticity it is necessary to take into account the symmetric load, Eq. (70), and consider the optimization problem (76). The equivalent symmetric load, Eq. (70), and resulting elastic states are listed in Tables 3 and 4.

Let us consider an elastic locus for rectangular cross-section (Fig. 2 a) and the multiplier η^E in the following form (cf. Eq. (73)):

$$F^E = \left|\frac{N}{N_o}\right| + \frac{3}{2}\left|\frac{M}{M_o}\right| - 1 = 0, \quad \eta^E = \frac{Pl}{M_o}\left(\left|\frac{N}{P}\right|\frac{H}{l} + \frac{3}{2}\left|\frac{M}{Pl}\right|\right). \tag{87}$$

Noting that the optimal solution of problem (76) exists for the following sets of load vertices and nodes

(1, I), (1, III), (4, I), (4, III)

Table 3. Loading program for symmetric loads

Load program	Loading	
	P_1	P_2
I	P/2	−P/2
II	P/2	P/2
III	−P/2	P/2
IV	−P/2	−P/2

we arrive for $H/l = 0.2$ at the value of the alternative multiplier η^{alt}, with respect to alternating plasticity

$$\eta^{alt} = \frac{Pl}{M_o}\left(\left|\frac{N}{P}\right|\frac{H}{l} + \frac{3}{2}\left|\frac{M}{Pl}\right|\right)_{(\xi_i=1, t_j=I)} = 0.2375\,\frac{Pl}{M_o}. \tag{88}$$

According to Eq. (77), the shakedown multiplier for the plane structure with rectangular cross-section is determined for the alternating plasticity mode of collapse and is equal to

$$\mu^{sh}P = \frac{P}{\eta^{alt}} = 4.21052M_o/l. \tag{89}$$

Let us compare this result with the respective one obtained for sandwich cross-section and yield locus independent of axial force (Fig. 3c). The shakedown multiplier is determined for incremental collapse mode and is equal to $\mu^{sh}P = 5.0526M_o/l$. It is seen that in the case of rectangular cross-section the shakedown multiplier (Eq. (89)) results from a quite different mode of collapse (alternating plasticity) and is about 16% smaller than that for sandwich cross-section.

Table 4. Elastic bending moments and axial forces

Load program	M^E/Pl at node			N^E/P at node	
	1	2, 3	4	1, 2	3, 4
I	8/64	0	−8/64	8/32	−8/32
II	5/64	−3/32	5/64	−11/32	−11/32
III	−8/64	0	8/64	−8/32	8/32
IV	−5/64	3/32	−5/64	11/32	11/32

6 Concluding remarks

The present paper provides an alternative approach to shakedown analysis by formulating it as the min-max problem. Such an approach may constitute a foundation for efficient numerical algorithms for an arbitrary nonlinear yield function, which otherwise would require a nonlinear programming approach with constraints. The first numerical solutions within this formulation indicate good efficiency of such procedures (cf. [10], [16]).

The generalized variable approach was considered by accounting for two mechanisms, namely incremental collapse (ratchetting) and alternating plasticity. The second case was reduced to a certain elastic optimization problem for a structure under symmetric portion of external loading with respect to the mean load value. In this way, the alternating plasticity mode is only a part of a more general solution with respect to the incremental collapse. This formulation, therefore, provides a uniform treatment of beam and surface structures with arbitrary cross-sectional properties.

References

[1] Hill, R.: The extremal stress-field concept. J. Mech. Phys. Solids **6**, 239–243 (1966).
[2] Koiter, W. T.: General theorems for elastic plastic solids. In: Progress in solid mechanics (Sneddon, I. N., Hill, R., eds.), pp. 165–221. Amsterdam: North-Holland 1960.
[3] Sawczuk, A.: Evaluation of upper bounds to shakedown loads of shells. J. Mech. Phys. Solids **17**, 291–301 (1969).
[4] Maier, G.: A matrix structural theory of piecewise-linear plasticity with interacting yield planes. Meccanica **7**, 51–66 (1970).
[5] Mróz, Z.: On the theory of steady plastic cycles in structures. Proc. of SMiRT 1, 6 part L, pp. 489–501, 1972.
[6] König, J. A.: A new method of shakedown structural analysis. Symp. Franco-Polonais, Problémes Non-Lineares de Mécanique, pp. 343–347. Kraków 1977.
[7] Gokhfeld, D. A., Cherniavsky, O. F.: Limit analysis of structures at thermal cycling. Sijthoff & Noordhoff 1980.
[8] König, J. A.: On exactness of the kinematical approach in the structural shakedown and limit analysis. Ing.-Arch. **52**, 318–326 (1982).
[9] König, J. A.: Shakedown of elastic-plastic structures. Warsaw: Elsevier & PWN-Polish Scientific Publishers 1987.
[10] Zwoliński, J., Bielawski, G.: An optimal selection of residual stress for determination limit and shakedown multiplier. Proc. Conf. Comp. Meth. Struct. Mech., pp. 459–462, Jadwisin 1987.
[11] Zarka, J., Frelat, J., Inglebert, G., Kasmai-Navidi, P.: A new approach to inelastic analysis of structures. Dordrecht: Martinus Nijhoff 1988.
[12] Genna, F.: A nonlinear inequality, finite element approach to the direct computation of shakedown load safety factors. Int. J. Mech. Sci. **30**, 769–789 (1988).
[13] Pycko, S.: Alternative solution of shakedown problem with nonlinear yield condition. Proc. 28th Polish Mechanics Conf., Kozubnik 1990.
[14] Stein, E., Zhang, G., Mahnkem, R., König, J. A.: Micromechanical modelling and computation of shakedown with nonlinear kinematic hardening including examples for 2-d problems. Proc. CSME Mechanical Engineering Forum, Toronto 1990.
[15] Orkisz, J., Orringer, O., Holowinski, M., Pazdanowski, M., Cecot, W.: Discrete analysis of actual residual stress resulting from cyclic loadings. Comp. Struct. **35**, 397–412 (1990).
[16] Pycko, S., König, J. A.: Determination of shakedown load as a solution of min-max problem. Proc. GAMM Conf., Kraków 1991.

Authors' address: S. Pycko and Z. Mróz, Polish Academy of Sciences, Institute of Fundamental Technological Research, ul. Świętokrzyska 21, 00-049 Warsaw, Poland