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Martin's Axiom Implies the Existence of Certain Slender Groups*

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§ 1. Introduction and Some Basic Definitions

a) On Growth Types

Let $\mathbf{P}=\mathbb{Z}^{\omega}$ be the *Baer-Specker group*, that is the set of all functions $f=\{f_n\}_{n\in\omega}=\{f_n\}=\{f(n)\}$ on the first infinite ordinal ω [represented by the non-negative integers] with values taken in the group $\mathbb{Z}$ of integers and addition defined co-ordinate wise. This group $\mathbf{P}$ was investigated first in a remarkable paper of Baer [1] in 1937. Some year later, in 1950, Specker [13] discovered some further interesting properties of $\mathbf{P}$, basically the idea of "*slenderness*". If $e_i\in\mathbf{P}$ is the function $e_i=\{\delta_{in}\}$ on ω defined by the Kronecker symbol δ_{in} for $i\in\omega$, it is quite common and sometimes convenient to write elements $f=\{f_n\}$ of $\mathbf{P}$ as infinite sums $\sum_{n=0}^{\infty} f_n e_n$; cf. Fuchs [5; p. 159].

If $\mathfrak{M}$ is the subset of $\mathbf{P}$ of all those sequences $\{f_n\}$ which are positive and monotonic, i.e. $1\leqq f_n\leqq f_{n+1}$ for all $n\in\omega$, Specker [13; p. 132] defines a *growth type* $\mathfrak{T}$ by the following two conditions:

(i) $\mathfrak{T}$ is a non-empty subset of $\mathfrak{M}$ which is minorant-closed, i.e. if $\{f_n\}\in\mathfrak{M}$, $\{t_n\}\in\mathfrak{T}$ and $f_n\leqq t_n$ for all $n\in\omega$, then $\{f_n\}\in\mathfrak{T}$.

(ii) $\mathfrak{T}$ is closed under sums, i.e. if $s,t\in\mathfrak{T}$, then $s+t\in\mathfrak{T}$.

In particular, $\mathfrak{M}$ and its subset $\mathfrak{B}$ of all bounded sequences are (the largest and the smallest respectively) growth types. This definition can be given in terms of a natural quasi-ordering on $\mathfrak{M}$.

($\mathfrak{M}$, $\leqq$) *Let be* $\{f_n\}\leqq\{g_n\}$ *if and only if there is a* $k\in\omega$ *such that* $f_n\leqq k\cdot g_n$ *for all* $n\in\omega$.

We shall distinguish between the following quite common order relations in this paper: A relation is a *quasi-ordering* if it is reflexive and transitive. The relation

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" $<$ " is a *partial ordering* if it is transitive and if the underlying set contains no elements x which satisfy $x<x$. Relations which are reflective, transitive and anti-symmetric are called *orderings*.

We can now redefine growth types omitting the algebraic notion of sums as:

Definition 1.1. *Then $\mathfrak{T} \neq \emptyset$ is a growth type if*

(i*) *$\mathfrak{T}$ is a cut of $\mathfrak{M}$, i.e. if $f \in \mathfrak{M}$, $t \in \mathfrak{T}$ and $f \leqq t$ then $f \in \mathfrak{T}$*

(ii*) *$\mathfrak{T}$ is directed (upwards) in $\mathfrak{M}$, i.e. if $t, u \in \mathfrak{T}$ there is a $v \in \mathfrak{T}$ such that $t \leqq v$ and $u \leqq v$.*

The growth type $\mathfrak{T}$ will be called Specker growth type if $\mathfrak{T}$ is bounded in $(\mathfrak{M}, \leqq)$, i.e. There is an upperbound $b \in \mathfrak{M}$ such that $t \leqq b$ for all $t \in \mathfrak{T}$.

Hence growth types are the ideals of $(\mathfrak{M}, \leqq)$ and Specker growth types are the bounded ideals in $(\mathfrak{M}, \leqq)$. Specker [13; p. 137, Satz V] constructed $2^{2^{\aleph_0}}$ different growth types which lead to $2^{2^{\aleph_0}}$ non-isomorphic monotone subgroups of $\mathbf{P}$; cf. part *b*. All such growth types $\neq \mathfrak{M}$ are bounded by $\{n^{n+1}\}$ and are therefore Specker growth types according to our definition. All Specker growth types are equivalent under the following equivalence relation " $\sim$ " between growth types. This equivalence relation can be motivated algebraically and it will allow us to distinguish between different types of slender groups; cf. part b. It is defined as follows.

Definition 1.2. *Denote by $\mathfrak{M}_s$ the subset of all strictly monotone sequences of $\mathfrak{M}$, i.e. all $\{f_n\} \in \mathfrak{M}$ with $f_n \neq f_{n+1}$ for all $n \in \omega$. Then $r \in \mathfrak{M}_s$ induces a monotone-stretching monomorphism* $r^{\#}\colon \mathbf{P} \to \mathbf{P}$ $\left(\sum_{n\in\omega} f_n e_n \to \sum_{n\in\omega} f_n \cdot \sum_{i=r_n}^{r_{n+1}-1} e_i\right)$ [*$=m-s$-monomorphism*]; cf. Göbel and Wald [7; p. 216]

Definition 1.3. *If $\mathfrak{T}$, $\mathfrak{U}$ are growth types, let $\mathfrak{T} \prec \mathfrak{U}$ if there is an $r \in \mathfrak{M}_s$ with $r^{\#}(\mathfrak{T}) \leqq \mathfrak{U}$ and let $\mathfrak{T} \sim \mathfrak{U}$ if $\mathfrak{T} \prec \mathfrak{U}$ and $\mathfrak{U} \prec \mathfrak{T}$. Then we denote by $\mathbb{P} = (\mathbb{P}, \prec)$ the set of all $\sim$-classes of growth types with the induced ordering $\prec$.*

Then all Specker growth types $\neq \mathfrak{B}$ are equivalent and thus represented by one element, e.g. the set $\mathfrak{L}$ of all monotone sequences which increase at most linearly. The single sets $\mathfrak{M}$ and $\mathfrak{B}$ form equivalence classes by themselves. Hence $3 \leqq |\mathbb{P}| \leqq 2^{2^{\aleph_0}}$. Because of a $1-1$ correspondence between the set $\mathbb{P}$ and different classes of slender groups we shall determine the cardinality of $\mathbb{P}$; c.f. Corollary 1.7 and Göbel and Wald [7; p. 203, *conjecture*]. In sections 2 through 4 we shall be working under ZFC together with *Martin's axiom* (**MA**) to prove our main result, namely

Theorem 1.4. *Martin's axiom implies* $|\mathbb{P}| = 2^{2^{\aleph_0}}$.

We refer the reader to Jech [10; pp. 229ff.] and § 2 for a discussion of Martin's axiom. **MA** is a trivial consequence of the continuum hypothesis (**CH**) and the opposite is not true; cf. Jech [10; p. 140, Lemma 16.1 and p. 232, Theorem 51]. Therefore our assumptions are relatively consistent with **ZFC**; cf. Jech [10; pp. 108ff]. The fact that $|\mathbb{P}| \geqq 4$ in **ZFC+CH** has already been shown by Wald [Ph-D-thesis, Essen 1979]. The methods applied here are refinements of ideas in this thesis. *We will show, that* $|\mathbb{P}|\geqq 4$ *is generally true using* **ZFC** *only.* In order to prove this, we will transform the problem $|\mathbb{P}|\geqq 4$ into an equivalent form without algebraic conditions:

($\mathfrak{M}, \sqsubset$) *Define* $\{f_n\}\sqsubset\{g_n\}$ *to mean that there is a* $k\in\omega$ *with* $f_n<g_n$ *for all* $n\in\omega$ *and* $n\geqq k$.

This order goes back to Hausdorff [8] and was recently investigated by Hechler [9] (denoted by $\prec$) whose motivation was purely set theoretical. (I.e. his theorems of the existence of certain scales.)

Then $\mathfrak{T}\leqq\mathfrak{M}$ is a $\sqsubset$-*growth type* (or $\sqsubset$-ideal) if $\mathfrak{T}$ is a cut and directed with respect to $\sqsubset$. Now our claim $|\mathbb{P}|\geqq 4$ can be formulated using the Hausdorff ordering $\sqsubset$ only. In addition it can be expressed in terms of subsets of ω. The following statements (a), (b) or (c) are equivalent with the opposite case $|\mathbb{P}|\leqq 3$:

Proposition. *The following three statements are equivalent:*

(a) $\mathfrak{M}$ *is the only unbounded* $\leqq$-*growth type in* $\mathfrak{M}$.

(b) $\mathfrak{M}$ *is the only unbounded* $\sqsubset$-*growth type in* $\mathfrak{M}$.

(c) *For any family* $\mathfrak{X}$ *of subsets of* ω *which satisfy the property*

(*) *intersections of finitly many members of* $\mathfrak{X}$ *are infinite*

there is a decomposition of ω *into finite non-empty subsets* A_n ($n\in\omega$) *such that* $X\cap A_n\neq\varnothing$ *for all* $X\in\mathfrak{X}$ *and almost all* $n\in\omega$.

The proof is given in (5.1). The last condition (*) shows, that ultrafilter will come into play. Therefore in (5.2) for an ultrafilter $\mathfrak{X}$ it will be shown, that (c) is not valid, which implies $|\mathbb{P}|\geqq 4$.

b) Connection Between Growth Types and Slender Groups

According to Specker [13; p. 132] a growth type $\mathfrak{T}$ can be associated with a subgroup $[\mathfrak{T}]$ of **P** which is called *monotone subgroup* after Fuchs [3; p. 51] and [5; p. 166, exercise 4]:

$$\{f_n\}\in[\mathfrak{T}] \quad \text{if and only if} \quad \left\{\max_{i=0}^{n}(1,|f_i|)\right\}\in\mathfrak{T}.$$

Obviously we get $\mathbf{P}=[\mathfrak{M}]$ and $\mathbf{B}=[\mathfrak{B}]$ is the set of all bounded sequences of **P**. Conversely a growth type $\mathfrak{T}$ satisfies $\mathfrak{T}=[\mathfrak{T}]\cap\mathfrak{M}$ and is determined by its monotone subgroup $[\mathfrak{T}]$. The group **B** is the *only* monotone subgroup which is free, as shown by Specker [13; p. 134, Satz II, p. 138, Satz VI with **CH**] and Nöbeling [11; without **CH**]. This result was generalized to ring theory; cf. Bergmann [2]. Because our results hold trivially for the monotone group **B**, *we shall exclude* **B** *from the class of monotone groups in the following discussion.*

For the special monotone group **P** the theory of slender groups was developed by Specker, Łoś, Sasiada, Fuchs and Nunke; cf. Fuchs [5; §94, 95]. Since monotone subgroups have many properties in common, it is natural to develop "slenderness" simultaneously for all monotone subgroups. This was carried out in Göbel and Wald [7]: Any monotone subgroup **U** of **P** contains the elements $e_i = \{\delta_{in}\}$ by definition. Hence we generalize slenderness in the sense of Łoś:

Definition 1.5 [7]. *A group G is called* **U**-*slender if any homomorphism σ:* $\mathbf{U} \to G$ *maps almost all e_i onto* 0.

Hence **P**-slender equals slender by definition. In addition G will be called **U**-*stout* if any homomorphism σ: $\mathbf{U} \to G$ is 0 if σ maps all e_i onto 0; cf. Göbel [6; p. 49, Theorem 4.1]. From our results in [7; p. 10, Satz 4.6] it follows that there is only one class of stout groups, i.e. **U**-*stout coincides with* **V**-*stout for all monotone subgroups* **U** *and* **V**. In Fuchs [3; p. 52, Theorem 2] it is shown that when the cardinality of the groups under discussion is restricted to be less than $2^{\aleph_0}$, there is only one class of slender groups. In general, we have the following

Theorem 1.6 [7; p. 17, Satz 5.5]: *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two growth types. Then the following three statements are equivalent:*

(1) $\mathfrak{X} \prec \mathfrak{Y}$ *in the sense of* (1.3)

(2) $[\mathfrak{X}]$-*slender groups are* $[\mathfrak{Y}]$-*slender.*

(3) $[\mathfrak{Y}]$ *is not* $[\mathfrak{X}]$-*slender.*

There is an immediate

Corollary 1.7. (a) $[\mathfrak{X}]$-*slender* $=$ $[\mathfrak{Y}]$-*slender if and only if* $\mathfrak{X} \sim \mathfrak{Y}$.

(b) *The set of all classes of slender groups defined by monotone subgroups of* **P** *is order isomorphic with* ($\mathbb{P}, \prec$) *from* (1.3).

(c) *All Specker growth types* ($\neq \mathfrak{B}$) *define the same class of slender groups;* cf. [7; p. 20, Satz 5.7(a)].

Combining (1.7) and our main result (1.4) of this paper, we obtain an answer to the question [7] about the number of different classes of slender groups:

Corollary 1.8. **MA** *implies the existence of precisely* $2^{2^{\aleph_0}}$ *different classes of slender groups defined by monotone subgroups of* **P**.

§ 2. Specker Growth Types and the Hausdorff Ordering $\sqsubset$

There is an obvious way to enlarge subsets of the set $\mathfrak{M}$ of all positive and monotone sequences to obtain growth types, following the

Definition 2.1. *If $X \leq \mathfrak{M}$, let $\bar{X}$ be the intersection of all growth types of $\mathfrak{M}$ containing X.*

The set $\bar{X}$ can also be described via elements. This will be accomplished by means of the following.

Definition 2.2. *If $x, y \in \mathfrak{M}$, let be $x \vee y \in \mathfrak{M}$ the point wise maximum, i.e.* $x \vee y = \{\max(x_n, y_n)\}$ *where* $x = \{x_n\}$ *and* $y = \{y_n\}$.

From (2.2) it follows that $x \vee y$ is a least upper bound of x and y with respect to $(\mathfrak{M}, \leq)$. We derive the immediate consequence.

Lemma 2.3. *Let* $X \leqq \mathfrak{M}$.

(a) $\bar{X}$ *is a growth type*

(b) $X \to \bar{X}$ *for all* $X \leqq \mathfrak{M}$ *is a closure operation, i.e.* $X \leqq \bar{X}$, $\bar{X} = \bar{\bar{X}}$

(c) $x \in \bar{X}$ *if and only if there are* $x_1, \ldots, x_n \in X$ *such that* $x \in \mathfrak{M}$ *and* $x \leqq x_1 \vee \ldots \vee x_n$.

Our next result (2.5) shows that many growth types $\bar{X}$ are bounded by certain elements of $\mathfrak{M}$, and so are Specker growth types. This result is used in § 4 and it will also illustrate that the proof of the existence of growth types which are not Specker growth types will be non-trivial. Corollary 2.5 will be shown under the hypothesis **ZFC+MA**.

If $(E, \leqq)$ is a quasi-ordered set, two elements $a, b \in E$ are *compatible*, if there exists $c \in E$ with $a \leqq c$ and $b \leqq c$ and if a, b are not compatible, they will be called *incompatible*. The set $(E, \leqq)$ satisfies the *countable anti-chain condition*, if all subsets of pairwise incompatible elements (anti-chains) are at most countable. It is customary to abbreviate the countable anti-chain condition by *c.a.c.*. A subset X of E is *dense* in $(E, \leqq)$, if for each $e \in E$ there is an $x \in X$ so that $e \leqq x$. **MA** reads as follows (cf. Solovay and Tennenbaum [12; p. 132]):

MA (topological version). *Let* $(P, \leqq)$ *be a quasi-ordered set satisfying c.a.c. and* $\mathbb{D}$ *be a family of less than* $2^{\aleph_0}$ *subsets of* P *then there exists a* $\mathbb{D}$*-generic subset* G *of* P, i.e. G satisfies:

(i) $G \leqq P$ is a cut
(ii) G is directed upwards
} i.e. G is an ideal in P (cf. 1.1).

(iii) If $D \in \mathbb{D}$ is dense in P, then $D \cap G \neq \varnothing$.

Theorem 2.4. (**ZFC+MA**): *If* $X \leqq \mathfrak{M}$, $|X| < 2^{\aleph_0}$ *there is an* $b \in \mathfrak{M}_s$ *such that* $x \sqsubset b$ *for all* $x \in X$.

The proof of (2.4) is given in Jech [10; p. 261, Lemma 24.12] where $b \in {}^{\omega}\omega$ is constructed. This bound b can be modified to be in $\mathfrak{M}_s$.

Since $\mathfrak{M}_s \subseteq \mathfrak{M}$ and $(f \sqsubset g \Rightarrow f \leqq g$ for $f, g \in \mathfrak{M}$, from (2.4) it follows that subsets $X \leqq \mathfrak{M}$ of cardinality $< 2^{\aleph_0}$ are bounded with respect to $\leqq$. Hence we get a

Corollary 2.5. (**ZFC+MA**): *If* $X \leqq \mathfrak{M}$, $|X| < 2^{\aleph_0}$ *then* $\bar{X}$ *is a Specker growth type.*

In [7] we associated with any strictly monotone sequence $r \in \mathfrak{M}_s$ a stretching-monomorphism ($=s$-monomorphism) $r^*\colon \mathbf{P} \to \mathbf{P}$. These monomorphisms where used to classify slender groups.

Since r^* never maps $\mathfrak{M}$ into itself if $r \neq \{n\}$, we introduced in (1.2) a second monomorphism $r^\#$ which does. We recall the notion of s-monomorphisms from [7].

Definition 2.6. *If* $r \in \mathfrak{M}_s$, *let be* $r^*\colon \mathbf{P} \to \mathbf{P}$ $(\sum_{n \in \omega} f_n e_n \to \sum_{n \in \omega} f_n e_{r_n})$ *the stretching monomorphism (s-monomorphsim) induced by r.*

The map $r^\#$ restricted to $\mathfrak{M}$ can be derived from r^* by making the latter monotone. This follows from

Lemma 2.7. *Let be* $r \in \mathfrak{M}_s$

(a) *If* $\tilde{f} = \left\{ \max_{i=1}^{n} (1, |f_i|) \right\}$ *for all* $f = \{f_n\} \in \mathbf{P}$ *then* $\tilde{f} = f$ *and* $r^{\#}(\tilde{f}) = \widetilde{r^*(f)}$ *for all* $f \in \mathfrak{M}$.

(b) *For growth types* $\mathfrak{T}$ *and* $\mathfrak{U}$ *of* $\mathfrak{M}$ *we have* $r^{\#}(\mathfrak{U}) \leqq \mathfrak{T}$ *if and only if* $r^*([\mathfrak{U}]) \leqq [\mathfrak{T}]$.

Proof. (a) follows from the definitions $r^{\#}$ and r^*.

(b) Let $r^{\#}(\mathfrak{U}) \leqq \mathfrak{T}$ and $f \in [\mathfrak{U}]$. Then $\tilde{f} \in \mathfrak{U}$ and consequently $\widetilde{r^*(f)} = r^{\#}(\tilde{f}) \in \mathfrak{T}$, using (a). Hence $r^*(f) \in [\mathfrak{T}]$. Conversely let $r^*([\mathfrak{U}]) \subseteq [\mathfrak{T}]$ and $f \in \mathfrak{U}$. Then $r^{\#}(f) \in \mathfrak{T}$ follows from $r^{\#}(f) = r^{\#}(\tilde{f}) = \widetilde{r^*(f)}$ und $r^*(f) \in [\mathfrak{T}]$.

In §1 we introduced $(\mathbb{P}, \prec)$ using the $m-s$-monomorphism $r^{\#}$. In [7] we applied s-monomorphisms.

Lemma 2.6 (b) shows that we obtain the same ordered set $[\mathbb{P}] = \{\mathbf{X} \subseteq \mathbf{P}$, *monoton*$\}/\sim$ in both cases. Hence the classifications (1.6), (1.7) remain the same if we interchange * and $^{\#}$. As in the case of s-monomorphisms, we shall need the following simple and essential

Lemma 2.8. *If* $g \in \mathfrak{M} \backslash \mathfrak{B}$ *and* $f \in \mathfrak{M}$ *there is an* $r \in \mathfrak{M}_s$ *such that* $r^{\#}(f) \leqq g$.

Proof. Since $\{g_n\}$ is not bounded, there is a strictly monotone positive sequence $r = \{r_n\}$ such that $f_n \leqq g_{r_n}$ for all $n \in \omega$. Hence $r^{\#}(f)_i = f_n \leqq g_{r_n} \leqq g_i$ for all $r_n \leqq i \leqq r_{n+1}$ and all $n \in \omega$. Therefore $r^{\#}(f) \leqq g$ by definition (§1) of "$\leqq$".

Lemma 2.9. *If* $r, s \in M_s$ *and* $r \sqsubset s$ *then* $s^{\#}(f) \leqq r^{\#}(f)$ *for all* $f \in \mathfrak{M}$.

Proof. Since $r \sqsubset s$, there is a $k \in \omega$ such that $r_n < s_n$ for all $n \geqq k$. Let be $n \geqq s_k$ and choose $e, m \in \omega$ such that $r_e \leqq n < r_{e+1}$ and $s_m \leqq n < s_{m+1}$. Since $m \geqq k$, we get $r_m < s_m \leqq n$ and $m \leqq e$. If $f \in \mathfrak{M}$, we obtain $s^{\#}(f)_n = f_m \leqq f_e = r^{\#}(f)_n$, i.e. $s^{\#}(f) \leqq r^{\#}(f)$.

Next we will show that there exists well ordered cofinal subsets of $\mathfrak{M}_s$.

Corollary 2.10. **(ZFC+MA)** $(\mathfrak{M}_s, \sqsubset)$ *has a cofinal well ordered chain of length* $2^{\aleph_0}$.

Proof. Since $|\mathfrak{M}_s| = 2^{\aleph_0}$, we will label the elements of $\mathfrak{M}_s$ in the form x_μ for all $\mu \in 2^{\aleph_0}$. Assume that we have constructed r_ν for $\nu \in \mu$ as well ordered chain already such that

$$(*) \qquad x_\nu \sqsubset r_\nu \quad \text{for all } \nu < \mu.$$

Let be $X = \{r_\nu, x_\mu;\ \nu \in \mu\}$. Since $|X| < 2^{\aleph_0}$, from (2.4) we obtain an element $r_\mu \in \mathfrak{M}_s$ which is upper bound of X. Then the set $\{r_\nu;\ \nu \in 2^{\aleph_0}\}$ is defined and by construction it is the required cofinal chain.

§3. Construction of Compatible and Incompatible Step Functions

The following quasi-ordering of *coarseness* "$\subseteq$" of functions in $\mathfrak{M} \backslash \mathfrak{B}$ will be very useful in this section. The proofs and the quasi-ordering $\subseteq$ while looking

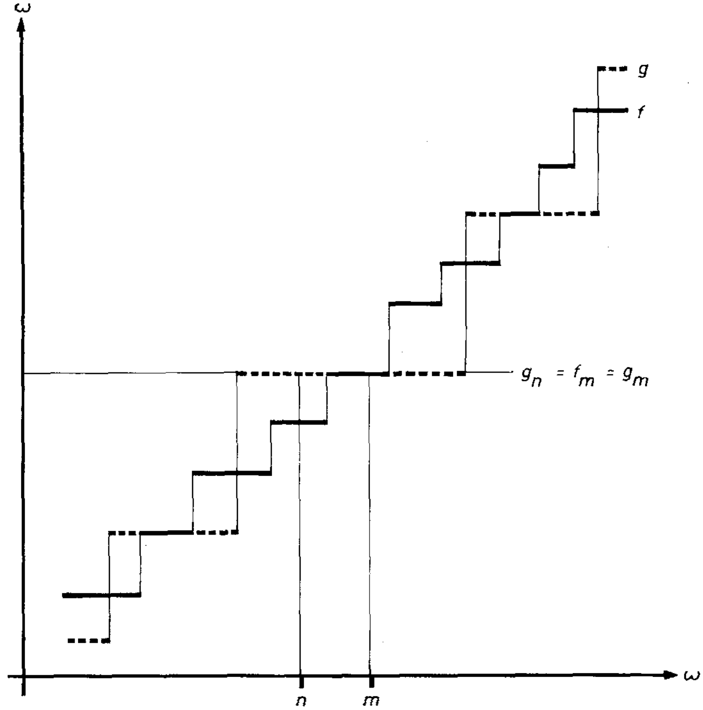

Fig. 1. Definition of $\subseteq$

some what technical are actually quite natural, when one has, in mind, a picture of the constructed functions. Hence we would like to ask the reader become familiar with the figures included in the proofs.

Definition 3.1. (cf. Fig. 1) *Let be $f, g \in \mathfrak{M} \setminus \mathfrak{B}$, then say $f \subseteq g$ (g is coarser than f) if and only if*

(i) *For almost all $n \in \omega$ there is an $m \in \omega$ such that $g_n = f_m$*

(ii) *For almost all $n \in \omega$ with $g_n = f_m$ we have $g_m = f_m$.*

We have the following

Lemma 3.2. *Let be $r \in \mathfrak{M}_s$, $f, y \in \mathfrak{M}$ and $f \notin \mathfrak{B}$. There is a $g \in \mathfrak{M}$ with $g \supseteq f$ and $r^{\#}(x) \nleq y$ for all $x \supseteq g$.*

Proof. First we construct the sequence $g \in \mathfrak{M}$ by induction: (cf. Fig. 2). Put $g_0 = f_0$ and assume g_k to be constructed already for all $k < n$ and some $0 \neq n \in \omega$. If $f_n \leqq g_{n-1}$, we choose $g_n = g_{n-1}$ and if $f_n > g_{n-1}$ choose $m \geqq n$ such that $f_m > r_n \cdot y_{r_n}$ and define $g_n = f_m$.

From $f \in \mathfrak{M} \setminus \mathfrak{B}$ it follows by construction that $g \in \mathfrak{M} \setminus \mathfrak{B}$, too. Next we will verify that $f \subseteq g$. Since (i) of (3.1) is true for all $n \in \omega$, we only have to prove (3.1) (ii): Therefore let $g_n = f_m$ and assume first $m \leqq n$. Then $g_m \leqq g_n$, since g is monotone. If $g_m < g_n$, we get $g_{m-1} \leqq g_m < g_n = f_m$ and therefore $g_m = f_k$ for some $k \geqq m$ by construction of g. Hence $g_m < g_n = f_m \leqq f_k = g_m$ is a contradiction, which

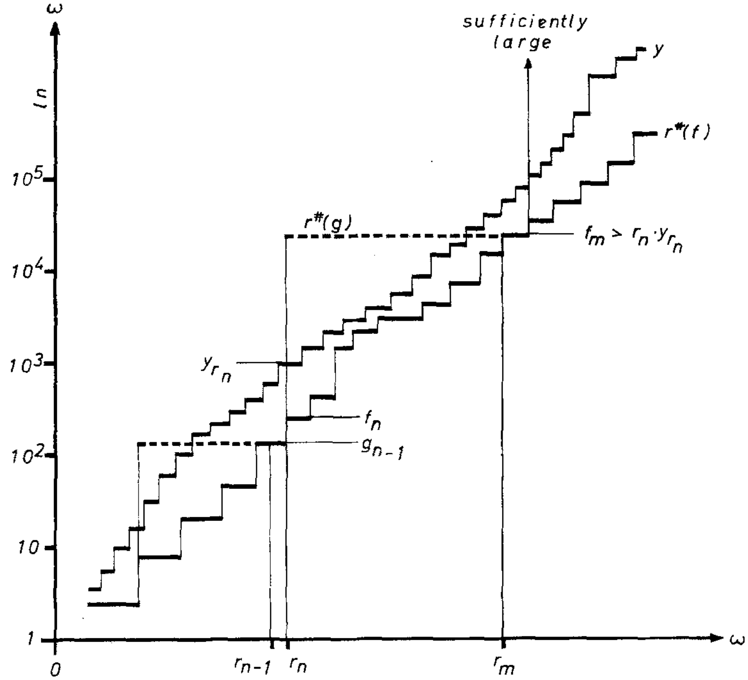

Fig. 2. Idea of the construction (Lemma 3.2)

shows $g_m = g_n$ in this case. If $m > n$, let $n < k \leqq m$ for $k \in \omega$. Therefore $f_k \leqq f_m = g_n \leqq g_{k-1}$ since f and g are monotone. From the construction of g it follows that $g_{k-1} = g_k$ and by induction on k is $f_m = g_n = g_m$. Finally we will show that $r^{\#}(x) \nleq y$ for $g \subseteq x$. The inequality has the explicit form

(*) For all $k \in \omega$ there is an $s \in \omega$ such that $r^{\#}(x)(s) > k \cdot y(s)$.

Because of (3.1) and $f \subseteq g \subseteq x$ there is an $e \in \omega$ such that $r_e \geqq k$ and

(i*) For all $n \geqq e$ there are $m, i \in \omega$ with $g_n = f_m$ and $x_n = g_i$.

(ii*) For all $n \geqq e$ if $g_n = f_m$ then $f_m = g_m$ moreover $x_n = g_m$ implies $x_m = g_m$.

Now we choose $m \in \omega$ with $m \geqq e$ and $x_m > x_e$ and $n \in \omega$ minimal with $x_m = g_n$. From $x_e < x_m = g_n = x_n$ it follows $e < n$ and therefore $r_e < r_n$. If $f_n \leqq g_{n-1}$ our n is no longer minimal with $x_m = g_n$ since $x_m = g_n = g_{n-1}$ by construction of g. Therefore $f_n > g_{n-1}$ and $g_n > r_n \cdot y_{r_n}$ by construction of g. From $r_e < r_n$ follows that $r^{\#}(x)(r_n) = x_n = g_n > r_n \cdot y_{r_n} > r_e\, y_{r_n} \geqq k \cdot y_{r_n}$ and (*) is satisfied for $s = r_n$.

Next we will show that $(\mathfrak{M}, \leqq)$ contains many subsets of incompatible functions derived from prescribed elements:

Lemma 3.3. *Let be $a, b, c \in \mathfrak{M} \setminus \mathfrak{B}$ and $c \nleq b$. Then there is an element $s \in \mathfrak{M} \setminus \mathfrak{B}$ such that $s \supseteq a$ and $c \nleq x \vee b$ for all $x \supseteq s$.*

Proof. First we construct $s \in \mathfrak{M} \setminus \mathfrak{B}$ and a sequence $\{i_n\} \in \mathfrak{M}$ by induction. Put $s_0 = a_0$, $i_0 = i_1 = 1$ and assume s_k, i_{k+1} has been constructed for $k < n$ (cf. Fig. 3).

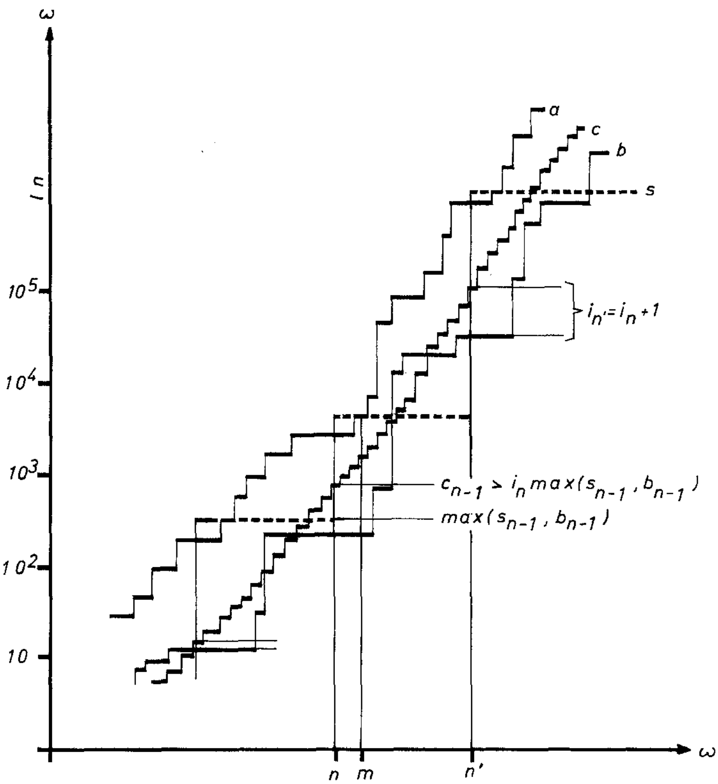

Fig. 3. Idea of the construction (Lemma 3.3)

There are two cases:

(1) $$c_{n-1} \leqq i_n \cdot \max\{s_{n-1}, b_{n-1}\} \quad \text{or} \quad s_{n-1} = a_m \quad \text{for some } m \geqq n$$

and

(2) $$i_n \cdot \max\{s_{n-1}, b_{n-1}\} < c_{n-1} \quad \text{and} \quad s_{n-1} \neq a_m \quad \text{for all } m \geqq n.$$

Put $s_n = s_{n-1}$ and $i_{n+1} = i_n$ in case (1) and choose $m \in \omega$ such that $a_{n-1} < a_m$ and put $s_n = a_m$ and $i_{n+1} = i_n + 1$ in case (2). Next we will show that the constructed element s satisfies the hypothesis of the lemma. To show that $s \in \mathfrak{M}$ we assume by way of contradiction that there exists $0 \neq n \in \omega$ with $s_n < s_{n-1}$.

From case (2) it follows that $a_{n-1} < s_n < s_{n-1}$. By construction there is an $m \in \omega$ with $s_{n-1} = a_m$. Hence $a_{n-1} < a_m$ and from $a \in \mathfrak{M}$ follows $n \leqq m$. Therefore $s_n = s_{n-1}$ by construction of s, which contradicts $s_n < s_{n-1}$. Consequently $s \in \mathfrak{M}$.

Now we assume $s \in \mathfrak{B}$. There is $k \in \omega$ such that $s_{n-1} = s_n$ and $i_n = i_{n+1}$ for all $n \geqq k$ by definition of $\mathfrak{B}$ and (1). Since $a, c \in \mathfrak{M} \setminus \mathfrak{B}$, $c \not\leqq b$ there exists $m \geqq k$ with $s_k < a_m$, $i_{k+1} \cdot s_k < c_m$ and $i_{k+1} \cdot b_m < c_m$. Hence $i_{m+1} \cdot \max\{s_m, b_m\} < c_m$ and $s_m \neq a_n$ for all $n > m$, so that we derive case (2) of the construction. This contradicts $s_k = s_{m+1}$ and $s \in \mathfrak{M} \setminus \mathfrak{B}$ is shown. In order to show $s \supseteq a$, the only condition of (4.1) which is not obvious, is (3.1) (ii). Hence let us assume $s_n = a_m$. If $n \leqq m$, the fact that $s_m = a_m$ follows inductively from the construction in case (1) and so condition (ii) holds. Hence we assume $m < n$ and by way of contradiction assume that $s_m < s_n$. From case (2) of the construction follows that there exists an $e \in \omega$ with $m < e \leqq n$ and $a_{e-1} < s_e$. Hence $a_{e-1} < s_e \leqq s_n = a_m$ implies $e \leqq m$ and the contradiction $m < e \leqq m$.

Finally we will show $c \nleqq x \vee b$ for $x \supseteq s$. Since $\{i_n\} \in \mathfrak{M} \backslash \mathfrak{B}$, for $k \in \omega$ there is an $n \in \omega$ with $i_n \geqq k$. Choose $t \geqq n$ such that $x_{t-1} < x_t$. Since $s \in \mathfrak{M} \backslash \mathfrak{B}$ and $s \subseteq x$ there is $m \geqq t$ with $x_m = s_m < s_{m+1}$.

If $s_m = a_j$ for some $j > m$, we get $s_j = s_m < s_{m+1}$ and so $j \leqq m$ which is a contradiction. Therefore $s_m \neq a_j$ for all $j > m$ and from our construction of s, we obtain $i_{m+1} \cdot \max\{s_m, b_m\} < c_m$ and consequently $c_m \nleqq k \cdot \max\{x_m, b_m\}$. Using the definitions of $\leqq$ and $\vee$ and the last inequality, we obtain $c \nleqq x \vee b$.

Next we will show the existence of upper bounds for chains in $\mathfrak{M} \backslash \mathfrak{B}$ of cardinality $< 2^{\aleph_0}$ with respect to the coarse ordering $\subseteq$.

Lemma 3.4. (ZFC+MA). *Let $\kappa \in 2^{\aleph_0}$ and $a_\nu \subseteq a_\mu$ for all $\nu \in \mu \in \kappa$ be a linearly ordered chain of elements of $\mathfrak{M} \backslash \mathfrak{B}$. There is an $a \in \mathfrak{M} \backslash \mathfrak{B}$ such that $a_\nu \subseteq a$ for all $\nu \in \kappa$.*

Proof. If F is a finite subset of the ordinal κ, choose $t(F) \in \omega$ minimal with respect to the condition:

For all $\nu, \mu \in F$ with $\nu \leqq \mu$ and all $n \geqq t(F)$ the following holds:

(i) There is an $m \in \omega$ such that $a_\mu(n) = a_\nu(m)$.

(ii) The equality $a_\mu(n) = a_\nu(m)$ implies $a_\mu(m) = a_\nu(m)$.

Let P be the set of all pairs (potential conditions) (s, F) of finite subsets F of κ and monotone functions $s: [0, k] \to \omega \backslash 0$ of initial segments $[0, k] = \{x \in \omega;\ 0 \leqq x \leqq k\}$ of ω and any $k \in \omega$. Next we will define a quasi-ordering $\leqq$ in P:

If $(s, F), (s', F') \in P$, we define $(s, F) \leqq (s', F')$ if and only if the following is satisfied:

(a) $s \subseteq s'$, i.e. the function s' extends s.

(b) $F \leqq F'$ (the usual subset relation)

(c) If $s \subset s'$ (s' extends s *properly*) and $F' \neq \varnothing$, for all $n \in \operatorname{dom}(s') \backslash \operatorname{dom}(s)$ there are $k_1, k_2 \in \omega$ and $\mu \in F'$ such that the following holds:

(1) $\max(\operatorname{dom}(s)) \leqq k_1 < n \leqq k_2 \leqq \max(\operatorname{dom}(s'))$

(2) $\nu \leqq \mu$ for all $\nu \in F$

(3) $k_2 \geqq t(F \cup \{\mu\})$

(4) $a_\mu(k_1) < a_\mu(k_2) < a_\mu(k_2 + 1)$

(5) $s'(m) = a_\mu(k_2)$ *if* $k_1 < m \leqq k_2$.

Using $t(F \cup \{\mu\}) \leqq t(F' \cup \{\mu\})$ for $F \leqq F'$, one easily checks that $(P, \leqq)$ is a quasi-ordering and $\leqq$ is in particular a transitive relation.

In order to show that $(P, \leqq)$ satisfies *c.a.c.*, let $C \leqq P$ and $|C| > \aleph_0$. The latter implies the existence of elements $(s, F), (s', F') \in C$ such that $s = s'$ and $F \neq F'$. If $F'' = F \cup F'$, then $(s, F) \leqq (s, F'')$, $(s', F') \leqq (s, F'')$ and $(s, F'') \in P$ follows trivially. Hence C contains compatible elements and antichains are at most countable, i.e. P

satisfies *c.a.c.* Next we claim the subsets of P

$$C_r = \{(s, F) \in P, \text{ there is an } x \in \mathrm{dom}(s) \text{ with } r \leqq s(x)\}$$
$$D_n = \{(s, F) \in P, n \in \mathrm{dom}(s)\}$$
$$E_\nu = \{(s, F) \in P, \nu \in F\}$$

to be dense in P for all $n, r \in \omega$ and $\nu \in \kappa$.

Proof of the density. Let $n, r \in \omega$, $\nu \in \kappa$, $p = (s, F) \in P$, $\mathrm{dom}(s) = [0, k]$, $F' = F \cup \{\nu\}$ and $\mu = \max F'$. Choose $k' \in \omega$ such that $n \leqq k'$, $k \leqq k'$, $t(F') \leqq k'$, $r \leqq a_\mu(k')$ and $a_\mu(k) < a_\mu(k') < a_\mu(k'+1)$. Next we define

$$s'(x) = \begin{cases} s(x) & \text{if } 0 \leqq x \leqq k \\ a_\mu(k') & \text{if } k < x \leqq k'. \end{cases}$$

Then $(s', F') \in C_r \cap D_n \cap E_\nu$ and $(s, F) \leqq (s', F')$. Therefore C_r, D_n, E_ν are dense in P.

An application of **MA** shows the existence of a generic set G of P such that

(1) For $p, p' \in G$ there is $q \in G$ with $p \leq q$ and $p' \leqq q$.
(2) $C_r \cap G \neq \varnothing$ for all $r \in \omega$
(3) $D_n \cap G \neq \varnothing$ for all $n \in \omega$
(4) $E_\nu \cap G \neq \varnothing$ for all $\nu \in \kappa$

We define $a = \bigcup_{(s,F) \in G} s$ where we interpret functions as graphs. Then the fact that a is a well-defined unbounded monotone function from ω to ω follows (1), (2), (3) and our definition of $(P, \leqq)$. Finally we will show $a_\nu \subseteq a$ for all $\nu \in \kappa$, which means that we must check the two conditions of definition (4.1):

Choose $p = (s, F) \in E_\nu \cap G$ and $k = \max(\mathrm{dom}(s))$. For any $n > k$ we will show

(α) There is an $m \in \omega$ such that $a(n) = a_\nu(m)$
(β) $a(n) = a_\nu(m)$ implies $a(m) = a_\nu(m)$.

Because D_n is dense, there is $p' = (s', F') \in G \cap D_n$. The filter condition (1) implies the existence of $p'' = (s'', f'') \in G$ such that $p \leqq p''$ and $p' \leqq p''$. Since $n \in \mathrm{dom}(s'') \setminus \mathrm{dom}(s)$ there are $k_1, k_2 \in \omega$ and $\mu \in F''$ such that $\max(\mathrm{dom}(s)) \leqq k_1 < n \leqq k_2 \leqq \max(\mathrm{dom}(s''))$, $\nu \leqq \mu$ for all $\nu \in F$, $k_2 \geqq t(F \cup \{\mu\})$, $a_\mu(k_1) < a_\mu(k_2) < a_\mu(k_2 + 1)$ and $s''(m) = a_\mu(k_2)$ if $k_1 < m \leqq k_2$ as follows from (c).

Since $\gamma \leqq \mu$ and $k_2 \geqq t(F \cup \{\mu\})$, there is an $m \in \omega$ such that $a_\mu(k_2) = a_\nu(m)$. Hence $a(n) = s''(n) = a_\mu(k_2) = a_\nu(m)$ and (α) is shown.

If $a(n) = a_\nu(m)$ for some $m \in \omega$, we obtain $a_\mu(k_2) = s''(n) = a(n) = a_\nu(m)$. Hence $a_\mu(m) = a_\nu(m) = a_\mu(k_2)$. In conjunction with $a_\mu(k_1) < a_\mu(k_2) < a_\mu(k_2+1)$ this implies $k_1 < m \leqq k_2$. Therefore $a(m) = s''(m) = a_\mu(k_2) = a_\nu(m)$ and (β) is shown.

Conditions (α) and (β) and (4.1) show $a_\nu \subseteq a$. **Q.E.D.**

§ 4. Proof of the Main Theorem

In this section we will prove our main result (1.4). Let us recall that $(\mathbb{P}, \prec)$ are the equivalence classes of growth types $\mathfrak{T}$, $\mathfrak{U}$ where $\mathfrak{T} \prec \mathfrak{U}$ if there is an $r \in \mathfrak{M}_s$ such that $r^{\#}(\mathfrak{T}) \leqq \mathfrak{U}$ and $\mathfrak{T} \sim \mathfrak{U}$ if $\mathfrak{T} \prec \mathfrak{U} \prec \mathfrak{T}$; cf. (1.2), (1.3) and (2.7).

Theorem 4.1. (ZFC+MA). $|\mathbb{P}|=2^{2^{\aleph_0}}$.

The major burden of proving (4.1) will be the following.

Construction 4.2. (ZFC+MA). *There is a subset* $K\subset\mathfrak{M}\setminus\mathfrak{B}$ *with the following properties.*

(*) $|K|=2^{\aleph_0}$.

(**) *If* X, Y *are any subsets of* K *with* $\bar{X}\prec\bar{Y}$, *then* $|X\setminus Y|<2^{\aleph_0}$.

We will prove (4.2) at the end of this section. In order to derive (4.1) from (4.2), we will use a set theoretic result.

Let $\mathfrak{I}_\aleph=\{X\leqq K_\aleph,\ |X|<\aleph\}$ the ideal of the boolean set $\mathfrak{P}(K_\aleph)$ of some set $K_\aleph$ of cardinality $\aleph$ and $\mathfrak{P}^*_\aleph=\mathfrak{P}(K_\aleph)/\mathfrak{I}_\aleph$ its quotient. We will write $\mathfrak{P}^*=\mathfrak{P}(K)/\mathfrak{I}$ if $\aleph=2^{\aleph_0}$ as in (4.2). We obtain the

Lemma 4.3. $|\mathfrak{P}^*_\aleph|=2^\aleph$ *and in particular* $|\mathfrak{P}^*|=2^{2^{\aleph_0}}$.

Decompose $K_\aleph$ into $\aleph$ subsets K_i of cardinality $\aleph$ with $i\in J$ and $|J|=\aleph$. If $T\leqq I$, let $K_T=\bigcup_{t\in T}K_t$. Then $K_T\equiv K_{T'} \mod \mathfrak{I}$ if and only if $T=T'$. Therefore, the set $\{K_T,\ T\leqq K\}$ represents a subset of $2^\aleph$ different elements of $\mathfrak{P}^*$.

Lemma 4.4. (ZFC+MA). *If* $X^*\in\mathfrak{P}^*$ *is represented by* $X<K$ *and* K *as in* (4.2), *the map* γ: $(\mathfrak{P}^*,\leqq)\to(\mathbb{P},\prec)(X^*\to\bar{X})$ *is an order preserving monomorphism.*

Proof. Lemma 4.4 is an immediate consequence of property (4.2), Lemma 2.8 and Theorem 2.4: For if $X^*\leqq Y^*$, then $|X\setminus Y|<2^{\aleph_0}$ by definition of $\mathfrak{P}^*$ and by (2.4) there is a $b\in\mathfrak{M}_s$ such that $u\leqq b$ for all $u\in X\setminus Y$. From (2.8) we have an $r\in\mathfrak{M}$ such that $r^\#(b)\leqq y$ for some $y\in Y$. Therefore $r^\#(X\setminus Y)\leqq\bar{Y}$ and trivially $r^\#(X\cap Y)\leqq\bar{Y}$ which proves $r^\#(\bar{X})\leqq\bar{Y}$. Hence $\bar{X}\prec\bar{Y}$ and γ is order preserving. From $\bar{X}\sim\bar{Y}$ it follows that $\bar{X}\prec\bar{Y}$ and $\bar{Y}\prec\bar{X}$, hence $|X\setminus Y|<2^{\aleph_0}$ and $|Y\setminus X|<2^{\aleph_0}$ taking $X, Y\leqq K$ according to (4.2)(**). Therefore $X^*=Y^*$ by definition of $\mathfrak{P}^*$ and γ is injective.

Proof of (4.1) *from* (4.2). Note that by (4.3) and (4.4) the following inequalities hold:

$$2^{2^{\aleph_0}}=|\mathfrak{P}^*|\leqq|\mathbb{P}|\leqq 2^{2^{\aleph_0}}.$$

Finally, we derive (4.2). First we choose a cofinal chain $\{r_\mu\}$ for $\mu\in 2^{\aleph_0}$ from the ordered set $\{\mathfrak{M}_s,\sqsubset)$, which exists by Corollary 2.10. To show that (4.2) holds we next construct K to be a sequence $x_\mu\in\mathfrak{M}\setminus\mathfrak{B}$ for $\mu\in 2^{\aleph_0}$, so that for each finite subset F of $2^{\aleph_0}$ and each element $v\in F$ the following property holds

$$(+)\qquad r_v^\#(x_v)\nleqq\bigvee_{\rho\in F\setminus\{v\}}x_\rho.$$

Property (+) will imply (4.2)(**). The construction is done by induction on the ordinal $\max(F)$. Let x_0 be an arbitrary element in $\mathfrak{M}\setminus\mathfrak{B}$. Now let $\mu\in 2^{\aleph_0}$ and assume that for $\sigma<\mu$ the element x_σ has been choosen such that (+) holds for every F with $\max(F)<\mu$ and every $v\in F$. Consider the set R_μ of all pairs (F,v) where F is a finite subset of $2^{\aleph_0}$ with $\max(F)=\mu$ and $v\in F$. Then $|R_\mu|=\kappa<2^{\aleph_0}$

and we can index R_μ by the ordinals less than κ, i.e. $R_\mu = \{p_\lambda;\ \lambda \in \kappa\}$. We will choose x_μ as an upper bound of a $\subseteq$-chain $\{y_\lambda\}$ for $\lambda \in \kappa$ were the latter is defined inductively as follows. We start with an arbitrary element $y_{-1} \in \mathfrak{M} \setminus \mathfrak{B}$. Assume $\lambda \in \kappa$ and that $y_{\lambda'}$ has already constructed for all $\lambda' < \lambda$.

By Lemma 3.4 we can find $y' \in \mathfrak{M} \setminus \mathfrak{B}$ so that $y_{\lambda'} \subseteq y'$ for all $\lambda' \in \{-1\} \cup \lambda$. Let $p_\lambda = (F, \nu)$. If $\nu = \mu = \max(F)$, we choose $y_\lambda \supseteq y'$ by Lemma 3.2 such that $r_\mu^\#(x) \nleqq \bigvee_{\rho \in F \setminus \{\mu\}} x_\rho$ for all $x \supseteq y_\lambda$. If $\nu \neq \mu$ we choose $y_\lambda \supseteq y'$ by Lemma 3.3 such that $r_\nu^\#(x_\nu) \nleqq \bigvee_{\rho \in F \setminus \{\mu, \nu\}} x_\rho \vee x$ for all $x \supseteq y_\lambda$ (By induction, we obtain $r_\nu^\#(x_\nu) \nleqq \bigvee_{\rho \in F \setminus \{\mu, \nu\}} x_\rho$.). Another application of Lemma 3.4 allows us to find x_μ coarser than y_λ for every $\lambda \in \kappa$. From the definition of the chain $\{y_\lambda\}$ we see that $(+)$ holds for every F with $\max(F) = \mu$ and every $\nu \in F$.

Let be $K = \{x_\mu;\ \mu \in 2^{\aleph_0}\}$ where the elements satisfy condition $(+)$ for all $\mu \in 2^{\aleph_0}$. Since $|K| = 2^{\aleph_0}$ by construction, we only need to show (4.2)(**): Let be X, Y subsets of K and $\bar{X} \prec \bar{Y}$. Assume $|X \setminus Y| = 2^{\aleph_0}$ for contradiction. By definition (1.3) of $\prec$ and (2.6)(b) there is an $r \in \mathfrak{M}_s$ such that $r^\#(\bar{X}) \leq \bar{Y}$. Since the chain $\{r_\nu;\ \nu \in 2^{\aleph_0}\}$ is a cofinal subset of $\mathfrak{M}_s, \sqsubset)$ there is a $\lambda \in 2^{\aleph_0}$ such that $r \sqsubset r_\lambda$. From $|X \setminus Y| = 2^{\aleph_0}$ follows by definition of cardinals the existence of $x_\nu \in X \setminus Y$ such that $\lambda \in \nu$. Since $r \sqsubset r_\lambda \sqsubset r_\nu$ we get from Lemma 2.9 that $r_\nu^\#(x_\nu) \leqq r^\#(x_\nu)$ and hence $r_\nu^\#(x_\nu) \in \bar{Y}$. Because of (2.3)(c) there are $x_{\nu_1}, \ldots, x_{\nu_n} \in Y$ such that $r_\nu^\#(x_\nu) \leqq x_{\nu_1} \vee \ldots \vee x_{\nu_n}$. Since $x_\nu \notin Y$ and $x_{\nu_i} \in Y$ it follows that $\nu \neq \nu_i$ for $1 \leqq i \leqq n$ and $(+)$ contradicts the last inequality between x_ν and x_{ν_i}'s.

§5. Existence of Growth Types Which are not Specker Without MA

First we will consider equivalent conditions in **ZFC** such that there is no unbounded growth type $\neq \mathfrak{M}$. We will derive:

Proposition 5.1. *The following conditions are equivalent*

(a) *There is no unbounded* $\leqq$*-growth-type* $\neq \mathfrak{M}$.

(b) *There is no unbounded* $\sqsubset$*-growth-type* $\neq \mathfrak{M}$.

(c) *For any family* $\mathfrak{X}$ *of subsets of* ω *which satisfies the property*

(*) *intersections of finitely many members of* $\mathfrak{X}$ *are infinite*

there is a decomposition of ω *into finite non-empty subsets* $A_n (n \in \omega)$ *such that* $X \cap A_n \neq \varnothing$ *for all* $X \in \mathfrak{X}$ *and almost all* $n \in \omega$.

Proof. (b)$\Rightarrow$(a). If $\mathfrak{T} \neq \mathfrak{M}$ is a $\leqq$-growth-type, then $\mathfrak{T}$ is a $\sqsubset$-growth-type as well, using the implication $(f \sqsubset g \Rightarrow f \leqq g)$. Therefore there is $b \in \mathfrak{M}$ such that $t \sqsubset b$ for all $t \in \mathfrak{T}$.

Using $(f \sqsubset g \Rightarrow f \leqq g)$ again, $\mathfrak{T}$ is a bounded $\leqq$-growth-type.

(a)$\Rightarrow$(c). Let $\mathfrak{X}$ be a family of subsets of ω satisfying (*) and $a = \{a_n\} \in \mathfrak{M}$ be any unbounded sequence. For $X \in \mathfrak{X}$ define $X' \in \mathfrak{M}$ as follows. If $n \in \omega$, let $X'(n) = a_m$ where $m = \min\{k \in X, k \geqq n\}$, which is well defined since X is infinite by (*). Then $\mathfrak{T} = \{X', X \in \mathfrak{X}\}$ is a $\leqq$-growth-type of $\mathfrak{M}$.

Assuming $(na_n\} \in \mathfrak{T}$, there are $X_1, \ldots, X_t \in \mathfrak{X}$ such that $(na_n\} \leqq X'_1 \vee \ldots \vee X'_t$ as follows from (2.3)(c).

Hence there is $k\in\omega$ with $na_n \leqq k\cdot\max\{X'_1(n),\dots,X'_t(n)\}$ for all $n\in\omega$. Since $X_1\cap\dots\cap X_t$ is infinite by (*), there is an $n\in X_1\cap\dots\cap X_t$ with $n>k$. Therefore $X'_i(n)=a_n$ and $na_n\leqq ka_n<na_n$ is a contradiction. Therefore $\{na_n\}\notin\mathfrak{T}$ in particular, $\mathfrak{T}\neq\mathfrak{M}$ and $\mathfrak{T}$ is bounded by assumption (a). Let $b=\{b_n\}\in\mathfrak{M}$ such that $t\leqq b$ for all $t\in\mathfrak{T}$. For $n\in\omega$ we define $k(n)=\max\{m\in\omega,\ n\leqq m,\ a_m\leqq nb_n\}$ if $\{m\in\omega,\ n\leqq m,\ a_m\leqq nb_n\}\neq\varnothing$ and $k(n)=n$ otherwise. Using the intervals $[n,k(n)]$ for $n\in\omega$ one can select inductively a decomposition of ω. Since $X'\leqq b$ for $X\in\mathfrak{X}$ there is $n_0\in\omega$ such that $X'(n)\leqq n_0 b_n$ for all $n\in\omega$. Now let $n\geqq n_0$, then $X'(n)=a_m$ for some $m\in X$ with $m\geqq n$. Therefore $a_m\leqq X'(m)\leqq n_0 b_n\leqq nb_n$ and $m\in\{i\in\omega,\ n\leqq i,\ a_i\leqq nb_n\}$. Hence $m\leqq k(n)$ and $m\in X\cap[n,k(n)]\neq\varnothing$, which shows (c).

(c)$\Rightarrow$(b). Let $\mathfrak{T}\neq\mathfrak{M}$ a $\sqsubset$-growth-type and $a=\{a_n\}\in\mathfrak{M}\setminus\mathfrak{T}$. If $x=\{x_n\}\in\mathfrak{T}$, let $x'=\{n\in\omega,\ x_n\leqq a_n\}$, which is infinite, since $a\not\sqsubset x$. The set $\mathfrak{X}=\{x',\ x\in\mathfrak{T}\}$ satisfies the intersection property (*) because of $x'_1\cap\dots\cap x'_t=(x_1\vee\dots\vee x_t)'$ for $x_1,\dots,x_t\in\mathfrak{T}$. Hence there is a decomposition $\omega=\bigcup_n A_n$ which satisfies the hypothesis of (c). Define an element $b=\{b_n\}\in\mathfrak{M}$ as $b_n=a_{\max(A_m)}+1$ where $m=\min\{i\in\omega,\ n\leqq k$ for all $k\in A_i\}$. Next we claim $x\sqsubset b$ for all $x\in\mathfrak{T}$ and $\mathfrak{T}$ is bounded.

Since $x\in\mathfrak{T}$, we get $A_m\cap x'\neq\varnothing$ for all $m>m_0$ and some $m_0\in\omega$. Let $k=\max\{A_0\cup\dots\cup A_{m_0})$ and take any $n>k$. If $m=\min\{i\in\omega,\ n\leqq k'$ for all $k'\in A_i\}$ then $m>m_0$ and $A_m\cap x'\neq\varnothing$. Let $j\in A_m\cap x'$ then $n\leqq j$ and therefore $x_n\leqq x_j\leqq a_j\leqq a_{\max(A_m)}<a_{\max(A_m)}+1=b_n$ and (5.3) is shown.

Proposition 5.2. (**ZFC**). *There are unbounded growth types* $\neq\mathfrak{M}$.

Proof. Let $\mathfrak{X}$ be an ultrafilter of ω which is not principal. Hence $\mathfrak{X}$ satisfies (*) of Proposition 5.1(c). Let A_n $(n\in\omega)$ be any decomposition of ω into finite non-empty subsets. If $\omega=N_1\cup N_2$ and $|N_1|=|N_2|=\infty$, let $Y_i=\bigcup_{n\in N_i}A_n$ for $i=1,2$. Since $\mathfrak{X}$ is an ultrafilter, either Y_1 of Y_2 belongs to $\mathfrak{X}$, and take $Y_1\in\mathfrak{X}$. Since $Y_1\cap A_n=\varnothing$ for all $n\in N_2$, condition (c) of (5.1) is not valid.

Finally we will remark, that the order-structure of growth types depends strongly on the underlying set theory as follows from a slightly modified version of a result of Hechler [9, p. 156, Theorem 1.1]:

Theorem 5.3 (Hechler [9]). *Let $\mathscr{X}$ be any countable standard model of* **ZFC** *and $A=(A,<)$ be any partial ordered set in $\mathscr{X}$ such that $|A|\leqq 2^{\aleph_0}$ (with respect to $\mathscr{X}$) and such that every countable subset (with respect to $\mathscr{X}$) of A has an upper bound. Then there is a normal extension $\mathscr{N}$ of $\mathscr{X}$ in which A is order-isomorphic to a cofinal subset of $(\mathfrak{M},\sqsubset)$.*

Remark. In order to obtain (5.3) from [9], change the potential p on p. 156 of [9] such that $p_a\in{}^n\omega$ is a function which is monotonic in addition. Then the image of A under the embeding $(a\to f_a)$ constructed by Hechler [9; p. 166] is already in $(\mathfrak{M},\sqsubset)$.

(5.3) can be used to obtain further unbounded growth types different from $\mathfrak{M}$ without the use of **MA**.

References

1. Baer, R.: Abelian groups without elements of finite order. Duke Math. J. **3**, 68–122 (1937)
2. Bergman, G.M.: Boolean rings of projection maps. J. London Math. Soc. (2) **4**, 593–598 (1972)
3. Fuchs, L.: Note on certain subgroups of products of infinite cyclic groups. Comment. Math. Univ. St. Paul **19**, 51–54 (1970)
4. Fuchs, L.: Infinite Abelian Groups, vol. I. New York-London: Academic Press 1970
5. Fuchs, L.: Infinite Abelian Groups, vol. II. New York-London: Academic Press 1974
6. Göbel, R.: On stout and slender groups. J. Algebra **53**, 39–55 (1975)
7. Göbel, R., Wald, B.: Wachstumstypen und schlanke Gruppen. Symposia Mathematica **23**, 201–239 (1979)
8. Hausdorff, F.: Untersuchungen über Ordnungstypen. Ber. Sächs. Akad. Wiss. **59**, 84–157 (1907)
9. Hechler, S.H.: On the existence of certain cofinal subsets of ${}^{\omega}\omega$. In: Axiomatic Set Theory. Proceedings of a Symposium (Los Angeles 1967), pp. 155–173. Proceedings of Symposia in Pure Mathematics **13**, Part 2. Providence, R.I.: American Mathematical Society 1974
10. Jech, T.J.: Set Theory. New York-London: Academic Press 1978
11. Nöbeling, G.: Verallgemeinerung eines Satzes von E. Specker. Invent. Math. **6**, 41–55 (1968)
12. Solovay, R.M., Tennenbaum, S.: Iterated Cohen extensions and Souslin's problem. Ann. of Math. (2) **94**, 201–245 (1971)
13. Specker, E.: Additive Gruppen von Folgen ganzer Zahlen. Portugal. Math. **9**, 131–140 (1950)

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