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Simplicial Sets from Categories

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1. Introduction

In this paper we show that, up to homotopy, the only "reasonable" functor which assigns a CW complex to every small category is the classifying space construction. This result is part of our attempt to better understand the relationship between the homotopic category of small categories and the homotopy category of CW complexes. This in turn is only part of the larger long-term program to develop the algebraic topology of small categories.

We prefer to compare the category of small categories, $\mathscr{Cat}$, with the category of simplicial sets, $\mathscr{K}$. The relationship between $\mathscr{K}$ and CW complexes is already well understood. In particular, if the maps between simplicial sets which induce isomorphisms of homotopy groups are inverted, the new homotopy category is called the homotopic category for $\mathscr{K}$. It is equivalent to the homotopy category of $\mathscr{W}$, the category of spaces of homotopy type of a CW complex [5; VII, 1]. The equivalence is given by Milnor realization $|\_|: \mathscr{K} \to \mathscr{W}$.

In $\mathscr{Cat}$, the objects are small categories, the morphisms are the functors, and homotopies are generated by natural transformation. Homotopy groups have been defined and so a homotopic category can be obtained for $\mathscr{Cat}$.

Latch [10] and Thomason [19] have shown that the homotopic categories for $\mathscr{Cat}$ and $\mathscr{K}$ are equivalent. The standard functor nerve, $N: \mathscr{Cat} \to \mathscr{K}$, gives the equivalence. The classifying space construction is $B\_ = |N\_|: \mathscr{Cat} \to \mathscr{W}$. This gives the equivalence between the homotopic category for $\mathscr{Cat}$ and the homotopy category for $\mathscr{W}$.

Categorical realization $c: \mathscr{K} \to \mathscr{Cat}$ is the left adjoint for nerve. Although $cN \simeq \mathrm{Id}_{\mathscr{Cat}}$, categorical realization is not a homotopy inverse for nerve because $Nc: \mathscr{K} \to \mathscr{K}$ is wildly wrong.

In [10], the functor $\Gamma: \mathscr{K} \to \mathscr{Cat}$ which gives the category of simplices ΓX for each simplicial set X, was shown to be an inverse functor to $N: \mathscr{Cat} \to \mathscr{K}$ for the equivalence of homotopic categories. Γ is the left adjoint for a functor S_γ:

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$\mathcal{C}at \to \mathcal{K}$. There is a natural transformation $N \xrightarrow{\cdot} S_\gamma$. Studying the relationship between these two functors was the main motivation for our work in this paper. The main theorem applies and shows that

$$N\mathbf{A} \to S_\gamma \mathbf{A}$$

always gives an isomorphism on homotopy groups. In particular, this and the Latch version of the equivalence of the homotopic categories for $\mathcal{C}at$ and $\mathcal{K}$ can be used to show that the adjunctions

$$\mathrm{Id}_{\mathcal{K}} \xrightarrow{\cdot} S_\gamma \Gamma \qquad \Gamma S_\gamma \xrightarrow{\cdot} \mathrm{Id}_{\mathcal{C}at}$$

for the adjoint pair $\Gamma \dashv S_\gamma$ induce isomorphisms for homotopy groups for every simplicial set and category respectively (see Corollary 4.7).

We state our main result now and explain the necessity of our hypotheses.

(4.1) **Theorem.** *Let $S_\theta: \mathcal{C}at \to \mathcal{K}$ be a representable functor with a natural transformation $N \xrightarrow{\cdot} S_\theta$. If each of the small categories $\theta[k]$ representing the k-simplices of $S_\theta(\_)$ are strongly contractible in $\mathcal{C}at$, then*

$$N\mathbf{A} \to S_\theta \mathbf{A}$$

induces an isomorphism of homotopy groups for all $\mathbf{A} \in \mathcal{C}at$. □

We need the natural transformation $N \xrightarrow{\cdot} S_\theta$ to be able to compare the two. For the other way, $S_\theta \xrightarrow{\cdot} N$, a simple extra condition is necessary (see Theorem 4.1′). We need representability to avoid cases such as $S_\theta \mathbf{A}$ equal to a point for all $\mathbf{A}$. The $\theta[k]$ are in some sense basic k-cells; so they should be contractible. It is curious that there is only one "homotopy" condition in the hypothesis, i.e. the contractability of the $\theta[k]$. This theorem gives conditions for S_θ to be an inverse to Γ for the equivalence between the homotopic categories for $\mathcal{C}at$ and $\mathcal{K}$.

In [4], conditions are given on the $\theta[k]$'s so that the left adjoint for S_θ is a homotopy inverse for nerve.

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In Section 2, we give the basic definitions and constructions that we need for $\mathcal{C}at$ and $\mathcal{K}$, while in Section 3 we develop the necessary homotopy theory in both places. We prove the main theorem and its immediate corollaries in Section 4. The final section is devoted to a list of examples.

2. Preliminaries

Let Δ be the category whose objects are finite total orders $[k]$, $k \geqq 0$, and whose morphisms are order preserving functions α: $[p] \to [k]$ ([5; II, 2]). It is well known (e.g. see [5; II, 2]) that Δ is generated by the collection of increasing injections δ^i: $[k-1] \rightarrowtail [k]$ with $i \notin \operatorname{Im} \delta^i$, $k>0$, $0 \leqq i \leqq k$; and by the collection of nondecreasing surjections σ^i: $[k+1] \twoheadrightarrow [k]$ which twice takes the value i, $0 \leqq i \leqq k$, $k \geqq 0$.

Let $\mathscr{E}ns$ represent the category of sets. The functor category $[\Delta^{\mathrm{op}}, \mathscr{E}ns]$ of simplicial sets is denoted by $\mathscr{K}$. For each $X \in \mathscr{K}$, $X: \Delta^{\mathrm{op}} \to \mathscr{E}ns$, let X_k represent the collection of k-simplices $X([k])$. The representable simplicial sets, $\Delta(\_, [k])$: $\Delta^{\mathrm{op}} \to \mathscr{E}ns$, are called the *standard simplicial sets*; and are denoted simply by $\Delta[k]$, $k \geqq 0$. Similarly, $\Delta(\alpha)$: $\Delta[p] \to \Delta[k]$ will denote the simplicial map $\Delta(\_, \alpha)$: $\Delta(\_, [p]) \to \Delta(\_, [k])$, for α: $[p] \to [k]$ in Δ.

Mor (X, Y) represents the set of all maps from X to Y, while $\mathscr{K}(X, Y)$ denotes the "internal-hom" *simplicial function space* whose collection of k-simplices is $\operatorname{Mor}(X \times \Delta[k], Y)$. Because the "internal-hom" functor is the right adjoint to "product" [5; II, 2],

(2.1) $\operatorname{Mor}(W \times X, Y) \cong \operatorname{Mor}(W, \mathscr{K}(X, Y))$

naturally in W, X and Y.

$\mathscr{C}at$ represents the category of small categories. The objects of a small category $\mathbf{A}$ form a set. Let $\operatorname{Mor}(\mathbf{A}, \mathbf{B})$ denote the set of functors from small category $\mathbf{A}$ to small category $\mathbf{B}$. The "internal-hom" category $\mathscr{C}at(\mathbf{A}, \mathbf{B})$ has objects, the functors F: $\mathbf{A} \to \mathbf{B}$ and morphisms, natural transformations ω: $F \xrightarrow{\cdot} G$. As above, the "internal-hom" functor is right adjoint to "product"; i.e.,

(2.2) $\operatorname{Mor}(\mathbf{C} \times \mathbf{A}, \mathbf{B}) \cong \operatorname{Mor}(\mathbf{C}, \mathscr{C}at(\mathbf{A}, \mathbf{B}))$

naturally in $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$.

For any functor θ: $\Delta \to \mathscr{C}at$, define the *θ-singular functor*

$$S_\theta: \mathscr{C}at \to [\Delta^{\mathrm{op}}, \mathscr{E}ns]$$

by the following "representable" construction: For each $\mathbf{A} \in \mathscr{C}at$,

(2.3) $S_\theta(\mathbf{A}) \equiv \operatorname{Mor}(\theta\_, \mathbf{A})$: $\Delta^{\mathrm{op}} \to \mathscr{E}ns$.

Hence a k-simplex in $S_\theta(\mathbf{A})$ is a functor r: $\theta[k] \to \mathbf{A}$.

Nerve, the standard example of a functor from $\mathscr{C}at$ to $\mathscr{K}$, is obtained as a singular functor in the following way: Consider the full inclusion functor ι: $\Delta \to \mathscr{C}at$, where $\iota[k] \equiv \mathbf{k}$ is the small category whose objects are u, $0 \leqq u \leqq k$, and having a unique morphism $u \to v$ for each $u \leqq v$. In fact, $\mathbf{0}$ is a terminal object of $\mathscr{C}at$; $\mathbf{1}$ the category with two objects and one nonidentity morphism $0 \to 1$. The *nerve functor* N: $\mathscr{C}at \to \mathscr{K}$ is the *ι-singular functor*; i.e., for each small category $\mathbf{A}$,

(2.4) $N(\mathbf{A}) \equiv \operatorname{Mor}(\iota\_, \mathbf{A})$: $\Delta^{\mathrm{op}} \to \mathscr{E}ns$.

Thus $\mathbf{A}$ is the simplicial set whose k-simplices, $(N\mathbf{A})_k = \operatorname{Mor}(\mathbf{k}, \mathbf{A})$, are diagrams in $\mathbf{A}$ of the form

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \to \cdots \to p_{k-1} \xrightarrow{a_k} p_k.$$

The i^{th}-face (resp. degeneracy) of this k-dimensional simplex is obtained by deleting the objects p_i (resp. replacing p_i by Id: $p_i \to p_i$) in the evident way. Since $\imath\colon \Delta \to \mathscr{C}\!at$ is full and faithful,

(2.5) $N(\mathbf{k}) \equiv \operatorname{Mor}(\imath\_, \imath[k]) \cong \Delta(\_, [k]) \equiv \Delta[k]$;

thus N preserves terminal objects, i.e.

(2.6) $N(\mathbf{0}) \cong \Delta[0]$.

The left adjoint of nerve is *categorical realization* $c\colon \mathscr{K} \to \mathscr{C}\!at$ [5; II, 4]; in fact the adjunction

(2.7) $cN \xrightarrow[\cong]{\cdot} \mathrm{Id}_{\mathscr{C}\!at}$

is invertible. Since N is a right adjoint, it preserves all limits; and in particular, N preserves all products, i.e.

(2.8) $N(\mathbf{A} \times \mathbf{B}) \cong N\mathbf{A} \times N\mathbf{B}$.

From (2.7), it follows that N is also full and faithful [5; I, 1], i.e.

(2.9) $\operatorname{Mor}(\mathbf{A}, \mathbf{B}) \cong \operatorname{Mor}(N\mathbf{A}, N\mathbf{B})$.

Actually, N also preserves "internal-Homs."

(2.10) **Lemma.** *N: $\mathscr{C}\!at \to \mathscr{K}$ commutes with the "internal-Hom" construction, i.e.*

$$N(\mathscr{C}\!at(\mathbf{A}, \mathbf{B})) \cong \mathscr{K}(N\mathbf{A}, N\mathbf{B})$$

naturally in $\mathbf{A}$ *and* $\mathbf{B}$. □

Proof. For each $[k] \in \Delta$,

$$\begin{aligned} N(\mathscr{C}\!at(\mathbf{A}, \mathbf{B}))_k &\equiv \operatorname{Mor}(\mathbf{k}, \mathscr{C}\!at(\mathbf{A}, \mathbf{B})), && \text{by (2.4)} \\ &\cong \operatorname{Mor}(\mathbf{k} \times \mathbf{A}, \mathbf{B}), && \text{by (2.2)} \\ &\cong \operatorname{Mor}(N(\mathbf{k} \times \mathbf{A}), N\mathbf{B}), && \text{by (2.9)} \\ &\cong \operatorname{Mor}(N\mathbf{A} \times \Delta[k], N\mathbf{B}), && \text{by (2.5) and (2.8)} \\ &\equiv (\mathscr{K}(N\mathbf{A}, N\mathbf{B}))_k. \end{aligned}$$

Since the above equivalences are all natural, the lemma follows. □

The nerve functor, N: $\mathscr{C}\!at \to \mathscr{K}$, and each general S_θ: $\mathscr{C}\!at \to \mathscr{K}$ have left adjoints, because of the following general Kan-type construction [8].

(2.11) **Lemma.** *Let $\mathscr{C}$ be a cocomplete (i.e. arbitrary colimits exist) category and θ: $\Delta \to \mathscr{C}$ a functor. Then there exists an adjoint pair $\hat{\theta} \dashv S_\theta$, where*

$$S_\theta\colon \mathscr{C} \to [\Delta^{\mathrm{op}}, \mathscr{E}\!ns] \equiv \mathscr{K}$$

*is the θ-*singular functor *defined by $S_\theta(A) \equiv \operatorname{Mor}(\theta\_, A)$ for $A \in \mathscr{C}$; and the θ-*realization functor *$\hat{\theta}$: $\mathscr{K} \to \mathscr{C}$ is its left adjoint. Lastly, $\hat{\theta}$: $\mathscr{K} \to \mathscr{C}$ is determined* *uniquely by the requirements that it preserve colimits and*

$$\hat{\theta}(\Delta[k]) \equiv \hat{\theta}(\Delta(\_,[k])) \cong \theta[k]. \quad \square$$

$\mathscr{C}at$ is a cocomplete category [5; Dic.], and thus satisfies the hypothesis of Lemma 2.11. Now consider an arbitrary $\theta\colon \Delta \to \mathscr{C}at$. Then $S_\theta\colon \mathscr{C}at \to \mathscr{K}$ is a right adjoint and hence, it preserves all limits; in particular, terminal objects

(2.12) $S_\theta(\mathbf{0}) \cong \Delta[0]$,

and products

(2.13) $S_\theta(\mathbf{A} \times \mathbf{B}) \cong S_\theta(\mathbf{A}) \times S_\theta(\mathbf{B})$.

3. Simplicial and Categorical Homotopy

Strong homotopy (SH) in $\mathscr{K}$ is the equivalence relation generated by the following elementary homotopies [13]: Let the "i^{th} vertex" inclusions $u_i\colon X \to X \times \Delta[1]$ correspond to the simplicial maps

$$X \cong X \times \Delta[0] \xrightarrow{\mathrm{Id} \times \Delta(\delta^{1-i})} X \times \Delta[1], \qquad i=0,1.$$

If f, $g \in \mathrm{Mor}(X,Y)$, $f \sim g$ iff there is a simplicial map $h\colon X \times \Delta[1] \to Y$ such that $h \cdot u_0 = f$ and $h \cdot u_1 = g$.

Similarly, the *strong homotopy* (*SH*) *relation* for $\mathscr{C}at$ is developed as follows: Suppose F, $G \in \mathrm{Mor}(\mathbf{A}, \mathbf{B})$. A natural transformation $\omega\colon F \overset{\cdot}{\longrightarrow} G$ is considered an elementary homotopy. Each one corresponds to a functor $\bar{\omega}\colon \mathbf{A} \times \mathbf{1} \to \mathbf{B}$ such that $\bar{\omega} \cdot (\mathrm{Id} \times \delta^1) = F$ and $\bar{\omega}(\mathrm{Id} \times \delta^0) = G$. Since N preserves products (2.8)

$$N\bar{\omega}\colon N\mathbf{A} \times N\mathbf{1} \cong N(\mathbf{A} \times \mathbf{1}) \to N\mathbf{B}.$$

As $N\mathbf{1} \cong \Delta[1]$ the standard 1-simplex, $N\bar{\omega}$ is a simplicial homotopy and $NF \sim NG$. Furthermore, since N is full and faithful by (2.9), $NF \sim NG$ in $\mathscr{K}$ insures the existence of a functor $\bar{\omega}\colon \mathbf{A} \times \mathbf{1} \to \mathbf{B}$, and thus the existence of a natural transformation $\omega\colon F \overset{\cdot}{\longrightarrow} G$. Hence, the *strong homotopy relation* in $\mathscr{C}at$, i.e. the equivalence relation generated by natural transformation, corresponds fully via N to the SH relation in $\mathscr{K}$.

(3.1) **Lemma.** *$NF \sim NG$ in $\mathscr{K}$ iff $F \sim G$ in $\mathscr{C}at$.* $\square$

Under mild hypotheses, $S_\theta\colon \mathscr{C}at \to \mathscr{K}$ will also, as does N, preserve strong homotopies.

(3.2) **Proposition.** *If there exists a natural transformation*

$\eta\colon \theta \overset{\cdot}{\longrightarrow} \iota\colon \Delta \to \mathscr{C}at$, then $S_\theta\colon \mathscr{C}at \to \mathscr{K}$ preserves strong homotopies. $\square$

Proof. Let

(3.3) $\tilde{\eta} \equiv \mathrm{Mor}(\eta_{-}, \_)\colon N \overset{\cdot}{\longrightarrow} S_\theta\colon \mathscr{C}at \to \mathscr{K}$

be the natural transformation induced from $\eta\colon \theta \xrightarrow{\cdot} \iota$; i.e. for each $\mathbf{A} \in \mathscr{C}\!at$, $[k] \in \Delta$

(3.4) $\quad \tilde{\eta}(\mathbf{A})_k \equiv \mathrm{Mor}(\eta([k]), \mathbf{A})\colon \mathrm{Mor}(\iota[k], \mathbf{A}) \to \mathrm{Mor}(\theta[k], \mathbf{A})$.

Suppose F, $G \in \mathrm{Mor}(\mathbf{A}, \mathbf{B})$ such that $F \sim G$; i.e., there is a functor $H\colon \mathbf{A} \times \mathbf{1} \to \mathbf{B}$ (equivalent to a natural transformation $F \xrightarrow{\cdot} G$) such that $H \circ (\mathrm{Id} \times \delta^1) = F$ and $H \circ (\mathrm{Id} \times \delta^0) = G$. Define $h\colon S_\theta(\mathbf{A}) \times \Delta[1] \to S_\theta(\mathbf{B})$ to be the following composition:

$$\begin{tikzcd}
S_\theta(\mathbf{A}) \times \Delta[1] \arrow[r, "\mathrm{Id} \times \tilde{\eta}(\mathbf{1})"] \arrow[dr, dotted, "h"'] & S_\theta(\mathbf{A}) \times S_\theta(\mathbf{1}) \cong S_\theta(\mathbf{A} \times \mathbf{1}) \arrow[d, "S_\theta(H)"] \\
 & S_\theta(\mathbf{B})
\end{tikzcd}$$

where $\tilde{\eta}([1])\colon N(\mathbf{1}) \cong \Delta[1] \to S_\theta(\mathbf{1})$. The naturality of $\tilde{\eta}\colon N \xrightarrow{\cdot} S_\theta$ and the fact that S_θ (2.13) (and thus N) preserves products, together guarantee that the following diagram commutes:

$$\begin{tikzcd}
S_\theta(\mathbf{A}) \times \Delta[1] \arrow[r, "\mathrm{Id} \times \tilde{\eta}(\mathbf{1})"] & S_\theta(\mathbf{A}) \times S_\theta(\mathbf{1}) \cong S_\theta(\mathbf{A} \times \mathbf{1}) \arrow[r, "S_\theta(H)"] & S_\theta(\mathbf{B}) \\
S_\theta(\mathbf{A}) \times \Delta[0] \arrow[u, "\mathrm{Id} \times N(\delta^1)"] \arrow[r, "\mathrm{Id} \times \tilde{\eta}(\mathbf{0})"] & S_\theta(\mathbf{A}) \times S_\theta(\mathbf{0}) \cong S_\theta(\mathbf{A} \times \mathbf{0}) \arrow[u, "\mathrm{Id} \times S_\theta(\delta^1)", "S_\theta(\mathrm{Id} \times \delta^1)"'] \arrow[r, "\cong"] & S_\theta(\mathbf{A}) \arrow[u, "S_\theta(F)"] \\
S_\theta(\mathbf{A}) \times \Delta[0] \arrow[u, "\cong"] \arrow[r, "\cong"] & S_\theta(\mathbf{A}) \times \Delta[0] \arrow[u, "\cong"] &
\end{tikzcd}$$

Hence $h \circ u_0 = h \circ (\mathrm{Id} \times N(\delta^1)) = S_\theta(F)$, and similarly, $h \circ u_1 = h \circ (\mathrm{Id} \times N(\delta^0)) = S_\theta(G)$. Thus $S_\theta(F) \sim S_\theta(G)$. Clearly, any "zig-zag" of natural transformations goes to a "zig-zag" of elementary homotopies in $\mathscr{K}$, and the proposition follows. $\square$

The *Milnor geometric realization* is a functor $|\_|\colon \mathscr{K} \to \mathscr{T}\!op$ [14], where $\mathscr{T}\!op$ is a convenient category (in the sense of Steenrod [18]) of compactly generated weak Hausdorff spaces which contains CW complexes. In fact, $|X|$ is a CW complex for every $X \in \mathscr{K}$. Geometric realization commutes with products; i.e., the canonical map

$$|X \times Y| \xrightarrow{\cong} |X| \times |Y|$$

is a homeomorphism. Hence since $|\Delta[1]| \cong \mathbf{I}$, the unit interval, the Milnor realization preserves strong homotopies.

A map $f\colon X \to Y$ in $\mathscr{K}$ is called a *weak homotopy equivalence* (*WHE*) if $|f|\colon |X| \to |Y|$ is a homotopy equivalence of CW complexes. We say a functor $F\colon \mathbf{A} \to \mathbf{B}$ in $\mathscr{C}\!at$ is a *weak homotopy equivalence* if $NF\colon N\mathbf{A} \to N\mathbf{B}$ is a WHE in $\mathscr{K}$; or equivalently, if $BF\colon B\mathbf{A} \to B\mathbf{B}$ is a homotopy equivalence in $\mathscr{T}\!op$, where $B_{\_} = |N_{\_}|\colon \mathscr{C}\!at \to \mathscr{T}\!op$ is the classifying space functor [15].

Remark. Notions of homotopy groups can be defined internally in $\mathscr{C}at$ (e.g. see [2], [6]), and in $\mathscr{K}$ (e.g. see [7], [9]). The functors $|\_|$: $\mathscr{K} \to \mathscr{T}op$ and N: $\mathscr{C}at \to \mathscr{K}$ relate these with each other and with the usual $\mathscr{T}op$ notion of homotopy groups. In each case, an analogue of Whitehead's theorem, which characterizes WHE's by the property of inducing isomorphisms on homotopy groups, holds.

A map f: $X \to Y$ in $\mathscr{C}at$ or $\mathscr{K}$ is said to be a *strong homotopy equivalence* (*SHE*) if f has a strong homotopy inverse; i.e., there is a g: $Y \to X$ such that fg and gf are strongly homotopic to Id_Y and Id_X, respectively.

Since the functors N: $\mathscr{C}at \to \mathscr{K}$ (see Lemma 3.1) and $|\_|$: $\mathscr{K} \to \mathscr{T}op$ preserve strong homotopies, f: $X \to Y$ a SHE in $\mathscr{C}at$ or $\mathscr{K}$ implies $|f|$ or $|Nf|$ is a homotopy equivalence; hence, f is a WHE. However, for X and Y CW complexes, WHE and SHE are the same (= *homotopy equivalence*, (*HE*)) ([17; p. 405]). In $\mathscr{C}at$ and $\mathscr{K}$ elementary examples show that not every WHE is a SHE.

(3.5) **Lemma.** *If f: $A \to B$ is a strong homotopy equivalence* (*SHE*) *in $\mathscr{K}$, then the map of simplicial function spaces*

$$\mathscr{K}(f, X)\colon \mathscr{K}(B, X) \to \mathscr{K}(A, X)$$

is a SHE in $\mathscr{K}$, for every $X \in \mathscr{K}$. □

Proof: See [5; IV, 1.5]. □

Remark. The condition SHE cannot be weakened to WHE. For example, if $B = \Delta[0]$ and $A = X$ is the simplicial real line (the infinite zig-zag), then $\mathscr{K}(B, X)$ has one component and $\mathscr{K}(A, X)$ has infinitely many components.

By Lemma 2.10, Lemma 3.1, and the definition of SHE, we have:

(3.6) **Lemma.** *If F: $\mathbf{A} \to \mathbf{B}$ is a SHE in $\mathscr{C}at$, then the map of "internal-hom" categories $\mathscr{C}at(F, \mathbf{X})$: $\mathscr{C}at(\mathbf{B}, \mathbf{X}) \to \mathscr{C}at(\mathbf{A}, \mathbf{X})$ is a SHE in $\mathscr{C}at$, for every $\mathbf{X} \in \mathscr{C}at$.* □

A small category $\mathbf{A}$ is called *strongly contractible* (*SC*) if it is SHE to the terminal category $\mathbf{0}$. A simple, but useful example of Lemma (3.6) follows from:

(3.7) **Proposition.** *If $\mathbf{A}$ has an initial (or terminal) object, then $\mathbf{A}$ is SC.* □

Proof. Let J: $\mathbf{0} \to \mathbf{A}$ be the inclusion functor such that $J(0) = u$, the initial object of $\mathbf{A}$. If T: $\mathbf{A} \to \mathbf{0}$ is the terminal functor (in $\mathscr{C}at$), then $T \circ J \cong \mathrm{Id}_{\mathbf{0}}$ and there is a natural transformation $JT \xrightarrow{\cdot} \mathrm{Id}_{\mathbf{A}}$ (by the fact that u is initial). Hence $\mathbf{A}$ is SHE to $\mathbf{0}$. □

The category $\mathbf{k}$ has both a terminal (k) and initial (0) object and so $\mathbf{0}$ and $\mathbf{k}$ are SHE. We will need the following special case of Lemma 3.6 in our proof of the main theorem.

(3.8) **Corollary.** *For $\mathbf{A} \in \mathscr{C}at$, the functor*

$$\mathscr{C}at(\mathbf{0}, \mathbf{A}) \cong \mathbf{A} \to \mathscr{C}at(\mathbf{p}, \mathbf{A})$$

induced by the unique functor $\mathbf{p} \to \mathbf{0}$, is a SHE. □

A *bisimplicial set* W is a functor $W\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathcal{E}ns$. The *diagonal functor*

$$\mathrm{diag}\colon [\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathcal{E}ns] \to [\Delta^{\mathrm{op}}, \mathcal{E}ns] \equiv \mathcal{K}$$

from the category of bisimplicial sets to simplicial sets, is defined by the rule

$$(\mathrm{diag}\, W)_k = W([k],[k]).$$

Similarly, a *bisimplicial space* T is a functor $T\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathcal{T}op$ with *diagonal simplicial space* diag T given by

$$(\mathrm{diag}\, T)_k = T([k],[k]).$$

In [15], Segal gives a realization functor $|\_|_{\mathcal{T}}\colon [\Delta^{\mathrm{op}}, \mathcal{T}op] \to \mathcal{T}op$, from the category of simplicial spaces to the convenient category $\mathcal{T}op$, which is similar to Milnor's construction $|\_|\colon [\Delta^{\mathrm{op}}, \mathcal{E}ns] \equiv \mathcal{K} \to \mathcal{T}op$. Both of these are constructed as special colimits called "coends" [12; IX, 6]. The next well known lemma follows from the special form of these realizations, and from the fact that colimits commute with each other (see [12; IX, 8]). We will frequently denote a simplicial set $X \in [\Delta^{\mathrm{op}}, \mathcal{E}ns]$ by $[k] \mapsto X([k]) \equiv X_k$, and its geometric realization by $|X| = |[k] \mapsto X_k|$.

(3.9) **Lemma.** *There are natural homeomorphisms*

$$\begin{aligned} &|[q] \mapsto |[p] \mapsto W([p],[q])||_{\mathcal{T}} \\ &\quad \cong |[p] \mapsto (\mathrm{diag}\, W)_p| \\ &\quad \cong |[p] \mapsto |[q] \mapsto W([p],[q])||_{\mathcal{T}} \end{aligned}$$

for any bisimplicial set $W\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathcal{E}ns$. $\square$

The following theorem is the essential "tool" used in the proof of our main theorem.

(3.10) **Theorem.** *Suppose* $g\colon W \xrightarrow{\cdot} V\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathcal{E}ns$ *is a map of bisimplicial sets satisfying the condition for each* p,

(3.11) $\quad g([p], \_)\colon W([p], \_) \xrightarrow{\cdot} V([p], \_)\colon \Delta^{\mathrm{op}} \to \mathcal{E}ns$

is a WHE. Then $(\mathrm{diag}\, g)\colon (\mathrm{diag}\, W) \xrightarrow{\cdot} (\mathrm{diag}\, V)\colon \Delta^{\mathrm{op}} \to \mathcal{E}ns$ *is a WHE.* $\square$

Proof. Suppose condition (3.11) holds; then for each p,

$$|[q] \mapsto g([p],[q])|\colon |[q] \mapsto W([p],[q])| \mapsto |[q] \mapsto V([p],[q])|$$

is a HE in $\mathcal{T}op$. Both

$$[p] \mapsto |[q] \mapsto W([p],[q])|$$

and

$$[p] \mapsto |[q] \mapsto V([p],[q])|$$

are "good" simplicial spaces (in the sense of Segal [16; App. A]), since

$$|[q]\mapsto W(\sigma^i,[q])|: |[q]\mapsto W([p],[q])|\to|[q]\mapsto W([p+1],[q])|$$

are always closed cofibrations ([5; III, 3]). Segal proves that

$$|[p]\mapsto|[q]\mapsto g([p],[q])||_{\mathscr{T}}$$

is a HE in $\mathscr{Top}$ in Proposition A.1 [16]. Thus from Lemma 3.9

$$|[p]\mapsto(\operatorname{diag} g)_p|$$

is also a HE in $\mathscr{Top}$; and hence $(\operatorname{diag} g)$ is a WHE in $\mathscr{K}$. □

(3.12) *Remark.* From the symmetry of Lemma 3.9, it is clear that Theorem 3.10 holds when condition (3.11) is replaced by:

(3.13) for each q,

$$g(\_,[q]): W(\_,[q])\to V(\_,[q]) \qquad \text{is a WHE.}$$

(3.14) *Remark.* Theorem 3.10 seems to have been proved independently by Bousfield and Kan [1, p. 335], Segal [16], and Tornehave.

4. The Main Theorem

(4.1) **Theorem.** *Let θ: $\Delta\to\mathscr{Cat}$ be a functor such that*

(i) *there exists a natural transformation*

$$\eta: \theta \stackrel{\cdot}{\longrightarrow} \iota: \Delta\to\mathscr{Cat}$$

(ii) $\eta([k])$: $\theta[k]\to\iota[k]\equiv\mathbf{k}$ *is a SHE in $\mathscr{Cat}$ for all k. Then the induced natural transformation* (3.3) *of singular functors*

$$\tilde{\eta}: N\stackrel{\cdot}{\longrightarrow} S_\theta: \mathscr{Cat}\to\mathscr{K}$$

is a WHE; i.e., for every small category $\mathbf{A}$,

$$\tilde{\eta}(\mathbf{A}): N\mathbf{A}\to S_\theta(\mathbf{A})$$

is a WHE in $\mathscr{K}$. □

(4.2) *Remark.* Note that the homotopy inverses (ii) for each $\eta([k])$ are not collectively required to be natural in k. In fact, none of the examples detailed in Section 5 have natural homotopy inverses.

Since f: $X\to Y$ is a WHE in $\mathscr{K}$ iff $|f|$: $|X|\to|Y|$ is a HE in $\mathscr{Top}$, the following corollary holds:

(4.3) **Corollary.** *If θ: $\Delta\to\mathscr{Cat}$ is a functor satisfying* (i) *and* (ii) *of Theorem* 4.1, *then for every small category* $\mathbf{A}$,

$$|\tilde{\eta}(\mathbf{A})|\colon B\mathbf{A} \to |S_\theta(\mathbf{A})|$$

is a HE in $\mathcal{T}op$, i.e. $B\mathbf{A}$ and $|S_\theta(\mathbf{A})|$ are naturally of the same homotopy type. □

(4.4) *Remark.* Theorem 4.1 is not true, in general, for natural transformations η: $\iota \xrightarrow{\cdot} \theta$. In particular, consider the terminal natural transformation τ: $\iota \xrightarrow{\cdot} \mathbf{0}$, where $\mathbf{0}$: $\Delta \to \mathcal{C}at$ is the constant functor taking value $\mathbf{0}$. However, a simple extra condition is enough to prove:

(4.1′) **Theorem.** *Let θ: $\Delta \to \mathcal{C}at$ be a functor such that*

(i) *there exists a natural transformation η: $\iota \xrightarrow{\cdot} \theta$: $\Delta \to \mathcal{C}at$*

(ii) *$\eta([k])$: $\iota[k] \to \theta[k]$ is a SHE in $\mathcal{C}at$ for all k and*

(iii) *$\theta[1](\theta\delta^0(a), \theta\delta^1(b)) = \emptyset$, for any objects a and b in $\theta[0]$. Then the induced natural transformation of singular functors*

$$\tilde{\eta}\colon S_\theta \xrightarrow{\cdot} N\colon \mathcal{C}at \to \mathcal{K}$$

is a WHE. □

Sketch Proof of Theorem 4.1′. The internal condition (iii) above is seen to be equivalent to the existence of a simplicial map $\hat{x}$ such that the following diagram commutes

$$\begin{tikzcd}
\Delta[1] \cong N\mathbf{1} \arrow[r, "\hat{x}"] & S_\theta(\mathbf{1}) \arrow[r, "\tilde{\eta}(\mathbf{1})"] & N(\mathbf{1}) \cong \Delta[1] \\
\Delta[0] \cong N\mathbf{0} \arrow[u, "N(\delta^i)"'] \arrow[r, leftrightarrow, "\cong"] & S_\theta(\mathbf{0}) \arrow[u, "S_\theta(\delta^i)"'] \arrow[r, "\tilde{\eta}(\mathbf{0})", "\cong"'] & N\mathbf{0} \cong \Delta[0] \arrow[u, "N(\delta^i)"']
\end{tikzcd}$$

for $i = 0, 1$. Now we use this $\hat{x}\colon N\mathbf{1} \to S_\theta(\mathbf{1})$ in the proof of Proposition 3.2 to show that S_θ: $\mathcal{C}at \to \mathcal{K}$ preserves strong homotopies. Thus, with appropriate modifications, the proof of Theorem 4.1, given below, adapts to show $\tilde{\eta}$: $S_\theta \xrightarrow{\cdot} N$ is a WHE. □

(4.5) *Remark.* If θ: $\Delta \to \mathcal{C}at$ satisfies the hypotheses (i) and (ii) of Theorem (4.1), then so does θ^{op}: $\Delta \to \mathcal{C}at$. Since $\mathbf{k}^{\mathrm{op}} \cong \mathbf{k}$ naturally,

(i) η^{op}: $\theta^{\mathrm{op}} \xrightarrow{\cdot} \iota^{\mathrm{op}} \cong \iota$: $\Delta \to \mathcal{C}at$ and

(ii) $\eta^{\mathrm{op}}([k])$: $(\theta[k])^{\mathrm{op}} \to \mathbf{k}$ is a SHE for all k.

Hence $\tilde{\eta}^{\mathrm{op}}$: $N \xrightarrow{\cdot} S_{\theta^{\mathrm{op}}}$: $\mathcal{C}at \to \mathcal{K}$ is a WHE.

(4.6) *Remark.* The hypotheses for Theorem 4.1 as stated in the introduction are equivalent to those stated here. The statement "representable functor S_θ" is equivalent to S_θ being the singular functor for a θ: $\Delta \to \mathcal{C}at$. Since $\mathbf{k}$ is SC, (ii) is equivalent to $\theta[k]$ being SC. By the Yoneda lemma, a natural transformation $N \xrightarrow{\cdot} S_\theta$ is equivalent to one: $\theta \xrightarrow{\cdot} \iota$.

Let γ: $\Delta \to \mathcal{C}at$ be as in Example 5.13. We denote $\hat{\gamma}$ by Γ (see Remark 5.15).

(4.7) **Corollary.** *The adjunctions*

$$\mathrm{Id}_{\mathcal{K}} \xrightarrow{\cdot} S_\gamma \Gamma \quad \textit{and} \quad \Gamma S_\gamma \xrightarrow{\cdot} \mathrm{Id}_{\mathcal{C}at}$$

induce WHE's

$$X \to S_\gamma \Gamma X \quad \textit{and} \quad \Gamma S_\gamma \mathbf{A} \to \mathbf{A}$$

for all $X \in \mathscr{K}$ *and* $\mathbf{A} \in \mathscr{C}at$. □

Proof. Categorical realization c: $\mathscr{K} \to \mathscr{C}at$ is the left adjoint to N: $\mathscr{C}at \to \mathscr{K}$, and the adjunction $cN \xrightarrow{\cdot} \mathrm{Id}_{\mathscr{C}at}$ is invertible (2.7). Because the natural transformation $\gamma \xrightarrow{\cdot} \iota$ induces $N \xrightarrow{\cdot} S_\gamma$ (3.3), there is also a natural transformation $\Gamma \xrightarrow{\cdot} c$ (from adjoint functor theory [12; IV, 7]). Consider the commutative diagram of natural transformations

$$(4.8) \qquad \begin{tikzcd} cN \arrow[d, "\cong"', "\cdot"] & \Gamma N \arrow[l, "\cdot"'] \arrow[d, "\cdot"] \\ \mathrm{Id}_{\mathscr{C}at} & \Gamma S_\gamma \arrow[l, "\cdot"'] \end{tikzcd}$$

coming from the natural transformation $\gamma \xrightarrow{\cdot} \iota$, and the two adjunctions. In [10], it was shown that $\Gamma N \xrightarrow{\cdot} cN$ is a WHE. Theorem 4.1 implies $N \xrightarrow{\cdot} S_\gamma$ is a WHE and in [10], it is shown that Γ preserves WHE's; so $\Gamma N \xrightarrow{\cdot} \Gamma S_\gamma$ is a WHE. By the commutativity of the diagram (4.8), $\Gamma S_\gamma \xrightarrow{\cdot} \mathrm{Id}_{\mathscr{C}at}$ must be a WHE.

The composition natural transformation

$$\Gamma \xrightarrow{\cdot} \Gamma S_\gamma \Gamma \xrightarrow{\cdot} \Gamma$$

given by the two adjunctions is the identity (for any adjoint pair); so is a WHE. By the above, $\Gamma S_\gamma \xrightarrow{\cdot} \mathrm{Id}_{\mathscr{C}at}$ is a WHE; thus $\Gamma S_\gamma \Gamma \xrightarrow{\cdot} \Gamma$ is also. These two facts show that $\Gamma \xrightarrow{\cdot} \Gamma S_\gamma \Gamma$ is a WHE. Latch [10] shows that f: $X \to Y$ is a WHE in $\mathscr{K}$ iff Γf: $\Gamma X \to \Gamma Y$ is a WHE in $\mathscr{C}at$; hence $\mathrm{Id}_{\mathscr{K}} \xrightarrow{\cdot} S_\gamma \Gamma$ is a WHE in $\mathscr{K}$. This concludes the proof. □

Proof of Theorem 4.1. The proof is done by applying Segal's Theorem 3.10 to several pairs of functors from $\mathscr{C}at$ to the category $[\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathscr{E}ns]$ of bisimplicial sets.

Step 1. Define F: $\mathscr{C}at \to [\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathscr{E}ns]$ by

$$\begin{aligned} (4.9) \quad F(\mathbf{A})([p],[q]) &\equiv N(\mathscr{C}at(\iota[q], \mathbf{A}))_p \\ &\equiv \mathrm{Mor}(\mathbf{p}, \mathscr{C}at(\iota[q], \mathbf{A})) \\ &\cong \mathrm{Mor}(\iota[q] \times \mathbf{p}, \mathbf{A}) \end{aligned}$$

where the last equivalence is simply the adjoint relation (2.2) for the "internal-hom", and the fact $\iota[q] \times \mathbf{p} \cong \mathbf{p} \times \iota[q]$. Similarly, G: $\mathscr{C}at \to [\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathscr{E}ns]$ is given by

$$(4.10)\quad \begin{aligned} G(\mathbf{A})([p],[q]) &\equiv N(\mathscr{C}at(\theta[q],\mathbf{A}))_p \\ &\equiv \mathrm{Mor}(\mathbf{p},\mathscr{C}at(\theta[q],\mathbf{A})) \\ &\cong \mathrm{Mor}(\theta[q]\times\mathbf{p},\mathbf{A}) \\ &\cong \mathrm{Mor}(\theta[q],\ \mathscr{C}at(\mathbf{p},\mathbf{A})) \\ &\equiv S_\theta(\mathscr{C}at(\mathbf{p},\mathbf{A}))_q. \end{aligned}$$

The natural transformation $\eta\colon \theta \xrightarrow{\cdot} \imath$ induces a natural transformation

$$(4.11)\quad \bar{\eta}\colon F \xrightarrow{\cdot} G\colon \mathscr{C}at \to [\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}},\mathscr{E}ns],$$

where

$$(4.12)\quad \begin{aligned} \bar{\eta}(\mathbf{A})([p],\ [q]) &\equiv N(\mathscr{C}at(\eta([q]),\mathbf{A}))_p \\ &\cong \mathrm{Mor}(\eta([q])\times\mathbf{p},\mathbf{A}). \end{aligned}$$

Next we show, for each $\mathbf{A}\in\mathscr{C}at$,

$$\bar{\eta}(\mathbf{A})\colon F(\mathbf{A})\to G(\mathbf{A})$$

satisfies the hypothesis of Theorem 3.10; i.e.

$$\bar{\eta}(\mathbf{A})(\_,[q])\colon F(\mathbf{A})(\_,[q])\to G(\mathbf{A})(\_,[q])$$

is a WHE for every q. Since $\eta\colon \theta[q]\to\imath[q]$ is a SHE, Lemma 3.6 shows that

$$\mathscr{C}at(\eta([q]),\mathbf{A})\colon \mathscr{C}at(\imath[q],\mathbf{A})\to\mathscr{C}at(\theta[q],\mathbf{A})$$

is always a SHE. Thus Lemma 3.1 implies

$$\begin{array}{ccccc} N(\mathscr{C}at(\eta([q]),\mathbf{A}))\colon & N(\mathscr{C}at(\imath[q],\mathbf{A})) & \to & N(\mathscr{C}at(\theta[q],\mathbf{A})) \\ ||| & ||| & & ||| \\ \bar{\eta}(\mathbf{A})(\_,[q]) \quad : & F(\mathbf{A})(\_,[q]) & \to & G(\mathbf{A})(\_,[q]) \end{array}$$

is a SHE, and thus a WHE. Hence by Theorem 3.10,

$$(4.13)\quad \operatorname{diag}\bar{\eta}(\mathbf{A})\colon \operatorname{diag} F(\mathbf{A})\to\operatorname{diag} G(\mathbf{A})$$

is a WHE.

Step 2. Define $\bar{G}\colon \mathscr{C}at\to[\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}},\mathscr{E}ns]$ by

$$(4.14)\quad \begin{aligned} \bar{G}(\mathbf{A})([p],[q]) &\equiv \mathrm{Mor}(\theta[q],\mathbf{A}) \\ &\equiv S_\theta(\mathbf{A})_q. \end{aligned}$$

Next we construct a natural transformation

$$(4.15)\quad \bar{\mu}\colon \bar{G}\xrightarrow{\cdot} G\colon \mathscr{C}at\to[\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}},\mathscr{E}ns].$$

Let $\theta_2\colon \Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}\to\mathscr{C}at$ be given by

$$\theta_2([p],[q])=\theta[q];$$

and

$$\mu\colon \theta \times \imath \xrightarrow{\cdot} \theta_2\colon \mathscr{C}at \to [\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathscr{E}ns]$$

be the natural transformation given by the projection functor

$$\mu([p],[q])\colon \theta[q] \times \mathbf{p} \to \theta[q].$$

Then μ induces the natural transformation

(4.16) $\quad \mathrm{Mor}(\mu, \_)\colon \mathrm{Mor}(\theta_2(\_,\_), \_) \xrightarrow{\cdot} \mathrm{Mor}(\theta \times \imath(\_,\_)\_).$

But by (4.10),

$$\mathrm{Mor}\,((\theta \times \imath)([p],[q]), \mathbf{A}) \cong G(\mathbf{A})([p],[q]).$$

Thus $\mathrm{Mor}(\mu, \_)$ is equivalent to a natural transformation

(4.17) $\quad \bar{\mu}\colon \bar{G} \xrightarrow{\cdot} G\colon \mathscr{C}at \xrightarrow{\cdot} [\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathscr{E}ns].$

To show

(4.18) $\quad \mathrm{diag}\,\bar{\mu}(\mathbf{A})\colon \mathrm{diag}\,\bar{G}(\mathbf{A}) \to \mathrm{diag}\,G(\mathbf{A})$

is a WHE, we prove, as above that $\bar{\mu}$ satisfies the hypothesis of Theorem 3.10; i.e. $\bar{\mu}(\mathbf{A})([p], \_)\colon \bar{G}(\mathbf{A})([p], \_) \to G(\mathbf{A})([p], \_)$ is a WHE for every p. From (4.10) and (4.14), it suffices to show

(4.19) $\quad \bar{\mu}(\mathbf{A})([p], \_)\colon S_\theta(\mathbf{A}) \to S_\theta(\mathscr{C}at(\mathbf{p}, \mathbf{A}))$

is a WHE for every p. By Corollary 3.8, $\mathbf{A} \to \mathscr{C}at(\mathbf{p}, \mathbf{A})$ is a SHE. By Proposition 3.2, S_θ preserves SHE's; and we have (4.19) is a SHE, and thus a WHE, completing Step 2.

Step 3. Define $\bar{F}\colon \mathscr{C}at \to [\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathscr{E}ns]$ by the rule

(4.20) $\quad \bar{F}(\mathbf{A})([p],[q]) \equiv \mathrm{Mor}\,(\imath[q], \mathbf{A})$

$$\equiv N(\mathbf{A})_q.$$

Then as in Step 2, there is a natural transformation

$$\bar{\rho}\colon \bar{F} \xrightarrow{\cdot} F\colon \mathscr{C}at \to [\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathscr{E}ns].$$

The same argument as in Step 2 with $\imath[q]$ in place of $\theta[q]$, proves that

(4.21) $\quad \mathrm{diag}\,\bar{\rho}\,(\mathbf{A})\colon \mathrm{diag}\,\bar{F}(\mathbf{A}) \to \mathrm{diag}\,F(\mathbf{A})$

is a WHE for every small category $\mathbf{A}$.

Final Step. Note that S_θ and $\mathrm{diag}\,\bar{G}$, and $S_\imath \equiv N$ and $\mathrm{diag}\,\bar{F}$ are naturally equivalent. Then for each small category $\mathbf{A}$, the following natural diagram commutes:

$$\begin{tikzcd}
S_\theta(\mathbf{A}) \arrow[r, "\cong"] & \operatorname{diag}\bar{G}(\mathbf{A}) \arrow[d, "\operatorname{diag}\bar{\mu}(\mathbf{A})"] \\
& \operatorname{diag}G(\mathbf{A}) \\
& \operatorname{diag}F(\mathbf{A}) \arrow[u, "\operatorname{diag}\bar{\eta}(\mathbf{A})"'] \\
N(\mathbf{A}) \arrow[uuu, "\tilde{\eta}(\mathbf{A})"] \arrow[r, "\cong"] & \operatorname{diag}\bar{F}(\mathbf{A}). \arrow[u, "\operatorname{diag}\bar{\rho}(\mathbf{A})"']
\end{tikzcd}$$

Since $\operatorname{diag}\bar{\mu}(\mathbf{A})$ (4.18), $\operatorname{diag}\bar{\rho}(\mathbf{A})$ (4.21), and $\operatorname{diag}\bar{\eta}(\mathbf{A})$ (4.13) are all WHE's,

$$\tilde{\eta}(\mathbf{A})\colon N\mathbf{A} \to S_\theta(\mathbf{A})$$

is a WHE; completing the proof of the theorem. □

5. Examples

Although the nerve functor $N \equiv S_\iota\colon \mathscr{C}at \to \mathscr{K}$ has a "simple" description, it has certain disadvantages. For example, $N(\mathbf{A})$ is a Kan complex iff $\mathbf{A}$ is a groupoid (e.g. see [11]). In particular, $N(\mathbf{k}) \cong \Delta[k]$ is not a Kan complex for $k \geqq 1$. The "straightforward" calculation of higher homotopy groups for $N\mathbf{A}$ using Kan's methods [9] and $N\mathbf{A}$'s "simple" structural definition is not practical, in general. The following catalogue of homotopy replacements for nerve is offered with the hope that some constructions in categorical and simplicial homotopy theory may become clearer.

The format for each of the examples is as follows. We specify each θ: $\Delta \to \mathscr{C}at$ by giving the representing category $\theta[k]$, for each k, and specifying $\theta(\delta^i)\colon \theta[k-1] \to \theta[k]$, $\theta(\sigma^i)\colon \theta[k+1] \to \theta[k]$. Next, we indicate why each $\theta[k]$ is SC. Lastly, we define the natural transformation $\eta\colon \theta \xrightarrow{\cdot} \iota\colon \Delta \to \mathscr{C}at$, leaving the details here to the reader. Special properties and remarks pertinent to each example will follow in the form of numbered remarks.

(5.1) *Example.* Let Δ_{face} denote the (non-full) subcategory of Δ whose objects are those of Δ and whose morphisms are all order-preserving injections α: $[p] \rightarrowtail [k]$. Define $\xi\colon \Delta \to \mathscr{C}at$ as follows:

(i) $\xi[k] = (\Delta_{\text{face}} \downarrow [k])^{\text{op}}$ with *objects* $\alpha\colon [p] \rightarrowtail [k]$ and *morphisms* $\mu\colon \alpha \to \beta$ iff

$$\begin{tikzcd}
{[q]} \arrow[r, tail, "\mu"] \arrow[dr, tail, "\beta"'] & {[p]} \arrow[d, tail, "\alpha"] \\
& {[k]}
\end{tikzcd}$$

commutes in Δ_{face}.

(ii) $\xi(\delta^i)$: $(\Delta_{\text{face}}\downarrow[k-1])^{\text{op}}\to(\Delta_{\text{face}}\downarrow[k])^{\text{op}}$ is given by $\xi(\delta^i)(\alpha)=\delta^i\circ\alpha$: $[p]\rightarrowtail[k]$.

(iii) $\xi(\sigma^i)$: $(\Delta_{\text{face}}\downarrow[k+1])^{\text{op}}\to(\Delta_{\text{face}}\downarrow[k])^{\text{op}}$ has a more complicated description. If α: $[p]\to[k+1]$, consider the mono-epi factorization of $\sigma^i\circ\alpha$:

(5.2)
$$\begin{tikzcd} {[p]} \arrow[r, tail, "\alpha"] \arrow[d, two heads, "(\sigma^i\alpha)^0"] & {[k+1]} \arrow[d, two heads, "\sigma^i"] \\ {[p']} \arrow[r, tail, "(\sigma^i\alpha)^+"] & {[k].} \end{tikzcd}$$

Then $\xi(\sigma^i)(\alpha)\equiv(\sigma^i\alpha)^+$.

For completeness, we give an equivalent description of ξ: $\Delta\to\mathscr{C}at$. Let $\mathbf{1}'$ be the category with two objects $\{-1,0\}$ and one nonidentity morphism $-1\to 0$. Then

$$\xi[k]\cong\left(\prod_{i=0}^{k}\mathbf{1}'\setminus\langle-1\rangle\right)^{\text{op}}$$

where $\langle-1\rangle=\langle-1,\ -1,\dots,-1\rangle$ in $\prod_{i=0}^{k}\mathbf{1}'$. The injection α: $[p]\rightarrowtail[k]$ is represented uniquely by $\langle u_0, u_1,\dots,u_k\rangle$ where

$$u_i=\begin{cases}-1, & \text{if } \alpha^{-1}(i)=\emptyset\\ \ \ 0, & \text{if } \alpha^{-1}(i)\neq\emptyset.\end{cases}$$

Also $\xi(\delta^i)$: $\left(\prod_{i=0}^{k-1}\mathbf{1}'\setminus\langle-1\rangle\right)^{\text{op}}\to\left(\prod_{i=0}^{k}\mathbf{1}'\setminus\langle-1\rangle\right)^{\text{op}}$ is given by

(5.3) $\quad \xi(\delta^i)\langle u_0,u_1,\dots,u_{k-1}\rangle\equiv\langle u_0,\dots,\underset{i}{-1},\dots,u_{k-1}\rangle;$

and $\xi(\sigma^i)$: $\left(\prod_{i=0}^{k+1}\mathbf{1}'\setminus\langle-1\rangle\right)^{\text{op}}\to\left(\prod_{i=0}^{k}\mathbf{1}'\setminus\langle-1\rangle\right)^{\text{op}}$ by

$$\xi(\sigma^i)\langle u_0,u_1,\dots,u_{k+1}\rangle\equiv\langle u_0,\dots,w_i,\dots,u_{k+1}\rangle,$$

where $w_i=\sup\{u_i,\ u_{i+1}\}$.

Each $\xi[k]$ has initial object $\mathrm{Id}_{[k]}$: $[k]\to[k]$, since α: $\mathrm{Id}_{[k]}\to\alpha$ uniquely in $(\Delta_{\text{face}}\downarrow[k])^{\text{op}}$. Hence $\xi[k]$ is SC, for every k by (3.7).

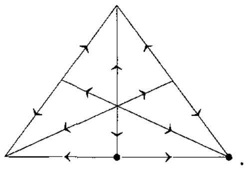
The natural transformation η: $\xi\overset{\bullet}{\longrightarrow}\iota$, relating ξ to ι, is "first" evaluation; i.e. for each k, $\eta([k])$: $\xi[k]\to\mathbf{k}$ is the functor defined by

(5.4) $\quad \eta([k])\,(\alpha\colon[p]\rightarrowtail[k])=\alpha(0)\in\mathbf{k}.$

(5.5) *Remark*. This example has strong geometric appeal. Each $\xi[k]$ has a definite "cell-like" structure. For example, $\xi[1]$ is pictured by

$$\bullet \longleftarrow \bullet \longrightarrow \bullet,$$

and $\xi[2]$ is depicted by



(5.6) *Remark.* From (2.9),

$$S_{\xi^{\mathrm{op}}}(\mathbf{A})_k \equiv \mathrm{Mor}(\xi^{\mathrm{op}}[k], \mathbf{A})$$
$$\cong \mathrm{Mor}(N\xi^{\mathrm{op}}[k], N\mathbf{A}),$$

naturally. But $N\xi^{\mathrm{op}}$: $\Delta \to \mathscr{K}$ is the functor Δ': $\Delta \to \mathscr{K}$ [7] used in Kan's construction of Ex: $\mathscr{K} \to \mathscr{K}$. Hence $S_{\xi^{\mathrm{op}}}$: $\mathscr{C}at \to \mathscr{K}$ is naturally equivalent to $\mathrm{Ex} \circ N$: $\mathscr{C}at \to \mathscr{K}$, i.e.

(5.7) $S_{\xi^{\mathrm{op}}}(\mathbf{A}) \cong \mathrm{Ex}(N\mathbf{A})$

for each $\mathbf{A} \in \mathscr{C}at$. Of course, the work of Kan [7] gives Theorem 4.1 for this case.

The next two remarks were observed by the first author and R. Fritsch.

(5.8) *Remark.* Note that $S_\xi(\mathbf{A})$ is a Kan complex iff $\mathbf{A}$ satisfies the following two conditions:

(i) For each diagram $p \xrightarrow{a} r \xleftarrow{b} q$ in $\mathbf{A}$, there exists a commutative square

$$\begin{tikzcd} p \arrow[r, "a"] & r \\ p' \arrow[u, dashed, "b'"] \arrow[r, dashed, "a'"] & q \arrow[u, "b"'] \end{tikzcd}$$

(ii) If $p \underset{a'}{\overset{a}{\rightrightarrows}} q \xrightarrow{b} r$, i.e. $ba = ba'$, then there exists a morphism $p' \xrightarrow{u} p$ such that $au = a'u$.

Conditions (i) and (ii) say that $\mathbf{A}$ admits a calculus of right fractions ([5; I, 2]). In particular, since the categories $\mathbf{k}$ clearly satisfy (i) and (ii), $S_\xi(\mathbf{k})$ are all Kan complexes. In an analogous fashion we see that $S_{\xi^{\mathrm{op}}}(\mathbf{k})$ and hence $\mathrm{Ex}(N\mathbf{k}) \cong \mathrm{Ex}(\Delta[k])$ are Kan complexes.

(5.9) *Remark.* Although $S_{\xi^{\mathrm{op}}}$: $\mathscr{C}at \to \mathscr{K}$ preserves WHE's, its left adjoint $\hat{\xi}^{\mathrm{op}}$: $\mathscr{K} \to \mathscr{C}at$ (see Lemma 2.11) does not preserve WHE's. See [4].

(5.10) *Example.* Let λ: $\Delta \to \mathscr{C}at$ be defined as follows:

(i) $\lambda[k]$ is the small category having as *objects* pairs (α, j), where α: $[q] \rightarrowtail [k]$ in Δ_{face} and $0 \leqq j \leqq q$. A *morphism* μ: $(\alpha, j) \to (\beta, l)$ of $\lambda[k]$ "is" an injection μ: $[p] \rightarrowtail [q]$ such that $\alpha \circ \mu = \beta$ in Δ_{face} and $\alpha(j) \leqq \beta(l)$.

(ii) $\lambda(\delta^i)$: $\lambda[k-1]\to\lambda[k]$ is defined by $\lambda(\delta^i)\,(\alpha,j)\equiv(\delta^i\circ\alpha,j)$.

(iii) $\lambda(\sigma^i)$: $\lambda[k+1]\to\lambda[k]$ is given by $\lambda(\sigma^i)(\alpha,j)\equiv((\sigma^i\,\alpha)^+,\ (\sigma^i\,\alpha)^0(j))$, where $(\sigma^i\,\alpha)^+\circ(\sigma^i\,\alpha)^0=\sigma^i\circ\alpha$ is the "mono-epi" factorization (5.2) of $\sigma^i\circ\alpha$.

As above, each $\lambda[k]$ has initial object $(\mathrm{Id}_{[k]},0)$, since α: $(\mathrm{Id}_{[k]},0)\to(\alpha,j)$ is always defined and is unique. Hence $\lambda[k]$ is SC, for every k.

The natural transformation η: $\lambda\overset{\cdot}{\longrightarrow}\iota$ is "evaluate", i.e. for each k, $\eta([k])$: $\lambda[k]\to\mathbf{k}$ is the functor given by evaluation

$$\eta([k])(\alpha,j)=\alpha(j)\in\mathbf{k}.$$

(5.11) *Remark.* Although S_λ: $\mathscr{C}at\to\mathscr{K}$ preserves SHE's (by Theorem 4.1), its left adjoint $\hat{\lambda}$: $\mathscr{K}\to\mathscr{C}at$ (see Lemma 2.11) does not preserve SHE's. See [4].

(5.12) *Remark.* $\lambda=c\circ\mathfrak{g}$: $\Delta\to\mathscr{C}at$, where the functor $\mathfrak{g}$: $\Delta\to\mathscr{K}$ was used by R. Fritsch in [3] to show that under certain conditions an isomorphism between the one-skeletons of $\hat{\mathfrak{g}}\,X$ and $\hat{\mathfrak{g}}\,Y$ in $\mathscr{K}$ implies the existence of an isomorphism between X and Y.

(5.13) *Example.* Define γ: $\Delta\to\mathscr{C}at$ as the "comma category" functor:

(i) $\gamma[k]$ is the small comma category $(\Delta\downarrow[k])^{\mathrm{op}}$ with *objects* all order preserving maps α: $[p]\to[k]$ and *morphisms* μ: $\alpha\to\beta$ such that

$$\begin{tikzcd}[q] \arrow[rr, "\mu"] \arrow[dr, "\beta"'] & & [p] \arrow[dl, "\alpha"] \\ & [k] & \end{tikzcd}$$

commutes in Δ.

(ii) $\gamma(\delta^i)\equiv(\Delta\downarrow\delta^i)^{\mathrm{op}}$: $(\Delta\downarrow[k-1])^{\mathrm{op}}\to(\Delta\downarrow[k])^{\mathrm{op}}$ is simply composition with δ^i, i.e.

$$\gamma(\delta^i)(\alpha\colon[p]\to[k-1])=\delta^i\circ\alpha.$$

(iii) Similarly, $\gamma(\sigma^i)\equiv(\Delta\downarrow\sigma^i)^{\mathrm{op}}$: $(\Delta\downarrow[k+1])^{\mathrm{op}}\to(\Delta\downarrow[k])^{\mathrm{op}}$.

The following equivalent formulation of γ: $\Delta\to\mathscr{C}at$ was developed by the first author and E. Cooper. Let Δ' be the category formed from Δ by formally adding an initial object $\{-1\}$. Then $\gamma[k]\cong\left(\prod_{i=0}^{k}\Delta'\smallsetminus\langle-1\rangle\right)^{\mathrm{op}}$, where $\langle-1\rangle=\langle-1,-1,\dots,-1\rangle$ of $\prod_{i=0}^{k}\Delta'$. Each α: $[p]\to[k]$ of $(\Delta\downarrow[k])^{\mathrm{op}}$ is represented in $\left(\prod_{i=0}^{k}\Delta'\smallsetminus\langle-1\rangle\right)^{\mathrm{op}}$ by $\langle v_0,v_i,\dots,v_k\rangle$, where

$$v_i=\begin{cases}[m_i], & \text{if the number of elements in } \alpha^{-1}(i) \text{ is } m_i+1\\ -1, & \text{if } \alpha^{-1}(i)=\emptyset.\end{cases}$$

The functor $\gamma(\delta^i)$: $\left(\prod_{i=0}^{k-1} \Delta' \searrow \langle -1 \rangle\right)^{\mathrm{op}} \to \left(\prod_{i=0}^{k} \Delta' \searrow \langle -1 \rangle\right)^{\mathrm{op}}$ is similar to the alternate description (5.3) of $\xi(\delta^i)$; i.e.

$$\gamma(\delta^i)(\langle v_0, \dots, v_{k-1} \rangle) = \left\langle v_0, \dots, \underset{i}{-1}, \dots, v_{k-1} \right\rangle.$$

However, the description for

$$\gamma(\sigma^i)\colon \left(\prod_{i=0}^{k+1} \Delta' \searrow \langle -1 \rangle\right)^{\mathrm{op}} \to \left(\prod_{i=0}^{k} \Delta' \searrow \langle -1 \rangle\right)^{\mathrm{op}}$$

is more complicated; i.e.

$$\gamma(\sigma^i)(\langle v_0, \dots, v_{k+1} \rangle) = \langle v_0, \dots, w_i, \dots, v_{k+1} \rangle$$

where

$$w_i = \begin{cases} [m_i + m_{i+1} + 1], & \text{if } v_i = [m_i], \quad v_{i+1} = [m_{i+1}] \\ [m_i], & \text{if } v_i = [m_i] \text{ and } v_{i+1} = -1 \\ [m_{i+1}], & \text{if } v_i = -1 \text{ and } v_{i+1} = [m_{i+1}] \\ -1, & \text{if } v_i = -1 = v_{i+1}. \end{cases}$$

Each $\gamma[k]$ has initial object $\mathrm{Id}_{[k]}\colon [k] \to [k]$ since $\alpha\colon \mathrm{Id}_{[k]} \longrightarrow \alpha$ uniquely in $(\Delta \downarrow [k])^{\mathrm{op}}$; thus $\gamma[k]$ is SC.

The natural transformation $\eta\colon \gamma \stackrel{\cdot}{\longrightarrow} \iota$ is "first" evaluation, as in the case for ξ; i.e. for each k, $\eta([k])\colon \gamma[k] \to \mathbf{k}$ is the functor defined by

$$\eta([k])\colon (\alpha\colon [p] \to [k]) \mapsto \alpha(0) \in \mathbf{k}.$$

(5.14) *Remark.* There is the "characteristic set" evaluation natural transformation $\chi\colon \gamma \stackrel{\cdot}{\longrightarrow} \xi$ given for each k by

$$\chi([k])(\langle v_0, \dots, v_k \rangle) = \langle u_0, \dots, u_k \rangle,$$

where $u_i = \begin{cases} 0, & \text{if } v_i \neq -1 \\ -1, & \text{if } v_i = -1. \end{cases}$

Clearly,

$$\begin{tikzcd} \gamma \arrow[rr, "\cdot"', "\chi"'] \arrow[dr, "\eta"', "\cdot"] & & \xi \arrow[dl, "\eta", "\cdot"'] \\ & \iota & \end{tikzcd}$$

is a commutative diagram of natural transformations which induces a corresponding commutative diagram of singular functors

$$\begin{tikzcd} S_\gamma & & S_\xi \arrow[ll, "\cdot", "\tilde{\chi}"'] \\ & N \arrow[ul, "\tilde{\eta}", "\cdot"'] \arrow[ur, "\tilde{\eta}"', "\cdot"] & \end{tikzcd}$$

If $\mathbf{A}$ is a one way delta (i.e. at most one of the morphism sets $\mathbf{A}(p,q)$, $\mathbf{A}(q,p)$ is nonempty for each pair of objects p, q in $\mathbf{A}$), then

$$\tilde{\chi}(\mathbf{A})\colon S_{\zeta}(\mathbf{A}) \xrightarrow{\cong} S_{\gamma}(\mathbf{A})$$

is an isomorphism of simplicial sets. In particular, it follows from Remark 5.8 that $S_{\gamma}(\mathbf{k})$ is Kan for every k, since $\mathbf{k}$ is clearly a one way delta.

(5.15) *Remark.* In contrast with the other left adjoint realization functors from $\mathscr{K}$ to $\mathscr{C}at$, $\hat{\gamma} \equiv \Gamma\colon \mathscr{K} \to \mathscr{C}at$ preserves WHE's (see [10]). Since Γ plays an important role in the study of the relationship between the homotopic categories of $\mathscr{C}at$ and $\mathscr{K}$, we give an explicit description for this "category of simplices" functor: For each simplicial set X, ΓX has as *objects* the collection $\bigsqcup_{k \geq 0} X_k$ of *all* simplices of X and as *morphisms* $\alpha\colon \langle x, [k]\rangle \to \langle X(\alpha)x, [p]\rangle$, for every α: $[p] \to [k]$ in Δ. If $f\colon X \to Y$ in $\mathscr{K}$, then $\Gamma f\colon \Gamma X \to \Gamma Y$ is defined by $\Gamma f(\langle x, [k]\rangle) = \langle f_k x, [k]\rangle$.

(5.16) *Example.* Let $U\colon \Delta_{\text{face}} \hookrightarrow \Delta$ be the (non-full) inclusion of Δ_{face} into Δ. Define $\omega\colon \Delta \to \mathscr{C}at$ as the "comma category" functor:

(i) $\omega[k]$ is the small comma category $(U\downarrow[k])^{\text{op}}$ with *objects* all order preserving maps $\alpha\colon [p] \to [k]$ and *morphisms* $\mu\colon \alpha \to \beta$ such that

$$\begin{tikzcd}[g] \arrow[rr, tail, "\mu"] \arrow[dr, "\beta"'] & & [p] \arrow[dl, "\alpha"] \\ & [k] & \end{tikzcd}$$

commutes in Δ and $\mu \in \Delta_{\text{face}}$.

(ii) $\omega(\delta^i) \equiv (U\downarrow\delta^i)^{\text{op}}\colon (U\downarrow[k-1])^{\text{op}} \to (U\downarrow[k])^{\text{op}}$ is simply composition with δ^i; i.e. $\omega(\delta^i)(\alpha\colon [p] \to [k-1]) = \delta^i \circ \alpha$

(iii) Similarly, $\omega(\sigma^i) \equiv (U\downarrow\sigma^i)^{\text{op}}\colon (U\downarrow[k+1])^{\text{op}} \to (U\downarrow[k])^{\text{op}}$.

To see that each $\omega[k]$ is SC, consider the functor $F[k]\colon \omega[k] \to \omega[k]$ which is defined as follows:

$$F[k](\beta\colon [p] \to [k]) \equiv \bar{\beta}\colon [p+k+1] \to [k]$$

where $[p+k+1]$ is represented by the totally ordered set having elements

$$\{0, 1, 2, \ldots, p, \bar{0}, \bar{1}, \ldots, \bar{k}\}$$

with ordering defined by

$$0 < 1 < \cdots < p,$$

$$\bar{0} < \bar{1} < \cdots < \bar{k},$$

$$r < \bar{s} \quad \text{if } \beta(r) \leqq s \quad \text{in } [k],$$

$$\bar{s} < r \quad \text{if } \beta(r) \nleqq s \quad \text{in } [k];$$

and with $\bar{\beta}(r) \equiv \beta(r)$ and $\bar{\beta}(\bar{s}) \equiv s$. There is a natural transformation

$$u[k]\colon F[k] \xrightarrow{\;\cdot\;} \mathrm{Id}_{\omega[k]}\colon \omega[k] \to \omega[k]$$

given by

$$u[k](\beta) = u\colon \bar{\beta} \to \beta,$$

where u: $[p] \rightarrowtail [p+k+1]$ is the unique injection making

$$\begin{tikzcd}[p] \arrow[rr, tail, "u"] \arrow[dr, "\beta"'] & & {[p+k+1]} \arrow[dl, "\bar{\beta}"] \\ & {[k]} & \end{tikzcd}$$

commute.

Similarly, there is a natural transformation

$$v[k]: F[k] \xrightarrow{\cdot} \Delta(\mathrm{ld}_{[k]}): \omega[k] \to \omega[k]$$

where $\Delta(\mathrm{Id}_{[k]})$ is the constant functor with value $\mathrm{ld}_{[k]}$: $[k] \to [k]$ in $\omega[k]$. Thus $\omega[k]$ is SC (via a two-stage homotopy).

The natural transformation η: $\omega \xrightarrow{\cdot} \iota$ is "first" evaluation, as in the case for γ and ξ.

(5.17) *Remark.* Using arguments similar to those for $\hat{\gamma} \equiv \Gamma$: $\to$ (See [10]), the left adjoint $\hat{\omega} \equiv \Gamma_\omega$: $\mathscr{K} \to \mathscr{C}at$ preserves WHE's. The dual $\Gamma_\omega^{\mathrm{op}}$: $\mathscr{K} \to \mathscr{C}at$ is the functor Λ: $\mathscr{K} \to \mathscr{C}at$ used by Lee in [11]. Furthermore, for each simplicial set X, $\Gamma_\omega X$ is the subcategory of ΓX with the same objects, but only having morphisms μ: $\langle x, [k] \rangle \to \langle X(\mu)x, [p] \rangle$ for each μ: $[p] \rightarrowtail [k]$ in Δ_{face}.

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