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Elastic constants and their admissible values for incompressible and slightly compressible anisotropic materials

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Summary. Constitutive relations for incompressible (slightly compressible) anisotropic materials cannot (could hardly) be obtained through the inversion of the generalized Hooke's law since the corresponding compliance tensor becomes singular (ill-conditioned) in this case. This is due to the fact that the incompressibility (slight compressibility) condition imposes some additional constraints on the elastic constants. The problem requires a special procedure discussed in the present paper. The idea of this procedure is based on the spectral decomposition of the compliance tensor but leads to a closed formula for the elasticity tensor without explicit using the eigenvalue problem solution. The condition of nonnegative (positive) definiteness of the material tensors restricts the elastic constants to belong to an admissible value domain. For orthotropic and transversely isotropic incompressible as well as isotropically compressible materials the corresponding domains are illustrated graphically.

1 Introduction

As a research object incompressible and slightly compressible anisotropic materials have drawn attention only recently which has mostly been motivated by interest in the numerical simulation of soft biological tissues and first of all human organs. Probably for this reason the constitutive equations for these structures are not well established in literature except for some special cases of anisotropy (see, e.g., [1], [2]).

The main problem in the formulation of the constitutive relations is due to the incompressibility condition imposing some constraints on the elastic constants such that the compliance tensor of the material becomes singular. Thus, the stress-strain (constitutive) relations cannot be obtained through the direct inversion of the Hooke's law which requires a special procedure to be discussed in the present paper in detail. The idea of this procedure is based on the spectral decomposition of the compliance tensor but leads to a closed formula for the elasticity tensor without explicit using the eigenvalue problem solution. Alternatively, this closed formula can also be obtained as the so-called generalized inverse of the compliance tensor [3].

Since the nature avoids any extremes the slight compressibility seems to be a more realistic assumption than ideal incompressibility. For a better insight into the slight compressibility we introduce here a notion of "weak" internal material constraint. This is a constitutive restriction (inequality) caused by one vanishing eigenvalue of the compliance tensor or, vice versa, large eigenvalue of the elasticity tensor.

Slightly compressible or in other words nearly incompressible materials are characterized by the bulk modulus which is large by comparison with other elastic moduli. In contrast to isotropy the bulk modulus of an anisotropic compressible material is a function of a stress and does not generally represent a material constant. Therefore one requires the above restriction to be satisfied for all admissible material states. This in turn can be achieved in an anisotropic solid by imposing not necessarily only one but also by a superposition of several weak internal constraints.

For the case of only one weak internal material constraint we will prove the volumetric response of a slightly compressible anisotropic solid to be purely isotropic. Such isotropically compressible anisotropic materials sometimes referred in literature to as *quasi-isotropic* [4] are frequently used as a constitutive model in the numerical analysis (see, e.g., [5]). It is worth mentioning that only for these solids the bulk modulus is independent of the stress state and, hence, represents a material constant. In this case the eigenstate corresponding to the weak internal constraint is a priori known (the identity tensor), such that we can formulate the closed formula for the elasticity tensor taking into account anisotropic materials with isotropic volumetric response. This formula enables to avoid the numerical inversion of the compliance tensor being ill-conditioned for nearly incompressible solids.

Finally, the closed formula presented is applied to derive the elasticity tensor for orthotropic and transversely isotropic materials being of special importance for engineering practice. By means of incompressibility (isotropic compressibility) condition the Poisson's ratio can be expressed in terms of the Young's (and bulk) moduli which reduces the number of independent material parameters. The requirement of positive definiteness (or in the case of incompressibility semi-definiteness) of the compliance tensor imposes some additional restrictions on the remaining elastic constants. These restrictions are well-established in literature for compressible orthotropic materials (see, e.g., [6], [7]) but, to our best knowledge, absolutely unknown for incompressible as well as isotropically compressible ones. Using eigenvalues of the compliance tensor we formulate these constraints yielding an admissible value domain the Young's moduli belong to. For orthotropic and transversely isotropic incompressible as well as isotropically compressible materials the corresponding domains are illustrated graphically.

2 Basic notations and definitions

Throughout the paper we will use absolute tensor notation and abide by the algebra of second- and fourth-order tensors by Itskov [8]. Some of important definitions to be exploited in the further derivations are recorded below.

Let Lin be a set of all linear mappings of a three-dimensional vector space V over reals into itself the elements of which are called second-order tensors (bold capitals). Fourth-order tensors (bold italic capitals) constitute in turn a set *L*in of all linear mappings of Lin into itself, such that:

$$\mathbf{B} = \boldsymbol{D} : \mathbf{A}, \qquad \mathbf{B} \in \mathbf{Lin}, \quad \forall \mathbf{A} \in \mathbf{Lin}, \quad \forall \boldsymbol{D} \in \boldsymbol{L}\mathrm{in}. \tag{2.1}$$

With respect to a dual basis $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_{ij}$ $(i, j = 1, 2, 3)$ in V second- and fourth-order tensors can be represented as follows:

$$\mathbf{A} = \mathrm{A}_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \mathrm{A}^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = \mathrm{A}^{i}_{\cdot j}\mathbf{g}_i \otimes \mathbf{g}^j = \dots, \tag{2.2}$$

$$\boldsymbol{D} = D_{ijkl}\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k \otimes \mathbf{g}^l = D^{ijkl}\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k \otimes \mathbf{g}_l = D^{i\ \ k}_{\cdot j \cdot l}\mathbf{g}_i \otimes \mathbf{g}^j \otimes \mathbf{g}_k \otimes \mathbf{g}^l = \dots, \tag{2.3}$$

such that the mapping (2.1) can be expressed by

$$\boldsymbol{D} : \mathbf{A} = D^{i\ k}_{.j.l} \mathrm{A}^{j}_{.k} \mathbf{g}_i \otimes \mathbf{g}^l \,. \tag{2.4}$$

Fourth-order tensors can be constructed from second-order ones by means of a tensor product "×" defined by

$$(\mathbf{A} \times \mathbf{B}) : \mathbf{C} = (\mathbf{B} : \mathbf{C})\, \mathbf{A} \,, \qquad \forall \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbf{Lin} \,. \tag{2.5}$$

Elasticity and compliance tensors to be considered in the paper belong to a subset $\boldsymbol{S}\mathrm{lin} \subset \boldsymbol{L}\mathrm{in}$ of super-symmetric fourth-order tensors for which holds:

$$\mathbf{A} : (\boldsymbol{H} : \mathbf{B}) = \mathbf{B} : (\boldsymbol{H} : \mathbf{A}) \,, \qquad \boldsymbol{H} : \mathbf{A} = \boldsymbol{H} : \mathbf{A}^{\mathrm{T}} \,, \qquad \boldsymbol{H} \in \boldsymbol{S}\mathrm{lin} \,, \quad \forall \mathbf{A}, \mathbf{B} \in \mathrm{Lin} \,. \tag{2.6}$$

In other words, $\boldsymbol{S}\mathrm{lin}$ constitute the set of all linear mapping of symmetric second-order tensors $\mathrm{Sym} = \{\mathbf{A} \in \mathbf{Lin} : \mathbf{A} = \mathbf{A}^{\mathrm{T}}\}$ into itself. Of special importance for the further derivation are two following elements of $\boldsymbol{S}\mathrm{lin}$: the super-symmetric identity tensor

$$\boldsymbol{I}^{\mathrm{s}} = \frac{1}{2}\, \mathbf{g}_i \otimes (\mathbf{g}^i \otimes \mathbf{g}_j + \mathbf{g}_j \otimes \mathbf{g}^i) \otimes \mathbf{g}^j \quad \Rightarrow \quad \boldsymbol{I}^{\mathrm{s}} : \mathbf{A} = \mathbf{A} \,, \quad \forall \mathbf{A} \in \mathbf{Sym} \tag{2.7}$$

and the super-summetric deviatoric projection tensor

$$\boldsymbol{P}^{\mathrm{s}}_{\mathrm{dev}} = \boldsymbol{I}^{\mathrm{s}} - \frac{1}{3}\, \mathbf{I} \times \mathbf{I} \quad \Rightarrow \quad \boldsymbol{P}^{\mathrm{s}}_{\mathrm{dev}} : \mathbf{A} = \mathrm{dev}\, \mathbf{A} \,, \quad \forall \mathbf{A} \in \mathbf{Sym} \,. \tag{2.8}$$

According to the spectral theorem a super-symmetric fourth-order tensors can be represented by

$$\boldsymbol{F} = \sum_{r=1}^{6} \lambda_r \mathbf{M}_r \times \mathbf{M}_r \,, \qquad \mathbf{M}_r : \mathbf{M}_t = \delta_{rt} \,, \qquad \mathbf{M}_r \in \mathbf{Sym} \,, \quad (r, t = 1, 2, \ldots, 6) \,, \tag{2.9.1–3}$$

where λ_r and $\mathbf{M}_r$ $(r = 1, 2, \ldots, 6)$ denote eigenvalues and associated eigentensors (eigenstates), respectively. In particularly, for the super-symmetric fourth-order identity tensor (2.7) the spectral decomposition (2.9) takes the form

$$\boldsymbol{I}^{\mathrm{s}} = \sum_{r=1}^{6} \mathbf{M}_r \times \mathbf{M}_r \,. \tag{2.10}$$

To complete this section we define the scalar product of fourth-order tensors as follows:

$$\boldsymbol{D} :: \boldsymbol{E} = D^{i\ k}_{.j.l} E^{j\ l}_{.i.k.} = D_{ijkl} E^{ijkl} = \ldots \tag{2.11}$$

3 Incompressibility condition for anisotropic materials

We start with the generalized Hooke's law establishing a linear relationship between stresses $\boldsymbol{\sigma}$ and infinitesimal strains $\boldsymbol{\varepsilon}$. The Hooke's law can be given in the tensor

$$\boldsymbol{\varepsilon} = \boldsymbol{H} : \boldsymbol{\sigma} \tag{3.1}$$

or alternatively in the classical matrix form (see, e.g. [9]) with respect to the Cartesian frame

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \sqrt{2}\,\varepsilon_{23} \\ \sqrt{2}\,\varepsilon_{31} \\ \sqrt{2}\,\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} H_{1111} & H_{1221} & H_{1331} & \sqrt{2}\,H_{1231} & \sqrt{2}\,H_{1311} & \sqrt{2}\,H_{1121} \\ H_{2112} & H_{2222} & H_{2332} & \sqrt{2}\,H_{2232} & \sqrt{2}\,H_{2312} & \sqrt{2}\,H_{2122} \\ H_{3113} & H_{3223} & H_{3333} & \sqrt{2}\,H_{3233} & \sqrt{2}\,H_{3313} & \sqrt{2}\,H_{3123} \\ \sqrt{2}\,H_{2113} & \sqrt{2}\,H_{2223} & \sqrt{2}\,H_{2333} & 2H_{2233} & 2H_{2313} & 2H_{2123} \\ \sqrt{2}\,H_{3111} & \sqrt{2}\,H_{3221} & \sqrt{2}\,H_{3331} & 2H_{3231} & 2H_{3311} & 2H_{3121} \\ \sqrt{2}\,H_{1112} & \sqrt{2}\,H_{1222} & \sqrt{2}\,H_{1332} & 2H_{1232} & 2H_{1312} & 2H_{1122} \end{bmatrix} \begin{bmatrix} \sigma^{11} \\ \sigma^{22} \\ \sigma^{33} \\ \sqrt{2}\,\sigma^{23} \\ \sqrt{2}\,\sigma^{31} \\ \sqrt{2}\,\sigma^{12} \end{bmatrix}. \tag{3.2}$$

In the general case of anisotropy the super-symmetric compliance tensor $\boldsymbol{H}$ is described by 21 independent components related to material constants. The incompressibility condition given in the linear case by

$$\operatorname{tr}\boldsymbol{\varepsilon} = \mathbf{I} : \boldsymbol{\varepsilon} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0\,, \tag{3.3}$$

where the component form is again related to the Cartesian frame, must be satisfied for arbitrary stresses $\boldsymbol{\sigma}$. In view of (3.1) the incompressibility condition (3.3) imposes 6 additional constraints on the compliance tensor

$$\boldsymbol{H} : \mathbf{I} = \mathbf{0} \quad \Rightarrow \quad \sum_{k=1}^{3} H_{kijk} = 0\,, \quad (i, j = 1, 2, 3) \tag{3.4.1–2}$$

and reduces the number of its independent components to 15.

For compressible materials the constitutive (stress-strain) relations at small strains can be obtained through the direct inversion of the Hooke's law (3.1) leading to

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon} = \boldsymbol{H}^{-1} : \boldsymbol{\varepsilon}\,, \tag{3.5}$$

where $\boldsymbol{C}$ denotes the super-symmetric fourth-order elasticity tensor. On the contrary, for incompressible materials the constitutive relations are given by

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon} - p\mathbf{I}\,, \tag{3.6}$$

where the unknown parameter p cannot basically be determined from the constitutive law. In this case the elasticity tensor $\boldsymbol{C}$ cannot be obtained through the direct inversion of the singular compliance tensor $\boldsymbol{H}$ which requires a special procedure to be considered in the following.

The key point of this procedure is the spectral decomposition of the material tensors, the idea of which goes back to Lord Kelvin [10].

The incompressibility condition (3.4) requires that at least one of eigenvalues of the compliance tensor is equal to zero $\lambda_1 = 0$ and the associated eigentensor is proportional to the identity tensor $\mathbf{M}_1 = \frac{1}{\sqrt{3}}\mathbf{I}$. Hence

$$\boldsymbol{H} = 0 \cdot \left(\frac{1}{\sqrt{3}}\mathbf{I}\right) \times \left(\frac{1}{\sqrt{3}}\mathbf{I}\right) + \sum_{r=2}^{6} \lambda_r \mathbf{M}_r \times \mathbf{M}_r\,. \tag{3.7}$$

Since $\boldsymbol{H}$ is symmetric all its eigenvalues are real. Henceforth we assume that the incompressibility condition is the only internal kinematical constraint of the material. The condition of positive semi-definiteness of the compliance tensor reads in this case as

$$\lambda_r > 0\,, \quad (r = 2, 3, \ldots, 6)\,. \tag{3.8}$$

It can easily be shown that the elasticity tensor $\boldsymbol{C}$ should also be singular. Otherwise the constitutive relations (3.6) could be inverted by $\boldsymbol{\varepsilon} = \boldsymbol{C}^{-1} : (\boldsymbol{\sigma} + p\mathbf{I})$. By comparing with (3.1) this, in turn, would lead to the condition $\boldsymbol{C}^{-1} = \boldsymbol{H}$ which cannot evidently be satisfied in view of singularity of the compliance tensor.

Since the tensor $\boldsymbol{C}$ is singular at least one of its eigenvalues is equal to zero. The corresponding eigentensor remains, however, undetermined such that the elasticity tensor can generally be given by

$$\boldsymbol{C} = 0 \cdot \mathbf{N}_1 \times \mathbf{N}_1 + \sum_{r=2}^{6} \Lambda_r \mathbf{N}_r \times \mathbf{N}_r . \tag{3.9}$$

Now, putting the relation (3.6) into (3.1) we obtain:

$$\boldsymbol{\varepsilon} = \boldsymbol{H} : \boldsymbol{C} : \boldsymbol{\varepsilon} \quad \Rightarrow \quad (\boldsymbol{H} : \boldsymbol{C} - \boldsymbol{I}^{\mathrm{s}}) : \boldsymbol{\varepsilon} = \mathbf{0} . \qquad (3.10.1\text{--}2)$$

Keeping in mind that the strain tensor $\boldsymbol{\varepsilon}$ underlies the incompressibility constraint (3.3) one can see that the identity (3.10.2) holds if only the fourth-order tensor $\boldsymbol{H} : \boldsymbol{C}$ has the form

$$\boldsymbol{H} : \boldsymbol{C} = \boldsymbol{I}^{\mathrm{s}} + \mathbf{A} \times \mathbf{I} , \qquad \forall \mathbf{A} \in \mathrm{Sym} , \tag{3.11}$$

where $\mathbf{A}$ is an arbitrary symmetric second-order tensor. Contracting Eq. (3.11) with $\mathbf{N}_1$ we obtain:

$$\mathbf{A} = -\frac{\mathbf{N}_1}{\mathrm{tr}\,(\mathbf{N}_1)} , \qquad \mathrm{tr}\,(\mathbf{N}_1) \neq 0 . \qquad (3.12.1\text{--}2)$$

The inequality (3.12.2) is the only condition (apart from (2.9.2) and (2.9.3)) imposed on the eigentensor $\mathbf{N}_1$. In other respect the choice of $\mathbf{N}_1$ is arbitrary.

Remark 3.1. It follows from (3.11) that the compliance and elasticity tensor of an incompressible material are in general (except for the special case $\mathbf{A} = -\mathbf{I}/3$) non-coaxial such that $\boldsymbol{H} : \boldsymbol{C} \neq \boldsymbol{C} : \boldsymbol{H}$. Since furthermore the eigenstate $\mathbf{N}_1$ remains undetermined the analysis of the elasticity tensor can only reveal that the material under consideration has some internal kinematical constraints (zero eigenvalues) but the character of these constraints (e.g., incompressibility or inextensibility) cannot basically be deduced. Thus, various solids obtained by imposing different internal kinematical constraints on one virgin material can be described by coinciding elasticity tensors. □

To avoid the non-uniqueness of the elasticity tensor (3.9) one needs an additional assumption concerning its zero-energy eigenmode. It is reasonable to assume that the only zero-energy eigenmode of an incompressible material is purely volumetric. This means that

$$\mathbf{N}_1 = \frac{1}{\sqrt{3}} \mathbf{I} . \tag{3.13}$$

Under this assumption the fourth-order tensor $\boldsymbol{H} : \boldsymbol{C}$ (3.11) coincides with the super-symmetric deviatoric projection tensor (2.8). Thus, the elasticity and compliance tensor become coaxial which is rather general assumption in the theory of internally constrained materials (see, e.g., [11], [12]):

$$\boldsymbol{H} : \boldsymbol{C} = \boldsymbol{C} : \boldsymbol{H} = \boldsymbol{P}^{\mathrm{s}}_{\mathrm{dev}} . \tag{3.14}$$

Now, under consideration of (3.7), (3.13) and (3.14) the spectral decomposition of the elasticity tensor takes the form

$$\boldsymbol{C} = 0 \cdot \left(\frac{1}{\sqrt{3}} \mathbf{I} \right) \times \left(\frac{1}{\sqrt{3}} \mathbf{I} \right) + \sum_{r=2}^{6} \lambda_r^{-1} \mathbf{M}_r \times \mathbf{M}_r . \tag{3.15}$$

The elasticity tensor (3.15) can also be expressed in terms of the compliance tensor without explicit use of the spectral decomposition

$$C = \left(H + \frac{1}{3}\,\mathbf{I}\times\mathbf{I} \right)^{-1} - \frac{1}{3}\,\mathbf{I}\times\mathbf{I}\,, \tag{3.16}$$

which enables to avoid the general solution of the eigenvalue problem. It can easily be proved that the expression (3.16) gives the elasticity tensor (3.15) after inserting the compliance tensor (3.7). The closed-formula for the elasticity tensor (3.16) can also be obtained by means of the so-called Moore-Penrose generalized inverse [3], [13] also assuming that the original and inverse matrix commute.

4 Slightly compressible anisotropic materials

The bulk modulus of an anisotropic material can be defined as a function of stress by

$$\varkappa(\boldsymbol{\sigma}) = -\frac{p}{\operatorname{tr}\boldsymbol{\varepsilon}} = \frac{1}{3}\,\frac{\operatorname{tr}\boldsymbol{\sigma}}{\mathbf{I} : (H : \boldsymbol{\sigma})}\,, \tag{4.1}$$

where p denotes the hydrostatic pressure. In the special case of isotropy the definition (4.1) leads to the classical relation $\varkappa = E/3(1-2\nu)$.

For slightly compressible solids $\varkappa$ is large in comparison with other stiffness moduli. This condition can be formulated by means of the inequality

$$\frac{1}{\varkappa(\boldsymbol{\sigma})} \ll \| H \|\,, \qquad \forall \boldsymbol{\sigma} \,|\, \operatorname{tr}\boldsymbol{\sigma} \neq 0\,, \tag{4.2}$$

where the quadratic norm $\| H \| = (H :: H)^{1/2}$ can be expressed using the spectral decomposition of the compliance tensor $H = \sum_{r=1}^{6} \lambda_r \mathbf{M}_r \times \mathbf{M}_r$ through its eigenvalues by

$$\| H \| = \left(\sum_{r=1}^{6} \lambda_r^2 \right)^{1/2}. \tag{4.3}$$

The condition of slight compressibility (4.2) must be fulfilled for all stress states satisfying the condition $p = (-1/3)\operatorname{tr}\boldsymbol{\sigma} \neq 0$.

Using the representation of the stress tensor

$$\boldsymbol{\sigma} = \sum_{r=1}^{6} \sigma_r \mathbf{M}_r\,, \tag{4.4}$$

where $\sigma_r = \operatorname{tr}(\boldsymbol{\sigma}\mathbf{M}_r)$, and taking into account (4.1) and (4.3) the condition (4.2) takes the form

$$\frac{1}{\varkappa} = \frac{3 \sum_{r=1}^{6} \lambda_r \sigma_r \operatorname{tr}\mathbf{M}_r}{\sum_{r=1}^{6} \sigma_r \operatorname{tr}\mathbf{M}_r} \ll \left(\sum_{r=1}^{6} \lambda_r^2 \right)^{1/2}. \tag{4.5}$$

Since the condition of slight compressibility (4.5) holds for arbitrary σ_k for which

$$\sum_{r=1}^{6} \sigma_r \operatorname{tr}\mathbf{M}_r \neq 0\,, \tag{4.6}$$

we can obtain setting $\sigma_k = 1, \sigma_i = 0 \; (i = 1, 2, \ldots, 6, i \neq k)$

$$3\lambda_k \ll \left(\sum_{r=1}^{6} \lambda_r^2\right)^{1/2}, \qquad \forall k = 1, 2, \ldots, 6 \,|\, \operatorname{tr} \mathbf{M}_k \neq 0 \,. \tag{4.7}$$

This inequality can be satisfied in the two following cases.

Case 1. The spectral decomposition of the compliance tensor *can* be given in terms of the eigentensors among which only one (say, $\mathbf{M}_1$) is not traceless:

$$\operatorname{tr} \mathbf{M}_1 \neq 0 \,, \qquad \operatorname{tr} \mathbf{M}_k = 0 \,, \quad (k = 2, 3, \ldots, 6) \,. \tag{4.8}$$

Considering the identities

$$\sum_{r=1}^{6} (\operatorname{tr} \mathbf{M}_r) \, \mathbf{M}_r = \mathbf{I} \,, \qquad \sum_{r=1}^{6} (\operatorname{tr} \mathbf{M}_r)^2 = 3 \tag{4.9.1–2}$$

following from (2.10) we immediately obtain for $\mathbf{M}_1$

$$\mathbf{M}_1 = \frac{1}{\sqrt{3}} \mathbf{I} \,. \tag{4.10}$$

It is observable that the volumetric response of the material is strictly isotropic in this case. Under consideration of (4.8) the condition (4.7) leads to

$$3\lambda_1 \ll \left(\sum_{r=1}^{6} \lambda_r^2\right)^{1/2} . \tag{4.11}$$

It is worth mentioning that in this case the bulk modulus (4.1) $\varkappa = 1/3\lambda_1$ is independent of the stress tensor and, hence, represents a material constant.

Case 2. The spectral decomposition of the compliance tensor *cannot* be given in terms of the eigentensors among which only one is not traceless. Thus, two or more (n) eigentensors are not traceless

$$\operatorname{tr} \mathbf{M}_k \neq 0 \,, \qquad \operatorname{tr} \mathbf{M}_l = 0 \,, \quad (k = 1, 2, \ldots, n; \; l = n+1, \ldots, 6; \; 1 < n < 6) \tag{4.12}$$

and as a result of (4.7) two ore more eigenvalues of the compliance tensor are of negligible magnitude order in comparison with the remaining ones and do not all coincide:

$$\begin{aligned} &\lambda_k \ll \lambda_l \,, \quad (k = 1, 2, \ldots, n; \; l = n+1, \ldots, 6; \; 1 < n < 6) \,, \\ &\exists r, k \in [1, 2, \ldots, n] \,, \qquad r \neq k : \; \lambda_r \neq \lambda_k \,. \end{aligned} \tag{4.13}$$

In this case none of the eigentensors coincides with the identity tensor. Thus, the slight compressibility is caused by the superposition of several weak internal material constraints, none of which alone is related to slight compressibility. A typical example of this is a material slightly extensible in three orthogonal to each other directions with different stiffness moduli. Thereby, the bulk modulus (4.1) depends on the stress tensor and does not represent a material constant. This case requires an a priori knowledge and a separate treatment of each weak internal material constraint and on account of this will not be considered in the following.

Remark 4.1. In the case of coalescence of eigenvalues $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$, we have in view of (4.9.1) and (4.12)

$$\boldsymbol{H} : \mathbf{I} = \sum_{k=1}^{n} \lambda_k \, (\operatorname{tr} \mathbf{M}_k) \, \mathbf{M}_k = \lambda \mathbf{I} \,. \tag{4.14}$$

This means that the identity tensor represents the eigentensor of the compliance tensor such that we again deal with the Case 1. For example, in the special case of orthotropy such a solid can be characterized by the material constants $E_1 = E_2 = E_3$, $\nu_{12} = \nu_{23} = \nu_{31} \approx 0.5$, $G_{12} \neq G_{23} \neq G_{31}$. □

5 Isotropically compressible anisotropic materials

In what follows we consider anisotropic materials with *strictly isotropic volumetric response* specified in the previous section under the Case 1. The isotropic volumetric response means that under a uniform hydrostatic pressure the deviatoric part of the strain tensor vanishes as it is also the case for incompressible materials. As we have shown in the previous section only in this case the slight compressibility can be considered as a result of only one weak internal material constraint. Furthermore, the bulk modulus of an anisotropic material is independent of the stress state and represents in this case a material constant.

The isotropic volumetric response is described by the condition

$$\boldsymbol{H} : \mathbf{I} = \frac{1}{3\varkappa}\mathbf{I} \quad \Rightarrow \quad \sum_{k=1}^{3} H_{kijk} = \frac{1}{3\varkappa}, \quad (i, j = 1, 2, 3). \tag{5.1.1–2}$$

Thus, for isotropically compressible anisotropic solids the number of independent components describing the compliance tensor (3.2) is one more ($\varkappa$) than for incompressible materials and is equal to 16.

In view of (5.1) and (3.5) the spectral decompositions for the compliance and elasticity tensor take the form

$$\begin{aligned} \boldsymbol{H} &= \frac{1}{3\varkappa}\left(\frac{1}{\sqrt{3}}\mathbf{I}\right) \times \left(\frac{1}{\sqrt{3}}\mathbf{I}\right) + \sum_{r=2}^{6} \lambda_r \mathbf{M}_r \times \mathbf{M}_r, \\ \boldsymbol{C} &= 3\varkappa\left(\frac{1}{\sqrt{3}}\mathbf{I}\right) \times \left(\frac{1}{\sqrt{3}}\mathbf{I}\right) + \sum_{r=2}^{6} \lambda_r^{-1} \mathbf{M}_r \times \mathbf{M}_r. \end{aligned} \tag{5.2}$$

Thus, the condition of isotropic compressibility (5.1) is given in terms of the elasticity tensor by

$$\boldsymbol{C} : \mathbf{I} = 3\varkappa\mathbf{I}. \tag{5.3}$$

In view of (4.11), for isotropically *slightly* compressible materials one eigenvalue of the compliance tensor is negligible by comparison with another ones. Thus, the compliance tensor becomes ill-conditioned and could hardly be inverted numerically according to (3.5). That is why it is reasonable to refer again to the closed formula solution (3.16) which can be specified for isotropically compressible materials as follows:

$$\boldsymbol{C} = \left[\boldsymbol{H} + \left(1 - \frac{1}{3\varkappa}\right)\frac{1}{3}\mathbf{I} \times \mathbf{I}\right]^{-1} + \left(\varkappa - \frac{1}{3}\right)\mathbf{I} \times \mathbf{I}. \tag{5.4}$$

This formula can easily be proved directly inserting the compliance tensor (5.2). If the slight compressibility is the only weak internal constraint of the material the tensor to be inverted in (5.4) is well-conditioned which avoids any difficulties by the numerical calculation of the elasticity tensor.

6 Special case: orthotropic material

6.1 General description of orthotropy

A material is said to be *orthotropic* if there exist three planes orthogonal to each other, by reflections with respect to which material properties remain invariant [1]. The axes normal to these planes are called principal material directions. With respect to the principal material directions the compliance tensor of an orthotropic solid can be represented in the matrix form by

$$\underline{\boldsymbol{H}} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_2} & -\frac{\nu_{13}}{E_3} & 0 & 0 & 0 \\ & \frac{1}{E_2} & -\frac{\nu_{23}}{E_3} & 0 & 0 & 0 \\ & & \frac{1}{E_3} & 0 & 0 & 0 \\ & & & \frac{1}{2G_{23}} & 0 & 0 \\ & \text{symm.} & & & \frac{1}{2G_{31}} & 0 \\ & & & & & \frac{1}{2G_{12}} \end{bmatrix} \tag{6.1}$$

and involves 9 independent material parameters, the so-called *engineering elastic constants*:

$E_i\ (i = 1, 2, 3)$: Young's moduli referred to the principal material directions;

$G_{ij} = G_{ji} \quad (i \neq j = 1, 2, 3)$: Lamé's shear moduli referred to the principal material planes;

$\nu_{ij} = \nu_{ji}\,\dfrac{E_j}{E_i}\ (i \neq j = 1, 2, 3)$: Poisson's ratios referred to the principal material planes.

For compressible materials the elasticity tensor $\boldsymbol{C} = \boldsymbol{H}^{-1}$ directly results from the inversion of (6.1). Thus,

$$\underline{\boldsymbol{C}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ & a_{22} & a_{23} & 0 & 0 & 0 \\ & & a_{33} & 0 & 0 & 0 \\ & & & 2G_{23} & 0 & 0 \\ & \text{symm.} & & & 2G_{31} & 0 \\ & & & & & 2G_{12} \end{bmatrix} \tag{6.2}$$

with the components $a_{ij}\ (i, j = 1, 2, 3)$ expressed by (see, e.g., [7]):

$$a_{ii} = E_i\,\frac{1 - \nu_{jk}\nu_{kj}}{\Delta}\,, \qquad a_{ij} = a_{ji} = E_i\,\frac{\nu_{ij} + \nu_{kj}\nu_{ik}}{\Delta}\,, \quad (i \neq j \neq k \neq i\,,\ i, j = 1, 2, 3)\,, \tag{6.3}$$

where

$$\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13}\,. \tag{6.4}$$

6.2 Incompressible orthotropic material

Three of six incompressibility constraints (3.4.2) are satisfied in this case identically. By means of the three remaining ones the Poisson's ratios can be expressed through the Young's moduli by [3]

$$\nu_{ij} = \frac{1}{2} + \frac{E_j}{2}\left(\frac{1}{E_i} - \frac{1}{E_k}\right), \quad (i \neq j \neq k \neq i\,,\ i,j = 1,2,3)\,, \tag{6.5}$$

which reduces the number of independent material constants of an orthotropic incompressible material to 6.

The closed formula (3.16) yields in this case the elasticity tensor with the matrix representation of the form (6.2), where

$$a_{ii} = \frac{1}{3D}\left(\frac{2}{E_j} + \frac{2}{E_k} - \frac{1}{E_i}\right), \qquad a_{ij} = \frac{1}{6D}\left(\frac{1}{E_i} + \frac{1}{E_j} - \frac{5}{E_k}\right), \quad (i \neq j \neq k \neq i\,,\ i,j = 1,2,3)\,, \tag{6.6}$$

with the abbreviation

$$D = \frac{3}{4}\left(\frac{2}{E_1E_2} + \frac{2}{E_2E_3} + \frac{2}{E_3E_1} - \frac{1}{E_1^2} - \frac{1}{E_2^2} - \frac{1}{E_3^2}\right). \tag{6.7}$$

The requirement of positive semi-definiteness of the compliance tensor imposes some additional restrictions on the engineering elastic constants. These restrictions are well-established in literature for compressible orthotropic materials (see, e.g., [6], [7]) but, to our best knowledge, absolutely unknown for incompressible ones.

The condition of positive semi-definiteness (3.8) is formulated in terms of the eigenvalues of the compliance tensor. For the compliance matrix of the form (6.1) the eigenvalue problem can readily be solved considering separately the upper left and lower right 3×3 sub-matrices. For the lower right sub-matrix in (6.1) we have

$$\lambda_4 = \frac{1}{2G_{23}}\,, \qquad \lambda_5 = \frac{1}{2G_{13}}\,, \qquad \lambda_6 = \frac{1}{2G_{12}}\,, \tag{6.8}$$

which leads to the well-known restrictions for the shear moduli

$$G_{ij} > 0\,, \qquad (i \neq j, i, j = 1,2,3)\,. \tag{6.9}$$

The eigenvalues of the upper left sub-matrix in (6.1) specified according to (6.5)

$$\mathbf{A} = \begin{bmatrix} \dfrac{1}{E_1} & \dfrac{1}{2}\left(\dfrac{1}{E_3} - \dfrac{1}{E_2} - \dfrac{1}{E_1}\right) & \dfrac{1}{2}\left(\dfrac{1}{E_2} - \dfrac{1}{E_3} - \dfrac{1}{E_1}\right) \\ & \dfrac{1}{E_2} & \dfrac{1}{2}\left(\dfrac{1}{E_1} - \dfrac{1}{E_2} - \dfrac{1}{E_3}\right) \\ \text{symm.} & & \dfrac{1}{E_3} \end{bmatrix} \tag{6.10}$$

can be determined from of the corresponding characteristic equation

$$\lambda^3 - \lambda^2 \mathrm{I}_{\mathbf{A}} + \lambda \mathrm{II}_{\mathbf{A}} - \mathrm{III}_{\mathbf{A}} = 0\,, \tag{6.11}$$

where $\mathrm{I}_{\mathbf{A}} = \mathrm{tr}\,\mathbf{A}$, $\mathrm{II}_{\mathbf{A}} = \dfrac{1}{2}\left[(\mathrm{tr}\,\mathbf{A})^2 - \mathrm{tr}(\mathbf{A}^2)\right]$ and $\mathrm{III}_{\mathbf{A}} = \det \mathbf{A}$.

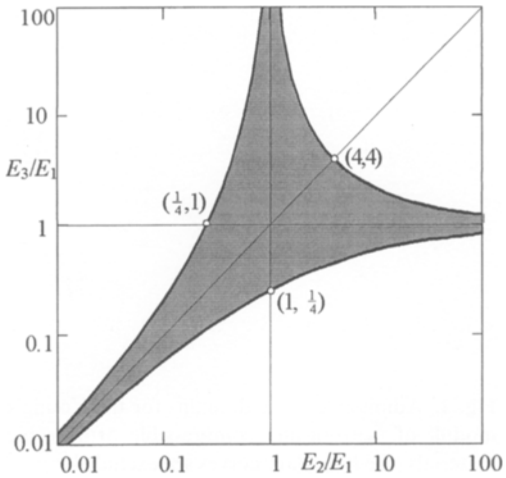

Fig. 1. The admissible value domain for the Young's moduli of incompressible orthotropic materials (the boundary curves are excluded)

Under consideration of the relation $\mathrm{III_A} = 0$ following from the incompressibility condition the solution of Eq. (6.11) yields:

$$\lambda_1 = 0\,, \qquad \lambda_{2,3} = \frac{1}{2}\left(\mathrm{I_A} \pm \sqrt{\mathrm{I_A^2} - 4\mathrm{II_A}}\right). \tag{6.12}$$

For positive Young's moduli $E_i > 0$ $(i = 1, 2, 3)$ the conditions $\lambda_{2,3} > 0$ are satisfied if only

$$\mathrm{II_A} = D = \frac{3}{4}\left(\frac{2}{E_1 E_2} + \frac{2}{E_2 E_3} + \frac{2}{E_3 E_1} - \frac{1}{E_1^2} - \frac{1}{E_2^2} - \frac{1}{E_3^2}\right) > 0\,. \tag{6.13}$$

This inequality yields an admissible value domain (Fig. 1) the Young's moduli E_i $(i = 1, 2, 3)$ of an orthotropic incompressible material belong to. Since this domain is bounded by the curves

$$f(x) = (1 \pm x^{-1/2})^{-2}\,, \tag{6.14}$$

the condition (6.13) can equivalently be represented by

$$\frac{1}{\sqrt{E_i}} + \frac{1}{\sqrt{E_j}} > \frac{1}{\sqrt{E_k}}\,, \quad (i \neq j \neq k \neq i\,,\ i, j, k = 1, 2, 3)\,. \tag{6.15}$$

6.3 *Isotropically compressible orthotropic material*

For isotropically compressible orthotropic materials the Poisson's ratios can be expressed by means of the conditions (5.1.2) and depend additionally on the bulk modulus $\varkappa$:

$$\nu_{ij} = \frac{1}{2} + \frac{E_j}{2}\left(\frac{1}{E_i} - \frac{1}{E_k} - \frac{1}{3\varkappa}\right), \quad (i \neq j \neq k \neq i\,,\ i, j = 1, 2, 3)\,. \tag{6.16}$$

By means of the closed formula (5.4) the components of the elasticity tensor (6.2) take the form:

$$\begin{aligned} a_{ii} &= \frac{1}{3D^*}\left(\frac{2}{E_j} + \frac{2}{E_k} - \frac{1}{E_i} - \frac{1}{3\varkappa}\right) + \varkappa\,, \\ a_{ij} &= \frac{1}{6D^*}\left(\frac{1}{E_i} + \frac{1}{E_j} - \frac{5}{E_k} + \frac{1}{3\varkappa}\right) + \varkappa\,, \end{aligned} \qquad (i \neq j \neq k \neq i\,,\ i, j = 1, 2, 3)\,, \tag{6.17}$$

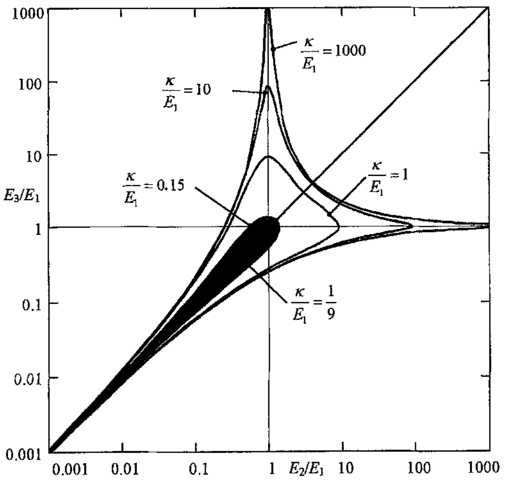

Fig. 2. Admissible value domains for the Young's moduli of isotropically compressible orthotropic materials (the boundary curves are excluded)

with the abbreviation

$$D^* = D - \frac{1}{6\varkappa}\left(\frac{1}{E_1}+\frac{1}{E_2}+\frac{1}{E_3}\right)+\frac{1}{36\varkappa^2}\,. \tag{6.18}$$

Additionally to (6.9) the conditions of positive definiteness of the compliance tensor can be obtained from the consideration of the upper left 3×3 sub-matrix $\mathbf{A}$ in (6.1) specified according to (6.16). One of the eigenvalues of this matrix is a priori known so that the solution of the corresponding characteristic equation (6.11) yields

$$\lambda_1 = \frac{1}{3\varkappa}\,, \qquad \lambda_{2,3} = \frac{1}{2}\left(\mathrm{I_A} - \frac{1}{3\varkappa}\right) \pm \sqrt{\frac{1}{4}\left(\mathrm{I_A} - \frac{1}{3\varkappa}\right)^2 - 3\varkappa\mathrm{III_A}}\,, \tag{6.19}$$

which requires in view of (3.8) that

$$\varkappa > 0\,, \qquad \mathrm{I_A} - \frac{\mathrm{I}}{3\varkappa} = \frac{1}{E_3}+\frac{1}{E_2}+\frac{1}{E_1}-\frac{1}{3\varkappa} > 0\,, \tag{6.20}$$

$$3\varkappa\mathrm{III_A} = D^* > 0 \quad \Rightarrow$$

$$\frac{3}{4}\left(\frac{2}{E_1E_2}+\frac{2}{E_2E_3}+\frac{2}{E_3E_1}-\frac{1}{E_1^2}-\frac{1}{E_2^2}-\frac{1}{E_3^2}\right)-\frac{1}{6\varkappa}\left(\frac{1}{E_1}+\frac{1}{E_2}+\frac{1}{E_3}\right)+\frac{1}{36\varkappa^2} > 0\,. \tag{6.21}$$

For the Young's moduli E_i $(i = 1, 2, 3)$ the inequalities (6.20) and (6.21) yield a set of admissible value domains depending on the bulk modulus $\varkappa$ (see Fig. 2).

From the consideration of the Fig. 2 one can observe, that for $\varkappa = (1/9)\,E_1$ the admissible value domain for the Young's moduli reduces to the line $E_2 = E_3$. Since the boundary curve is excluded the domain becomes empty in this case. Physically this means that there exists no isotropically compressible orthotropic solid with these material parameters. From the conditions (6.20) and (6.21) it follows moreover that

$$\frac{1}{3\varkappa} < \left(\frac{1}{E_1}+\frac{1}{E_2}+\frac{1}{E_3}\right) - 2\left(\frac{1}{E_1^2}+\frac{1}{E_2^2}+\frac{1}{E_3^2}-\frac{1}{E_1E_2}-\frac{1}{E_2E_3}-\frac{1}{E_3E_1}\right)^{1/2} \le \frac{3}{E_i}\,, \quad (i = 1, 2, 3)\,,$$

and hence

$$\varkappa > \frac{E_i}{9}\,, \quad (i = 1, 2, 3)\,, \tag{6.22}$$

which is completely analogical to isotropy, where $\varkappa > E/9$ due to the thermodynamic restriction for the isotropic Poisson's ratio $\nu > -1$.

In the context of thermodynamic restrictions strongly anisotropic materials are of special interest. In the case of orthotropy they can be obtained setting e.g., $E_2/E_1 \to 0$ or vice versa $E_2/E_1 \to \infty$. Under consideration of (6.20) and (6.21) (see also Fig. 2) we obtain in the last case

$$\frac{E_2}{E_1} \to \infty \quad \Rightarrow \quad E_3 = E_1\,, \qquad \frac{\varkappa}{E_1} \to \infty\,. \tag{6.23}$$

By considering the restriction (6.22) it can also be observed that the bulk modulus of a strongly anisotropic material must be at least of the magnitude order of the largest Young's modulus.

7 Special case: transversely isotropic material

7.1 General description of transverse isotropy

Transverse isotropy is characterized by the material symmetry with respect to one selected direction (the principal material direction). By rotations about as well as reflections with respect to planes orthogonal or parallel to the principal direction the material properties remain unchanged.

Constitutive relations for transversely isotropic materials can be obtained specifying those ones for orthotropy. Thereby, the following relationships between engineering elastic constants should be taken into account:

$$E_2 = E_3\,, \qquad \nu_{12} = \nu_{13} \quad (\nu_{21} = \nu_{31})\,, \qquad G_{12} = G_{31}\,, \qquad G_{23} = \frac{E_2}{2(1 + \nu_{23})}\,, \tag{7.1}$$

where the index 1 corresponds to the principal material direction. Further, it holds

$$\nu_{12} = \nu_{21}\,\frac{E_2}{E_1}\,, \qquad \nu_{23} = \nu_{32}\,. \tag{7.2}$$

The components of the elasticity tensor (6.2) become in view of (6.3)–(6.4)

$$\begin{aligned} a_{11} &= E_1\,\frac{1 - \nu_{23}^2}{\Delta}\,, & a_{22} &= a_{33} = E_2\,\frac{1 - \nu_{12}\nu_{21}}{\Delta}\,, \\ a_{23} &= E_2\,\frac{\nu_{23} + \nu_{12}\nu_{21}}{\Delta}\,, & a_{12} &= a_{13} = E_1\,\frac{\nu_{12}(1 + \nu_{23})}{\Delta} \end{aligned} \tag{7.3}$$

with

$$\Delta = (1 + \nu_{23})\,(1 - \nu_{23} - 2\nu_{12}\nu_{21})\,. \tag{7.4}$$

7.2 Incompressible transversely isotropic material

In this case, only two of six incompressibility conditions (3.4.2) are not identically satisfied. Under consideration of (6.5) and (7.1) one obtains

$$\nu_{21} = \nu_{31} = \frac{1}{2}\,, \qquad \nu_{12} = \nu_{13} = \frac{E_2}{2E_1}\,, \qquad \nu_{23} = \nu_{32} = 1 - \frac{E_2}{2E_1}\,, \qquad G_{23} = \frac{E_2 E_1}{4E_1 - E_2}\,, \tag{7.5}$$

such that only 3 material constants remain independent.
The components of the elasticity tensor (6.2) take in view of (6.6)–(6.7) the form

$$\begin{aligned} a_{11} &= \frac{4}{9}\,E_1\,, & a_{22} &= a_{33} = \frac{1}{3D}\left(\frac{2}{E_1} + \frac{1}{E_2}\right), \\ a_{23} &= \frac{1}{6D}\left(\frac{2}{E_2} - \frac{5}{E_1}\right), & a_{12} &= a_{13} = -\frac{2}{9}\,E_1\,, \end{aligned} \tag{7.6}$$

where

$$D = \frac{3}{4E_1}\left(\frac{4}{E_2} - \frac{1}{E_1}\right). \tag{7.7}$$

Under consideration of (7.7) the condition of positive definiteness (6.13) can be simplified as follows (see also [2]):

$$E_2 < 4E_1\,. \tag{7.8}$$

7.3 Isotropically compressible transversely isotropic material

In this case the number of independent material constants is one more ($\varkappa$) and equal to 4. By virtue of (7.1) and (6.16) we have

$$\begin{aligned} &\nu_{21} = \nu_{31} = \frac{1}{2} - \frac{E_1}{6\varkappa}\,, \qquad \nu_{12} = \nu_{13} = \frac{E_2}{2}\left(\frac{1}{E_1} - \frac{1}{3\varkappa}\right), \qquad \nu_{23} = \nu_{32} = 1 - \frac{E_2}{2}\left(\frac{1}{E_1} + \frac{1}{3\varkappa}\right), \\ &G_{23} = \frac{E_1 E_2}{4E_1 - E_2 - E_1 E_2/3\varkappa}\,. \end{aligned} \tag{7.9}$$

The components of the elasticity tensor (6.2) take in view of (6.17)–(6.18) the form:

$$\begin{aligned} a_{11} &= \frac{1}{3D^*}\left(\frac{4}{E_2} - \frac{1}{E_1} - \frac{1}{3\varkappa}\right) + \varkappa\,, & a_{22} &= a_{33} = \frac{1}{3D^*}\left(\frac{2}{E_1} + \frac{1}{E_2} - \frac{1}{3\varkappa}\right) + \varkappa\,, \\ a_{23} &= \frac{1}{6D^*}\left(\frac{2}{E_2} - \frac{5}{E_1} + \frac{1}{3\varkappa}\right) + \varkappa\,, & a_{12} &= a_{13} = \frac{1}{6D^*}\left(\frac{1}{E_1} - \frac{4}{E_2} + \frac{1}{3\varkappa}\right) + \varkappa\,, \end{aligned} \tag{7.10}$$

where

$$D^* = \frac{3}{4E_1}\left(\frac{4}{E_2} - \frac{1}{E_1}\right) - \frac{1}{6\varkappa}\left(\frac{1}{E_1} + \frac{2}{E_2}\right) + \frac{1}{36\varkappa^2}\,. \tag{7.11}$$

The positive definiteness of the compliance tensor requires in view of (6.20)–(6.21) that

$$\frac{1}{3\varkappa} < \left(\frac{4}{E_2} - \frac{1}{E_1}\right), \qquad \frac{1}{3\varkappa} < \frac{3}{E_1}\,. \tag{7.12}$$

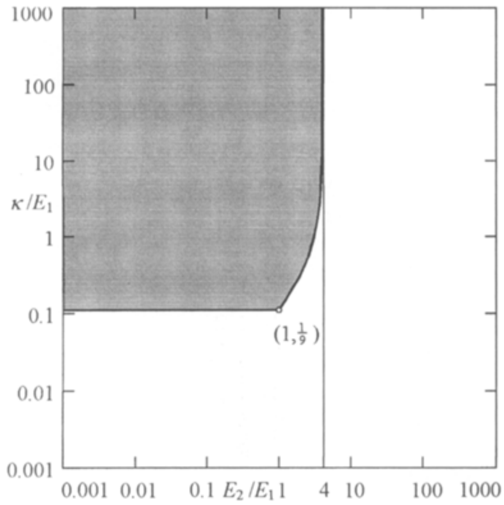

Fig. 3. The admissible value domain for the material constants E_1, E_2 and $\varkappa$ of transversely isotropic materials with isotropic volumetric response (the boundary curves are excluded)

Accordingly, the material constants E_1, E_2 and $\varkappa$ belong to the open domain of admissible values represented graphically in the Fig. 3.

8 Concluding remarks

The constitutive (stress-strain) relations for incompressible anisotropic materials cannot be obtained through the direct inversion of the generalized Hooke's law since the corresponding compliance tensor becomes singular in this case. The problem requires a special procedure discussed in the paper. The procedure is based on the spectral decomposition of the material tensors but finally leads to a closed formula for the elasticity tensor without explicit use of the eigenvalue problem solution.

Considering the material tensors in the case of incompressibility (this is valid also for any other internal kinematical constraints) we have shown that the elasticity tensor is defined only up to the zero-energy eigenmode. Thus, the material tensors can be in general non-coaxial. The analysis of the elasticity tensor can only reveal that the material under consideration possesses internal kinematical constraints (zero eigenvalues) but the character of these constraints cannot basically be deduced and requires knowledge of the compliance tensor. Moreover, various solids obtained by imposing different internal kinematical constraints on one virgin material can be described by coinciding elasticity tensors.

For slightly compressible anisotropic materials the case of isotropic volumetric response is of special attention. We have shown that only in this case the slight compressibility can be considered as a result of only one weak internal material constraint. Furthermore, the bulk modulus of an anisotropic material is independent in this case of the stress state and represents a material constant. To obtain stress-strain relations for isotropically slightly compressible materials without numerical inversion of the ill-conditioned compliance tensor we have specified the closed formula for the elasticity tensor such that it additionally involves the bulk modulus.

Applying these solutions to the important special cases of orthotropic and transversely isotropic materials one can easily obtain the corresponding elasticity tensors in analytical form. Further, examining these tensors on the positive definiteness (or semi-definiteness in the case of strict incompressibility) we have formulated the resulting restrictions imposed on the elastic constants. These restrictions yield an admissible value domain the elastic constants must belong to. For the Young's moduli of orthotropic and transversely isotropic incompressible as well as isotropically compressible materials these domains are illustrated graphically. One important result should also be mentioned in this context. In the case of isotropy the Poisson's ratio lies between -1 and 0.5 which requires that $\varkappa > E/9$. In complete analogy we have also obtained for isotropically compressible orthotropic materials that $\varkappa > E_i/9$ $(i = 1, 2, 3)$.

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