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Wave propagation in sheared rubber

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Summary. The speeds of propagation and polarization amplitudes are presented for finite amplitude plane shear waves propagating in rubber which is maintained in a state of static finite simple shear. The Mooney-Rivlin form of the stored-energy function is used to model the mechanical behaviour of the material. General relations are obtained between the speed of propagation of the fastest and slowest waves and the speed of propagation of the finite amplitude circularly polarized waves which may propagate along the acoustic axes. The slowness and ray surfaces are also presented.

1 Introduction

The propagation of finite amplitude waves in finitely deformed rubber is considered in the context of the theory of finite deformation of homogeneous isotropic incompressible elastic materials. The Mooney–Rivlin form of the stored energy function is used to model the properties of the rubber. It has been shown [1] that two linearly polarized finite amplitude plane shear waves polarized in directions orthogonal to each other and to the direction of propagation may propagate in any direction in a Mooney-Rivlin material which is maintained in a state of arbitrary finite static homogeneous deformation. Also, even though the theory is non-linear and the two waves are of finite amplitude, they propagate independently of each other, i.e., they do not interact. There are in general two directions, the directions of the acoustic axes, such that for each of them the two waves propagate with the same speed. These directions are determined soley by the basic static deformation and do not depend upon the two material constants which occur in the stored-energy function. Finite amplitude circularly polarized plane waves may propagate along the acoustic axes.

Here, the special case when the basic static deformation is a simple shear is considered. Then there is only one parameter, the amount of shear, K, determining the deformation. It is equivalent to the case when one principal stretch has the value one and the product of the two other principal stretches is also one. All these principal stretches are different from each other provided $K \neq 0$. The speeds of propagation and the amplitudes of the waves are written down explicity for an arbitrary propagation direction. Some general results, independent of the material constants describing the material, are obtained relating the maximum and minimum wave speeds and the speed of propagation of circularly polarized waves. It is shown that irrespective of the amount of shear, the speed of propagation is constant for circularly polarized waves, whether of finite amplitude or infinitesimal amplitude. It is also seen that the product of the fastest and slowest wave speeds is equal to the square of the speed of circularly polarized waves. Thus this product is independent of the amount of shear – like the speed of circularly polarized waves, it is an invariant of the material. Finally, explicit equations for the slowness and ray surfaces are written down. The sections of these surfaces by the plane of shear, the shearing plane, and the plane orthogonal to both are presented.

2 Basic equations

Incompressible homogeneous isotropic elastic materials of the Mooney-Rivlin type are characterized by a strain-energy W per unit volume given by

$$2W = C(I-3) + D(II-3). \tag{1}$$

Here, C and D are material constants assumed to satisfy $C \geqq 0, D > 0$ or $C > 0, D \geqq 0$ (strong ellipticity conditions). If $D = 0$, the material is said to be "neo-Hookean". Also I and II are principal invariants of the left Cauchy-Green strain tensor given by

$$I = \mathrm{tr}\mathbf{B}, \quad 2II = (\mathrm{tr}\mathbf{B})^2 - \mathrm{tr}\mathbf{B}^2. \tag{2}$$

If the deformation in which a particle at $\mathbf{X}$ is displaced to $\mathbf{x}$ is given by $\mathbf{x} = \mathbf{x}(\mathbf{X})$, or $x_i = x_i(X_A)$ in terms of components with respect to a rectangular Cartesian coordinate system, the components of $\mathbf{B}$ are

$$B_{ij} = (\partial x_i/\partial X_A)\,(\partial x_j/\partial X_A). \tag{3}$$

The constitutive equation is

$$\mathbf{t} = -p\mathbf{1} + C\mathbf{B} - D\mathbf{B}^{-1}, \tag{4}$$

where $\mathbf{t}$ is the Cauchy stress tensor, and p is an indeterminate pressure corresponding to the incompressibility constraint det $\mathbf{B} = 1$.

If the material is maintained in a state of static homogeneous deformation, two finite amplitude plane transverse waves may propagate in any direction $\mathbf{n}$. The two possible unit amplitude vectors, denoted by $\mathbf{a}$ and $\mathbf{b}$, form an orthogonal triad with $\mathbf{n}$ and are along conjugate directions with respect to the "$\mathbf{B}^{-1}$-ellipsoid", $\mathbf{x} \cdot \mathbf{B}^{-1}\mathbf{x} = 1$, associated with the strain tensor $\mathbf{B}^{-1}$ of the static homogeneous deformation. Indeed, $\mathbf{a}$ and $\mathbf{b}$ are along the principal axes of the elliptical section of the $\mathbf{B}^{-1}$-ellipsoid by the plane $\mathbf{n} \cdot \mathbf{x} = 0$. The phase speeds of the waves propagating along $\mathbf{n}$ with amplitudes along $\mathbf{a}$ and $\mathbf{b}$ are denoted by $v(\mathbf{n};\mathbf{a})$ and $v(\mathbf{n};\mathbf{b})$, respectively, and are given by [1]

$$\begin{aligned} \varrho v^2(\mathbf{n};\,\mathbf{a}) &= C\mathbf{n}\cdot\mathbf{B}\mathbf{n} + D\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a}, \\ \varrho v^2(\mathbf{n};\,\mathbf{b}) &= C\mathbf{n}\cdot\mathbf{B}\mathbf{n} + D\mathbf{b}\cdot\mathbf{B}^{-1}\mathbf{b}, \end{aligned} \tag{5}$$

where ϱ is the material density. These two speeds of propagation are equal ($\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a} = \mathbf{b}\cdot\mathbf{B}^{-1}\mathbf{b}$) if and only if $\mathbf{n}$ is along the normal to a plane of central circular section of the $\mathbf{B}^{-1}$-ellipsoid [1]. When the three principal stretches λ_1, λ_2, λ_3 of the basic static homogeneous deformation are all unequal, as is the case in simple shear, there are two such plane central circular sections. The unit normals to them are $\mathbf{n}^{\pm}$, given by [1]

$$\sqrt{\lambda_3^{-2} - \lambda_1^{-2}}\,\mathbf{n}^{\pm} = \sqrt{\lambda_2^{-2} - \lambda_1^{-2}}\,\mathbf{e}_1 \pm \sqrt{\lambda_3^{-2} - \lambda_2^{-2}}\,\mathbf{e}_3, \tag{6}$$

where $\mathbf{e}_\alpha$ are the unit vectors along the principal axes of strain, corresponding to λ_α, $(\alpha = 1, 2, 3)$, assumed to be ordered $\lambda_1 > \lambda_2 > \lambda_3$. The directions of $\mathbf{n}^+$ and $\mathbf{n}^-$ are the acoustic axes of the deformed Mooney-Rivlin material.

Also, it has been shown [3] that the velocities corresponding to the direction of propagation **n** are given in terms of **n** by

$$2\varrho v^2 = 2C\mathbf{n}\cdot\mathbf{Bn} + D(\operatorname{tr}\mathbf{B}^{-1} - \mathbf{n}\cdot\mathbf{B}^{-1}\mathbf{n}) \pm D\sqrt{(\operatorname{tr}\mathbf{B}^{-1} - \mathbf{n}\cdot\mathbf{B}^{-1}\mathbf{n})^2 - 4\mathbf{n}\cdot\mathbf{Bn}}, \tag{7}$$

and the equation for the slowness surface is [3]

$$\Omega(\mathbf{s}) \equiv C^2(\mathbf{s}\cdot\mathbf{Bs})^2 + CD(\mathbf{s}\cdot\mathbf{Bs})\left\{(\operatorname{tr}\mathbf{B}^{-1})\,\mathbf{s}\cdot\mathbf{s} - \mathbf{s}\cdot\mathbf{B}^{-1}\mathbf{s}\right\} + D^2(\mathbf{s}\cdot\mathbf{s})\,(\mathbf{s}\cdot\mathbf{Bs}) - 2\varrho C(\mathbf{s}\cdot\mathbf{Bs}) - \varrho D\left\{(\operatorname{tr}\mathbf{B}^{-1})\,\mathbf{s}\cdot\mathbf{s} - \mathbf{s}\cdot\mathbf{B}^{-1}\mathbf{s}\right\} + \varrho^2 = 0, \tag{8}$$

where $\mathbf{s} = \mathbf{n}/v$ is the slowness vector.

The energy flux velocity **g** of a wave with given slowness is **s** is normal to the slowness surface at the extremity of the vector **s**, and such that $\mathbf{g}\cdot\mathbf{s} = 1$ (see [2] and [8] for details). Thus

$$\mathbf{g} = \frac{\partial\Omega}{\partial\mathbf{s}}\left(\mathbf{s}\cdot\frac{\partial\Omega}{\partial\mathbf{s}}\right)^{-1}. \tag{9}$$

For homogeneous waves of infinitesimal amplitude, **g** is the group velocity (see [8] and [9]). When **s** is varied, the extremity of the vector **g**, located at the origin, describes a surface called the "ray surface". It is the envelope of the planes $\mathbf{s}\cdot\mathbf{g} = 1$ (regarding g_i as the independent variables) for all the possible **s** satisfying $\Omega(\mathbf{s}) = 0$. It has been shown [3] that the equation for the ray surface is

$$\Psi(\mathbf{g}) \equiv \varrho^2(\mathbf{g}\cdot\mathbf{B}^{-1}\mathbf{g})\,(C\mathbf{g}\cdot\mathbf{B}^{-1}\mathbf{g} + D\mathbf{g}\cdot\mathbf{g}) - \varrho\left[2C^2\mathbf{g}\cdot\mathbf{B}^{-1}\mathbf{g} + CD\left\{(\operatorname{tr}\mathbf{B})\,\mathbf{g}\cdot\mathbf{B}^{-1}\mathbf{g} + \mathbf{g}\cdot\mathbf{g}\right\} + D^2\left\{(\operatorname{tr}\mathbf{B})\,\mathbf{g}\cdot\mathbf{g} - \mathbf{g}\cdot\mathbf{Bg}\right\}\right] + C^3 + C^2D\operatorname{tr}\mathbf{B} + CD^2\operatorname{tr}\mathbf{B}^{-1} + D^3 = 0. \tag{10}$$

Finally, we recall that for the two waves propagating in the direction **n** with phase speeds $v(\mathbf{n};\mathbf{a})$ and $v(\mathbf{n};\mathbf{b})$ given by (5) the energy flux velocities, denoted by $\mathbf{g}(\mathbf{n};\mathbf{a})$ and $\mathbf{g}(\mathbf{n};\mathbf{b})$, respectively, are given by [3]

$$\varrho v(\mathbf{n};\mathbf{a})\,\mathbf{g}(\mathbf{n};\mathbf{a}) = C\mathbf{Bn} - D\mathbf{b}\times\mathbf{B}^{-1}\mathbf{a},$$
$$\varrho v(\mathbf{n};\mathbf{b})\,\mathbf{g}(\mathbf{n};\mathbf{b}) = C\mathbf{Bn} + D\mathbf{a}\times\mathbf{B}^{-1}\mathbf{b}. \tag{11}$$

3 Simple shear

Let the basic static deformation be a simple shear of amount K. Thus let

$$x = X + KY, \quad y = Y, \quad z = Z, \tag{12}$$

in a rectangular Cartesian coordinates system $0xyz$. Without loss of generality we take $K > 0$. Then

$$\mathbf{B} = \begin{pmatrix} K^2+1 & K & 0 \\ K & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{B}^{-1} = \begin{pmatrix} 1 & -K & 0 \\ -K & K^2+1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{13}$$

The principal stretches, λ_α, are given by

$$2\lambda_1 = \sqrt{K^2+4}+K, \quad \lambda_2 = 1, \quad 2\lambda_3 = \sqrt{K^2+4}-K. \tag{14.1,2,3}$$

We note $\lambda_1 > \lambda_2 > \lambda_3$, because $K > 0$. Also, $\lambda_3 = 1/\lambda_1$. The unit vectors $\mathbf{e}_\alpha$, along the corresponding principal axes of strain are given by

$$\begin{aligned}\sqrt{2}\,(K^2+4)^{1/4}\,\mathbf{e}_1 &= \sqrt{\sqrt{K^2+4}+K}\,\mathbf{i} + \sqrt{\sqrt{K^2+4}-K}\,\mathbf{j},\\ \mathbf{e}_2 &= \mathbf{k},\\ \sqrt{2}\,(K^2+4)^{1/4}\,\mathbf{e}_3 &= \sqrt{\sqrt{K^2+4}-K}\,\mathbf{i} - \sqrt{\sqrt{K^2+4}+K}\,\mathbf{j},\end{aligned} \tag{15}$$

where **i**, **j**, and **k** are mutually orthogonal unit vectors along the rectangular Cartesian coordinate axes $0xyz$.

The stretches λ_x and λ_y along the x and y-axes (that is the stretches of material line elements which are along the x and y-axes in the deformed state of the material), are given by

$$\lambda_x{}^2 = \frac{1}{B^{-1}_{xx}} = 1, \quad \lambda_y{}^2 = \frac{1}{B^{-1}_{yy}} = \frac{1}{(K^2+1)}. \tag{16}$$

The unit normals $\mathbf{n}^\pm$ to the planes of the central circular sections of the $\mathbf{B}^{-1}$-ellipsoid, $\mathbf{x}\cdot\mathbf{B}^{-1}\mathbf{x} = 1$, are given by

$$\sqrt{K^2+4}\,\mathbf{n}^+ = 2\mathbf{i} - K\mathbf{j}, \quad \mathbf{n}^- = \mathbf{j}. \tag{17}$$

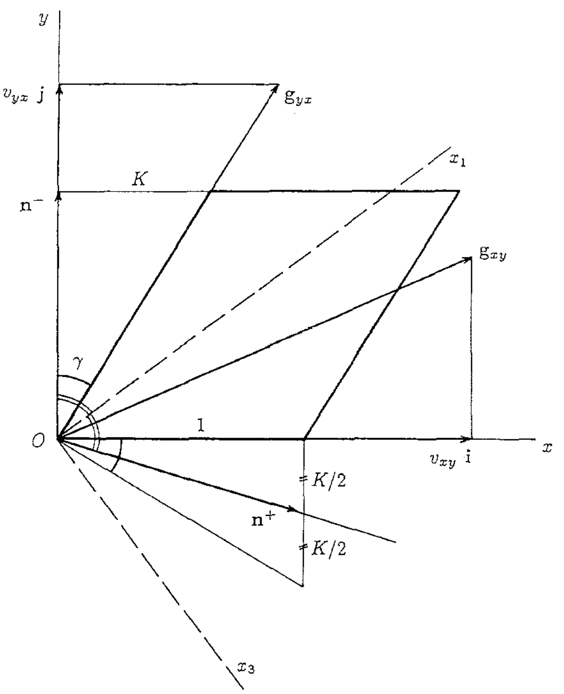

Fig. 1. Directions in the plane of shear (xy-plane) depending solely on the amount of shear K: principal axes of strain x_1, x_3 corresponding to λ_1, λ_3; acoustic axes $\mathbf{n}^+$ and $\mathbf{n}^-$; energy flux velocities $\mathbf{g}_{xy}$ and $\mathbf{g}_{yx}$. $K = 0.614$

They are determined solely by the amount of shear. Note that $\mathbf{e}_1$, $\mathbf{e}_3$ are along the internal and external bisectors of $\mathbf{n}^+$ and $\mathbf{n}^-$ (see Fig. 1). The angle Ψ between $\mathbf{n}^+$ and $\mathbf{n}^-$ is given by $\cot \Psi = -K/2$.

Also, from the constitutive Eq. (4), we have

$$t_{xy} = (C + D)\,K, \quad t_{xx} - t_{yy} = (C + D)\,K^2, \tag{18.1,2}$$

and from Eq. (18.1) it follows that $C + D = \mu$ (say) may be interpreted as the shear modulus.

4 Wave speed and polarization

Here, the speeds of propagation are written down explicitly for arbitrary propagation directions $\mathbf{n}$. Special attention is given to the wave speeds and energy flux velocities for propagation along a coordinate axis. Also using the results of Boulanger and Hayes [5] the directions of the amplitude polarization are written down explicitly.

4.1 Wave speeds. Waves propagating along the coordinate axes

For an arbitrary propagation direction $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k}$, we have

$$\begin{aligned} &\mathbf{n}\cdot\mathbf{B}\mathbf{n} = 1 + 2Kn_xn_y + K^2n_x{}^2, \\ &\mathrm{tr}\mathbf{B}^{-1} - \mathbf{n}\cdot\mathbf{B}^{-1}\mathbf{n} = 2 + 2Kn_xn_y + K^2(1 - n_y{}^2), \end{aligned} \tag{19}$$

and then from Eq. (7), the speeds of propagation of the two waves propagating along $\mathbf{n}$ are given by

$$\begin{aligned} \varrho v^2 &= C(1 + 2Kn_xn_y + K^2n_x{}^2) \\ &\quad + \frac{D}{2}\left\{2 + 2Kn_xn_y + K^2(1 - n_y{}^2) \pm K\sqrt{1 - n_y{}^2}\sqrt{4 + K^2 - (2n_x - Kn_y)^2}\right\}. \end{aligned} \tag{20}$$

We consider in particular propagation along the axes $0xyz$.

For $\mathbf{n} = \mathbf{i}$, it follows from Eq. (13) that the elliptical section of the $\mathbf{B}^{-1}$-ellipsoid, $\mathbf{x}\cdot\mathbf{B}^{-1}\mathbf{x} = 1$, by the yz-plane has its principal axes along the y and z-axes. Hence $\mathbf{a} = \mathbf{j}$, $\mathbf{b} = \mathbf{k}$. Using Eq. (5) or Eq. (20), we have

$$\varrho v^2(\mathbf{i};\mathbf{j}) = (C + D)\,(K^2 + 1), \quad \varrho v^2(\mathbf{i};\mathbf{k}) = C(K^2 + 1) + D, \tag{21.1,2}$$

and using Eq. (11), we have for the corresponding energy flux velocities

$$\mathbf{g}(\mathbf{i};\mathbf{j}) = v(\mathbf{i};\mathbf{j})\,\{\mathbf{i} + K(K^2 + 1)^{-1}\mathbf{j}\}, \quad \mathbf{g}(\mathbf{i};\mathbf{k}) = v(\mathbf{i};\mathbf{k})\,\{\mathbf{i} + CK[C(K^2 + 1) + D]^{-1}\mathbf{j}\}. \tag{22.1,2}$$

Similarly, for propagation along $\mathbf{n} = \mathbf{j}$, we have

$$\varrho v^2(\mathbf{j};\mathbf{k}) = \varrho v^2(\mathbf{j};\mathbf{i}) = C + D = \mu, \tag{23}$$

with

$$\mathbf{g}(\mathbf{j};\mathbf{k}) = v(\mathbf{j};\mathbf{k})\,\{\mathbf{j} + C(C + D)^{-1}K\mathbf{i}\}, \quad \mathbf{g}(\mathbf{j};\mathbf{i}) = v(\mathbf{j};\mathbf{i})\,\{\mathbf{j} + K\mathbf{i}\}. \tag{24.1,2}$$

Also, for $\mathbf{n} = \mathbf{k} = \mathbf{e}_2$, we have $\mathbf{a} = \mathbf{e}_3$, $\mathbf{b} = \mathbf{e}_1$, and

$$\varrho v^2(\mathbf{k};\mathbf{e}_3) = C + D\lambda_3{}^{-2}, \quad \varrho v^2(\mathbf{k};\mathbf{e}_1) = C + D\lambda_1{}^{-2}, \tag{25}$$

with

$$\mathbf{g}(\mathbf{k};\,\mathbf{e}_3) = v(\mathbf{k};\,\mathbf{e}_3)\,\mathbf{k}, \qquad \mathbf{g}(\mathbf{k};\,\mathbf{e}_1) = v(\mathbf{k};\,\mathbf{e}_1)\,\mathbf{k}. \tag{26}$$

We note that for the waves propagating along the axes $0x$ and $0y$, the energy flux velocity is not along the propagation direction, whilst for the wave propagating along the principal axis $0z$, the energy flux velocity is along the propagation direction and thus is the same as the phase velocity.

Writing now $v_{xy} = v(\mathbf{i};\,\mathbf{j})$ and $v_{yx} = v(\mathbf{j};\,\mathbf{i})$ for convenience, we have $\varrho v_{xy}^2 = \mu(K^2 + 1)$ and $\varrho v_{yx}^2 = \mu$ (see Eqs. (21.1) and (23)), and recalling Eq. (18.2) it follows that

$$\frac{\varrho v_{xy}^2}{\lambda_x{}^2} = \frac{\varrho v_{yx}^2}{\lambda_y{}^2} = \frac{t_{xx} - t_{yy}}{\lambda_x{}^2 - \lambda_y{}^2}. \tag{27}$$

This result is analogous to Ericksen's relations for principal waves [4], although here the x and y axes are not principal axes of strain. Writing also $\mathbf{g}_{xy} = \mathbf{g}(\mathbf{i};\,\mathbf{j})$ and $\mathbf{g}_{yx} = \mathbf{g}(\mathbf{j};\,\mathbf{i})$, we note from Eqs. (22.1) and (24.2) that the directions of $\mathbf{g}_{xy}$ and $\mathbf{g}_{yx}$ depend solely on the amount of shear, and that

$$\lambda_x \mathbf{j}\cdot\mathbf{g}_{xy} = \lambda_y \mathbf{i}\cdot\mathbf{g}_{yx}, \qquad \lambda_y \mathbf{i}\cdot\mathbf{g}_{xy} = \lambda_x \mathbf{j}\cdot\mathbf{g}_{yx}. \tag{28}$$

Also, $\mathbf{g}_{yx} = v_{yx}(\mathbf{j} + K\mathbf{i})$ is along the direction into which the y-axis is transformed by the simple shear (by the simple shear, $X = Z = 0$ is transformed into $x = Ky$, $z = 0$; see Fig. 1).

4.2 Polarization directions

The polarization directions $\mathbf{a}$ and $\mathbf{b}$ for an arbitrary propagation direction $\mathbf{n}$ are along the principal axes of the central elliptical section of the $\mathbf{B}^{-1}$-ellipsoid by the plane $\mathbf{n}\cdot\mathbf{x} = 0$. Using the results of Boulanger & Hayes [5] these may be written explicitly.

There are two cases:

Case (i) $\mathbf{n}\cdot(\mathbf{n}^+ \times \mathbf{n}^-) \neq 0$.

Here, $\mathbf{n}$ is not coplanar with $\mathbf{n}^+$ and $\mathbf{n}^-$, or equivalently $\mathbf{n}$ is not in the plane of shear (xy-plane). Then, $\mathbf{a}$ and $\mathbf{b}$ are unit vectors along the vectors $\mathbf{a}'$ and $\mathbf{b}'$ given by

$$\left.\begin{matrix}\mathbf{a}'\\ \mathbf{b}'\end{matrix}\right\} = \mathbf{n}\times\mathbf{n}^+/\delta_+ \pm \mathbf{n}\times\mathbf{n}^-/\delta_-$$
$$= \left\{\frac{Kn_z\mathbf{i} + 2n_z\mathbf{j} - (Kn_x + 2n_y)\,\mathbf{k}}{\sqrt{K^2+4}}\right\}\delta_+{}^{-1}$$
$$\pm\{-n_z\mathbf{i} + n_x\mathbf{k}\}\,\delta_-{}^{-1}, \tag{29}$$

where δ_+ and δ_- are given by

$$\delta_+{}^2 = \frac{(K^2+4)\;n_z{}^2 + K^2 n_x{}^2 + 4n_y{}^2 + 4Kn_xn_y}{K^2+4}, \tag{30}$$

$$\delta_-{}^2 = n_x{}^2 + n_z{}^2.$$

Case (ii) $\mathbf{n}\cdot(\mathbf{n}^+ \times \mathbf{n}^-) = 0$.

Here, $\mathbf{n}$ lies in the plane of shear, so that $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j}$, $n_z = 0$. Then, $\mathbf{a}$ and $\mathbf{b}$ are unit vetors along $\mathbf{a}'$ and $\mathbf{b}'$ given by

$$\mathbf{a}' = \mathbf{n}^+ \times \mathbf{n}^- = \frac{2}{\sqrt{K^2+4}}\mathbf{k},$$
$$\mathbf{b}' = \mathbf{n} \times \mathbf{B}^{-1}(\mathbf{n}^+ \times \mathbf{n}^-) = 2\frac{(n_y\mathbf{i} - n_x\mathbf{j})}{\sqrt{K^2+4}}, \tag{31}$$

and hence,

$$\mathbf{a} = \mathbf{k}, \quad \mathbf{b} = n_y\mathbf{i} - n_x\mathbf{j} = \mathbf{n} \times \mathbf{k}. \tag{32}$$

Thus, for example, if $\mathbf{n}$ lies in the shearing plane (zx-plane), $\mathbf{n} = n_x\mathbf{i} + n_z\mathbf{k}$, $n_y = 0$, then from Eqs. (29), (30),

$$\delta_+{}^2 = \frac{K^2 + 4n_z{}^2}{K^2+4}, \quad \delta_-{}^2 = 1, \tag{33}$$

$$\left.\begin{matrix}\mathbf{a}'\\ \mathbf{b}'\end{matrix}\right\} = \alpha_\mp(n_z\mathbf{i} - n_x\mathbf{k}) + \frac{2n_z}{\sqrt{K^2 + 4n_z{}^2}}\mathbf{j}, \tag{34}$$

where

$$\alpha_\mp = \frac{K}{\sqrt{K^2 + 4n_z{}^2}} \mp 1. \tag{35}$$

We note from Eq. (32) that for any direction of propagation $\mathbf{n}$, lying in the plane of shear, one wave is always polarized along the z-axis (orthogonal to the plane of shear) and the other is polarized along the direction perpendicular to $\mathbf{n}$ in the plane of shear. From Eq. (5) or (20) we also note that the wave speeds of these two waves are given by

$$\varrho v^2(\mathbf{n}; \mathbf{k}) = C(1 + K^2 n_x{}^2 + 2Kn_x n_y) + D,$$
$$\varrho v^2(\mathbf{n}; \mathbf{n} \times \mathbf{k}) = (C + D)(1 + K^2 n_x{}^2 + 2Kn_x n_y), \tag{36}$$

with $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j}$.

5 General relations amongst wave speeds

On using Eqs. (20) and (17) we note that the wave speed $v_\odot$ (say) of the waves which may propagate along $\mathbf{n}^+$ or $\mathbf{n}^-$, the acoustic axes, is given by

$$\varrho v_\odot{}^2 = C + D = \mu. \tag{37}$$

We note that $v_\odot$ is independent of the amount of shear K, and is precisely the speed of the waves that may propagate along any direction in the undeformed Mooney-Rivlin material. We recall [1] that finite amplitude circularly polarized plane waves, of either handedness, and finite amplitude elliptically polarized plane waves, of either handedness, may propagate along $\mathbf{n}^+$ and $\mathbf{n}^-$ with speed $v_\odot$. All of these waves travel with the same speed as do waves in the undeformed material.

Thus, if two specimens of the same piece of rubber are stretched by different amounts, both with one principal stretch λ_2 fixed, with $\lambda_2 = 1$, then finite amplitude circularly polarized waves may propagate with the same speed in each specimen, which is also the speed of waves in an unstretched specimen of the same piece of rubber. Equivalently, the speed of propagation of finite amplitude circularly polarized plane waves is not altered as the amount of shear is altered. Of course, one of the acoustic axes, $\mathbf{n}^+$, is altered as K is altered, but the other, $\mathbf{n}^- = \mathbf{j}$ remains unchanged.

It has been shown [3] that the fastest wave propagates in the direction of greatest stretch with its amplitude in the direction of least stretch. Thus, in the present case, the fastest wave propagates along $\mathbf{n} = \mathbf{e}_1$ and is polarized along $\mathbf{a} = \mathbf{e}_3$. Hence, from Eq. (5) the greatest speed, v_{max} (say), is given by

$$\varrho v_{\text{max}}^2 = (C + D)\,\lambda_1{}^2 = \mu\lambda_1{}^2, \tag{38}$$

with λ_1 given by Eq. $(14)_1$. Similarly, the slowest wave propagates along the direction $\mathbf{n} = \mathbf{e}_3$ of least stretch and is polarized along the direction $\mathbf{a} = \mathbf{e}_1$ of greatest stretch. Hence, the least speed, v_{min} (say), is given by

$$\varrho v_{\text{min}}^2 = (C + D)\,\lambda_3{}^2 = \mu\lambda_3{}^2, \tag{39}$$

with λ_3 given by Eq. (14.3).

From Eqs. (38) and (39), we note, using the constitutive Eq. (4) and recalling $\lambda_1\lambda_3 = 1$,

$$\varrho\lambda_1{}^{-2}v_{\text{max}}^2 = \varrho\lambda_3{}^{-2}v_{\text{min}}^2 = \frac{t_1 - t_3}{\lambda_1{}^2 - \lambda_3{}^2} = \mu, \tag{40}$$

where t_1 and t_3 and the greatest and least principal stresses, respectively. This result is in agreement with results known for infinitesimal waves [6]. More suggestively, if we write $\lambda_1 = \lambda_{\text{max}}$ and $\lambda_3 = \lambda_{\text{min}}$, Eq. (40) yields

$$\lambda_{\text{min}}v_{\text{max}} = \lambda_{\text{max}}v_{\text{min}}. \tag{41}$$

Using the fact that $\lambda_1\lambda_3 = 1$, we also note from Eqs. (38) and (39),

$$v_{\text{max}}v_{\text{min}} = v_{\odot}{}^2 = \frac{\mu}{\varrho}. \tag{42}$$

Thus the product of the largest and smallest speeds of propagation of finite-amplitude plane waves propagating in a Mooney-Rivlin material which is maintained in a state of simple shear is independent of the amount of shear.

Also, we have

$$\begin{aligned} v_{\text{max}}^2 + v_{\text{min}}^2 &= (K^2 + 2)\,v_{\odot}{}^2, \\ v_{\text{max}}^2 - v_{\text{min}}^2 &= K\sqrt{K^2 + 4}\,v_{\odot}{}^2, \end{aligned} \tag{43}$$

and, using Eq. (14), Eq. (41) may also be written in the form

$$\sqrt{K^2 + 4}\,(v_{\text{max}} - v_{\text{min}}) = K(v_{\text{max}} + v_{\text{min}}). \tag{44}$$

Using Eqs. (36) and (37), we note that the constitutive constants C and D may be expressed in terms of $v_\odot$ and of the speeds of the two waves propagating in any direction $\mathbf{n}$ (other than $\mathbf{n}^+$ and $\mathbf{n}^-$) in the plane of shear. Indeed, we have

$$C = \varrho v_\odot{}^2 \frac{v^2(\mathbf{n};\ \mathbf{k}) - v_\odot{}^2}{v^2(\mathbf{n};\ \mathbf{n}\times\mathbf{k}) - v_\odot{}^2},$$
$$D = \varrho v_\odot{}^2 \frac{v^2(\mathbf{n};\ \mathbf{n}\times\mathbf{k}) - v^2(\mathbf{n};\ \mathbf{k})}{v^2(\mathbf{n};\ \mathbf{n}\times\mathbf{k}) - v_\odot{}^2}, \tag{45}$$

where $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j}$.

Finally, Knowles [7] has shown that the period T_0 of large amplitude free oscillations of a very thin hollow tube of Mooney-Rivlin material of undeformed radius r_0 is

$$T_0 = \pi \sqrt{\frac{\varrho}{\mu}}\, r_0. \tag{46}$$

Thus,

$$T_0 = \frac{\pi r_0}{v_\odot}, \tag{47}$$

so that twice the period is the time it would take a finite amplitude circularly polarized wave to travel the length of the circumference of the tube in the undeformed state.

6 Slowness and ray surfaces

Here, the equations for the slowness and ray surfaces are written down explicitly, and special attention is given to the sections of these surfaces by the coordinate planes.

6.1 Slowness surface

Recalling that $\mathbf{s} = s_x\mathbf{i} + s_y\mathbf{j} + s_z\mathbf{k}$ is the slowness vector, from Eq. (13) we have

$$(\mathrm{tr}\ \mathbf{B}^{-1})\ \mathbf{s}\cdot\mathbf{s} - \mathbf{s}\cdot\mathbf{B}^{-1}\mathbf{s} = \mathbf{s}\cdot\mathbf{B}\mathbf{s} + \mathbf{s}\cdot\mathbf{s} + K^2 s_z{}^2. \tag{48}$$

Using this in Eq. (8), we obtain the equation of the slowness surface in the form

$$\Omega(\mathbf{s}) \equiv \{(C+D)\ \mathbf{s}\cdot\mathbf{B}\mathbf{s} - \varrho\}\ \{C\mathbf{s}\cdot\mathbf{B}\mathbf{s} + D(\mathbf{s}\cdot\mathbf{s} + K^2 s_z{}^2) - \varrho\} - D^2K^2 s_z{}^2(\mathbf{s}\cdot\mathbf{B}\mathbf{s}) = 0, \tag{49}$$

where $\mathbf{s}\cdot\mathbf{B}\mathbf{s}$ is given by

$$\mathbf{s}\cdot\mathbf{B}\mathbf{s} = \mathbf{s}\cdot\mathbf{s} + K^2 s_x{}^2 + 2K s_x s_y. \tag{50}$$

We briefly consider the intersections of the slowness surface with the coordinate planes. From Eq. (49), it is clear that the intersection with the xy-plane (plane of shear) is made of the two ellipses

$$(C+D)\ \{s_x{}^2(1+K^2) + s_y{}^2 + 2K s_x s_y\} = \varrho,$$
$$C\{s_x{}^2(1+K^2) + s_y{}^2 + 2K s_x s_y\} + D(s_x{}^2 + s_y{}^2) = \varrho, \tag{51}$$

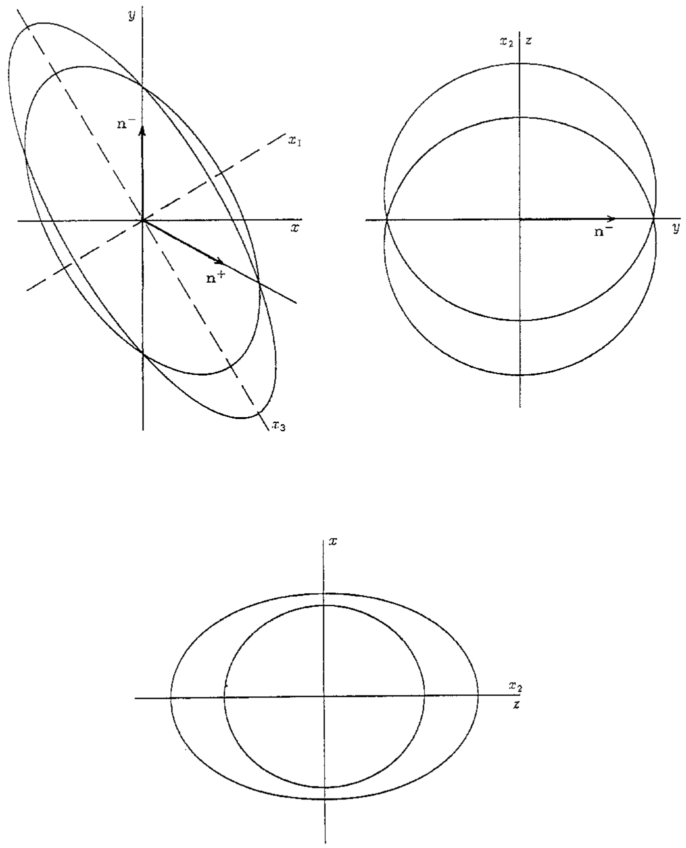

Fig. 2. Sections of the slowness surface by the xy-plane (plane of shear), the yz-plane, and the zx-plane (shearing plane). The axes x_1, x_2, x_3 are the principal axes of strain; $\mathbf{n}^+$ and $\mathbf{n}^-$ are the acoustic axes. $C/D = 1.5$, $K = 1.1$

both of which have their major axis along $\mathbf{e}_3$ and their minor axis along $\mathbf{e}_1$, where $\mathbf{e}_1$ and $\mathbf{e}_3$ are the principal axes of strain given by Eq. (15). These two ellipses intersect in the directions of the acoustic axes $\mathbf{n}^+$ and $\mathbf{n}^-$ given by Eq. (17) (see Fig. 2).

The equation of the intersection of the slowness surface with the yz-plane is

$$\{\varrho - (C + D)\,(s_y^2 + s_z^2)\}^2 - DK^2 s_z^2 \{\varrho - C(s_y^2 + s_z^2)\} = 0\,. \tag{52}$$

This curve, represented in Fig. 2, has two singular points (self-intersection) on the y-axis.

The equation of the intersection of the slowness surface with the zx-plane is

$$\{\varrho - (C + D)\,[s_x^2(1 + K^2) + s_z^2]\}\,\{\varrho - C[s_x^2(1 + K^2) + s_z^2] - D(s_x^2 + s_z^2)\}$$
$$- DK^2 s_z^2 \{\varrho - C[s_x^2(1 + K^2) + s_z^2]\} = 0\,. \tag{53}$$

It consists of two non intersecting closed curves (see Fig. 2).

6.2 Ray surface

Recalling that $\mathbf{g} = g_x\mathbf{i} + g_y\mathbf{j} + g_z\mathbf{k}$ is the energy flux velocity, from Eq. (13) we have

$$(\mathrm{tr}\,\mathbf{B})\,\mathbf{g}\cdot\mathbf{g} - \mathbf{g}\cdot\mathbf{B}\mathbf{g} = \mathbf{g}\cdot\mathbf{B}^{-1}\mathbf{g} + \mathbf{g}\cdot\mathbf{g} + K^2 g_z^2\,. \tag{54}$$

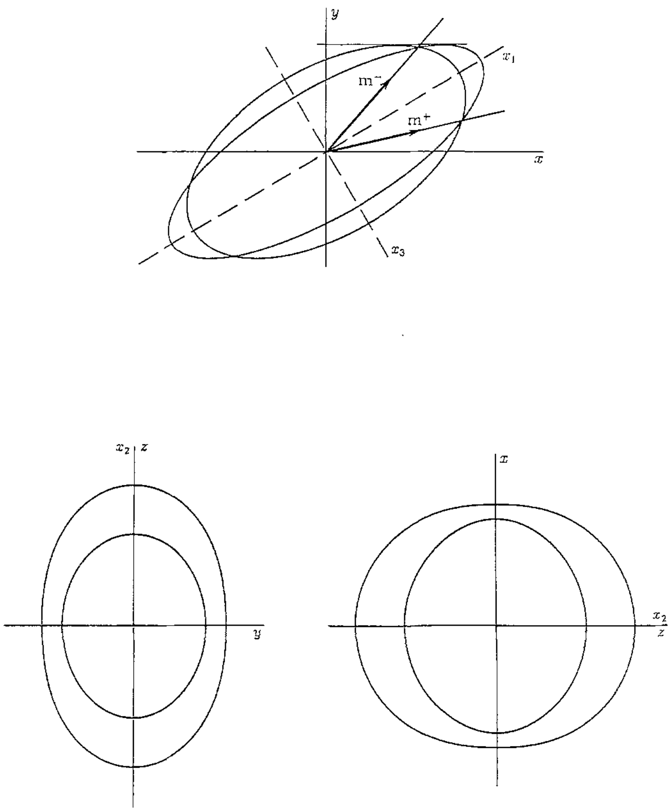

Fig. 3. Sections of the ray surface by the xy-plane (plane of shear), the yz-plane, and the zx-plane (shearing plane). The axes x_1, x_2, x_3 are the principal axes of strain; $\mathbf{m}^+$ and $\mathbf{m}^-$ are the ray axes. $C/D = 1.5, K = 1.1$

Using this in Eq. (10), we obtain the equation of the ray surface in the form

$$\Psi(\mathbf{g}) \equiv \{C^2 + CD(2 + K^2) + D^2 - \varrho(C\mathbf{g}\cdot\mathbf{B}^{-1}\mathbf{g} + D\mathbf{g}\cdot\mathbf{g})\}\ \{C + D - \varrho\mathbf{g}\cdot\mathbf{B}^{-1}\mathbf{g}\} - D^2K^2\varrho g_z{}^2 = 0, \tag{55}$$

where $\mathbf{g}\cdot\mathbf{B}^{-1}\mathbf{g}$ is given by

$$\mathbf{g}\cdot\mathbf{B}^{-1}\mathbf{g} = \mathbf{g}\cdot\mathbf{g} + K^2 g_y{}^2 - 2Kg_xg_y. \tag{56}$$

We also briefly consider the intersections of the ray surface with the coordinate planes. From Eq. (55), it is clear that the intersection with the xy-plane (plane of shear) consists of the two ellipses

$$\begin{aligned} &\varrho C\{g_x{}^2 + (1 + K^2)\, g_y{}^2 - 2Kg_xg_y\} + \varrho D(g_x{}^2 + g_y{}^2) = (C + D)^2 + K^2CD, \\ &\varrho\{g_x{}^2 + (1 + K^2)g_y{}^2 - 2Kg_xg_y\} = C + D, \end{aligned} \tag{57}$$

both of which have their major axis along $\mathbf{e}_1$, and their minor axis along $\mathbf{e}_3$ (see Fig. 3). These two ellipses intersect in the directions given by

$$KC\left(\frac{g_x}{g_y}\right) = C(1 + K^2) + D \pm \sqrt{(C + D)^2 + K^2CD} \tag{58}$$

These directions are called “ray axes” [2]. Unlike the acoustic axes, these depend on the constitutive coefficients C, D as well as on the amount of shear K. The duality between slowness surface, acoustic axes and ray surface, ray axes, is explained in detail in [3].

The equation of the intersection of the ray surface with the yz-plane is

$$\{(C+D)^2 + K^2CD - \varrho C[(1+K^2)\,g_y{}^2 + g_z{}^2]$$

$$-\varrho D(g_y{}^2 + g_z{}^2)\}\,\{C + D - \varrho[(1+K^2)\,g_y{}^2 + g_z{}^2]\} - D^2K^2\varrho g_z{}^2 = 0. \tag{59}$$

It consists of two non intersecting closed curves (see Fig. 3).

The equation of the intersection of the ray surface with the zx-plane is

$$\{(C+D)^2 + K^2CD - \varrho(C+D)\,(g_x{}^2 + g_z{}^2)\}\,\{C + D - \varrho(g_x{}^2 + g_z{}^2)\} - D^2K^2\varrho g_z{}^2 = 0. \tag{60}$$

It also consists of two non intersecting closed curves (see Fig. 3).

7 Concluding remarks

Explicit expressions have been given for the speeds of propagation and the polarization amplitudes for the two finite amplitude plane transverse waves which may propagate in any direction in rubber maintained in a state of finite static simple shear. Explicit expressions have also been given for the slowness surface and the ray surface.

A striking result relates to the fastest and slowest plane transverse finite amplitude waves, for a given state of shear. The slowest wave propagates along the direction of least stretch and is polarized along the direction of greatest stretch while the fastest wave propagates along the direction of greatest stretch and is polarized along the direction of least stretch. The product of the speed of the fastest and the slowest wave is a constant of the material (μ/ϱ), and is independent of the amount of (finite) shear.

For ‘small’ shear, i.e. $K^2 \ll K$, it follows from (42) and (44) that

$$v_{\min} = \left(1 - \frac{1}{2}K\right)\sqrt{\frac{\mu}{\varrho}}, \qquad v_{\max} = \left(1 + \frac{1}{2}K\right)\sqrt{\frac{\mu}{\varrho}}, \tag{61}$$

both values coalescing to the speed $(\mu/\varrho)^{1/2}$ as $K \to 0$, in the limit of no shear. On the other hand, for large values of K,

$$v_{\min} = \left(\frac{1}{K}\right)\sqrt{\frac{\mu}{\varrho}}, \qquad v_{\max} = K\sqrt{\frac{\mu}{\varrho}}. \tag{62}$$

The larger the shear the greater the value of $v_{\max}$ and the smaller the value of $v_{\min}$.

It is remarkable that the speed of propagation of circularly polarized finite amplitude waves is constant, irrespective of the amount of shear. In the virgin state, finite amplitude circularly polarized waves may propagate in every direction, with speed $\sqrt{\mu/\varrho}$. In the state of simple shear, there are two acoustic axes both of which lie in the plane of shear and are determined only by the amount of shear. Circularly polarized waves may propagate along either acoustic axis with the same speed $\sqrt{\mu/\varrho}$, as in the undeformed state.

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