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Some lattice models of bilinear logic

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Dedicated to the memory of Alan Day

The aim of this article is to present to universal algebraists a generalization of Boolean algebras which do not obey Gentzen's three structural rules. These so-called *Grishin algebras* are models of *classical bilinear propositional logic*, a non-commutative version of linear logic which allows two negations. Examples in which these negations coincide are easy to come by, but examples in which they are distinct are more elusive. To this purpose, it was found necessary to generalize the notion of a group to that of a *lattice ordered monoid with adjoints.* While the left inverse and the right inverse of a group element necessarily coincide this is not so for the left adjoint and the right adjoint of an element in a lattice ordered monoid.

1. Preliminaries

Some surprising algebraic varieties arise from considerations in logic. Here we shall confine attention to a class of algebras which arise as models of *bilinear* propositional logic, by which I mean logic without the usual structural rules postulated by Gentzen [Kleene 1952]: interchange, weakening and contraction. In the presence of these structural rules, the models are well-known algebras, namely Heyting algebras and Boolean algebras, depending on whether our logic is intuitionistic or classical. When the contraction rule is dropped, the intuitionistic models are called BCK algebras. Computer scientists appear to be interested in models of classical *linear* logic, in which both weakening and contraction are dropped, but interchange is retained, while some philosophers are interested in relevance logic, in which only weakening is forbidden. From now on we shall concentrate on bilinear logic, a non-commutative variant of Girard's [1987, 1989] linear logic, but we will ignore quantifiers and modalities.

The intuitionistic version of bilinear logic has been called "syntactic calculus", because of its application to linguistics [L 1958, 1989]. Its models are "residuated

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lattices" (see Birkhoff's Lattice Theory [1967]), which are said to have been introduced by Dilworth. A *residuated lattice ordered monoid*, as it is more commonly called now, is a poset with operations (I, $\otimes$, $/$, $\backslash$, $\top$, $\wedge$, $\perp$, $\vee$), where the first two describe a monoid, the last four a lattice, and the remaining two operations, called "over" and "under", satisfy

$$A \otimes B \leq C \quad \Leftrightarrow \quad A \leq C/B \quad \Leftrightarrow \quad B \leq A \backslash C. \tag{$*$}$$

These conditions assert, in the language of category theory, that $-/B$ is right adjoint to $- \otimes B$ and that $A \backslash -$ is right adjoint to $A \otimes -$. From this it follows, in particular, that $A \leq B$ and $C \leq D$ imply $A \otimes C \leq B \otimes D$ and $A/D \leq B/C$ (functoriality).

Complete residuated lattice ordered monoids are also known as "quantales" (see Rosenthal [1990]). Our reason for using capital letters to denote the elements of a residuated lattice ordered monoid is that, in the most familiar examples, they are subsets of a monoid or, more generally, sets of arrows of a category. Somewhat less familiar are the downward closed subsets of a partially ordered monoid studied by Došen [1985] and Buszkowski [1986], of which the subsets of a monoid are a special case, and, more generally, the downward closed sets of arrows of a category.

Why is the class of residuated lattice ordered monoids an equational class of algebras? That is, why is a residuated lattice ordered monoid a set with operations satisfying certain identities, and not just a poset with additional structure? Evidently, a lattice can be presented equationally if the partial order is defined as usual by

$$A \leq B \quad \Leftrightarrow \quad A \wedge B = A. \tag{$**$}$$

Moreover, the conditions ($*$) can be replaced by the inequalities

$$\begin{aligned} &(C/B) \otimes B \leq C, \qquad A \leq (A \otimes B)/B, \\ &A \otimes (A \backslash C) \leq C, \qquad B \leq A \backslash (A \otimes B), \end{aligned} \tag{$***$}$$

which themselves can be replaced by equations with the help of ($**$), and functoriality, which can also be presented by equations such as $(A \vee B) \otimes C = (A \otimes C) \vee (B \otimes C)$.

Models of intuitionistic bilinear logic are thus seen to be well-known; so let us turn to the classical version of bilinear logic, that is, noncommutative linear logic, again without quantifiers and modalities.

2. Grishin algebras

Models of classical bilinear logic, in several different variants, first turn up in the work of Grishin [1983]. They also appear in various guises in Abrusci [1991], Ono [1993] and Blute and Seely [1991]. These authors speak of "pure noncommutative linear logic", "Gentzen linear logic without interchange" and "non-commutative planar $*$-autonomous categories" respectively. Such models were also studied in [L1993] as "models of BL2".

Let us define a *Grishin algebra* as a poset with operations (I, $\otimes$, ${}^{\perp}$, ${}^{\top}$, O, $\top$, $\wedge$, $\perp$, $\vee$), where the first two operations describe a monoid, the last four a lattice, while the remaining operations satisfy the equations

$$A^{\perp\top} = A = A^{\top\perp} \tag{$\dagger$}$$

and the conditions

$$A \leq B \quad \Leftrightarrow \quad A \otimes B^{\perp} \leq O \quad \Leftrightarrow \quad B^{\top} \otimes A \leq 0. \tag{$\dagger\dagger$}$$

Of course, we expect that these conditions can also be replaced by equations.

PROPOSITION 2.1. *A Grishin algebra may be described as a residuated lattice ordered monoid with two unary operations* ${}^{\perp}$ *and* ${}^{\top}$ *satisfying the identities* ($\dagger$) *and*

$$I^{\perp} = I^{\top}, \qquad C/B = (B \otimes C^{\perp})^{\top}, \qquad A \backslash C = (C^{\top} \otimes A)^{\perp}.$$

Proof. In any Grishin algebra we have $I \leq I$, hence $I^{\perp} = I \otimes I^{\perp} \leq O$. But also $I^{\perp\top} \otimes O = I \otimes O \leq O$, hence $O \leq I^{\perp}$. Therefore $I^{\perp} = O$ and similarly $I^{\top} = O$, so $I^{\perp} = I^{\top}$. Moreover,

$$\begin{aligned} A \otimes B \leq C \quad &\Leftrightarrow \quad A \otimes B \otimes C^{\perp} \leq O \\ &\Leftrightarrow \quad A \leq (B \otimes C^{\perp})^{\top}. \end{aligned}$$

This shows that we may write $(B \otimes C^{\perp})^{\top} = C/B$, and the remaining equation follows from symmetry, yielding a residuated lattice.

Conversely, assume as given a residuated lattice ordered monoid satisfying ($\dagger$) and the new identities. We write $I^{\perp} = O = I^{\top}$ and argue as follows:

$$\begin{aligned} &A \otimes B^{\perp} \leq O = I^{\top} \\ &\quad \Leftrightarrow \quad A \leq I^{\top}/B^{\perp} = (B^{\perp} \otimes I^{\top\perp})^{\top} = (B^{\perp} \otimes I)^{\top} = B^{\perp\top} = B, \end{aligned}$$

and similarly $B^{\top} \otimes A \leq O \Leftrightarrow A \leq B$, so ($\dagger\dagger$) holds.

LEMMA 2.2. *In any Grishin algebra,*

$$(B^{\perp} \otimes A^{\perp})^{\top} = (B^{\top} \otimes A^{\top})^{\perp}.$$

Proof. For any element C,

$$\begin{aligned} C \leq (B^{\perp} \otimes A^{\perp})^{\top} &\Leftrightarrow C \otimes B^{\perp} \otimes A^{\perp} \leq O \\ &\Leftrightarrow C \otimes B^{\perp} \leq A \\ &\Leftrightarrow A^{\top} \otimes C \otimes B^{\perp} \leq O \\ &\Leftrightarrow A^{\top} \otimes C \leq B \\ &\Leftrightarrow B^{\top} \otimes A^{\top} \otimes C \leq 0 \\ &\Leftrightarrow C \leq (B^{\top} \otimes A^{\top})^{\perp}. \end{aligned}$$

This suggests the definition

$$A \oplus B = (B^{\perp} \otimes A^{\perp})^{\top} = (B^{\top} \otimes A^{\top})^{\perp},$$

making $\oplus$ a kind of De Morgan dual of $\otimes$.

LEMMA 2.3. *In any Grishin algebra,*

$$A \leq B \quad \Leftrightarrow \quad B^{\perp} \leq A^{\perp} \quad \Leftrightarrow \quad B^{\top} \leq A^{\top}.$$

Proof. From $A \leq B$ we obtain $A \otimes B^{\perp} \leq O$, that is, $A^{\perp\top} \otimes B^{\perp} \leq O$, hence $B^{\perp} \leq A^{\perp}$. The other implications are shown similarly.

PROPOSITION 2.4. *The opposite of a Grishin algebra, in which $\leq$ is replaced by $\geq$, is also a Grishin algebra, with operations $(O, \oplus, {}^{\top}, {}^{\perp}, I, \perp, \vee, \top, \wedge)$. Both ${}^{\perp}$ and ${}^{\top}$ are anti-isomorphisms between the Grishin algebra and its opposite.*

Proof. The second assertion follows from Lemma 2.3 and facts established earlier. We illustrate the proof of the first assertion by proving two of the identities it comprises:

$$A \oplus O = A, \qquad A \geq B \quad \Leftrightarrow \quad A \oplus B^{\top} \geq I.$$

Indeed,

$$A \oplus O = (O^{\perp} \otimes A^{\perp})^{\top} = (I \otimes A^{\perp})^{\top} = A^{\perp\top} = A$$

and, by Lemma 2.3,

$$O^{\top}=I\leq A\oplus B^{\top}=(B\otimes A^{\perp})^{\top} \quad\Leftrightarrow\quad B\otimes A^{\perp}\leq O \quad\Leftrightarrow\quad B\leq A.$$

3. Cyclic Grishin algebras

If we postulate $A^{\perp}=A^{\top}$ in classical bilinear logic, we obtain the "cyclic (non-commutative) linear logic" of Yetter [1990], called BL3 in [L1993]. Examples of Grishin algebras with $A^{\perp}=A^{\top}$ are easily found [ibid.], e.g., the set of subsets of a group, with O defined as the complement of $\{1\}$. More generally, one has the set of subsets of a Brandt groupoid, i.e., the set of sets of arrows of a category in which all arrows are isomorphisms. All these are instances of an abstract relation algebra as axiomatized by Tarski and Givant [1987]. Of course, the usual algebra of binary relations on a set is a concrete example of their axioms. Let me discuss here only one class of models, which generalize the known concrete relation algebras and which appear not yet to have been considered.

EXAMPLE 3.1. With any poset A we associate the poset $P(A)$ consisting of all downward closed subsets of A, ordered by inclusion. Alternatively, $P(A)$ may be described as the set of all monotone mappings from A to the two-element poset. Now consider the monoid $M(A)$ of all monotone mappings $f^{\dagger}$ of $P(A)$ into itself which have a right adjoint f^{*}:

$$f^{\dagger}(X)\leq Y \quad\Leftrightarrow\quad X\leq f^{*}(Y),$$

for all X, $Y\in P(A)$. An equivalent assertion is that $f^{\dagger}$ preserves suprema. As a poset, $M(A)$ is isomorphic with $P(A^{\circ}\times A)$, where A° is the opposite poset of A, or with the poset of all monotone mappings $f\colon A\to P(A)$. Indeed, for any $X\in P(A)$,

$$f^{*}(X)=\{a\in A\mid f(a)\leq X\},$$

$$f^{\dagger}(X)=\bigcup\{f(a)\mid a\in X\}.$$

Conversely, $f=f^{\dagger}I$, where

$$I(a)=\{b\in A\mid b\leq a\}.$$

We define $g \otimes f$ by

$$(g \otimes f)^\dagger = g^\dagger f^\dagger$$

(the categorically trained reader will smell a Kleisli category here) and

$$O(a) = \{b \in A \mid a \nleq b\},$$
$$f^\perp(a) = \{b \in A \mid a \notin f(b)\}$$

and leave the reader to check that we have a cyclic Grishin algebra.

If A is a set, viewed as a discrete poset, $M(A)$ is of course the usual algebra of binary relations on A.

Example 3.1 suggests several generalizations, to which I hope to return on another occasion. In place of the *monoid* $M(A)$ one may look at the *category* of all posets, with arrows $A \to B$ defined as monotone mappings $A \to P(B)$. Moreover, one may replace the category of posets by the metacategory of locally small V-categories for a suitable closed category V, so that the arrows $A \to B$ turn out to be Lawvere's [1973] *bimodules*. The consideration of these has already been suggested by Trimble [1993] and by Rosenthal [1993] in a similar context. Finally, one may give an axiomatic treatment of this whole setup in the language of 2-categories.

4. Lattice ordered monoids with adjoints

To find examples of Grishin algebras in which $A^\perp \neq A^\top$ we have to go beyond groups or groupoids.

DEFINITION 4.1. By a *lattice ordered monoid with adjoints* we shall mean a lattice ordered monoid in which every element a has both a right adjoint a^r and a left adjoint a^l, in the following sense:

$$a \cdot a^r \leq 1 \leq a^r \cdot a, \qquad a^l \cdot a \leq 1 \leq a \cdot a^l.$$

It is easily seen that both a^r and a^l are uniquely determined by a. Yet, in general, $a^r \neq a^l$, as is seen from the following example.

EXAMPLE 4.2. Consider the monoid of all unbounded monotone mappings $f\colon \mathbf{Z} \to \mathbf{Z}$. Each such mapping has a right adjoint f^r and a left adjoint f^l, both of which are again unbounded, namely

$$f^r(y) = \max\{x \in \mathbf{Z} \mid f(x) \le y\},$$
$$f^l(x) = \min\{y \in \mathbf{Z} \mid x \le f(y)\}.$$

Indeed, it is easily seen that

$$ff^r \le 1 \le f^r f, \qquad f^l f \le 1 \le ff^l.$$

To show that $f^r \neq f^l$ in general take $f(x) = 2x$; then

$$f^r(y) = [y/2], \qquad f^l(y) = [(y+1)/2].$$

It is an amusing exercise [L 1994] to prove that, for any unbounded monotone mapping $f\colon \mathbf{Z} \to \mathbf{Z}$, the following two sets are complementary subsets of $\mathbf{Z}$:

$$\{f(x) + x \mid x \in \mathbf{Z}\}, \qquad \{f^r(y) + y + 1 \mid y \in \mathbf{Z}\}.$$

It is easily seen that a lattice ordered monoid M with adjoints is a Grishin algebra, in which

$$I = O = 1, \qquad a^r = a^{\perp}, \qquad a^l = a^{\top},$$
$$a \otimes b = a \oplus b = a \cdot b.$$

To obtain a Grishin algebra in which $I \neq O$ and $\otimes$ does not coincide with $\oplus$, we shall pass to $P(M)$.

PROPOSITION 4.3. *The lattice of downward closed subsets of a lattice ordered monoid with adjoints is a Grishin algebra.*

Proof. We define

$$A \otimes B = \{c \in M \mid \exists_{a \in A} \exists_{b \in B}\, c \le a \cdot b\},$$
$$I = \{c \in M \mid c \le 1\},$$
$$O = \{c \in M \mid 1 \not\le c\},$$
$$A^{\perp} = \{c \in M \mid \forall_{a \in A}\, 1 \not\le a \cdot c\},$$
$$A^{\top} = \{c \in M \mid \forall_{a \in A}\, 1 \not\le c \cdot a\},$$

The fact that $\otimes$ and I determine a monoid is easily seen. We will show (i) that $A^{\perp\top}=A$ and (ii) that $A\otimes B^{\perp}\leq O \Leftrightarrow A\leq B$; the other properties that follow by symmetry.

(i) Suppose $c\in A^{\perp\top}$, that is, $\forall_{d\in A^{\perp}}1\nleq c\cdot d$. Thus, for all $d\in M$, $1\leq c\cdot d$ implies $d\notin A^{\perp}$. However, $1\leq c\cdot c^{l}$, hence $c^{l}\notin A^{\perp}$, that is, there exists $a\in A$ such that $1\leq a\cdot c^{l}$. Therefore, $c\leq a\cdot c^{l}\cdot c\leq a$ and so $c\in A$. Thus $A^{\perp\top}\leq A$. The converse inclusion is straightforward.

(ii) Suppose $A\otimes B^{\perp}\leq O$, that is, $1\notin A\otimes B^{\perp}$, that is, for all $a\in A$ and $c\in B^{\perp}$, $1\nleq a\cdot c$. Given any $a\in A$, it follows that, for any $c\in M$, $1\leq a\cdot c$ implies $c\notin B^{\perp}$. But $1\leq a\cdot a^{l}$, hence $a^{l}\notin B^{\perp}$, that is, for some $b\in B$, $1\leq b\cdot a^{l}$. Therefore, $a\leq b\cdot a^{l}\cdot a\leq b$, and so $a\in B$. Thus $A\leq B$. The converse implication follows from $B\otimes B^{\perp}\leq O$, so we have to show that $1\notin B\otimes B^{\perp}$. Thus, supposing $1\leq b\cdot c$ with $b\in B$, we wish to show that $c\notin B^{\perp}$. But this is evident from the definition of $B^{\perp}$.

A somewhat tedious calculation shows that

$$A\oplus B=\{c\in M\mid \forall_{a,b\in M}(a\cdot b\leq c \quad\Rightarrow\quad (a\in A\vee b\in B))\},$$

which is not the same as $A\otimes B$.

Proposition 4.3 can be generalized in two ways: by replacing "lattice ordered" by "partially ordered" and by replacing "monoid" by "category". A *partially ordered category* is a special kind of 2-category, namely a category where, for any two objects α and β, $\mathrm{Hom}(\alpha,\beta)$ is a poset such that, for any $f,g:\alpha\to\beta$, $h:\gamma\to\alpha$ and $k:\beta\to\delta$, $f\leq g$ implies $fh\leq gh$ and $kf\leq kg$. In a partially ordered category *with adjoints*, for every arrow $f:\alpha\to\beta$ there are given arrows $f^{r},f^{l}:\beta\to\alpha$ such that

$$ff^{r}\leq 1_{\beta},\qquad 1_{\alpha}\leq f^{r}f,\qquad f^{l}f\leq 1_{\alpha},\qquad 1_{\beta}\leq ff^{l}.$$

PROPOSITION 4.4. *The lattice of downward closed subsets of a partially ordered category with adjoints is a Grishin algebra.*

The proof is the same as for Proposition 4.3, only now M is the set of all arrows of the category and the definitions of I, O, $A^{\perp}$ and $A^{\top}$ have to be modified as follows:

$$c\in I \quad\Leftrightarrow\quad \exists_{\alpha}c\leq 1_{\alpha},$$

$$c\in O \quad\Leftrightarrow\quad \forall_{\alpha}1_{\alpha}\nleq c,$$

$$c\in A^{\perp} \quad\Leftrightarrow\quad \forall_{a\in A}\forall_{\alpha}1_{\alpha}\nleq ac,$$

$$c\in A^{\top} \quad\Leftrightarrow\quad \forall_{a\in A}\forall_{\alpha}1_{\alpha}\nleq ca.$$

5. Some final remarks

We have discussed certain *lattice* models of bilinear logic, both intuitionistic and classical. Had we not insisted on their forming a variety, we could equally well have talked about *poset* models, by dismissing the lattice operations $\top$, $\wedge$, $\perp$, $\vee$ from consideration. (These are called "additives" by computer scientists, following Girard.)

As mentioned before, bilinear logic differs from ordinary logic by the omission of Gentzen's three structural rules. These may however be added to our lattice or poset models as follows:

$A \leq I$ (*weakening*),

$A \leq A \otimes A$ (*contraction*),

$A \otimes B \leq B \otimes A$ (*interchange*).

If all three rules are postulated, a residuated lattice becomes a Heyting algebra and a Grishin algebra becomes a Boolean algebra. Of course, one is at liberty to postulate only some of these rules, as is the case for BCK-algebras, where only contraction is absent, and for models of linear logic, where only interchange is present (which follows from the other two rules in this form). If one postulates weakening and contraction, but not interchange, one obtains a noncommutative version of Heyting or Boolean algebras. A noncommutative generalization of Boolean algebras, along apparently quite different lines, has recently been investigated by Diers and Koudsi [1992].

The anti-isomorphism in Proposition 2.4 can be replaced by an isomorphism, provided we write $B \oplus A$ in place of $A \oplus B$; but this would contradict the notation of [L 1993].

The proof of Proposition 4.3 can be refined to yield a stronger result.

PROPOSITION 5.1. *The lattice of downward closed subsets of a Grishin algebra is a Grishin algebra.*

To show this one should replace the multiplication symbol appearing in the definitions of $A \otimes B$, $A^{\perp}$ and $A^{\top}$ by $\oplus$. Hopefully, this is not the last word on the subject.

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