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Nonlinear Small Data Scattering for the Wave and Klein-Gordon Equation

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§ 0. Introduction and Notation

In this paper the scattering operator which belongs to the pair of equations

$$u_{tt}+Au+f(u)=0 \tag{1}$$

and

$$u_{tt}+Au=0 \tag{2}$$

is studied. Here A denotes the operator $-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}+m^2$, $m\in\mathbb{R}$ and $f(u)=\lambda|u|^{\rho-1}u$, where $\lambda\in\mathbb{R}$, $\rho>1$.

Given arbitrary initial data (ϕ^-,ψ^-) of finite small energy it is shown that there exists a solution u in $\mathbb{R}^n\times\mathbb{R}$ of (1) which behaves asymptotically as $t\to-\infty$ like the solution $u_0^-(t)$ of the Cauchy problem for (2) with data (ϕ^-,ψ^-) at $t=0$ in the sense that $\|u(t)-u_0^-(t)\|_e\to 0$ as $t\to-\infty$, and moreover there exists a solution $u_0^+(t)$ of (2) with corresponding data (ϕ^+,ψ^+) at $t=0$ such that $\|u(t)-u_0^+(t)\|_e\to 0$ $(t\to+\infty)$. Here $\|\cdot\|_e$ denotes the energy norm, defined by

$$\|v(t)\|_e^2:=\tfrac{1}{2}(\|A^{\frac{1}{2}}v(t)\|^2+\|v_t(t)\|^2).$$

The mapping S: $(\phi^-,\psi^-)\to(\phi^+,\psi^+)$ is called the scattering operator. The existence of this operator is shown, if (ϕ^-,ψ^-) are given in a whole neighbourhood of the origin in energy space, provided the following conditions are fulfilled:

a) in the case $m=0$ (nonlinear wave equations):

$$3\leqq n\leqq 5 \quad\text{and}\quad \rho=1+\frac{4}{n-2}$$

b) in the case $m\neq 0$ (*nonlinear Klein-Gordon equations*):

$$2\leqq n\leqq 5 \quad\text{and}\quad 1+\frac{4}{n-1}\leqq\rho\leqq 1+\frac{4}{n-2},\qquad 2<\rho<\infty.$$

In the case of the nonlinear Klein-Gordon equation the range $1+\frac{4}{n}\leqq\rho\leqq 1+\frac{4}{n-1}$ was considered for arbitrary n by Strauss in [5, 6], and the same results were proven. For smooth and small data most of the results were known before. Brenner proved in [1] the existence of everywhere defined scattering operators (i.e. for possibly large data) in the Klein-Gordon case for $n=3$ or 4 and $\rho_0(n)<\rho\leqq 1+\frac{4}{n-1}$, provided $\lambda\geqq 0$, where $\rho_0(n)>1+\frac{4}{n}$ has to be chosen appropriately. He remarks that the result remains true for some $\rho>1+\frac{4}{n-1}$.

Moreover it is shown that the ordinary Cauchy problem for the wave and Klein-Gordon equation has a unique global solution for small data of finite energy under the same assumption on the nonlinearity f and the dimension n and there exist so-called asymptotic scattering states in the sense of energy norms. Finally a uniqueness result for the Cauchy problem is proven for arbitrary data of finite energy within a regularity class V which has the property that a solution of the corresponding linear problem with data of finite energy belongs to it.

The key estimates for the results presented here are the decay properties (in time) and a space-time estimate for solutions of the linear problems, which were essentially known before (see § 1 below).

We use the following notation: $\hat{}$ and $\mathscr{F}^{-1}$ denote the Fourier transform and its inverse resp. with respect to space variables. For $s\in\mathbb{R}$, $1\leqq p\leqq\infty$ let $H^{s,p}(\mathbb{R}^n)=H^{s,p}$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to $\|\mathscr{F}^{-1}((1+|\xi|^2)^{\frac{s}{2}}\hat{f}(\xi))\|_{L^p(\mathbb{R}^n)}$, and $\dot{H}^{s,p}(\mathbb{R}^n)=\dot{H}^{s,p}$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to $\|\mathscr{F}^{-1}(|\xi|^s\hat{f}(\xi))\|_{L^p(\mathbb{R}^n)}$. Conjugate exponents are denoted by p,p'; q,q'; etc. Constants are denoted by c and change from line to line.

§ 1. Estimates for the Linear Wave and Klein-Gordon Equation

For convenience of the reader we repeat first of all well-known results on decay of solutions of the linear wave and Klein-Gordon equation.

Theorem 0. a) *Let A denote $-\Delta:=-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}$. Then for any $\psi\in C_0^\infty(\mathbb{R}^n)$ and $t>0$, $1<p\leqq 2\leqq p'<\infty$, the following estimate holds:*

$$\|A^{-\frac{1}{2}}\sin(A^{\frac{1}{2}}t)\psi\|_{L^{p'}(\mathbb{R}^n)}\leqq c t^{-(n-1)\left(\frac{1}{2}-\frac{1}{p'}\right)}\|\psi\|_{\dot{H}^{\frac{n-1}{2}-\frac{n+1}{p'},p}(\mathbb{R}^n)}.$$

b) *Let A denote $-\Delta+m^2$, $m\neq 0$. Then for any $\psi\in C_0^\infty(\mathbb{R}^n)$, $1<p\leqq 2\leqq p'<\infty$ the following estimate holds:*

$$\|A^{-\frac{1}{2}}\sin(A^{\frac{1}{2}}t)\psi\|_{L^{p'}(\mathbb{R}^n)}\leqq K(t)\|\psi\|_{H^{\frac{n-1+\theta}{2}-\frac{n+1+\theta}{p'},p}(\mathbb{R}^n)},$$

where

$$K(t)=c\begin{cases} t^{-(n-1-\theta)\left(\frac{1}{2}-\frac{1}{p'}\right)} & 0<t\leqq 1 \\ t^{-(n-1+\theta)\left(\frac{1}{2}-\frac{1}{p'}\right)} & t\geqq 1 \end{cases} \quad 0\leqq\theta\leqq 1.$$

The *proof* of a) and b) is done e.g. by the stationary phase method and can be found in [3], Theorem 2.2, and in [1], respectively.

The following theorem is a generalization of a result due to Strichartz [7] and is closely related to the arguments of Marshall [2] in the case of the Klein-Gordon equation.

Theorem 1. *Let* A *denote* $-\Delta:=-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}$, *and assume* $f\in L^2(\mathbb{R}^n)$. *Then the solution* u_0 *of the Cauchy problem*

$$u_{0_{tt}}+Au_0=0, \quad u_0(0)=0, \quad u_{0_t}(0)=f$$

fulfills the estimate

$$\|A^{\frac{1-b}{4}}u_0\|_{L^r(\mathbb{R},L^q(\mathbb{R}^n))}\leqq c\|f\|_{L^2(\mathbb{R}^n)},$$

provided $2<r<\infty$, $\frac{1}{2}-\frac{2}{(n-1)r}=\frac{1}{q}$, $b:=\frac{n-1}{2}-\frac{n+1}{q}$.

Corollary. *If* u_0 *is the solution of the Cauchy problem*

$$u_{0_{tt}}+Au_0=0, \quad u_0(0)=g, \quad u_{0_t}(0)=f,$$

and $2\leqq q<\frac{2(n-1)}{n-3}$, $q<\infty$, *the following estimate holds:*

$$\|u_0\|_{L^{\frac{4q}{(n-1)(q-2)}}(\mathbb{R},\dot{H}^{\frac{2(n+1)-(n-3)q}{4q},q}(\mathbb{R}^n))}\leqq c(\|f\|_{L^2(\mathbb{R}^n)}+\|g\|_{\dot{H}^{1,2}(\mathbb{R}^n)}).$$

Proof of Theorem 1. In the sequel we denote by $\hat{}$ and $*$ the Fourier transform and convolution resp. with respect to space variables whereas $\sim$ and $\#$ denote the Fourier transform and convolution resp. with respect to space and time variables.

With $\hat{K}(\xi,t):=\frac{e^{-it|\xi|}}{|\xi|^{b+1}}$ we have

$$\|K\# F\|_{L^r(\mathbb{R},L^q(\mathbb{R}^n))}\leqq\left\|\int_{-\infty}^{+\infty}\left\|\mathscr{F}_\xi^{-1}\left(\frac{e^{-i(t-s)\xi}}{|\xi|^{b+1}}\hat{F}(\xi,s)\right)\right\|_{L^q(\mathbb{R}^n)}ds\right\|_{L^r(\mathbb{R})}$$

$$\leqq c\left\|\int_{-\infty}^{+\infty}(t-s)^{-\frac{n-1}{2}+\frac{n-1}{q}}\left\||\xi|^{-b}|\xi|^{\frac{n-1}{2}-\frac{n+1}{q}}F(\xi,s)\right\|_{L^{q'}(\mathbb{R}^n)}ds\right\|_{L^r(\mathbb{R})}$$

$$=c\left\|\int_{-\infty}^{+\infty}(t-s)^{-\frac{n-1}{2}+\frac{n-1}{q}}\|F(\cdot,s)\|_{L^{q'}(\mathbb{R}^n)}ds\right\|_{L^r(\mathbb{R})},$$

where we used Theorem 0 a). The generalized Young's inequality is now applied and gives an estimate by $c\|F\|_{L^{r'}(\mathbb{R},L^{q'}(\mathbb{R}^n))}$, provided $\frac{1}{r}=\frac{1}{r'}-\frac{1}{\rho'}$, where ρ

$=\left(\frac{n-1}{2}-\frac{n-1}{q}\right)^{-1}$. This condition is fulfilled under the assumption of the theorem.

Consequently Plancherel's theorem in $n+1$ variables gives:

$$\left|\int_{-\infty}^{+\infty}\int_{\mathbb{R}^n}|\tilde{F}(\xi,\tau)|^2\tilde{K}(\xi,\tau)\,d\xi\,d\tau\right|=\left|\int_{-\infty}^{+\infty}\int_{\mathbb{R}^n}\bar{F}(x,t)(K\# F)(x,t)\,dx\,dt\right|$$
$$\leqq c\|F\|^2_{L^{r'}(\mathbb{R},L^{q'}(\mathbb{R}^n))}.$$

Because of

$$\tilde{K}(\xi,\tau)=\frac{1}{|\xi|^{b+1}}\int_{-\infty}^{+\infty}e^{-it(|\xi|-\tau)}\,dt=\frac{1}{|\xi|^{b+1}}\delta(|\xi|-\tau)$$

we arrive at

$$\int_{\mathbb{R}^n}|\tilde{F}(\xi,|\xi|)|^2\frac{d\xi}{|\xi|^{b+1}}\leqq c\|F\|^2_{L^{r'}(\mathbb{R},L^{q'}(\mathbb{R}^n))}.$$

Thus the following duality argument holds by Plancherel's theorem:

$$\left|\int_{-\infty}^{+\infty}\int_{\mathbb{R}^n}(K*f)(x,t)F(x,t)\,dx\,dt\right|=\left|\int_{-\infty}^{+\infty}\int_{\mathbb{R}^n}\hat{f}(\xi)\tilde{K}(\xi,\tau)\tilde{F}(\xi,\tau)\,d\xi\,d\tau\right|$$
$$\leqq\left(\int_{\mathbb{R}^n}|\tilde{F}(\xi,|\xi|)|^2\frac{d\xi}{|\xi|^{b+1}}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}|\hat{f}(\xi)|^2\frac{d\xi}{|\xi|^{b+1}}\right)^{\frac{1}{2}}$$
$$\leqq c\|F\|_{L^{r'}(\mathbb{R},L^{q'}(\mathbb{R}^n))}\left(\int_{\mathbb{R}^n}|\hat{f}(\xi)|^2\frac{d\xi}{|\xi|^{b+1}}\right)^{\frac{1}{2}},$$

consequently

$$\|K*f\|_{L^r(\mathbb{R},L^q(\mathbb{R}^n))}\leqq c\left(\int_{\mathbb{R}^n}|\hat{f}(\xi)|^2\frac{d\xi}{|\xi|^{b+1}}\right)^{\frac{1}{2}}=c\|A^{-\frac{b+1}{4}}f\|_{L^2(\mathbb{R}^n)}.$$

The same estimate holds for $\hat{K}_1(\xi,t):=\frac{e^{it|\xi|}}{|\xi|^{b+1}}$, thus

$$A^{-\frac{b}{2}}u_0=\mathscr{F}_\xi^{-1}\left(\frac{\hat{K}+\hat{K}_1}{2i}(\xi,t)\hat{f}(\xi)\right)$$

fulfills

$$\|A^{-\frac{b}{2}}u_0\|_{L^r(\mathbb{R},L^q(\mathbb{R}^n))}\leqq c\|A^{-\frac{b+1}{4}}f\|_{L^2(\mathbb{R}^n)}.$$

The proof is complete.

Remark. In exactly the same manner for solutions u_0 of the Klein-Gordon equation $u_{0_{tt}}+Au_0=0$, $u_0(0)=g$, $u_t(0)=f$ with $A:=-\Delta+m^2$, $m\neq 0$, the following estimate can be proven:

$$\|u_0\|_{L^{\frac{4q}{(n-1)(q-2)}}(\mathbb{R},H^{\frac{2(n+1)-(n-3)q}{4q},q}(\mathbb{R}^n))}\leqq c(\|f\|_{L^2(\mathbb{R}^n)}+\|g\|_{H^{1,2}(\mathbb{R}^n)}),$$

if $2\leqq q<\frac{2(n-1)}{n-3}$, $q<\infty$. For details we refer to Marshall's paper [2].

§ 2. The Wave Equation with Power Nonlinearity

We consider the Cauchy problem for nonlinear wave equations of the type

$$u_{tt}-\Delta u+\lambda|u|^{\rho-1}u=0 \quad \text{where } \lambda\in\mathbb{R}.$$

We use the abbreviations $f(u)=\lambda|u|^{\rho-1}u$, $A:=-\Delta$, and assume throughout $\rho=1+\frac{4}{n-2}$ and $3\leqq n\leqq 5$.

Theorem 2. *Let* $(\phi^-,\psi^-)\in\dot{H}^{1,2}(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ *be given with* $\|\phi^-\|_{\dot{H}^{1,2}}+\|\psi^-\|_{L^2}<\delta$, *where* $\delta>0$ *is sufficiently small, and let* $u_0^-(t)$ *denote the solution in* $\dot{H}^{1,2}(\mathbb{R}^n)$ *with* $\nabla u_0^-(0)=\nabla\phi^-$, $u_{0_t}^-(t)=\psi^-$ *of the linear wave equation* $u_{0_{tt}}^- - \Delta u_0^- =0$. *Under these assumptions there exists a unique solution u of the integral equation*

$$u(t)=u_0^-(t)+\int_{-\infty}^{t} A^{-\frac{1}{2}}\sin[A^{\frac{1}{2}}(t-\tau)]f(u(\tau))\,d\tau$$

with $(u,u_t)\in V\times L^\infty(\mathbb{R},L^2(\mathbb{R}^n))$, *where* $V:=L^\rho(\mathbb{R},\dot{H}^{s,r}(\mathbb{R}^n))\cap L^\infty(\mathbb{R},\dot{H}^{1,2}(\mathbb{R}^n))$, *where* $s:=\frac{(n-1)\rho-(n+1)}{(n-1)\rho}$, $r:=\frac{2(n-1)\rho}{(n-1)\rho-4}$.

Proof. We want to apply the contraction mapping principle and first remark that $u_0^-\in V$ if we choose $q=\frac{2(n-1)\rho}{(n-1)\rho-4}$ in the Corollary to Theorem 1, and moreover $\|u_0^-\|_V\leqq c\delta$. This choice of q is possible because $2<\rho<\infty$ for $3\leqq n\leqq 5$. Now consider $u,\tilde{u}\in V$ and estimate as follows:

$$\left\|\int_{-\infty}^{t} A^{-\frac{1}{2}}\sin[A^{\frac{1}{2}}(t-\tau)][f(u(\tau))-f(\tilde{u}(\tau))]\,d\tau\right\|_{\dot{H}^{s,r}}$$
$$\leqq c\int_{-\infty}^{t}(t-\tau)^{-\frac{2}{\rho}}\|f(u(\tau))-f(\tilde{u}(\tau))\|_{\dot{H}^{\bar{s},r'}}\,d\tau, \quad \text{where } \bar{s}=\frac{n+1}{(n-1)\rho}.$$

Here we used Theorem 0a). The imbedding $\dot{H}^{1,\tilde{r}'}\subset\dot{H}^{\bar{s},r'}$ holds provided $\frac{1}{\tilde{r}'}=\frac{(n+2)\rho+2}{2n\rho}$ (cf. Stein [4], p. 119, Theorem 1), and thus

$$\|f(u)-f(\tilde{u})\|_{\dot{H}^{\bar{s},r'}}\leqq c\sum_{|\alpha|=1}\|f'(u)D^\alpha u-f'(\tilde{u})D^\alpha\tilde{u}\|_{L^{\tilde{r}'}}$$
$$\leqq c\sum_{|\alpha|=1}(\|(u-\tilde{u})(|u|^{\rho-2}+|\tilde{u}|^{\rho-2})D^\alpha u\|_{L^{\tilde{r}'}}+\|\tilde{u}|\tilde{u}|^{\rho-2}(D^\alpha u-D^\alpha\tilde{u})\|_{L^{\tilde{r}'}}).$$

The first term is estimated using Hölder's inequality by

$$\|u-\tilde{u}\|_{L^{\tilde{r}'\hat{p}}}(\|u\|^{\rho-2}_{L^{\tilde{r}'(\rho-2)\hat{q}}}+\|\tilde{u}\|^{\rho-2}_{L^{\tilde{r}'(\rho-2)\hat{q}}})\|D^\alpha u\|_{L^{\tilde{r}'\hat{r}}}.$$

The choice $\hat{r}=\frac{2}{\tilde{r}'}$ and $\hat{p}=(\rho-2)\hat{q}$ leads to $\tilde{r}'\hat{p}=\tilde{r}'(\rho-2)\hat{q}=\frac{n\rho(\rho-1)}{\rho+1}$. Now the imbedding $\dot{H}^{s,r}\subset L^{\frac{n\rho(\rho-1)}{\rho+1}}$ holds if and only if $\rho=\frac{n+2}{n-2}$ so that we have

$$\|f(u)-f(\tilde{u})\|_{\dot{H}^{s,r'}}$$
$$\leqq c(\|u-\tilde{u}\|_{\dot{H}^{s,r}}(\|u\|_{\dot{H}^{s,r}}^{\rho-2}+\|\tilde{u}\|_{\dot{H}^{s,r}}^{\rho-2})\|u\|_{\dot{H}^{1,2}}+\|\tilde{u}\|_{\dot{H}^{s,r}}^{\rho-1}\|u-\tilde{u}\|_{\dot{H}^{1,2}}).$$

Therefore the following inequality holds:

$$\left\|\int_{-\infty}^{t} A^{-\frac{1}{2}}\sin[A^{\frac{1}{2}}(t-\tau)][f(u(\tau))-f(\tilde{u}(\tau))]\,d\tau\right\|_{\dot{H}^{s,r}}$$
$$\leqq c\int_{-\infty}^{t}(t-\tau)^{-\frac{2}{\rho}}[\|u(\tau)-\tilde{u}(\tau)\|_{\dot{H}^{s,r}}(\|u(\tau)\|_{\dot{H}^{s,r}}^{\rho-2}+\|\tilde{u}(\tau)\|_{\dot{H}^{s,r}}^{\rho-2})\|u\|_{L^\infty(\mathbb{R},\dot{H}^{1,2}(\mathbb{R}^n))}$$
$$+\|\tilde{u}(\tau)\|_{\dot{H}^{s,r}}^{\rho-1}\|u-\tilde{u}\|_{L^\infty(\mathbb{R},\dot{H}^{1,2}(\mathbb{R}^n))}]\,d\tau.$$

Now we use the generalized Young's inequality

$$\|K*g\|_{L^q(\mathbb{R})}\leqq c\|g\|_{L^p(\mathbb{R})} \quad \text{for } K(t)=\frac{1}{|t|^\alpha}, \quad 0<\alpha<1,$$

where $\tilde{\rho}=\frac{1}{\alpha}$ and $\frac{1}{p}=\frac{1}{q}+\frac{1}{\tilde{\rho}'}$. We have $\tilde{\rho}=\frac{\rho}{2}$, $q=\rho$, $p=\frac{\rho}{\rho-1}$, thus

$$\left\|\int_{-\infty}^{t} A^{-\frac{1}{2}}\sin[A^{\frac{1}{2}}(t-\tau)][f(u(\tau))-f(\tilde{u}(\tau))]\,d\tau\right\|_{L^\rho(\mathbb{R},\dot{H}^{s,r}(\mathbb{R}^n))}$$
$$\leqq c\left\{\left[\int_{-\infty}^{+\infty}\|u(\tau)-\tilde{u}(\tau)\|_{\dot{H}^{s,r}}^{\frac{\rho}{\rho-1}}(\|u(\tau)\|_{\dot{H}^{s,r}}^{\frac{\rho}{\rho-1}(\rho-2)}+\|\tilde{u}(\tau)\|_{\dot{H}^{s,r}}^{\frac{\rho}{\rho-1}(\rho-2)})\,d\tau\right]^{\frac{\rho-1}{\rho}}\right.$$
$$\left.\cdot\|u\|_{L^\infty(\mathbb{R},\dot{H}^{1,2}(\mathbb{R}^n))}+\left[\int_{-\infty}^{+\infty}\|\tilde{u}(\tau)\|_{\dot{H}^{s,r}}^{\rho}\,d\tau\right]^{\frac{\rho-1}{\rho}}\|u-\tilde{u}\|_{L^\infty(\mathbb{R},\dot{H}^{1,2}(\mathbb{R}^n))}\right\}$$
$$\leqq c\|u-\tilde{u}\|_V(\|u\|_V^{\rho-1}+\|\tilde{u}\|_V^{\rho-1}). \tag{1}$$

Furthermore one has

$$\left\|\int_{-\infty}^{t} A^{-\frac{1}{2}}\sin[A^{\frac{1}{2}}(t-\tau)][f(u(\tau))-f(\tilde{u}(\tau))]\,d\tau\right\|_{L^\infty(\mathbb{R},\dot{H}^{1,2}(\mathbb{R}^n))}$$
$$\leqq c\int_{-\infty}^{+\infty}\|f(u(\tau))-f(\tilde{u}(\tau))\|_{L^2(\mathbb{R}^n)}\,d\tau$$
$$\leqq c\int_{-\infty}^{+\infty}(\|u(\tau)\|_{L^{2\rho}}^{\rho-1}+\|\tilde{u}(\tau)\|_{L^{2\rho}}^{\rho-1})\|u(\tau)-\tilde{u}(\tau)\|_{L^{2\rho}}\,d\tau$$
$$\leqq c(\|u\|_V^{\rho-1}+\|\tilde{u}\|_V^{\rho-1})\|u-\tilde{u}\|_V,$$

because the imbedding $\dot{H}^{s,r}\subset L^{2\rho}$ holds if and only if $\rho=\frac{n+2}{n-2}$.

If we denote the transformation which maps u into the right hand side of the considered integral equation by T we have shown

$$\|T(u)-T(\tilde{u})\|_V\leqq c(\|u\|_V^{\rho-1}+\|\tilde{u}\|_V^{\rho-1})\|u-\tilde{u}\|_V$$

as well as

$$\|T(u)\|_V\leqq c\delta+c\|u\|_V^\rho.$$

Assume now that $c\delta \leqq \frac{1}{2}\delta_1$ and $2c\,\delta_1^{\rho-1} \leqq \frac{1}{2}$. If then $\|u\|_V$, $\|\tilde{u}\|_V \leqq \delta_1$ we conclude that

$$\|T(u) - T(\tilde{u})\|_V \leqq \tfrac{1}{2}\|u-\tilde{u}\|_V \quad \text{and} \quad \|T(u)\|_V \leqq \delta_1.$$

The contraction mapping principle shows the existence of a unique solution within the ball $\|u\|_V \leqq \delta_1$. For uniqueness within the whole of V we remark that any two solutions u and $\tilde{u}$ satisfy the estimate (in analogy to (1)):

$$\begin{aligned}\|u-\tilde{u}\|&_{L^\rho(I,\dot{H}^{s,r}(\mathbb{R}^n))}\\ &\leqq c[\|u\|_{L^\infty(I,\dot{H}^{1,2}(\mathbb{R}^n))}(\|u\|^{\rho-2}_{L^\rho(I,\dot{H}^{s,r}(\mathbb{R}^n))} + \|\tilde{u}\|^{\rho-2}_{L^\rho(I,\dot{H}^{s,r}(\mathbb{R}^n))})\\ &\quad \cdot \|u-\tilde{u}\|_{L^\rho(I,\dot{H}^{s,r}(\mathbb{R}^n))} + \|\tilde{u}\|^{\rho-1}_{L^\rho(I,\dot{H}^{s,r}(\mathbb{R}^n))}\|u-\tilde{u}\|_{L^\infty(I,\dot{H}^{1,2}(\mathbb{R}^n))}]\end{aligned}$$

as well as

$$\begin{aligned}\|u-\tilde{u}\|&_{L^\infty(I,\dot{H}^{1,2}(\mathbb{R}^n))}\\ &\leqq c(\|u\|^{\rho-1}_{L^\rho(I,\dot{H}^{s,r}(\mathbb{R}^n))} + \|\tilde{u}\|^{\rho-1}_{L^\rho(I,\dot{H}^{s,r}(\mathbb{R}^n))})\|u-\tilde{u}\|_{L^\rho(I,\dot{H}^{s,r}(\mathbb{R}^n))},\end{aligned}$$

where $I=(-\infty, \bar{T})$.

The condition $u, \tilde{u} \in L^\rho(\mathbb{R}, \dot{H}^{s,r}(\mathbb{R}^n))$ allows us to choose $|\bar{T}|$ so large that

$$\|u-\tilde{u}\|_{V_{\bar{T}}} \leqq \tfrac{1}{2}\|u-\tilde{u}\|_{V_{\bar{T}}},$$

where $V_{\bar{T}} = L^\rho(I, \dot{H}^{s,r}(\mathbb{R}^n)) \cap L^\infty(I, \dot{H}^{1,2}(\mathbb{R}^n))$, so that $u=\tilde{u}$ in $V_{\bar{T}}$. Step by step it is possible to replace $\bar{T}$ by $\bar{T}+\varepsilon$ where ε depends possibly on u and $\tilde{u}$, but is the same for all steps. Thus $u=\tilde{u}$ in V, and the proof is complete.

Corollary. $\|u(t)-u_0^-(t)\|_e \to 0$ $(t\to -\infty)$.

Proof. $\|u(t)-u_0^-(t)\|_e \leqq c\int_{-\infty}^{t} \|f(u(\tau))\|_{L^2}\,d\tau \leqq c\int_{-\infty}^{t} \|u(\tau)\|^\rho_{L^{2\rho}}\,d\tau$. This tends to zero, because $L^\rho(\mathbb{R}, (\dot{H}^{s,r}(\mathbb{R}^n)) \subset L^\rho(\mathbb{R}, L^{2\rho}(\mathbb{R}^n))$.

Define now

$$u_0^+(t) := u(t) + \int_t^\infty A^{-\frac{1}{2}}\sin[A^{\frac{1}{2}}(t-\tau)]f(u(\tau))\,d\tau, \quad \text{and} \quad u_0^+(0)=\phi^+,\ u_{0_t}^+(0)=\psi^+.$$

Then we show in exactly the same manner as above $\|u(t)-u_0^+(t)\|_e \to 0$ $(t\to\infty)$, so that we have proven

Theorem 3. *The scattering operator* $S: (\phi^-, \psi^-) \to (\phi^+, \psi^+)$ *exists in the sense of energy norms in a whole neighbourhood of the origin in* $\dot{H}^{1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

§ 3. The Nonlinear Klein-Gordon Equation

In this section we consider the Cauchy problem for equations of the type

$$u_{tt} + Au + f(u) = 0,$$

where $A := -\Delta + m^2$, $m \neq 0$, $f(u) := \lambda |u|^{\rho-1}u$, $\lambda \in \mathbb{R}$.

We assume $2 \leqq n \leqq 5$ and $1+\frac{4}{n-1} \leqq \rho \leqq 1+\frac{4}{n-2}$, $2<\rho<+\infty$.

Theorem 4. *Let* $(\phi^-, \psi^-) \in H^{1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *be given with* $\|\phi^-\|_{H^{1,2}(\mathbb{R}^n)} + \|\psi^-\|_{L^2(\mathbb{R}^n)} \leqq \delta$, $\delta>0$ *sufficiently small, and let* $u_0^-(t)$ *denote the unique solution in* $H^{1,2}(\mathbb{R}^n)$ *of* $u_{0_{tt}}^- + Au_0^- = 0$, $u_0^-(0)=\phi^-$, $u_{0_t}^-(0)=\psi^-$. *Under these assumptions there exists a unique solution u of the integral equation*

$$u(t)=u_0^-(t)+\int_{-\infty}^{t} A^{-\frac{1}{2}} \sin[A^{\frac{1}{2}}(t-\tau)] f(u(\tau))\, d\tau$$

with $(u, u_t) \in V \times L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))$, *where* $V := L^\rho(\mathbb{R}, H^{s,r}(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}, H^{1,2}(\mathbb{R}^n))$ *and* s, r *as in Theorem 2.*

Proof. In analogy to the proof of Theorem 2 we have $u_0^- \in V$ and $\|u_0^-\|_V \leqq c\delta$ by use of the remark after Theorem 1, and moreover

$$\left\| \int_{-\infty}^{t} A^{-\frac{1}{2}} \sin[A^{\frac{1}{2}}(t-\tau)][f(u(\tau)) - f(\tilde{u}(\tau))]\, d\tau \right\|_{H^{s,r}} \leqq c \int_{-\infty}^{t} (t-\tau)^{-\frac{2}{\rho}} \|f(u(\tau)) - f(\tilde{u}(\tau))\|_{H^{\bar{s},r'}}\, d\tau$$

by Theorem 0b) (notation as in the proof of Theorem 2).

We now use Sobolev's imbedding $H^{1,\tilde{r}'} \subset H^{\bar{s},r'}$ which holds for

$$\frac{1}{\tilde{r}'} \geqq \frac{1}{r'} \geqq \frac{1}{\tilde{r}'} - \frac{1-\bar{s}}{n}$$

and arrive as in the proof of Theorem 2 at

$$\begin{aligned} &\|f(u)-f(\tilde{u})\|_{H^{\bar{s},r'}} \\ &\quad \leqq c \sum_{|\alpha| \leqq 1} [\|u-\tilde{u}\|_{L^{\tilde{r}'\hat{p}}} (\|u\|^{\rho-2}_{L^{\tilde{r}'(\rho-2)\hat{q}}} + \|\tilde{u}\|^{\rho-2}_{L^{\tilde{r}'(\rho-2)\hat{q}}}) \|D^\alpha u\|_{L^{\tilde{r}'\hat{r}}} \\ &\qquad + \|\tilde{u}\|_{L^{\tilde{r}'\hat{p}}} \|\tilde{u}\|^{\rho-2}_{L^{\tilde{r}'(\rho-2)\hat{q}}} \|D^\alpha u - D^\alpha \tilde{u}\|_{L^{\tilde{r}'\hat{r}}}]. \end{aligned}$$

We choose $\hat{p}$, $\hat{q}$, $\hat{r}$ as follows: $\tilde{r}'\hat{r}=2$, $\hat{p}=(\rho-2)\hat{q}$.

This gives $\tilde{r}'\hat{p}=(\rho-1)\frac{2\tilde{r}'}{2-\tilde{r}'}$. Now the imbedding $H^{s,r} \subset L^{(\rho-1)\frac{2\tilde{r}'}{2-\tilde{r}'}}$ holds provided $\frac{1}{r} \geqq \frac{1}{\rho-1}\left(\frac{1}{\tilde{r}'}-\frac{1}{2}\right) \geqq \frac{1}{r}-\frac{s}{n}$. This condition can be fulfilled taking into account the range for $\frac{1}{\tilde{r}'}$ above if and only if $1+\frac{4}{n-1} \leqq \rho \leqq 1+\frac{4}{n-2}$ as a simple calculation shows. Thus the following estimate holds:

$$\left\| \int_{-\infty}^{t} A^{-\frac{1}{2}} \sin[A^{\frac{1}{2}}(t-\tau)][f(u(\tau))-f(\tilde{u}(\tau))]\,d\tau \right\|_{H^{s,r}}$$

$$\leqq c \int_{-\infty}^{t} (t-\tau)^{-\frac{2}{\rho}} [\|u(\tau)-\tilde{u}(\tau)\|_{H^{s,r}} (\|u(\tau)\|_{H^{s,r}}^{\rho-2} + \|\tilde{u}(\tau)\|_{H^{s,r}}^{\rho-2}) \|u\|_{L^\infty(\mathbb{R}, H^{1,2}(\mathbb{R}^n))}$$

$$+ \|\tilde{u}(\tau)\|_{H^{s,r}}^{\rho-1} \|u-\tilde{u}\|_{L^\infty(\mathbb{R}, H^{1,2}(\mathbb{R}^n))}]\,d\tau$$

$$\leqq c\|u-\tilde{u}\|_V (\|u\|^{\rho-1} + \|\tilde{u}\|_V^{\rho-1}),$$

if we proceed as in the proof of Theorem 2.

Moreover

$$\left\| \int_{-\infty}^{t} A^{-\frac{1}{2}} \sin[A^{\frac{1}{2}}(t-\tau)][f(u(\tau))-f(\tilde{u}(\tau))]\,d\tau \right\|_{L^\infty(\mathbb{R}, H^{1,2}(\mathbb{R}^n))}$$

$$\leqq c \int_{-\infty}^{+\infty} (\|u(\tau)\|_{L^{2\rho}}^{\rho-1} + \|\tilde{u}(\tau)\|_{L^{2\rho}}^{\rho-1}) \|u(\tau)-\tilde{u}(\tau)\|_{L^{2\rho}}\,d\tau$$

$$\leqq c(\|u\|_V^{\rho-1} + \|\tilde{u}\|_V^{\rho-1}) \|u-\tilde{u}\|_V,$$

where we used the imbedding $H^{s,r} \subset L^{2\rho}$ which holds for $1+\frac{4}{n-1} \leqq \rho \leqq 1+\frac{4}{n-2}$ again. Now the claim follows as in Theorem 2.

Consequently as in the case of the wave equation we have

Theorem 5. *The scattering operator* $S: (\phi^-, \psi^-) \to (\phi^+, \psi^+)$ *exists in the sense of energy norms in a whole neighbourhood of the origin in* $H^{1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

§ 4. Further Results

By the same methods as above it is possible to prove the following result for the initial-value problem under the same assumptions on f and n.

Theorem 6. *Let* $(\phi, \psi) \in \dot{H}^{1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *(or* $H^{1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$*) be given with sufficiently small norm, and let* $u_0(t)$ *denote the solution in* $\dot{H}^{1,2}(\mathbb{R}^n)$ *(or* $H^{1,2}(\mathbb{R}^n)$ *resp.) with* $u_{0_t}(t) \in L^2(\mathbb{R}^n)$ *of the problem* $u_{0_{tt}} + Au_0 = 0$, $u_0(0) = \phi$, $u_{0_t}(0) = \psi$, *where* $A = -\Delta$ *(or* $-\Delta + m^2$, $m \neq 0$, *resp.). Under these assumptions the integral equation*

$$u(t) = u_0(t) - \int_0^t A^{-\frac{1}{2}} \sin[A^{\frac{1}{2}}(t-\tau)] f(u(\tau))\,d\tau$$

has a unique solution in V *with* $u_t \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))$. *Moreover there exist solutions* $u_\pm$ *of* $u_{\pm_{tt}} + Au_\pm = 0$ *such that* $\|u(t) - u_\pm(t)\|_e \to 0$ $(t \to \pm\infty)$.

Proof. The existence of u is shown as before, and

$$u_\pm(t) := u(t) + \int_{\pm\infty}^{t} A^{-\frac{1}{2}} \sin[A^{\frac{1}{2}}(t-\tau)] f(u(\tau))\,d\tau$$

has the claimed properties.

Moreover, the following uniqueness result is valid under the same assumptions on f and n as before (proof as above):

Theorem 7. *Let* $(\phi, \psi) \in \dot{H}^{1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *(or* $H^{1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$*) be given arbitrarily, and let* $u_0(t)$ *denote the solution in* $\dot{H}^{1,2}(\mathbb{R}^n)$ *(or* $H^{1,2}(\mathbb{R}^n)$ *resp.) with* $u_{0_t}(t) \in L^2(\mathbb{R}^n)$ *of the problem* $u_{0_{tt}} + Au_0 = 0$, $u_0(0) = \phi$, $u_{0_t}(0) = \psi$, *where* $A = -\Delta$ *(or* $A = -\Delta + m^2$, $m \neq 0$, *resp.). Then there exists at most one solution of the integral equation*

$$u(t) = u_0(t) - \int_0^t A^{-\frac{1}{2}} \sin[A^{\frac{1}{2}}(t-\tau)] f(u(\tau)) d\tau$$

in the space V.

Concluding Remark. The results of §§ 2–4 hold in the case of a more general nonlinearity $f \in C^1(\mathbb{R})$, which fulfills $f(0) = f'(0) = 0$ and $|f'(u_1) - f'(u_2)| \leqq c(|u_1|^{\rho-2} + |u_2|^{\rho-2})|u_1 - u_2|$ for $u_1, u_2 \in \mathbb{R}$, where ρ is restricted as above.

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