Informational Interpretation of Substructural Propositional Logics

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Abstract. This paper deals with various substructural propositional logics, in particular with substructural subsystems of Nelson's constructive propositional logics $\mathbf{N}^-$ and $\mathbf{N}$. Došen's groupoid semantics is extended to these constructive systems and is provided with an informational interpretation in terms of information pieces and operations on information pieces.

Key words: substructural logics, constructive negation, groupoid semantics, informational interpretation

1. INTRODUCTION

Many logical systems admit of an interpretation in semantical models based on abstract information structures, that is to say, models based on a non-empty set I viewed as set of *information pieces* or *information states* represented by pieces of information together with certain relations or operations on I and possibly some designated pieces of information. Well-known examples of such *informational interpretations* are among others Kripke's (1965) interpretation of intuitionistic propositional logic IPL by means of quasi-ordered information states $\langle I, \sqsubseteq\rangle$ and Urquhart's (1972) interpretation of relevant implicational logic $R_{\supset}$ in terms of the addition of information pieces and the empty piece of information.[1] The abstract information structures to be considered in the present paper are semilattice-ordered monoids (*slomos*),

[1] The notion of an informational interpretation by means of models based on abstract information structures probably cannot be captured by a precise definition. A non-trivial constraint would be to require that there are intended models and that every intended model is complete. Then, for instance, Grzegorczyk's (1964) "philosophically plausible" interpretation of IPL as a logic of scientific research would *fail* to be informational. Although Grzegroczyk's 'researches' for IPL form the class of concrete, intended models, it can readily be verified that there exists no complete research for IPL. Every research as defined by Grzegorczyk induces a Kripke model $\langle I, \sqsubseteq_P, 1, v_0\rangle$ for IPL based on a tree. Each $a \in I$ is a finite set of propositional variables, and for every propositional variable p, the basic valuation v_0 is defined by $v_0(p) = \{a \mid p \in a\}$. A basic valuation v_0 is extended to a valuation function v as in Kripke models for IPL, and validity in a research is defined as truth at every information piece. Now take any research $\mathcal{R} = \langle I, \sqsubseteq_P, 1, v_0\rangle$ for IPL. Then the set $\Gamma = \bigcup\{a \mid a \in I\}$ is finite, and for all propositional variables p, q such that $q \notin \Gamma$, we have that $q \supset p$ is valid in $\mathcal{R}$. But $q \supset p$ is not a theorem of IPL and hence $\mathcal{R}$ is not complete.

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see (Došen, 1989, p. 43 ff.). Models based on *slomos* will becalled *monoid models*. We suggest the following understanding of *slomos* $\langle I, \cdot, \sqcap, 1\rangle$:

- I is a set of information pieces
- $\cdot$ is the *addition* of information pieces
- $\sqcap$ is the *intersection* of information pieces
- 1 is the *initial*, ideally the *empty* piece of information.

The aim of this paper is to develop an informational interpretation by means of monoid models for a broad range of substructural propositional logics, i.e., logics with a restricted set of structural inference rules. Section 2 introduces various families of substructural propositional logics. The monoid models are presented in Section 3, and the informational interpretation is dealt with in Section 4.

2. FOUR FAMILIES OF SUBSTRUCTURAL PROPOSITIONAL LOGICS

In this section we will introduce sequent calculus presentations for a number of substructural subsystems of Johansson's intuitionistic minimal logic MPL, IPL and Nelson's constructive propositional logics $\mathbf{N}^-$ and $\mathbf{N}$, see (Almukdad and Nelson, 1984). Preparatory to this, we shall first briefly motivate going substructural and going constructive.[2]

Going substructural

If we think of deductive information processing, the premises form the database (DB) and the consequence relation $\rightarrow$ is the information-processing mechanism. In a certain sense all standard logics constitute *maximal* conceptions of deductive information processing. Consider for instance IPL. In IPL the sequent arrow represents a Tarskian syntactic consequence relation between finite *sets* of premises and single formulas. This consequence relation is monotonic: if formula A is derivable from Γ, then A is derivable from every finite superset of Γ. It is a commonplace by now that everyday reasoning (often) is nonmonotonic. We are deriving conclusions which may turn out wrong in the light of new, additonal information and, accordingly, we are willing to retract, if necessary, such 'provisional' conclusions. In sequent-style presentations the fact that one is dealing with *monotonic* inferences of

[2] One might ask why we do not consider substructural subsystems of classical propositional logic CPL. CPL may be regarded as inappropriate, because it validates *tertium non datur*. As Urquhart (1972, p. 166) emphasizes, information can be incomplete: "[w]ith no information whatever about, say, Smith, we can neither conclude 'Smith is tall' nor 'Smith is not tall'. Thus we would *not* expect the law of excluded middle to be valid in a semantics involving pieces of information."

single formulas from finite *sets* of premises can explicitly be stated by means of *structural* inference rules, i.e. inference rules which govern the manipulation of premises (or contexts).[3] Besides the monotonicity rule **M** there are structural rules allowing for permuting (**P**) and contracting (**C**) premise occurrences. If one now considers a systematic variation of such structural rules of inference, the standard package $\{\mathbf{P}, \mathbf{C}, \mathbf{M}\}$ breaks down into a more differentiated ensemble of rules, and DBs need not only be conceived of as sets of sentences with a monotonic inference operation defined on them (for a general framework of *structured* consequence relations see (Gabbay, 1991).

Although in the most commonly used logics the structural rules **P**, **C** and **M** are assumed, giving up all or part of them has a long tradition. For example, relevant implicational logic $R_{\supset}$ developed by Church (1950) and Moh (1950) is nothing but intuitionistic implicational logic $IPL_{\supset}$ without the monotonicity rule, and in general not to accept the full strength of monotonicity forms the basic idea of relevance logic, cf. (Dunn, 1986). Conceptions of deductive information processing weaker than the intuitionistic one can also typically be found within logical syntax, i.e. Categorial Grammar. The ('product-free' version of the) syntactic calculus of Lambek (1958) turns out to be intuitionistic implicational logic without any structural rules of inference (but restricted to derivations from non-empty sequences). This syntactic calculus is an order-sensitive logic of *occurrences*, since in syntax every occurrence of a linguistic item to which a syntactic type (logically speaking, a premise) is assigned matters. If the product-free Lambek Calculus is extended by the structural rule of permutation, one obtains the so-called non-directional Lambek Calculus of syntactic categories, see (van Benthem, 1986, 1988). In this case one is concerned with nonmonotonic inferences of single formulas from finite, non-empty *multisets* of formulas, i.e. collections in which every occurrence matters but the order of occurrences is irrelevant. Allowing for derivations from the empty multiset, the non-directional Lambek Calculus turns out to be the implicational fragment of Girard's intuitionistic linear logic without 'exponentials' (Girard, 1987), in other words, intuitionisitic logic without the rules of monotonicity and contraction (cf. also (Avron, 1988), (Troelstra, 1992)).

Going Constructive

[3] In this respect also the rules

$(id)\quad \vdash A \rightarrow A$ and
$(cut)\quad Y \rightarrow A\ \ XAZ \rightarrow B \vdash XYZ \rightarrow B$

are structural rules. Following Girard (Girard *et al.*, 1989), we will, however, regard (id) and (cut) as logical rules available in any (ordinary) sequent calculus.

Any theory of information processing, in order to be viewed as adequate, will be expected to allow for representing both positive as well as negative information. According to Gurevich (1977) intuitionistic logic does *not* provide an adequate treatment of negative information. In intuitionistic logic (in the language with $\supset$, $\wedge$, $\vee$, and $\bot$) a negated sentence $\neg A$ abbreviates $A \supset \bot$, that is to say, $\neg A$ is understood as "A implies absurdity". Gurevich remarks that "[i]n many cases the falsehood of a simple scientific sentence can be ascertained as directly (or undirectly) as its truth" (1977, p. 49). Gurevich therefore would like to have available a primitive *strong* negation in order to express explicit falsity. It is instructive to reformulate Gurevich's point of view in semantical terms. In an intuitionistic Kripke model $\langle I, \sqsubseteq, v_0 \rangle$, $\neg p$ is true at an information state $a \in I$ iff p is not verified at any information state into which a may develop.Thus, while verifying p at $a \in I$ does not involve considering other information states than a, verifying $\neg p$ involves inspection of all information states $b \in I$ such that $a \sqsubseteq b$. Gurevich's remark amounts to the complaint that there is no possibility of direct falsification of p on the spot.[4] Now, the idea of taking negative information seriously and putting it on a par with positive information leads Gurevich to intuitionistic logic with strong negation, as developed by Nelson (1949). Recent pleas for the relevance of negative information and the usefulness of strong negation for representing negative reasoning can be found in (Pearce and Wagner, 1990), (Pearce, 1991) and (Wagner, 1991).

Nelson's strong negation $\sim$ is also called *constructive* negation. Indeed, although intuitionistic logic is often referred to as 'constructive logic', intuitionistic negation exhibits certain non-constructive features. Whereas on the one hand, in contrast to classical logic, intuitionistic logic enjoys the disjunction property (or principle of constructible truth):

$(A \vee B)$ is provable iff A is provable or B is provable,

it fails to satisfy the principle of *constructible falsity*, which one should expect to hold for a truly constructive negation:

$\neg(A \wedge B)$ is provable iff $\neg A$ is provable or $\neg B$ is provable.

In Nelson's systems of constructive logic constructible falsity holds wrt $\sim$.

Introducing Nelson's strong, constructive negation $\sim$ into positive intuitionistic logic also can be and has been motivated by supplementing the proof interpretation of the intuitionistic connectives $\supset$, $\wedge$, and $\vee$ by a *disproof interpretation*, cf. (López-Escobar, 1972):

[4] Obviously, Gurevich's insistance on falsification has a famous precursor in Popper's philosophy of science (see e.g. (Popper, 1963)) according to which *falsification* is even the more important epistemological principle as compared to *verification*.

- Π is a disproof of a conjunction $A \wedge B$ iff Π is either a disproof of A or a disproof a B
- Π is a disproof of a disjunction $A \vee B$ iff Π is a pair $\langle \Pi_1, \Pi_2 \rangle$ such that Π_1 disproves A and Π_2 disproves B
- Π is a disproof of an implication $A \supset B$ iff Π is a pair $\langle \Pi_1, \Pi_2 \rangle$ such that Π_1 proves A and Π_2 disproves B
- Π is a disproof of a negation $\sim A$ iff Π is a proof of A.

A proof of $\sim A$ then amounts to a disproof (or refutation) of A (and not to a proof of $A \supset \bot$).

The proof systems. Consider the propositional language L in the following vocabulary:

a denumerable set $PROP$ of propositional variables;
propositional constants: $\mathbf{t}$, $\top$, $\bot$;
binary connectives: $/$, $\backslash$, $\circ$, $\wedge$, $\vee$;
auxilliary symbols: (,) .

We use $p, q, p_1, p_2, \ldots$ resp. $A, B, C, A_1, A_2, \ldots$ resp. $X, Y, Z, X_1, X_2, \ldots$ as schematic letters for propositional variables, formulas, and finite, possibly empty sequences of formula occurrences, respectively. An expression $X \rightarrow A$ is called a *sequent*. We say that a formula A is provable, if the sequent $\rightarrow A$ is provable; two formulas A, B are said to be interderivable, if the sequents $A \rightarrow B$, $B \rightarrow A$ are provable.

DEFINITION 1. (i) The rules of intuitionistic minimal sequential propositional logic $MSPL$ (i.e. Johansson's MPL without structural inference rules) are:

(id) $\vdash A \rightarrow A$;
(cut) $Y \rightarrow A \quad XAZ \rightarrow B \vdash XYZ \rightarrow B$;
$(\rightarrow \mathbf{t})$ $\vdash X \rightarrow \mathbf{t}$;
$(\rightarrow \top)$ $\vdash \rightarrow \top$;
$(\top \rightarrow)$ $XY \rightarrow A \vdash X\top Y \rightarrow A$;
$(\rightarrow /)$ $XA \rightarrow B \vdash X \rightarrow (B/A)$;
$(/ \rightarrow)$ $Y \rightarrow A \quad XBZ \rightarrow C \vdash X(B/A)YZ \rightarrow C$;
$(\rightarrow \backslash)$ $AX \rightarrow B \vdash X \rightarrow (A \backslash B)$;

$$
\begin{array}{ll}
(\backslash \rightarrow) & Y \rightarrow A \quad XBZ \rightarrow C \vdash XY(A \backslash B)Z \rightarrow C; \\
(\rightarrow \circ) & X \rightarrow A \quad Y \rightarrow B \vdash XY \rightarrow (A \circ B); \\
(\circ \rightarrow) & XABY \rightarrow C \vdash X(A \circ B)Y \rightarrow C; \\
(\rightarrow \wedge) & X \rightarrow A \quad X \rightarrow B \vdash X \rightarrow (A \wedge B); \\
(\wedge \rightarrow) & XAY \rightarrow C \vdash X(A \wedge B)Y \rightarrow C, \\
 & XBY \rightarrow C \vdash X(A \wedge B)Y \rightarrow C; \\
(\rightarrow \vee) & X \rightarrow A \vdash X \rightarrow (A \vee B), \\
 & X \rightarrow B \vdash X \rightarrow (A \vee B); \\
(\vee \rightarrow) & XAY \rightarrow C \quad XBY \rightarrow C \vdash X(A \vee B)Y \rightarrow C.
\end{array}
$$

(ii) The rules of intuitionistic sequential propositional logic $ISPL$[5] (i.e. IPL without structural rules of inference) are those of $MSPL$ plus

$$(\bot \rightarrow) \quad \vdash X \bot Y \rightarrow A.$$

Let $L^{\sim}$ denote the result of enriching L by a new unary connective $\sim$ which denotes *strong*, *constructive* negation.

DEFINITION 2. (i) The rules of constructive minimal sequential propositional logic $COSPL^{-}$ (i.e. Nelson's constructive minimal propositional logic $\mathbf{N}^{-}$ (cf. (Almukdad and Nelson, 1984), (von Kutschera, 1969), (López-Escobar, 1972), (Routley, 1974)) without structural inference rules) are the rules of $MSPL$ together with:

$$
\begin{array}{ll}
(\rightarrow\sim /) & X \rightarrow\sim B \quad Y \rightarrow A \vdash XY \rightarrow\sim (B/A); \\
(\sim / \rightarrow) & X \sim BAY \rightarrow C \vdash X \sim (B/A)Y \rightarrow C; \\
(\rightarrow\sim \backslash) & X \rightarrow A \quad Y \rightarrow\sim B \vdash XY \rightarrow\sim (A \backslash B); \\
(\sim \backslash \rightarrow) & XA \sim BY \rightarrow C \vdash X \sim (A \backslash B)Y \rightarrow C; \\
(\rightarrow\sim \circ) & X \rightarrow\sim A \quad Y \rightarrow\sim B \vdash XY \rightarrow\sim (A \circ B); \\
(\sim \circ \rightarrow) & X \sim A \sim BY \rightarrow C \vdash X \sim (A \circ B)Y \rightarrow C, \\
(\rightarrow\sim \wedge) & X \rightarrow\sim A \vdash X \rightarrow\sim (A \wedge B), \\
 & X \rightarrow\sim B \vdash X \rightarrow\sim (A \wedge B); \\
(\sim \wedge \rightarrow) & X \sim AY \rightarrow C \quad X \sim BY \rightarrow C \vdash X \sim (A \wedge B)Y \rightarrow C,
\end{array}
$$

[5] Also called "non-commutative intuitionistic linear propositional logic without exponentials".

$$(\rightarrow\sim\vee)\quad X\rightarrow\sim A\quad X\rightarrow\sim B\vdash X\rightarrow\sim(A\vee B);$$
$$(\sim\vee\rightarrow)\quad X\sim BY\rightarrow C\vdash X\sim(A\vee B)Y\rightarrow C,$$
$$X\sim AY\rightarrow C\vdash X\sim(A\vee B)Y\rightarrow C;$$
$$(\rightarrow\sim\sim)\quad X\rightarrow A\vdash X\rightarrow\sim\sim A;$$
$$(\sim\sim\rightarrow)\quad XAY\rightarrow B\vdash X\sim\sim AY\rightarrow B.$$

(ii) The rules of constructive sequential propositional logic $COSPL$ (i.e. Nelson's propositional logic **N** (see e.g. (Nelson, 1949), (Markov, 1950), (von Kutschera, 1969), (Thomason, 1969), (Routley, 1974), (Gurevich, 1977), (Almukdad and Nelson, 1984)) without structural rules of inference) are those of $COSPL^-$ together with $(\bot\rightarrow)$ and:

$$(\sim\mathbf{t}\rightarrow)\vdash X\sim\mathbf{t}Y\rightarrow A;(\sim\top\rightarrow)\vdash X\sim\top Y\rightarrow A;$$
$$(\rightarrow\sim\bot)\vdash X\rightarrow\sim\bot.$$

The idea behind these sequent rules involving $\sim$ is that they directly reflect refutability conditions for main connectives or constants in the scope of $\sim$. If $(\bot\rightarrow)$, $(\sim\mathbf{t}\rightarrow)$, and $(\sim\top\rightarrow)$ are assumed, we say that $\bot$, $\sim\mathbf{t}$, and $\sim\top$ act or are treated as falsum constants; otherwise $\bot$, $\sim\mathbf{t}$, and $\sim\top$ are regarded as propositional variables.

Now that we have defined four base logics, families of propositional logics can be built up on top of these basic systems by successively adding certain structural inference rules. One may, for example, select any combination taken from the following collection of rules (cf. also (Došen, 1988)):[6]

permutation $(\mathbf{P})$: $XABY\rightarrow C\vdash XBAY\rightarrow C;$

contraction $(\mathbf{C})$: $XAAY\rightarrow B\vdash XAY\rightarrow B;$

cancellation $(\mathbf{C}')$: $XAYAZ\rightarrow B\vdash XAYZ\rightarrow B,$
$XAYAZ\rightarrow B\vdash XYAZ\rightarrow B;$

expansion $(\mathbf{E})$: $XAY\rightarrow B\vdash XAAY\rightarrow B;$

duplication $(\mathbf{E}')$: $XAYZ\rightarrow B\vdash XAYAZ\rightarrow B,$
$XYAZ\rightarrow B\vdash XAYAZ\rightarrow B;$

monotonicity $(\mathbf{M})$: $XY\rightarrow B\vdash XAY\rightarrow B.$

In the absence of structural rules the premises are conceived of as *sequences of occurrences*, whereas in the presence of **P** resp. **P**, **C**, and **E** one is dealing

[6] This collection sticks closely to the "standard package" and is by no means meant to be the only reasonable choice. A structural inference rule which is prominent in Artificial Intelligence is *cautious monotonicity*; a sequential version of this rule would be: $XY\rightarrow B\quad XY\rightarrow A\vdash XAY\rightarrow B.$

with *multisets* resp. *sets* of premises. Different grades of monotonicity of inference are provided by selecting among **E**, **E**′, and **M**.

Let $\Delta \subseteq \{\mathbf{P}, \mathbf{C}, \mathbf{C}', \mathbf{E}, \mathbf{E}', \mathbf{M}\}$, let Ξ range over $\{MSPL, ISPL, COSPL^-, COSPL\}$, and let Ξ_Δ denote the extension of Ξ by the rules in Δ. Note that **P** is derivable in $\Xi_{\{\mathbf{C},\mathbf{M}\}}$ and that in $ISPL_\Delta$ we can define t as $(\bot/\bot)$ (or as $(\bot \setminus \bot)$). The negations $\neg^r$, $\neg^l$ are defined by $\neg^r A = (\bot/A)$, $\neg^l A = (A \setminus \bot)$. Whereas in the systems based on $ISPL$ or $COSPL$ one can deduce everything from a database containing contradictions $\neg^r A \circ A$ or $A \circ \neg^l A$, this is not the case for systems based on $MSPL$ resp. $COSPL^-$. In the presence of **M** resp. **M** and **C** the constants t and $\top$ resp. $A \circ B$ and $A \wedge B$ are interderivable. In extensions of $MSPL$ and $ISPL$ with **M**, $\top$ can be defined as $p \setminus p$ (or as p/p), for some propositional variable p. In the presence of **P**, A/B and $B \setminus A$ resp. $\neg^r A$ and $\neg^l A$ are interderivable. Using suitable translations (see (Wansing, 1993)), it can be seen that $MSPL_{\{\mathbf{P},\mathbf{C},\mathbf{M}\}}$, $ISPL_{\{\mathbf{P},\mathbf{C},\mathbf{M}\}}$, $COSPL^-_{\{\mathbf{P},\mathbf{C},\mathbf{M}\}}$, $COSPL_{\{\mathbf{P},\mathbf{C},\mathbf{M}\}}$, respectively, in fact can be identified as MPL, IPL, $\mathbf{N}^-$, **N**, respectively. By means of (cut) it can readily be shown that in every system Ξ_Δ instead of the rules $(\rightarrow\ \top)$, $(/\ \rightarrow)$, $(\setminus\ \rightarrow)$, $(\rightarrow\ \circ)$, $(\wedge\ \rightarrow)$, and $(\rightarrow\ \vee)$ one may equivalently use, respectively:

$(\top\ \uparrow)$ $X\top Y \rightarrow A \vdash XY \rightarrow A$;

$(\uparrow\ /)$ $X \rightarrow (B/A) \vdash XA \rightarrow B$;

$(\uparrow\ \setminus)$ $X \rightarrow (A \setminus B) \vdash AX \rightarrow B$;

$(\circ\ \uparrow)$ $X(A \circ B)Y \rightarrow C \vdash XABY \rightarrow C$;

$(\uparrow\ \wedge)$ $X \rightarrow (A \wedge B) \vdash X \rightarrow A$,
$X \rightarrow (A \wedge B \vdash X \rightarrow B$;

$(\vee\ \uparrow)$ $X(A \vee B)Y \rightarrow C \vdash XAY \rightarrow C$,
$X(A \vee B)Y \rightarrow C \vdash XBY \rightarrow C$.

If A and B are interderivable in Ξ_Δ, this will be abbreviated by $\vdash_{\Xi_\Delta} A \leftrightarrow B$. We have

$$\vdash_{\Xi_\Delta} A \circ (B \circ C) \leftrightarrow (A \circ B) \circ C;$$

$$(\dagger)\ \vdash_{\Xi_\Delta} (A \vee B) \circ C \leftrightarrow (A \circ C) \vee (B \circ C),$$

$$\vdash_{\Xi_\Delta} A \circ (B \vee C) \leftrightarrow (A \circ B) \vee (A \circ C).$$

A well-known peculiarity of Nelson's systems $\mathbf{N}^-$ and **N** is the failure of intersubstitutivity of provable equivalents. Let $(A \rightleftharpoons^+ B)$ be defined as $(A \setminus B) \wedge (B \setminus A) \wedge (A/B) \wedge (B/A)$. Using the terminology of Pearce and

Rautenberg (1991), $\rightleftharpoons^+$ may be called acceptance-equivalence. In each system $COSPL^-_\Delta$ and $COSPL_\Delta$, provable acceptance-equivalence is an equivalence relation but *not* a congruence relation, that is to say, one cannot prove the Replacement Theorem wrt it. For instance: $\vdash_{COSPL^-_\Delta} \rightarrow\, \sim (p \setminus q) \rightleftharpoons^+ p \circ \sim q$, but $\not\vdash_{COSPL^-_\Delta} \rightarrow\, \sim\sim (p \setminus q) \rightleftharpoons^+ \sim (p \circ \sim q)$. (This can easily be verified using the monoid semantics presented in Section 3).

Rejection-equivalence $\rightleftharpoons^-$ can then be defined by $(A \rightleftharpoons^- B) = (\sim B \setminus \sim A) \wedge (\sim A \setminus \sim B) \wedge (\sim A / \sim B) \wedge (\sim B / \sim A)$. Clearly, $\vdash\ \rightarrow A \rightleftharpoons^+ B$ iff $\vdash A \leftrightarrow B$ and $\vdash \rightarrow A \rightleftharpoons^- B$ iff $\vdash\ \sim A \leftrightarrow\ \sim B$, and clearly also provable rejection-equivalence fails to be a congruence relation in $COSPL^-_\Delta$ or $COSPL_\Delta$. If one defines strong equivalence $(A \rightleftharpoons B)$ as $(A \rightleftharpoons^+ B) \wedge (A \rightleftharpoons^- B)$, then provable strong equivalence *is* a congruence relation in $COSPL^-_\Delta$ and $COSPL_\Delta$.[7]Let C_A denote an $L^\sim$-formula that contains a certain occurrence of A as a subformula, and let C_B denote the result of replacing this occurrence of A in C by B. The degree of A ($d(A)$) is the number of occurrences of propositional constants and connectives in A.

THEOREM 1. (replacement) If $\rightarrow A \rightleftharpoons B$ is provable in $COSPL^-_\Delta$ or $COSPL_\Delta$, then so is $\rightarrow C_A \rightleftharpoons C_B$.

Proof. By induction on $l = d(C_A) - d(A)$. If $l = 0$, the proof is trivial. Assume that the claim holds for every $l \leq m$, and $l = m + 1$.

$C_A = \sim D$: Assume that $d(D_A) \leq l$ and $\vdash \rightarrow A \rightleftharpoons B$. By the induction hypothesis, $\vdash \rightarrow D_A \rightleftharpoons D_B$, and therefore the following formulas are provable: $D_A \setminus D_B$, $D_B \setminus D_A$, D_A/D_B, D_B/D_A, $\sim D_A \setminus \sim D_B$, $\sim D_B \setminus \sim D_A$, $\sim D_A / \sim D_B$, and $\sim D_B / \sim D_A$. By (cut), $(\setminus \rightarrow)$, $(\uparrow \setminus)$, $(/ \rightarrow)$, $(\uparrow /)$, $(\rightarrow\sim\sim)$, and $(\sim\sim\rightarrow)$, also $\sim\sim D_A \setminus \sim\sim D_B$, $\sim\sim D_B \setminus \sim\sim D_A$, $\sim\sim D_A / \sim\sim D_B$, and $\sim\sim D_B / \sim\sim D_A$ are provable and thus $\vdash \rightarrow C_A \rightleftharpoons C_B$.

$C_A = D_1 \nabla D_2$, $\nabla \in \{/, \setminus, \wedge, \circ, \vee\}$. We consider the case for $\nabla = \wedge$. Here we have the following derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{\rightarrow D_{1A} \setminus D_{1B}}{D_{1A} \rightarrow D_{1B}}}{D_{1A} \wedge D_2 \rightarrow D_{1B}} \quad \dfrac{D_2 \rightarrow D_2}{D_{1A} \wedge D_2 \rightarrow D_2}}{D_{1A} \wedge D_2 \rightarrow D_{1B} \wedge D_2}}{\rightarrow C_A \setminus C_B;}$$

[7] The distinction between positive and negative (semantic) consequence is well-known from partial logic, see e.g. (Fenstad *et al.*, 1990), (Thijsse, 1990). Note, however, that for the variety of notions of semantic consequence considered by Thijsse "logical equivalence [...] turns out as mutual consequence" as Thijsse (1990, p. 29) quotes from (Blamey, 1986). In other words, intersubstitutivity of provable equivalents holds.

$$\dfrac{\dfrac{\dfrac{\dfrac{\rightarrow\sim D_{1A}\backslash\sim D_{1B}}{\sim D_{1A}\rightarrow\sim D_{1B}}}{\sim D_{1A}\rightarrow\sim(D_{1B}\wedge D_2)} \quad \dfrac{\dfrac{}{\sim D_2\rightarrow\sim D_2}}{\sim D_2\rightarrow\sim(D_{1B}\wedge D_2)}}{\sim(D_{1A}\wedge D_2)\rightarrow\sim(D_{1B}\wedge D_2)}}{\rightarrow\sim C_A\backslash\sim C_B.}$$

Analogously we obtain $C_B\backslash C_A$, $\sim C_B\backslash\sim C_A$, C_A/C_B, C_B/C_A, $\sim C_A/\sim C_B$, and $\sim C_B/\sim C_A$. The remaining cases are similar. □

The following collection of equivalences in terms of $\rightleftharpoons^+$ which are provable in $COSPL^-_\Delta$ without using (cut) will turn out useful:

$$\begin{array}{lll}(\mathbf{r}\,1) & \sim(A\wedge B)\rightleftharpoons^+(\sim A\vee\sim B), & \sim(A\vee B)\rightleftharpoons^+(\sim A\wedge\sim B),\\ & \sim(B/A)\rightleftharpoons^+(\sim B\circ A), & \sim(A\backslash B)\rightleftharpoons^+(A\circ\sim B),\\ & \sim(A\circ B)\rightleftharpoons^+\sim A\circ\sim B, & \sim\sim A\rightleftharpoons^+ A.\end{array}$$

In $COSPL_\Delta$ also the following acceptance-equivalences are provable without resort to (cut):

$$(\mathbf{r}\,2)\quad \sim\bot\rightleftharpoons^+\mathbf{t},\ \sim\mathbf{t}\rightleftharpoons^+\bot,\ \sim\top\rightleftharpoons^+\bot.$$

These provable acceptance equivalences describe a procedure for associating to each $L^\sim$-formula A one $L^\sim$-formula B such that $\vdash_{COSPL_\Delta}\ \rightarrow A\rightleftharpoons^+ B$, resp. $\vdash_{COSPL^-_\Delta}\ \rightarrow A\rightleftharpoons^+ B$, and B has occurrences of $\sim$ only in front of propositional variables resp. propositional variables or constants, if for $COSPL^-_\Delta$, $\sim\bot$, $\sim\mathbf{t}$, and $\sim\top$ are associated to themselves. In $COSPL^-_\Delta$, $(A\wedge B)\rightleftharpoons^+\sim(\sim A\vee\sim B)$ and $(A\vee B)\rightleftharpoons^+\sim(\sim A\wedge\sim B)$ are provable (again without using (cut)). Together with $\vdash\ \sim(A\wedge B)\rightleftharpoons^+\sim A\vee\sim B$, $\vdash\ \sim(A\vee B)\rightleftharpoons^+\sim A\wedge\sim B$ this shows that in $COSPL^-_\Delta$ and $COSPL_\Delta$, $\wedge$ resp. $\vee$ can be defined by means of $\vee$ and $\sim$ resp. $\wedge$ and $\sim$. Moreover, from $(\mathbf{r}\,2)$ we know that in $COSPL_\Delta$, $\bot$ can be defined as $\sim\mathbf{t}$.

The provable acceptance equivalences $(\mathbf{r}\,1)$ and $(\mathbf{r}\,2)$ specify the refutability (or rejectability) conditions referred to above. The refutability conditions for $(A\wedge B)$, $(A\vee B)$, and $\sim A$ are identified as the provability conditions for $\sim A\vee\sim B$, $\sim A\wedge\sim B$ and A, respectively, which is very natural. Also the refutability conditions for the directional implications are convincing, because in the absence of structural inference rules they are provability conditions of direction-sensitive, non-commutative $\circ$-conjunctions. Less clear are the rejectability conditions for $A\circ B$, since there is no 'intensional' disjunction corresponding to the 'intensional' conjunction $\circ$. What does it mean to refute a concatenation, i.e. a text, $A \circ B$? To assume that $\sim (A \circ B)$ is provably acceptance equivalent to $\sim A \vee \sim B$ is problematic, since we could then e.g. prove $A(A\backslash\sim A) \rightarrow \sim (A \circ (A \backslash \sim A))$ as well as $A(A\backslash\sim A) \rightarrow (A \circ (A\backslash\sim A))$. Moreover, $\vdash \rightarrow \sim (A \circ B) \rightleftharpoons^{+} \sim A \wedge \sim B$ would, if no structural rules are assumed, make the provability conditions of a non-directional connective the refutability conditions of a direction-sensitive connective, which, as a kind of mismatch, would be rather surprising. In contrast to this, we may regard it as plausible that the refutation of a concatenation $A \circ B$ is provably acceptance equivalent to a refutation of each component of the concatenation: refuting a text means refuting every sentence in the text. The equivalences (r 1) may thus be viewed as a justification of $COSPL^{-}_{\Delta}$'s sequent rules involving $\sim$; and indeed the very formulation of these rules is *induced* in an obvious way by (r 1). The rejection equivalence of $\mathbf{t}$, $\top$, and $\sim \bot$ in $COSPL_{\Delta}$ can be elaborated as follows: Since every sequence of premise occurrences proves $\mathbf{t}$, $\mathbf{t}$ cannot be disproved; there is no sequence of premise occurrences that refutes $\mathbf{t}$. Similarly, intuitionistic falsum cannot be proved; therefore in the constructive case, $\sim \mathbf{t}$ should be added as a falsum. The truth constant $\top$ is a theorem. Although there are sequences X of premise occurrences such that $X \rightarrow \top$ is not provable in the absence of **M**, this does not mean that X refutes $\top$. On the contrary, it is hardly imaginable that a theorem is refutable. In this way we arrive at the same refutability conditions for $\mathbf{t}$, $\top$ and $\sim \bot$.

Let us state a number of known facts about the substructural logics we have introduced.

THEOREM 2. Let $\Theta \subseteq \{\mathbf{P}, \mathbf{C}, \mathbf{C}', \mathbf{M}\}$.

(i) (cut-elimination) Applications of (cut) can be eliminated from proofs in Ξ_{Θ}.

(ii) (disjunction property) If $A \vee B$ is provable in Ξ_{Θ}, then A is provable or B is provable.

(iii) (constructible falsity) If $\sim (A \wedge B)$ is provable in $COSPL^{-}_{\Theta}$ or $COSPL_{\Theta}$, then $\sim A$ is provable or $\sim B$ is provable.

(iv) $COSPL^{-}_{\Theta}$ is a conservative extension of $MSPL_{\Theta}$ and $COSPL_{\Theta}$ is a conservative extension of $ISPL_{\Theta}$.

(v) Interpolation holds for Ξ_{Θ}.

(vi) Provability of sequents in Ξ_{Θ} is decidable.

(vii) Cut-elimination fails for Ξ_{Δ}, if **E** or **E**$'$ are in Δ, but **M** is not.

Proof. See (Wansing, 1993). (i) is proved by a standard argument; (ii), (iii) and (iv) are corollaries of (i). Also the proofs of (v) and (vi) make use of cut-elimination. In the proof of (vi) Kripke's method for proving the decidability of sequents in $R_{\supset}$ is applied. Some counterexamples to cut-eliminability rely on this decision procedure. □

Note that in the presence of (cut) the expansion rule **E** is interreplaceable with the rule

$$\text{mingle (MI)} : X \rightarrow A \quad Y \rightarrow A \vdash XY \rightarrow A.$$

If **M** is not available, also **MI** blocks cut-elimination.[8] Attempts to eliminate applications of (cut) give rise to problematic cases like the following:

$$\left[\dfrac{\dfrac{\dfrac{\Pi_1}{X\rightarrow A}\quad\dfrac{\Pi_2}{Y\rightarrow A}}{XY\rightarrow A}\quad\dfrac{\Pi_3}{X_1AZ\rightarrow B}}{X_1XYZ\rightarrow B}\right]\ \text{is converted into}$$

$$\left[\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1}{X\rightarrow A}\quad\dfrac{\Pi_3}{X_1AZ\rightarrow B}}{X_1XZ\rightarrow B}}{\vdots}}{X\rightarrow(\ldots(A_n\backslash\ldots(A_1\backslash(\ldots(B/B_m)/\ldots/B_1))\ldots)}\quad\dfrac{\dfrac{\dfrac{\dfrac{\Pi_2}{Y\rightarrow A}\quad\dfrac{\Pi_3}{X_1AZ\rightarrow B}}{X_1YZ\rightarrow B}}{\vdots}}{Y\rightarrow(\ldots(A_n\backslash\ldots(A_1\backslash(\ldots(B/B_m)/\ldots/B_1))\ldots)}}{XY\rightarrow(\ldots(A_n\backslash\ldots(A_1\backslash(\ldots(B/B_m)/\ldots/B_1))\ldots)}}{\dfrac{\vdots}{X_1XYZ\rightarrow B,}}\right]$$

where $X_1 = A_1 \ldots A_n$ and $Z = B_1 \ldots B_m$. The steps from $XY \rightarrow (\ldots(A_n \backslash \ldots (A_1 \backslash (\ldots (B/B_m)/ \ldots /B_1)) \ldots)$ to $X_1XYZ \rightarrow B$ involve applications of (cut). In the converted proof we still have an application of **MI** that is immediately followed by an application of (cut), and unfortunately this constellation may *loop*, as can be tested with the following example:

$$\dfrac{\dfrac{\dfrac{\Pi_1}{(p_1\backslash p_2)(p_2\backslash p_3)\rightarrow p_1\backslash p_3}\quad p_1\backslash p_3\rightarrow p_1\backslash p_3}{(p_1\backslash p_2)(p_2\backslash p_3)(p_1\backslash p_3)\rightarrow p_1\backslash p_3}\qquad\dfrac{\Pi_2}{p_1(p_1\backslash p_3)\rightarrow p_3}}{p_1(p_1\backslash p_2)(p_2\backslash p_3)(p_1\backslash p_3)\rightarrow p_3.}$$

3. MONOID MODELS

We shall reproduce a version of Došen's (1989) groupoid semantics. This *monoid semantics*, which will turn out to be adequate for $MSPL_{\Delta}$ and $ISPL_{\Delta}$, is then generalized to $COSPL_{\Delta}^{-}$ and $COSPL_{\Delta}$.

DEFINITION 3. A semilattice-ordered monoid (*slomo*) is a structure $\langle I, \cdot, \cap, 1\rangle$ such that $1 \in I$, I is closed under the binary operations $\cdot$ and $\cap$, $\cdot$ is associative, $\cap$ is associative, commutative, and idempotent, for every $a \in I$,

[8] Nuel Belnap pointed out to me that in the presence of (cut), **MI** is interreplaceable with the rule $X \rightarrow A \quad Y \rightarrow A \quad Z_1AZ_2 \rightarrow B \vdash Z_1XYZ_2 \rightarrow B$. This rule allows to eliminate (cut) also in the absence of **M**.

$a \cdot 1 = a = 1 \cdot a$, and $\cdot$ distributes over $\cap$ (i.e. for every $a, b, c \in I$: $a \cdot (b \cap c) = (a \cdot b) \cap (a \cdot c)$ and $(b \cap c) \cdot a = (b \cdot a) \cap (c \cdot a)$), cf. (Dosen, 1989, p. 43). In a slomo the partial order $\leq$ is defined by $a \leq b$ iff $a = a \cap b$.

DEFINITION 4. (i) A structure $\langle I, \cdot, \cap, 1, v_0 \rangle$ is called a monoid model for $MSPL$ iff $\langle I, \cdot, \cap, 1 \rangle$ is a *slomo* and v_0 is a mapping from $PROP \cup \{\bot\}$ into 2^I such that for every $q \in PROP \cup \{\bot\}$ the following holds:

($\cap$ Heredity v_0) $\quad a_1 \cap a_2 \in v_0(q)$ iff $(a_1 \in v_0(q)$ and $a_2 \in v_0(q))$.

(ii) A monoid model $\langle I, \cdot, \cap, 1, v_0 \rangle$ for $MSPL$ is a monoid model for $ISPL$ if for every $p \in PROP$ and every $a, b \in I$:

$(\star)$ $\quad a \in v_0(\bot)$ implies $a \in v_0(p)$, $a \leq a \cdot b$, $a \leq b \cdot a$, $a \cdot a \leq a$, and $1 \leq a$.

DEFINITION 5. The valuation v induced by a monoid model for $MSPL$ $\langle I, \cdot, \cap, 1, v_0 \rangle$ is the function from the set of all L-formulas into 2^I inductively defined as follows (where $q \in PROP \cup \{\bot\}$):

$$
\begin{aligned}
v(q) &= v_0(q),\\
v(\mathbf{t}) &= I,\\
v(\top) &= \{a \mid 1 \leq a\},\\
v(B/A) &= \{a \mid (\forall b \in v(A))\, a \cdot b \in v(B)\},\\
v(A \setminus B) &= \{a \mid (\forall b \in v(A))\, b \cdot a \in v(B)\},\\
v(A \circ B) &= \{a \mid (\exists b_1 \in v(A))(\exists b_2 \in v(B))\, b_1 \cdot b_2 \leq a\},\\
v(A \wedge B) &= \{a \mid a \in v(A) \text{ and } a \in v(B)\},\\
v(A \vee B) &= \{a \mid (\exists b_1 \in v(A))(\exists b_2 \in v(B))\, b_1 \cap b_2 \leq a \text{ or}\\
&\qquad a \in v(A) \text{ or } a \in v(B)\}.
\end{aligned}
$$

DEFINITION 6. (semantic consequence) Let $\mathcal{M} = \langle I, \cdot, \cap, 1, v_0 \rangle$ be a monoid model for $MSPL$. If X is a non-empty sequence $A_1 \ldots A_n$, let $v(X) = v(A_1 \circ \ldots \circ A_n)$.

$$X \to A \text{ holds (or is valid) in } \mathcal{M} \text{ iff } \begin{cases} v(X) \subseteq v(A) & \text{if } X \text{ is nonempty,}\\ 1 \in v(A) & \text{otherwise.} \end{cases}$$

LEMMA 1. (i) For every monoid model for $MSPL$ $\langle I, \cdot, \cap, 1, v_0 \rangle$, every a, $b \in I$, and every L-formula A:

($\cap$ Heredity v) $\quad a \cap b \in v(A)$ iff $(a \in v(A)$ and $b \in v(A))$.

(ii) For every monoid model for $ISPL$ $\langle I, \cdot, \cap, 1, v_0 \rangle$ and every L-formula A:

$v(\bot) \subseteq v(A)$.

Proof. By induction on the complexity of A, using $(\star)$ for (ii). □

By the definition of $\leq$, it immediately follows from ($\cap$ Heredity v) that for every monoid model for $MSPL$ $\langle I, \cdot, \cap, 1, v_0\rangle$, every a, $b \in I$, and every L-formula A:

(Heredity) if $a \leq b$, then $(a \in v(A)$ implies $b \in v(A))$.

DEFINITION 7. (i) A structure $\langle I, \cdot, \cap, 1, v_0^+, v_0^-\rangle$ is called a monoid model for $COSPL^-$ iff $\langle I, \cdot, \cap, 1\rangle$ is a *slomo* and v_0^+, v_0^- are mappings from $PROP \cup \{\bot, \sim \mathbf{t}, \sim \top\}$ into 2^I such that for every $q \in PROP \cup \{\bot, \sim \mathbf{t}, \sim \top\}$ the following holds:

($\cap$ Heredity v_0^+) $a_1 \cap a_2 \in v_0^+(q)$ iff $(a_1 \in v_0^+(q)$ and $a_2 \in v_0^+(q))$.

($\cap$ Heredity v_0^-) $a_1 \cap a_2 \in v_0^-(q)$ iff $(a_1 \in v_0^-(q)$ and $a_2 \in v_0^-(q))$.

(ii) A monoid model for $COSPL$ is a monoid model for

$COSPL^-$ $\langle I, \cdot, \cap, 1, v_0^+, v_0^-\rangle$,

where $v_0^+(\bot) = v_0^+(\sim \mathbf{t}) = v_0^+(\sim \top)$, $v_0^-(\bot) = I$, and for every $p \in PROP$ and every $a, b \in I$, $(\star)$ holds.

DEFINITION 8. The valuation functions v^+, v^- induced by a monoid model for $COSPL^-$ $\langle I, \cdot, \cap, 1, v_0^+, v_0^-\rangle$ are the functions from the set of $L^{\sim}$-formulas into 2^I which are inductively defined as follows (where $q \in PROP \cup \{\bot, \sim \mathbf{t}, \sim \top\}$):

$$
\begin{aligned}
v^+(q) &= v_0^+(q),\\
v^-(q) &= v_0^-(q),\\
v^+(\mathbf{t}) &= I,\\
v^+(\top) &= \{a \mid 1 \leq a\},\\
v^+(B/A) &= \{a \mid (\forall b \in v^+(A))a \cdot b \in v^+(B)\},\\
v^-(B/A) &= \{a \mid (\exists b_1 \in v^-(B))(\exists b_2 \in v^+(A))b_1 \cdot b_2 \leq a\},\\
v^+(A \setminus B) &= \{a \mid (\forall b \in v^+(A))b \cdot a \in v^+(B)\},\\
v^-(A \setminus B) &= \{a \mid (\exists b_1 \in v^-(B))(\exists b_2 \in v^+(A))b_2 \cdot b_1 \leq a\},\\
v^+(A \circ B) &= \{a \mid (\exists b_1 \in v^+(A))(\exists b_2 \in v^+(B))b_1 \cdot b_2 \leq a\},
\end{aligned}
$$

$$
\begin{aligned}
v^-(A \circ B) &= \{a \mid (\exists b_1 \in v^-(A))(\exists b_2 \in v^-(B)) b_1 \cdot b_2 \leq a\}, \\
v^+(A \wedge B) &= \{a \mid a \in v^+(A) \text{ and } a \in v^+(B)\}, \\
v^-(A \wedge B) &= \{a \mid (\exists b_1 \in v^-(A))(\exists b_2 \in v^-(B)) b_1 \cap b_2 \leq a \text{ or} \\
&\qquad a \in v^-(A) \text{ or } a \in v^-(B)\}, \\
v^+(A \vee B) &= \{a \mid (\exists b_1 \in v^+(A))(\exists b_2 \in v^+(B)) b_1 \cap b_2 \leq a \text{ or} \\
&\qquad a \in v^+(A) \text{ or } a \in v^+(B)\}, \\
v^-(A \vee B) &= \{a \mid a \in v^-(A) \text{ and } a \in v^-(B)\}, \\
v^+(\sim A) &= v^-(A), \\
v^-(\sim A) &= v^+(A).
\end{aligned}
$$

Thus, the definition of a valuation v in a monoid model for $MSPL$ resp. $ISPL$ agrees with the definition of v^+ in monoid models for $COSPL^-$ resp. $COSPL$. Moreover, the clauses $v^-(A)$ directly reflect the provable equivalences (r 1) and (r 2) in terms of $\rightleftharpoons^+$ listed in Section 2 (for instance, $v^-(B/A)$ reflects $\sim (B/A) \rightleftharpoons^+ (\sim B \circ A)$). Therefore, to each $L^\sim$-formula A, one can find a (provably acceptance-equivalent) $L^\sim$-formula B such that $v^-(A) = v^+(\sim A) = v^+(B)$, and $\sim$ occurs in B only in front of propositional variables or constants. This fact can be used to simplify inductive proofs.

DEFINITION 9. (semantic consequence) Let $\mathcal{M} = \langle I, \cdot, \cap, 1, v_0^+, v_0^- \rangle$ be a monoid model for $COSPL^-$. If X is a non-empty sequence $A_1 \ldots A_n$, let $v^+(X) = v^+(A_1 \circ \ldots \circ A_n)$.

$$
X \rightarrow A \text{ holds (or is valid) in } \mathcal{M} \text{ iff} \begin{cases} v^+(X) \subseteq v^+(A) & \text{if } X \text{ is nonempty,} \\ 1 \in v^+(A) & \text{otherwise.} \end{cases}
$$

LEMMA 2. (i) For every monoid model for $COSPL^-$ $\langle I, \cdot, \cap, 1, v_0^+, v_0^- \rangle$, every a, $b \in I$, and every $L^\sim$-formula A:

($\cap$ Heredity v^+) $\quad a \cap b \in v^+(A)$ iff ($a \in v^+(A)$ and $b \in v^+(A)$);

($\cap$ Heredity v^-) $\quad a \cap b \in v^-(A)$ iff ($a \in v^-(A)$ and $b \in v^-(A)$).

(ii) For every monoid model for $COSPL$ $\langle I, \cdot, \cap, 1, v_0^+, v_0^- \rangle$ and every $L^\sim$-formula A:

$v^+(\bot) \subseteq v^+(A)$.

Proof. By (simultaneous) induction on the complexity of A. For ($\cap$ Heredity v^-) it is enough to consider the cases where A is a propositional variable, $\bot$, t, or $\top$. □.

It can readily be verified that ($\sqcap$ Heredity v^+) resp. ($\sqcap$ Heredity v^-) implies that for every monoid model for $COSPL^-$ $\langle I, \cdot, \sqcap, 1, v_0^+, v_0^- \rangle$, every a, b $\in I$, and every $L^{\sim}$-formula A:

(Heredity $^+$) if $a \leq b$, then ($a \in v^+(A)$ implies $b \in v^+(A)$),

(Heredity $^-$) if $a \leq b$, then ($a \in v^-(A)$ implies $b \in v^-(A)$).

THEOREM 3. (soundness) If $\vdash_\Xi X \rightarrow A$, then $X \rightarrow A$ holds in every monoid model for Ξ.

Proof. By induction on the complexity of proofs in Ξ. All cases are straightforward, except for ($\top \rightarrow$), ($\vee \rightarrow$) and ($\sim \wedge \rightarrow$). In the latter two cases one has to make use of both ($\sqcap$ Heredity v^+) and (Heredity $^+$) (cf. (Došen, 1989, p. 48)). Let us here consider ($\top \rightarrow$) and ($\sim \wedge \rightarrow$). Let $\mathcal{M} = \langle I, \cdot, \sqcap, 1, v_0 \rangle$ resp. $\mathcal{M} = \langle I, \cdot, \sqcap, 1, v_0^+, v_0^- \rangle$ be any monoid model for Ξ. ($\top \rightarrow$): It is enough to show that $v^+(\top \circ A) \subseteq v^+(A)$ and $v^+(A \circ \top) \subseteq v^+(A)$. Consider the latter. Note that if $a \leq b$, then $c \cdot a \leq c \cdot b$, for every a, b, $c \in I$. Suppose $c \in v^+(A \circ \top)$. Then $(\exists b_1 \in v^+(A))$ $(\exists b_2 \in v^+(\top))$ $b_1 \cdot b_2 \leq c$. Now, $1 \leq b_2$. Therefore $b_1 = b_1 \cdot 1 \leq b_1 \cdot b_2$. By transitivity of $\leq$, $b_1 \leq c$. Hence, by (Heredity $^+$), $c \in v^+(A)$. ($\sim \wedge \rightarrow$): Let $C = C_1 \circ \ldots \circ C_n$, $D = D_1 \circ \ldots \circ D_n$, and suppose that $v^+(C \sim AD) \subseteq v^+(E)$,$v^+(C \sim BD) \subseteq v^+(E)$. Then $v^+(C \sim A) \subseteq v^+((E/D))$ and $v^+(C \sim B) \subseteq v^+((E/D))$. Hence$(\forall a \in v^+(C))$ $(\forall b \in v^+(\sim A))$ $a \cdot b \in v^+((E/D))$, $(\forall a \in v^+(C))$ $(\forall b \in v^+(\sim B))$ $a \cdot b \in v^+((E/D))$. Therefore,

	$a \in v^+(C), b_1 \in v^+(\sim A), b_2 \in v^+(\sim B), b_1 \sqcap b_2 \leq b$	
only if	$a \cdot b_1 \in v^+((E/D))$ and $a \cdot b_2 \in v^+((E/D))$	
only if	$(a \cdot b_1) \sqcap (a \cdot b_2) \in v^+((E/D))$	($\sqcap$ Heredity v^+)
only if	$a \cdot (b_1 \sqcap b_2) \in v^+((E/D))$	
only if	$(a \cdot b) \in v^+((E/D))$	(Heredity $^+$).

Since also ($a \in v^+(C)$ and ($b \in v^+(\sim A)$ or $b \in v^+(\sim B)$)) only if $a \cdot b \in v^+((E/D))$, we obtain

	$a \in v^+(C)$	
only if	$(\forall b \in v^+(\sim A \vee \sim B))\ a \cdot b \in v^+((E/D))$	
iff	$(\forall b \in v^+(\sim (A \wedge B))\ a \cdot b \in v^+((E/D))$	
only if	$v^+(C \sim (A \wedge B)D) \subseteq v^+(C)$.	□

Using ($\sqcap$ Heredity v), Došen (1989, p. 52 f.) proves a number of correspondences between structural rules of inference and conditions on *slomos* in the sense that a given structural rule R is validity preserving in a monoid model

$\mathcal{M}$ iff the condition on *slomos* corresponding to R is satisfied by the *slomo* on which $\mathcal{M}$ is based. In the present context we have the following correspondences:

	for every $a, b \in I$	
P	$a \cdot b \leq b \cdot a$	
C	$a \cdot a \leq a$	
C′	$a \cdot b \cdot a \leq a \cdot b,$	$a \cdot b \cdot a \leq b \cdot a$
E	$a \leq a \cdot a$	
E′	$a \cdot b \leq a \cdot b \cdot a,$	$b \cdot a \leq a \cdot b \cdot a$
M	$1 \leq a$	

Note that the behaviour of $\sim$ is not reflected in these structural conditions on *slomos*; it is completely captured by the valuations v^- and v^+. By the above correspondences, Ξ_Δ is sound wrt the class of monoid models for Ξ whose underlying *slomos* satisfy the conditions that correspond to the rules in Δ. Let us call this class of models M_{Ξ_Δ}.

4. INFORMATIONAL INTERPRETATION

By itself, the monoid semantics of the previous section might convey the impression of being just a technical tool without very much explanatory value and intuitive appeal. In this section we shall, however, develop an informational interpretation for the systems Ξ_Δ that is based on the above mentioned understanding of *slomos* $\langle I, \cdot, \sqcap, 1 \rangle$, viz. I is a set of information pieces or information states, $\cdot$ is the addition of information pieces, $\sqcap$ is the intersection of information pieces and 1 is the initial, ideally the empty piece of information. The partial order $\leq$ can, as in Kripke's interpretation of IPL, be understood as the 'possible development' or 'possible prolongation' (in the sense of 'possible expansion') of information states or pieces. Of course, this reading is illuminating only if we are willing to attach some explanatory power to the notions of addition, intersection and development of information pieces. We claim that under the suggested reading the properties which in *slomos* are postulated for $\cdot$, $\sqcap$, and 1 are intuitively plausible. Or, to state it the other way around, if we have a set I of information pieces including one initial piece of information together with one addition and one intersection operation on I, then it is plausible to assume that these components should form a *slomo*.

What can we say about the evaluation of formulas in monoid models? The valuation function v resp. v^+ in monoid models for $MSPL$ resp. $COSPL^-$ specifies truth conditions; the valuation v^- in monoid models for $COSPL^-$ in addition specifies falsity conditions, where falsity is falsity in the sense of refutation. Thus, in contrast to the minimal intuitionistic and the intuitionistic case, in the minimal constructive and the constructive systems truth and falsity are regarded as prima facie independent notions. Now, do the valuation clauses emerge as plausible under the given interpretation of *slomos*? Let us first consider the valuations v and v^+. The evaluation of elements from $PROP$ resp., in the minimal cases,$PROP \cup \{\bot\}$ or $PROP \cup \{\bot, \sim \mathrm{t}, \sim \top\}$, is unproblematic. ($\cap$ Heredity v_0), ($\cap$ Heredity v_0^+), and ($\cap$ Heredity v_0^-) are natural requirements. If a formula A is true on the strength of the intersection $a \cap b$ of information pieces a, b, then A should be true on the strength of both pieces, and conversely.If $\bot$, $\sim \mathrm{t}$ and $\sim \top$ are treated as falsum constants, then any formula should be derivable from premises containing at least one of these falsum constants. This is guaranteed by ($\star$) and the fact that ($\uparrow$ /) and ($\uparrow$ \) preserve validity. The evaluation of t and $\top$ is without doubt reasonable: we may distinguish between a truth constant which is true at every information state and another truth constant which is true at every information state into which the initial piece of information may develop. The clauses for / and \ are just directional versions of Urquhart's (1972) truth definition. Moreover, it is rather natural to say that if A is true at information state b_1 and B is true at information state b_2, then $(A \circ B)$, which is a conjunction in the sense of juxtaposition, is true at every information piece into which $b_1 \cdot b_2$ may develop. The case of $\wedge$ is again unproblematic. In the case of $\vee$ it makes perfectly good sense to require that $(A \vee B)$ is true not only at pieces of information a at which A is true or at which B is true but also at pieces of information which prolong the intersection of pieces of information b_1 and b_2 such that A is true at b_1 and B is true at b_2. Thus, $(A \vee B)$ should also be true at information pieces which prolong so to speak the common content of information pieces b_1, b_2 with A true at b_1 and B true at b_2.[9]Finally, the evaluation of $\sim A$ by means of v^+ is intuitively convincing. We have that $\sim A$ is true at a piece of information a iff A is false at a. Turning to v^- we can thus say that the definition of $v^-(\sim A)$ is intuitively sound. In general the definition of v^- can be justified by the naturalness of the above-

[9] Došen (1989, p. 45) motivates the evaluation clause for disjunction by pointing out an analogous clause in Birkhoff's and Frink's representation of lattices by sets.

listed provable equivalences in terms of $\rightleftharpoons^+$.[10] Moreover the definition of semantic consequence is in accordance with what we have said so far. The information states into which the initial (or empty) piece of information 1 may develop should take precedence over the set of all information pieces.

We want to show that the present interpretation of monoid models is informational in the strong sense that there is a model which can arguably be talked about as the intended model under the given interpretation and which is a complete model for the logic in question. Now, the following assumptions seem to be natural: (i) Think of information pieces as finite sequences of formula occurrences, since in our basic calculi the databases are juxtapositions of such finite sequences. (ii) Identify those pieces of information which are interderivable (identifying $A_1 \ldots A_n$ with $A_1 \circ \ldots \circ A_n$, if $n > 1$). This is enough from the point of view of deductive information processing, although the representatives need not be synonymous in the sense of being intersubstitutable in all deductive contexts: if A and B are interderivable, by (cut) we have in Ξ_Δ, $\vdash XAY \to C$ iff $\vdash XBY \to C$ and $\vdash X \to A$ iff $X \to B$. The formulas interderivable with $\top$ can then be viewed as representing the empty piece of information, since we have that $\vdash_{\Xi_\Delta} \to A$ iff $\vdash_{\Xi_\Delta} \top \to A$. Next, let a and b be two pieces of information, and let A resp. B be a representative of a resp. b. The addition $a \cdot b$ of a and b should be the equivalence class of $(A \circ B)$ wrt interderivability, and the intersection $a \cap b$ of a and b should be the equivalence class of $(A \vee B)$ wrt interderivability. These considerations naturally lead us to the following definition of intended models.

DEFINITION 10. Let $\overset{\circ}{X} = A_1 \circ \ldots \circ A_n$, if $X = A_1 \ldots A_n$ $(n > 1)$, and let $\overset{\circ}{X} = A$, if $X = A$. For every formula A, the equivalence class of A modulo $\leftrightarrow$ will be denoted by $\mid A \mid$. The intended model $\mathcal{M}_{\Xi_\Delta} = \langle I, \cdot, \cap, 1, v_0 \rangle$ resp. $\mathcal{M}_{\Xi_\Delta} = \langle I, \cdot, \cap, 1, v_0^+, v_0^- \rangle$ for Ξ_Δ is defined as follows, where $q \in PROP$ $\cup \{\bot, \sim \mathbf{t}, \sim \top\}$:

- $I = \{|\overset{\circ}{X}| \mid X \text{ is a non-empty sequence of formula occurrences}\}$;
- $|\overset{\circ}{X}_1| \cdot |\overset{\circ}{X}_2| = |\overset{\circ}{X}_1 \circ \overset{\circ}{X}_2|$;
- $|\overset{\circ}{X}_1| \cap |\overset{\circ}{X}_2| = |\overset{\circ}{X}_1 \vee \overset{\circ}{X}_2|$;
- $1 = \mid \top \mid$;
- $v_0(q) = \{|\overset{\circ}{X}| \mid \vdash_{\Xi_\Delta} X \to q\}$;
- $v_0^+(q) = v_0(q)$, with $v_0^+(\bot) = v_0^+(\sim \mathbf{t}) = v^+(\sim \top)$, if $\Xi = COSPL$;
- $v_0^-(q) = \{|\overset{\circ}{X}| \mid \vdash_{\Xi_\Delta} X \to \sim q\}$, with $v_0^-(\bot) = I$, if $\Xi = COSPL$.

[10] We have already commented upon the falsity conditions of $(A \circ B)$, $\mathbf{t}$, and $\top$ in Section 1.

This construction clearly has an algebraic twist; note, however, that it does *not* yield the so-called Lindenbaum-algebra for Ξ_Δ. In the constructive case, for instance, there are no algebraic operations corresponding to $\rightleftharpoons$, /, \, **t**, and $\sim$ (cf. Rasiowa's (1974, p. 68) quasi-pseudo-Boolean algebras for **N**). The present semantics is close to syntax, but it is not 'syntax in disguise'.

LEMMA 3. $\mathcal{M}_{\Xi_\Delta}$ is in fact a monoid model for Ξ.

Proof. Obviously, $1 \in I$ and I is closed under $\cdot$ and $\cap$. Associativity of $\cdot$ and associativity, commutativity, and idempotence of $\cap$ are immediate. To see that 1 is a neutral element wrt $\cdot$, observe that $\vdash_{\Xi_\Delta} (\top \circ A) \leftrightarrow A$ and $\vdash_{\Xi_\Delta} (A \circ \top) \leftrightarrow A$. Distributivity of $\cdot$ over $\cap$ follows by $(\dagger)$. Eventually, we have to check ($\cap$ Heredity v_0^+) and ($\cap$ Heredity v_0^-). We check the latter property, to check the former is completely analogous:

$$
\begin{array}{lll}
 & |\overset{\circ}{X}_1| \in v_0^-(q) \text{ and } |\overset{\circ}{X}_2| \in v_0^-(q) & \\
\text{iff} & \vdash X_1 \to\sim q \text{ and } \vdash X_2 \to\sim q & \text{Def. } v_0^- \\
\text{iff} & \vdash \overset{\circ}{X}_1 \vee \overset{\circ}{X}_2 \to\sim q & (\circ \to), (\circ \uparrow), (\vee \to), (\vee \uparrow) \\
\text{iff} & |\overset{\circ}{X}_1 \vee \overset{\circ}{X}_2| \in v_0^-(q) & \text{Def. } v_0^- \\
\text{iff} & |\overset{\circ}{X}_1| \cap |\overset{\circ}{X}_2| \in v_0^-(q) & \text{Def. } \cap .
\end{array}
$$

It is straightforward to verify that in $\mathcal{M}_{ISPL_\Delta}$ and $\mathcal{M}_{COSPL_\Delta}$ for every $|\overset{\circ}{X}|, |\overset{\circ}{Y}| \in I$ and every $p \in PROP$, $|\overset{\circ}{X}| \in v_0(\bot)$ implies $|\overset{\circ}{X}| \in v_0(p)$, $|\overset{\circ}{X}| \leq |\overset{\circ}{X}| \cdot |\overset{\circ}{Y}|$, $|\overset{\circ}{X}| \leq |\overset{\circ}{Y}| \cdot |\overset{\circ}{X}|$, $|\overset{\circ}{X}| \cdot |\overset{\circ}{X}| \leq |\overset{\circ}{X}|$, and $1 \leq |\overset{\circ}{X}|$ □.

We shall now show that $\mathcal{M}_{\Xi_\Delta}$ is a complete, i.e., canonical model for Ξ_Δ.

LEMMA 4. (Truthlemma) For every $|\overset{\circ}{X}| \in I$ and every L-formula resp. $L^\sim$-formula A, the intended model $\mathcal{M}_{\Xi_\Delta}$ satisfies:

$$|\overset{\circ}{X}| \in v(A) \text{ resp. } v^+(A) \text{ iff} \vdash_{\Xi_\Delta} X \to A.$$

Proof. By induction on the complexity of A.

- $A = q \in PROP \cup \{\bot, \sim \mathbf{t}, \sim \top\}$: by the definition of v_0 resp. v_0^+.
- $A = \mathbf{t}$: $|\overset{\circ}{X}| \in v(\mathbf{t})$ iff $|\overset{\circ}{X}| \in I$ iff $\vdash X \to \mathbf{t}$.
- $A = \top$:

$$
\begin{array}{lll}
 & |\overset{\circ}{X}| \in v(\top) & \\
\text{iff} & 1 \leq |\overset{\circ}{X}| & \\
\text{iff} & |\top| \cap |\overset{\circ}{X}| = |\top| & \text{def. } \leq \\
\text{iff} & |\top \vee \overset{\circ}{X}| = |\top| & \text{def. } \cap \\
\text{iff} & \vdash X \to \top & (\vee \uparrow), (id), (\vee \to).
\end{array}
$$

– $A = \sim B$: It is enough to consider the case where B is a propositional variable or constant. Then the claim holds by the definition of v_0^-.

– $A = (B \setminus C)$:

$$\begin{array}{lll}
 & |\mathring{X}| \in v(B \setminus C) & \\
\text{iff} & (\forall\, |\mathring{Y}| \in v(B))\ |\mathring{Y}| \circ |\mathring{X}| \in v(C) & \\
\text{iff} & (\forall\, |\mathring{Y}| \in v(B))\ |\mathring{Y} \circ \mathring{X}| \in v(C) & \\
\text{iff} & \forall Y (\text{if } \vdash Y \rightarrow B, \text{ then } \vdash YX \rightarrow C) & \text{ind. hyp.} \\
\text{iff} & \vdash BX \rightarrow C & (cut), (id) \\
\text{iff} & \vdash X \rightarrow (B \setminus C) & (\uparrow \setminus), (\rightarrow \setminus).
\end{array}$$

– $A = (C/B)$: analogous to the previous case.

– $A = (B \circ C)$:

$$\begin{array}{lll}
 & \vdash X \rightarrow (B \circ C) & \\
\text{iff} & \forall A (\text{if } \vdash B \circ C \rightarrow A, \text{ then } \vdash X \rightarrow A) & (cut), (id) \\
\text{iff} & \forall A (\text{if } \vdash BC \rightarrow A, \text{ then } \vdash X \rightarrow A) & (\circ \uparrow), (\circ \rightarrow) \\
\text{iff} & \exists Y_1 \exists Y_2 (\vdash Y_1 \rightarrow B \text{ and } \vdash Y_2 \rightarrow C \text{ and} & \\
 & \forall A (\text{if } \vdash Y_1 Y_2 \rightarrow A, \text{ then } \vdash X \rightarrow A)) & (id) \\
\text{iff} & (\exists\, |\mathring{Y}_1| \in v(B))(\exists\, |\mathring{Y}_2| \in v(C)) \vdash \mathring{X} \rightarrow \mathring{Y}_1 \circ \mathring{Y}_2 & \text{ind. hyp.}, (cut) \\
\text{iff} & (\exists\, |\mathring{Y}_1| \in v(B))(\exists\, |\mathring{Y}_2| \in v(C))\ |\mathring{Y}_1 \circ \mathring{Y}_2| \leq |\mathring{X}| & \text{def. } \leq, (\vee \uparrow), (\rightarrow \vee) \\
\text{iff} & |\mathring{X}| \in v(B \circ C). &
\end{array}$$

– $A = (B \wedge C)$: use the induction hypothesis.

– $A = (B \vee C)$:

$$\begin{array}{lll}
 & |\mathring{X}| \in v(B \vee C) & \\
\text{iff} & (\exists\, |\mathring{Y}_1| \in v(B))(\exists\, |\mathring{Y}_2| \in v(C))\,(|\mathring{Y}_1 \vee \mathring{Y}_2| \cap |\mathring{X}| = & \\
 & |\mathring{Y}_1 \vee \mathring{Y}_2| \text{ or } |\mathring{X}| \in v(B) \text{ or } |\mathring{X}| \in v(C)) & \\
\text{iff} & \exists\, \mathring{Y}_1\, \exists\, \mathring{Y}_2\, (\vdash \mathring{Y}_1 \rightarrow B \text{ and } \mathring{Y}_2 \rightarrow C \text{ and} & \\
 & \vdash (\mathring{Y}_1 \vee \mathring{Y}_2) \vee \mathring{X} \leftrightarrow \mathring{Y}_1 \vee \mathring{Y}_2) \text{ or} & \\
 & \vdash \mathring{X} \rightarrow B \text{ or } \mathring{X} \rightarrow C & \text{ind. hyp., def. } \cap \\
\text{iff} & \vdash \mathring{X} \rightarrow (B \vee C) & (\vee \uparrow), (\vee \rightarrow), (cut), (id)\, (\rightarrow \vee) \\
\text{iff} & \vdash X \rightarrow (B \vee C) & (\circ \uparrow)\, (\circ \rightarrow). \quad \square
\end{array}$$

With the Truthlemma in our hands, we are in a position to prove completeness.

THEOREM 4. (completeness) If $X \rightarrow A$ holds in every monoid model from M_{Ξ_Δ}, then $\vdash_{\Xi_\Delta} X \rightarrow A$.

Proof. Suppose that $\not\vdash_{\Xi_\Delta} X \rightarrow A$. By the Truthlemma, this is the case iff in $\mathcal{M}_{\Xi_\Delta}$ we have that $|\mathring{X}| \notin v^+(A)$. But this implies that $v^+(\mathring{X}) \not\subseteq v(A)$, since $|\mathring{X}| \in v(\mathring{X})$. It remains to be shown that in each case the underlying *slomo* of the intended model satisfies the conditions which correspond to the rules in Δ, i.e. $\mathcal{M}_{\Xi_\Delta} \in \mathsf{M}_{\Xi_\Delta}$.But this is a completely straightforward matter.

Consider by way of example the case of the structural rule **E**. It has to be shown that $\mid A \mid \leq \mid A \mid \cdot \mid A \mid$, i.e. $\mid A \mid \cap \mid A \cdot A \mid = \mid A \mid$. Now, using **E** it can easily be seen that $\vdash_{\Xi_\Delta} A \vee (A \circ A) \leftrightarrow A$. □

Next, consider the following conception of an intended monoid model. Think of a piece of information as the deductive closure of a finite sequence of formula occurrences. The intersection of information pieces should then be nothing but set intersection, and the addition $a_1 \cdot a_2$ of two pieces of information a_1, a_2 with finite representations X_1, X_2 should be the deductive closure of X_1X_2, i.e. the deductive closure of the juxtaposition of their representations. The empty piece of information would be represented by the set of all theorems, i.e. the deductive closure of the empty sequence. Where these considerations lead us is Došen's (1989) construction of a canonical monoid model:

DEFINITION 11. The canonical monoid model $\mathcal{M}'_{\Xi_\Delta} = \langle I', \cdot', \cap', 1', v_0 \rangle$ resp. $\mathcal{M}'_{\Xi_\Delta} = \langle I', \cdot', \cap', 1', v_0^+, v_0^- \rangle$ for Ξ_Δ is defined as follows, where $q \in PROP \cup \{\perp, \sim \mathbf{t}, \sim \top\}$:

- $I' = \{a \mid \exists X, a = \{A \mid \vdash_{\Xi_\Delta} X \rightarrow A\}\}$;
- if $a_1 = \{A \mid \vdash_{\Xi_\Delta} X_1 \rightarrow A\}$ and $a_2 = \{A \mid \vdash_{\Xi_\Delta} X_2 \rightarrow A\}$, then $a_1 \cdot' a_2 = \{A \mid \vdash_{\Xi_\Delta} X_1X_2 \rightarrow A\}$;
- $\cap'$ is set-intersection;
- $1' = \{A \mid \vdash_{\Xi_\Delta} \rightarrow A\}$;
- $v_0(q) = \{a \in I \mid q \in a\}$;
- $v_0^+(q) = v_0(q)$, with $v_0^+(\perp) = v_0^+(\sim \mathbf{t}) = v^+(\sim \top)$, if $\Xi = COSPL$;
- $v_0^-(q) = \{a \in I \mid \sim q \in a\}$, with $v_0^-(\perp) = I$, if $\Xi = COSPL$.

It can easily be shown that $\mathcal{M}'_{\Xi_\Delta}$ is in fact a monoid model for Ξ and that $\mathcal{M}'_{\Xi_\Delta} \in \mathsf{M}_{\Xi_\Delta}$. Assume for example $\mathbf{C} \in \Delta$. Suppose that $a = \{A \mid \vdash X \rightarrow A\}$, $X = A_1 \ldots A_n$. Since

$$\dfrac{X \rightarrow A_1 \circ \ldots \circ A_n \qquad \dfrac{\dfrac{A_1 \ldots A_n A_1 \ldots A_n \rightarrow A}{A_1 \circ \ldots \circ A_n A_1 \circ \ldots \circ A_n \rightarrow A}}{A_1 \circ \ldots \circ A_n \rightarrow A}}{X \rightarrow A}$$

$\{A \mid \vdash XX \rightarrow A\} \subseteq \{A \mid \vdash X \rightarrow A\}$, that is to say, $a \cdot a \leq a$.

As it turns out, both constructions of canonical models are isomorphic and can therefore be identified:

OBSERVATION 1. There exists an isomorphism between the underlying *slomos* of $\mathcal{M}'_{\Xi_\Delta}$ and $\mathcal{M}_{\Xi_\Delta}$.

Proof. The function $h : I' \longrightarrow I$ defined by $h(\{A \mid\vdash_{\Xi_\Delta} X \rightarrow A\}) = |\mathring{X}|$ is such an isomorphism. $1-1$: Suppose that $a = \{A \mid\vdash_{\Xi_\Delta} X \rightarrow A\} \neq b = \{B \mid\vdash_{\Xi_\Delta} Y \rightarrow B\}$, but $h(a) = h(b)$. Then $\vdash_{\Xi_\Delta} \mathring{X} \leftrightarrow \mathring{Y}$ and by (cut) and $(\uparrow \circ)$, $a = b$. Onto: obvious. The homomorphism property is easy to establish. Let $a = \{A \mid\vdash_{\Xi_\Delta} X \rightarrow A\}$, $b = \{B \mid\vdash_{\Xi_\Delta} Y \rightarrow B\}$. $h(a \cdot' b) = |\mathring{X} \circ \mathring{Y}| = h(a) \cdot h(b)$. $h(a \cap' b) = \{A \mid\vdash_{\Xi_\Delta} X \rightarrow A\} \cap \{B \mid\vdash_{\Xi_\Delta} Y \rightarrow B\} = \{C \mid\vdash_{\Xi_\Delta} X \rightarrow C$ and $\vdash_{\Xi_\Delta} Y \rightarrow C\} = \{C \mid\vdash_{\Xi_\Delta} \mathring{X} \vee \mathring{Y} \rightarrow C\}$, by $(\uparrow \circ)$, $(\vee \rightarrow)$. Thus $h(a \cap' b) = |\mathring{X} \vee \mathring{Y}| = |\mathring{X}| \cap |\mathring{Y}| = h(A) \cap h(B)$. □

In conclusion we may say that the monoid semantics provides an informational interpretation for a broad range of substructural propositional logics, including the limiting cases N^-, N, MPL, and IPL. Different conceptions of deductive information processing within one family of formal systems naturally correspond to different conceptions of *slomos* as abstract information structures. Moreover, in (Wansing, 1993) it is shown that *slomos* constitute an *exhaustive* format of abstract information structures in so far as every propositional connective which is definable in a certain higher-level proof-theoretic semantics is also explicitly definable in every model from an appropriate class of monoid models.

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