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On the Second Cohomology Groups of Semidirect Products

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Introduction

Let G be a finite group, then $H^3(G, \mathbb{Z})$ is canonically isomorphic to $H^2(G, \mathbb{Q}/\mathbb{Z})$, which is called the Schur multiplicator of the group G. In general we study $H^2(G, A)$, where A is a G-module with trivial G-action. The second cohomology group $H^2(G, A)$ is the additive group of 2-cocycles taken modulo the subgroup of 2-coboundaries, where a map $f\colon G\times G\to A$ is a 2-cocycle on G if and only if f satisfies the following:

$$f(\sigma,\tau)-f(\rho\sigma,\tau)+f(\rho,\sigma\tau)-f(\rho,\sigma)=0 \qquad (\rho,\sigma,\tau\in G),$$

and f is a 2-coboundary on G if and only if there exists a map $g\colon G\to A$ such that

$$f(\sigma,\tau)=g(\tau)-g(\sigma\tau)+g(\sigma) \qquad (\sigma,\tau\in G).$$

Lyndon [5] proved that if a finite group G is the direct product of its normal subgroups N and T, then the cohomology group $H^n(G, A)$ of G with a coefficient module A with trivial G-action is related to the compound cohomology groups $H^k(T, H^{n-k}(N, A))$ $(0\leqq k\leqq n)$.

In this paper we show that, when G is the semidirect product of a normal subgroup N and a subgroup T, then

$$H^2(G,A)\cong H^2(T,A)\oplus \tilde{H}^2(G,A),$$

and we have the following canonical exact sequence

$$0\to H^1(T,H^1(N,A))\to \tilde{H}^2(G,A)\xrightarrow{\text{res}} H^2(N,A)^T$$
$$\xrightarrow{d_2} H^2(T,H^1(N,A))\to \tilde{H}^3(G,A),$$

where $\tilde{H}^i(G,A)$ is the kernel of the restriction homomorphism $H^i(G,A)\to H^i(T,A)$. When both N and T are finite cyclic subgroups, the homomorphism d_2, and $\tilde{H}^2(G,A)$ hence $H^2(G,A)$ can be determined completely for an arbitrary G-module A with trivial G-action. As its application, the Schur multiplicator of a semidirect product of cyclic subgroups is concretely computed.

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§ 1. Theorems

Let G be a finite group, and let A be a G-module with trivial G-action. Then it was shown by Eilenberg and MacLane [3] that the cohomology group $H^n(G, A)$ is unaffected even if we restrict ourselves only to those n-cochains f which satisfy $f(g_1, g_2, \ldots, g_n)=0$ whenever one of the arguments g_i is 1 (=the identity element of G). Since this condition is very useful in computation, we henceforth assume that all cochains satisfy the condition.

If $G=N\cdot T$ is the semidirect product of a normal subgroup N and a subgroup T, then every element g of G is uniquely represented in the form $g=nt$ ($n\in N$, $t\in T$), and for any $n\in N$, $t\in T$,

$$tn={}^t n t \qquad \text{with } {}^t n=tnt^{-1}.$$

Proposition 1. *Let G be the semidirect product of a normal subgroup N and a subgroup T, and let A be a G-module with trivial G-action.*

(I) *Let a map $f\colon G\times G\to A$ be a 2-cocycle on G. Then f can always be normalized up to coboundaries as follows:*

$$(*) \qquad f(N, T)=0,$$

and hence

(a) $f(nt, n't')=f(t, t')+f(t, n')+f(n', {}^t n') \quad (n, n'\in N,\ t, t'\in T)$.

We call such a 2-cocycle f a normal 2-cocycle. Thus a normal 2-cocycle f on G is determined uniquely by $f|_{N\times N}$, $f|_{T\times T}$ and $f|_{T\times N}$.

(II) *The data $f|_{N\times N}$, $f|_{T\times T}$ and $f|_{T\times N}$ determine a normal 2-cocycle on G if and only if they satisfy the following:*

(b) *f is a 2-cocycle on N,*

(c) *f is a 2-cocycle on T,*

(d) $f(tt', n)=f(t', n)+f(t, {}^{t'}n) \quad (n\in N,\ t, t'\in T)$,

(e) $f(n, n')-f({}^t n, {}^t n')=f(t, n)-f(t, nn')+f(t, n') \quad (n, n'\in N,\ t\in T)$.

Proof. For the discussion in this proof, it is more convenient to treat the law of composition of A as multiplication. There is a one-to-one correspondence between the elements of $H^2(G, A)$ and the classes of equivalent central extensions

$$1\to A\to \Pi\to G\to 1,$$

and giving a 2-cocycle f representing a cohomology class corresponds to giving a section $\pi\colon G\to\Pi$, via the relation

$$f(g,g')\,\pi(gg')=\pi(g)\,\pi(g')\qquad (g,g'\in G).$$

It is clear that a 2-cochain f satisfies $(*)$ if and only if the corresponding section satisfies

$$(**)\qquad \pi(nt)=\pi(n)\,\pi(t)\qquad (n\in N,\ t\in T).$$

Hence the normalization $(*)$ of 2-cocycles on G is obvious, and since

$$\pi(nt)\,\pi(n't')=f(nt,n't')\,\pi(ntn't')=f(nt,n't')\,\pi(n{}^{t}n'tt')\quad (n,n'\in N,\ t,t'\in T),$$

and the left hand side is equal to

$$\begin{aligned}\pi(n)\,\pi(t)\,\pi(n')\,\pi(t')&=\pi(n)\,f(t,n')\,\pi({}^{t}n')\,\pi(t)\,\pi(t')\\&=f(t,n')\,f(n,{}^{t}n')\,f(t,t')\,\pi(n{}^{t}n'tt'),\end{aligned}$$

we have (a). To prove (II), suppose f satisfies $(*)$, and let π be the corresponding section satisfying $(**)$. Since

$$\pi(t)\,\pi(n)=f(t,n)\,\pi(tn)=f(t,n)\,\pi({}^{t}nt)=f(t,n)\,\pi({}^{t}n)\,\pi(t)\qquad (n\in N,\ t\in T),$$

it follows that

$$\pi(t)\,\pi(n)\,\pi(t)^{-1}=f(t,n)\,\pi({}^{t}n)\qquad (n\in N,\ t\in T).$$

By replacing t by tt' or n by nn' in the above relation, we easily get (d) and (e), respectively.

Conversely it is easy to show that a map $f\colon G\times G\to A$ satisfying the relations (a)$\sim$(e) is a 2-cocycle on G. Q.E.D.

Let $\tilde{H}^{i}(G,A)$ be the kernel of the restriction homomorphism

$$\mathrm{res}\colon H^{i}(G,A)\to H^{i}(T,A).$$

Then by the splitting of the exact sequence

$$\begin{array}{c}1\to N\to G\to G/N\to 1,\\ \uparrow\ \nearrow\\ T\end{array}$$

we see that

$$H^{i}(G,A)\cong H^{i}(T,A)\oplus\tilde{H}^{i}(G,A)\qquad (i\geqq 1),$$

canonically.

Theorem 2. *Let G be the semidirect product of a normal subgroup N and a subgroup T, and let A be a G-module with trivial G-action. Then*

(I) *$H^{2}(G,A)\cong H^{2}(T,A)\oplus\tilde{H}^{2}(G,A)$ canonically.*

(II) *We have a canonical exact sequence*

$$0\to H^1(T, H^1(N,A))\to \tilde{H}^2(G,A)\xrightarrow{\text{res}} H^2(N,A)^T$$
$$\xrightarrow{d_2} H^2(T, H^1(N,A))\to \tilde{H}^3(G,A),$$

where T acts on $H^1(N,A)=\mathrm{Hom}(N,A)$ and $H^2(N,A)$ via the canonical action induced by conjugation of T on N, and $H^2(N,A)^T$ is the subgroup of T-invarients. The homomorphism res: $\tilde{H}^2(G,A)\to H^2(N,A)^T$ *is induced by the restriction map* res: $H^2(G,A)\to H^2(N,A)$.

Proof. Let $Z_*^2=Z_*^2(G,A)$ and $B_*^2=B_*^2(G,A)$ be the groups of normal 2-cocycles and 2-coboundaries, respectively. By Proposition 1, we have $H^2(G,A)=Z_*^2/B_*^2$. Let $\tilde{Z}_*^2$ be the subgroup of elements f of Z_*^2 for which $f(T,T)=0$. Put $\tilde{B}_*^2=\tilde{Z}_*^2\cap B_*^2$. Then by Proposition 1, it is easy to see that $\tilde{H}^2(G,A)\cong\tilde{Z}_*^2/\tilde{B}_*^2$, because f is an element of $\tilde{B}_*^2$ if and only if there exists a map $v\colon N\to A$ with $v(1)=0$ such that

$$f(t,n)=v(n)-v({}^t n) \qquad (n\in N,\ t\in T),$$

and

$$f(n,n')=v(n)-v(nn')+v(n') \qquad (n,n'\in N).$$

By Proposition 1, the homomorphism sending f to $(f|_{T\times N}, f|_{N\times N})$ is an injection

$$\tilde{Z}_*^2\to Z^1(T, C^1(N,A))\oplus Z^2(N,A).$$

Its image consists of elements (u,h) satisfying

$$h(n,n')-h({}^t n, {}^t n')=u(t,n)-u(t,nn')+u(t,n') \qquad (n,n'\in N,\ t\in T). \quad (1)$$

We identify $\tilde{Z}_*^2$ with its image in the right hand side. The subgroup consisting of elements $(u,0)$ with $u\in Z^1(T, H^1(N,A))$ is obviously contained in $\tilde{Z}_*^2$. We identify $Z^1(T, H^1(N,A))$ with its image in

$$Z^1(T, C^1(N,A))\oplus Z^2(N,A).$$

It is straightforward to see that

$$\tilde{B}_*^2\cap Z^1(T, H^1(N,A))=B^1(T, H^1(N,A)).$$

Thus we have a canonical injection

$$H^1(T, H^1(N,A))\to \tilde{H}^2(G,A).$$

Consider the homomorphism $p\colon \tilde{Z}_*^2\to Z^2(N,A)$ sending (u,h) to h.

Lemma 3. $$p^{-1}(B^2(N,A))=Z^1(T, H^1(N,A))+\tilde{B}_*^2.$$

Proof. Let (u,h) be an element of $\tilde{Z}_*^2$ with $h\in B^2(N,A)$. Then there exists a map $v\colon N\to A$ with $v(1)=0$ such that

$$h(n,n')=v(n)-v(nn')+v(n') \qquad (n,n'\in N).$$

By (1), we get

$$\{u(t, n)-v(n)+v({}^t n)\}+\{u(t, n')-v(n')+v({}^t n')\}$$
$$=\{u(t, nn')-v(nn')+v({}^t(nn'))\} \qquad (n, n' \in N, t \in T).$$

Thus $u'(t, n)=u(t, n)-v(n)+v({}^t n)$ is in $Z^1(T, H^1(N, A))$ and $(u', 0)$ is congruent to (u, h) modulo $\tilde{B}^2_*$. Q.E.D.

Proof of Theorem 2 *(continued)*. So far we obtained a canonical exact sequence

$$0 \to H^1(T, H^1(N, A)) \to \tilde{H}^2(G, A) \xrightarrow{\ p\ } H^2(N, A).$$

Moreover p is obviously induced by the restriction map res: $H^2(G, A) \to H^2(N, A)$. By (1), the image of p is contained in $H^2(N, A)^T$.

Next we define a canonical homomorphism

$$d_2 \colon H^2(N, A)^T \to H^2(T, H^1(N, A)).$$

Let $h\colon N \times N \to A$ be a 2-cocycle on N representing a cohomology class of $H^2(N, A)^T$. Then there is a map $u\colon T \times N \to A$ with $u(1, n)=u(t, 1)=0$ $(n \in N, t \in T)$ such that

$$h(n, n')-h({}^t n, {}^t n')=u(t, n)-u(t, nn')+u(t, n') \qquad (n, n' \in N, t \in T).$$

Consider

$$(d_2 h)(t, t', n)=u(t', n)-u(tt', n)+u(t, {}^{t'} n).$$

$d_2 h$ is easily seen to be contained in $Z^2(T, H^1(N, A))$. We now show that this d_2 gives rise to the sought-for homomorphism. Suppose $h'\colon N \times N \to A$ is another 2-cocycle cohomologous to h. Then there exists a map $v\colon N \to A$ with $v(1)=0$ such that

$$h'(n, n')=h(n, n')+v(n)-v(nn')+v(n') \qquad (n, n' \in N).$$

Let $u'\colon T \times N \to A$ be a map with $u'(1, n)=u'(t, 1)=0$ $(n \in N, t \in T)$ such that

$$h'(n, n')-h'({}^t n, {}^t n')=u'(t, n)-u'(t, nn')+u'(t, n') \qquad (n, n' \in N, t \in T).$$

Then $w(t, n)=u'(t, n)-u(t, n)-v(n)+v({}^t n)$ is easily seen to be contained in $C^1(T, H^1(N, A))$. It is straightforward to see that

$$(d_2 h')(t, t', n)=(d_2 h)(t, t', n)+w(t', n)-w(tt', n)+w(t, {}^{t'} n) \quad (n \in N, t, t' \in T).$$

Thus $d_2\colon H^2(N, A)^T \to H^2(T, H^1(N, A))$ is well defined.

To show the exactness of the sequence, suppose $d_2 h$ is cohomologous to 0. Then there exists $w \in C^1(T, H^1(N, A))$ with

$$(d_2 h)(t, t', n)=w(t', n)-w(tt', n)+w(t, {}^{t'} n) \qquad (n \in N, t, t' \in T).$$

Then

$$u(t,n)=w(t,n)+z(t,n) \qquad (n\in N,\ t\in T),$$

with $z(t',n)-z(tt',n)+z(t,{}^{t'}n)=0$ $(n\in N,\ t,t'\in T)$, and

$$h(n,n')-h({}^{t}n,{}^{t}n')=z(t,n)-z(t,nn')+z(t,n') \qquad (n,n'\in N,\ t\in T).$$

Thus (z,h) is in $\tilde{Z}^2_*$.

Finally we consider a canonical homomorphism $H^2(T,H^1(N,A))\to \tilde{H}^3(G,A)$. Let $f\colon T\times T\to H^1(N,A)$ be a 2-cocycle on T representing a cohomology class of $H^2(T,H^1(N,A))$. Then f is considered as a map from $T\times T\times N$ to A, and it is easy to see that f is a 3-cocycle on G with

$$f(nt,n't',n''t'')=f(t,t',n'') \qquad (n,n',n''\in N,\ t,t',t''\in T).$$

Suppose f is cohomologous to zero. Then there exists a map $v\colon T\times N\to A$ with $v(1,n)=v(t,1)=0$ $(n\in N,\ t\in T)$ and

$$v(t,nn')=v(t,n)+v(t,n') \qquad (t\in T,\ n,n'\in N)$$

such that

$$f(t,t',n'')=v(t',n'')-v(tt',n'')+v(t,{}^{t'}n'') \qquad (t,t'\in T,\ n''\in N).$$

Putting $v(nt,n't')=v(t,n')$ $(n,n'\in N,\ t,t'\in T)$, we have easily

$$f(g,g',g'')=v(g',g'')-v(gg',g'')+v(g,g'g'')-v(g,g') \qquad (g,g',g''\in G).$$

Thus the canonical homomorphism $H^2(T,H^1(N,A))\to\tilde{H}^3(G,A)$ is well defined.

To show the last exactness of the sequence, let $h\colon N\times N\to A$ be a 2-cocycle on N representing a cohomology class of $H^2(N,A)^T$. Then there is a map $u\colon T\times N\to A$ with $u(1,n)=u(t,1)=0$ $(n\in N,\ t\in T)$ such that

$$h(n,n')-h({}^{t}n,{}^{t}n')=u(t,n)-u(t,nn')+u(t,n') \qquad (n,n'\in N,\ t\in T).$$

Putting $v(nt,n't')=u(t,n')+h(n,{}^{t}n')$ $(n,n'\in N,\ t,t'\in T)$, we have easily

$$(d_2 h)(g,g',g'')=v(g',g'')-v(gg',g'')+v(g,g'g'')-v(g,g') \qquad (g,g',g''\in G).$$

Thus its image in $\tilde{H}^3(G,A)$ is zero. Conversely let $f\colon T\times T\to H^1(N,A)$ be a 2-cocycle on T representing a cohomology class of $H^2(T,H^1(N,A))$, and let its image in $\tilde{H}^3(G,A)$ be zero. Then there exists a map $v\colon G\times G\to A$ with $v(1,g)=v(g,1)$ $(g\in G)$ such that

$$f(g,g',g'')=v(g',g'')-v(gg',g'')+v(g,g'g'')-v(g,g') \qquad (g,g',g''\in G).$$

Putting $u(t,n)=v(t,n)-v({}^{t}n,t)$ $(t\in T,\ n\in N)$ and

$$h(n,n')=v(n,n') \qquad (n,n'\in N),$$

we have easily

$$h(n', n'') - h(nn', n'') + h(n, n'n'') - h(n, n') = 0 \qquad (n, n', n'' \in N),$$

$$h(n, n') - h({}^t n, {}^t n') = u(t, n') - u(t, nn') + u(t, n) \qquad (n, n' \in N, t \in T),$$

and

$$f(t, t', n'') = u(t, n'') - u(tt', n'') + u(t, {}^{t'}n'') \qquad (t, t' \in T, n'' \in N).$$

Thus h determine an element of $H^2(N, A)^T$, and the cohomology class of $d_2 h$ in $H^2(T, H^1(N, A))$ is equal to that of f. Q.E.D.

Remark 1. In general there is the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G/N, H^q(N, A)) \Rightarrow H^{p+q}(G, A).$$

When $G \to G/N$ splits, $E_2^{p,q}$ with $q > 0$ converges to $\tilde{H}^{p+q}(G, A)$. The exact sequence we obtained in Theorem 2 is nothing but the exact sequence of terms of low degree in this latter spectral sequence. Our proof gives us concrete descriptions of $\tilde{H}^2(G, A)$ hence $H^2(G, A)$, and d_2 (cf. Theorem 7, 8 below).

We denote by $H^2(N, A)^*$ the image of res: $\tilde{H}^2(G, A) \to H^2(N, A)^T$. By Theorem 2, $H^2(N, A)^* = \operatorname{Ker} d_2$.

Corollary. *If $G = N \times T$ is the direct product of N and T, then canonically*

$$H^2(G, A) \cong H^2(T, A) \oplus H^1(T, H^1(N, A)) \oplus H^2(N, A).$$

Proof. T acts trivially on N, hence $H^2(N, A)^T = H^2(N, A)$. Thus we have an exact sequence by Theorem 2,

$$0 \to H^2(T, A) \oplus H^1(T, H^1(N, A)) \to H^2(G, A) \to H^2(N, A).$$

By the splitting of the exact sequence

$$\begin{array}{ccccccccc} 1 & \to & T & \to & G & \to & G/T & \to & 1 \\ & & & & \uparrow & \nearrow\wr & & & \\ & & & & N & & & & \end{array}$$

we have

$$H^2(G, A) \cong H^2(T, A) \oplus H^1(T, H^1(N, A)) \oplus H^2(N, A). \quad \text{Q.E.D.}$$

Remark 2 (Akagawa). In general $H^2(N, A)^*$ is not equal to $H^2(N, A)^T$. Here is an example. Let G be the group generated by elements n, t with defining relations:

$$n^9 = t^3 = 1, \qquad tnt^{-1} = n^4.$$

Then G is the semidirect product of the normal subgroup $N = \{n\}$ and the subgroup $T = \{t\}$. Let $A = \mathbb{Z}/3\mathbb{Z}$ be the G-module with trivial G-action.

Then easily

$$H^2(T, A) = \mathbb{Z}/3\mathbb{Z}, \quad H^1(T, H^1(N, A)) = \mathbb{Z}/3\mathbb{Z},$$

and

$$H^2(N, A)^T = \mathbb{Z}/3\mathbb{Z}, \quad H^2(T, H^1(N, A)) = \mathbb{Z}/3\mathbb{Z}.$$

By Theorem 8 below, we have $d_2 = \text{identity}$ and hence $H^2(N, A)^* = 0$ and

$$H^2(G, A) \cong H^2(T, A) \oplus H^1(T, H^1(N, A)) = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}.$$

Remark 3. In general $H^2(G, A)$ is not equal to

$$H^2(T, A) \oplus H^1(T, H^1(N, A)) \oplus H^2(N, A)^*,$$

namely the canonical exact sequence

$$0 \to H^1(T, H^1(N, A)) \to \tilde{H}^2(G, A) \to H^2(N, A)^* \to 0$$

is not split. Here is an example. Let G be the group generated by elements n, t with defining relations:

$$n^9 = t^9 = 1, \quad t n t^{-1} = n^7.$$

Then G is the semidirect product of the normal subgroup $N = \{n\}$ and the subgroup $T = \{t\}$. Let $A = \mathbb{Z}/27\mathbb{Z}$ be the G-module with trivial G-action. Then easily

$$H^2(T, A) = \mathbb{Z}/3\mathbb{Z}, \quad H^1(T, H^1(N, A)) = \mathbb{Z}/3\mathbb{Z},$$

and

$$H^2(N, A)^T = \mathbb{Z}/3\mathbb{Z}, \quad H^2(T, H^1(N, A)) = \mathbb{Z}/3\mathbb{Z}.$$

By Theorem 8 below, we have $d_2 = 3 \cdot \text{identity} = 0$, and hence

$$H^2(N, A)^* = H^2(N, A)^T = \mathbb{Z}/3\mathbb{Z}.$$

On the other hand we have by Theorem 7 below,

$$\tilde{H}^2(G, A) = \mathbb{Z}/9\mathbb{Z} \neq H^1(T, H^1(N, A)) \oplus H^2(N, A)^* = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}.$$

We now determine completely $\tilde{H}^2(G, A)$ hence $H^2(G, A)$, and identify the homomorphism d_2, when N and T are both finite cyclic groups and G is their semidirect product.

The following is well known, and may be proved in the same way as in Proposition 1.

Lemma 4. *Let S be a cyclic group of order s generated by an element σ, and let A be a left S-module. Then a 2-cocycle $f\colon S \times S \to A$ can always be normalized as*

$$f(\sigma^i, \sigma^j) = \left(\left[\frac{i+j}{s}\right] - \left[\frac{i}{s}\right] - \left[\frac{j}{s}\right]\right) f(\sigma, \sigma^{-1}) \quad (i, j \in \mathbb{Z}),$$

where [ ] *is the Gauss symbol. Moreover the homomorphism sending* f *to* $f(\sigma, \sigma^{-1}) \left(\bmod \left(\sum_{l=0}^{s-1} \sigma^l\right) A\right)$ *gives a canonical isomorphism*

$$H^2(S, A) \cong A^S \Big/ \left(\sum_{l=0}^{s-1} \sigma^l\right) A.$$

Lemma 5. *Let* $G = N \cdot T$ *be the semidirect product of a normal subgroup* N *and a finite cyclic subgroup* $T = \{t\} \cong \mathbb{Z}/m\mathbb{Z}$, *and let* A *be a* G-*module with trivial* G-*action. Let a map* $f \colon N \times N \to A$ *be a 2-cocycle on* N *representing an element of* $H^2(N, A)^T$. *Then there exists a map* $v \colon T \times N \to A$ *with* $v(1, n) = v(t', 1) = 0$ $(n \in N,\ t' \in T)$ *which satisfies the following:*

$$\text{(i)} \qquad v(t^i, n) = \sum_{l=0}^{i-1} v(t, {}^{t^l}n) \qquad (0 \leqq i < m,\ n \in N),$$

and

$$\text{(ii)} \quad f(n, n') - f({}^{t'}n, {}^{t'}n') = v(t', n) - v(t', nn') + v(t', n') \quad (n, n' \in N,\ t' \in T).$$

Moreover $d_2 \colon H^2(N, A)^T \to H^2(T, H^1(N, A))$ *sends the cohomology class of* f *to that of a 2-cocycle on* T *with coefficients in* $H^1(N, A)$:

$$\left(\left[\frac{i+j}{m}\right] - \left[\frac{i}{m}\right] - \left[\frac{j}{m}\right]\right) \sum_{l=0}^{m-1} v(t, {}^{t^l}n) \qquad (i, j \in \mathbb{Z},\ n \in N).$$

Proof. Let a map $f \colon N \times N \to A$ be a 2-cocycle representing an element of $H^2(N, A)^T$, then there exists a map $u \colon T \times N \to A$ with $u(1, n) = u(t', 1) = 0$ $(n \in N,\ t' \in T)$ such that

$$f(n, n') - f({}^{t'}n, {}^{t'}n') = u(t', n) - u(t', nn') + u(t', n') \qquad (n, n' \in N,\ t' \in T).$$

Consider

$$(d_2 f)(t', t'', n) = u(t'', n) - u(t't'', n) + u(t', {}^{t''}n) \qquad (t', t'' \in T,\ n \in N),$$

which represents an element of $H^2(T, H^1(N, A))$. Since T is cyclic, there exists a map $w \colon T \times N \to A$ with $w(1, n) = w(t', 1) = 0$ $(n \in N,\ t' \in T)$ and $w(t', nn') = w(t', n) + w(t', n')$ $(t' \in T,\ n, n' \in N)$, which normalizes $d_2 f$ as in Lemma 4, i.e.

$$\begin{aligned}
&(d_2 f)(t^i, t^j, n) + \{w(t^j, n) - w(t^{i+j}, n) + w(t^i, {}^{t^j}n)\} \\
&\quad = \{u(t^j, n) - u(t^{i+j}, n) + u(t^i, {}^{t^j}n)\} + \{w(t^j, n) - w(t^{i+j}, n) + w(t^i, {}^{t^j}n)\} \\
&\quad = \left(\left[\frac{i+j}{m}\right] - \left[\frac{i}{m}\right] - \left[\frac{j}{m}\right]\right) \{(d_2 f)(t, t^{-1}, n) + w(t^{-1}, n) + w(t, {}^{t^{-1}}n)\} \\
&\qquad\qquad (i, j \in \mathbb{Z}).
\end{aligned} \tag{2}$$

Then the map $v = u + w \colon T \times N \to A$ satisfies the relations (i) and (ii). The rest follows immediately. Q.E.D.

Lemma 6. *Let $G=N\cdot T$ be the semidirect product of a cyclic normal subgroup $N=\{n\}\cong\mathbb{Z}/k\mathbb{Z}$ and a subgroup T, and let A be a G-module with trivial G-action. Let a map $f\colon N\times N\to A$ be a 2-cocycle on N of the form $f(n^i,n^j)=\left(\left[\frac{i+j}{k}\right]-\left[\frac{i}{k}\right]-\left[\frac{j}{k}\right]\right)f(n,n^{-1})$ $(i,j\in\mathbb{Z})$ such that*

(iii) $f(n',n'')-f({}^tn',{}^tn'')=u(t,n')-u(t,n'n'')+u(t,n'')$ $(n',n''\in N,\ t\in T)$

for a map $u\colon T\times N\to A$ with $u(1,n')=u(t,1)=0$ $(n'\in N,\ t\in T)$. Then

$$\text{(iv)}\quad u(t,n^i)=\left(\left[\frac{ir(t)}{k}\right]-\left[\frac{i}{k}\right]\right)f(n,n^{-1})+iu(t,n)\qquad (i\in\mathbb{Z},\ t\in T),$$

where $r(t)$ is an integer with ${}^tn=n^{r(t)}$ and $[\ ]$ is the Gauss symbol.

Proof. Obvious.

Let $G_{k,m}$ be the group of order km generated by elements n, t with defining relations:

$$n^k=1,\qquad t^m=1,\qquad tnt^{-1}=n^r\quad\text{with } r^m\equiv 1 \pmod{k}.$$

Then $G_{k,m}$ is the semidirect product of the cyclic subgroups $N=\{n\}$ and $T=\{t\}$.

Theorem 7. *Let $G=G_{k,m}$ be as above, and let A be a G-module with trivial G-action. Then $\tilde{H}^2(G,A)$ is isomorphic to the additive group consisting of elements (a,b) in $A\times A$ with the relation:*

$$\text{(v)}\quad \left(k,r-1,\sum_{l=0}^{m-1}r^l,\frac{r^m-1}{k}\right)\left\{\frac{r-1}{(k,r-1)}a+\frac{k}{(k,r-1)}b\right\}=0,$$

taken modulo the subgroup consisting of elements

$$\text{(vi)}\qquad (kc,(1-r)c)\quad\text{with } c\in A,$$

via the map sending a cohomology class of f to the element (a,b) with $a=f(n,n^{-1})$ and $b=f(t,n)$, provided f is normalized as

$$f(n^i,n^j)=\left(\left[\frac{i+j}{k}\right]-\left[\frac{i}{k}\right]-\left[\frac{j}{k}\right]\right)f(n,n^{-1})\qquad (i,j\in\mathbb{Z}).$$

Here $\left(k,r-1,\sum_{l=0}^{m-1}r^l,\frac{r^m-1}{k}\right)$ and $(k,r-1)$ are the greatest common divisors of k, $r-1$, $\sum_{l=0}^{m-1}r^l$ and $\frac{r^m-1}{k}$; and k and $r-1$, respectively.

Proof. Let f be a normal 2-cocycle on G representing a cohomology class of $\tilde{H}^2(G,A)$. By Proposition 1, f is a 2-cocycle on N, hence by

Lemma 4, f can be normalized as follows:

$$f(n^i, n^j)=\left(\left[\frac{i+j}{k}\right]-\left[\frac{i}{k}\right]-\left[\frac{j}{k}\right]\right) f(n, n^{-1}) \qquad (i, j \in \mathbb{Z}). \tag{3}$$

But by Proposition 1,

$$f(n', n'')-f({}^{t'}n', {}^{t'}n'')=f(t', n')-f(t', n'n'')+f(t', n'') \qquad (n', n'' \in N, t' \in T),$$

hence by Lemma 6,

$$f(t, n^j)=\left(\left[\frac{jr}{k}\right]-\left[\frac{j}{k}\right]\right) f(n, n^{-1})+j f(t, n) \qquad (j \in \mathbb{Z}). \tag{4}$$

On the other hand by Proposition 1,

$$f(t' t'', n')=f(t'', n')+f(t', {}^{t''}n') \qquad (t', t'' \in T, n' \in N).$$

Then

$$f(t^i, n')=\sum_{l=0}^{i-1} f(t, {}^{t^l}n') \qquad (i \in \mathbb{Z}),$$

hence by (4),

$$f(t^i, n^j)=\left(\left[\frac{j r^i}{k}\right]-\left[\frac{j}{k}\right]\right) f(n, n^{-1})+j \sum_{l=0}^{i-1} r^l f(t, n) \qquad (i, j \in \mathbb{Z}). \tag{5}$$

Since $f(1, n')=f(t', 1)=0$ $(n' \in N, t' \in T)$, we have

$$(r-1) f(n, n^{-1})+k f(t, n)=0 \tag{6}$$

and

$$\frac{r^m-1}{k} f(n, n^{-1})+\sum_{l=0}^{m-1} r^l f(t, n)=0, \tag{7}$$

by setting $i=1, j=k$, and $i=m, j=1$ in (5), respectively. Moreover (6) and (7) can be unified to be a single equality

$$\left(k, r-1, \sum_{l=0}^{m-1} r^l, \frac{r^m-1}{k}\right)\left\{\frac{r-1}{(k, r-1)} f(n, n^{-1})+\frac{k}{(k, r-1)} f(t, n)\right\}=0, \tag{8}$$

which is (v) with $a=f(n, n^{-1})$ and $b=f(t, n)$.

Conversely if $f(n, n^{-1})$ and $f(t, n)$ satisfy (8), then $f(n^i, n^j)$ and $f(t^i, n^j)$ are well-defined by (3) and (5), respectively. Moreover they satisfy (b)$\sim$(e) in Proposition 1, hence determine a normal 2-cocycle on G with $f(T, T)=0$.

Let f be a normal 2-coboundary on G with $f(T, T)=0$, then there exists a map $v\colon G \to A$ with $v(1)=0$ such that

$$f(g, g')=v(g)-v(g g')+v(g') \qquad (g, g' \in G).$$

Then it is straightforward to see that

$$f(n, n^{-1}) = k v(n) \quad \text{and} \quad f(t, n) = (1-r)\, v(n),$$

which is (vi) with $a = f(n, n^{-1})$, $b = f(t, n)$ and $c = v(n)$.

Conversely if $f(n, n^{-1}) = kc$ and $f(t, n) = (1-r)c$ with $c \in A$, then f is a normal 2-cobundary on G with $f(T, T) = 0$. Q.E.D.

Let A be a $G_{k,m}$-module with trivial $G_{k,m}$-action. Then by Lemma 4, the canonical isomorphism

$$H^2(N, A)^T \cong {}_{(1-r)}(A/kA)$$

sends a cohomology class of f to $f(n, n^{-1})$ (mod ${}_kA$) with $(1-r) f(n, n^{-1}) + k u(t, n) = 0$, provided f is normalized as follows:

$$f(n^i, n^j) = \left(\left[\frac{i+j}{k}\right] - \left[\frac{i}{k}\right] - \left[\frac{j}{k}\right]\right) f(n, n^{-1}) \qquad (i, j \in \mathbb{Z}),$$

and

$$f(n', n'') - f({}^{t'}n', {}^{t'}n'') = u(t', n') - u(t', n'n'') + u(t', n'') \qquad (n', n'' \in N,\ t' \in T),$$

for a map $u\colon T \times N \to A$ with $u(1, n') = u(t', 1) = 0$ $(n' \in N,\ t' \in T)$.

Moreover by the same lemma, the canonical isomorphism

$$H^2(T, H^1(N, A)) \cong {}_{(r-1)}({}_kA) \Big/ \left(\sum_{l=0}^{m-1} r^l\right) {}_kA = {}_{(k,\, r-1)}A \Big/ \left(\sum_{l=0}^{m-1} r^l\right) {}_kA$$

sends a cohomology class of h to $h(t, t^{-1}, n)$ $\left(\text{mod } \left(\sum_{l=0}^{m-1} r^l\right) {}_kA\right)$, provided h is normalized as

$$h(t^i, t^j, n') = \left(\left[\frac{i+j}{m}\right] - \left[\frac{i}{m}\right] - \left[\frac{j}{m}\right]\right) h(t, t^{-1}, n') \qquad (i, j \in \mathbb{Z},\ n' \in N).$$

Theorem 8. *Let $G = G_{k,m}$ be as above, and let A be a G-module with trivial G-action. Then the homomorphism*

$$d_2\colon {}_{(1-r)}(A/kA) \to {}_{(k,\, r-1)}A \Big/ \left(\sum_{l=0}^{m-1} r^l\right) {}_kA$$

sends $a\, (\text{mod } kA)$ with $(1-r)\, a = kb$ to

$$\frac{r^m - 1}{k}\, a + \sum_{l=0}^{m-1} r^l\, b \left(\text{mod } \left(\sum_{l=0}^{m-1} r^l\right) {}_kA\right).$$

Proof. Let f, with $f(n^i, n^j) = \left(\left[\frac{i+j}{k}\right] - \left[\frac{i}{k}\right] - \left[\frac{j}{k}\right]\right) f(n, n^{-1})$ $(i, j \in \mathbb{Z})$, be a 2-cocycle on N representing a cohomology class of $H^2(N, A)^T$, then by Lemma 5, there exists a map $v\colon T \times N \to A$ with $v(1, n') = v(t', 1) = 0$

$(n' \in N, t' \in T)$ such that

$$v(t^i, n') = \sum_{l=0}^{i-1} v(t, {}^{t^l}n') \qquad (0 \leqq i < m, n' \in N),$$

$$f(n^i, n^j) - f({}^t n^i, {}^t n^j) = v(t, n^i) - v(t, n^{i+j}) + v(t, n^j) \qquad (i, j \in \mathbb{Z}),$$

and that the cohomology class of $d_2 f$ is

$$\sum_{l=0}^{m-1} v(t, {}^{t^l}n) \left(\bmod \left(\sum_{l=0}^{m-1} r^l\right) {}_k A\right).$$

On the other hand by Lemma 6,

$$v(t, n^j) = \left(\left[\frac{jr}{k}\right] - \left[\frac{j}{k}\right]\right) f(n, n^{-1}) + j\, v(t, n) \qquad (j \in \mathbb{Z}).$$

Hence

$$(1-r) f(n, n^{-1}) + k\, v(t, n) = 0,$$

and

$$\sum_{l=0}^{m-1} v(t, {}^{t^l}n) = \frac{r^m - 1}{k} f(n, n^{-1}) + \sum_{l=0}^{m-1} r^l v(t, n). \quad \text{Q.E.D.}$$

§ 2. An Application: Schur Multiplicator

As usual $\mathbb{Z}, \mathbb{Q}$ are the ring of integers, the field of rational numbers, respectively. Then

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \quad \text{(exact)}.$$

Let G be a finite group. Then

$$H^3(G, \mathbb{Z}) \cong H^2(G, \mathbb{Q}/\mathbb{Z}).$$

This group is called the Schur multiplicator of the group G.

Let G be a finite abelian group, then $H^3(G, \mathbb{Z})$ is trivial if and only if G is cyclic. We now compute $H^3(G, \mathbb{Z})$ when G is a semidirect product of its finite cyclic subgroups. Let $G_{k,m}$ be the group of order km generated by elements n, t with defining relations:

$$n^k = 1, \qquad t^m = 1, \qquad t n t^{-1} = n^r \quad \text{with } r^m \equiv 1 \pmod{k}.$$

Then $G_{k,m}$ is the semidirect product of the cyclic subgroups $N = \{n\}$ and $T = \{t\}$.

Proposition 9.

$$H^3(G_{k,m}, \mathbb{Z}) = \mathbb{Z} \Big/ \left(k, r-1, \sum_{l=0}^{m-1} r^l, \frac{r^m - 1}{k}\right) \mathbb{Z},$$

where $\left(k, r-1, \sum_{l=0}^{m-1} r^l, \frac{r^m-1}{k}\right)$ is the greatest common divisor of $k, r-1$, $\sum_{l=0}^{m-1} r^l$ and $\frac{r^m-1}{k}$.

Proof. Since the Schur multiplicator of a cyclic group is trivial, we have by Theorem 7

$$H^3(G_{k,m}, \mathbb{Z}) = H^1(T, H^1(N, \mathbb{Q}/\mathbb{Z})) = \mathbb{Z} \Big/ \left(k, r-1, \sum_{l=0}^{m-1} r^l, \frac{r^m-1}{k}\right) \mathbb{Z}. \quad \text{Q.E.D.}$$

We list below the Schur multiplicators of all non-abelian groups of order $\leqq 30$, but in this table there are some groups whose ones are computed by the well-known methods, e.g. the restriction map and the transfer map. We use the notation of Coxeter-Moser [2] (p. 134).

Order	Symbol	$H^3(G,\mathbb{Z})$	Order	Symbol	$H^3(G,\mathbb{Z})$
6	D_3	0	21	R''	0
8	D_4	$\mathbb{Z}/2\mathbb{Z}$	22	D_{11}	0
	$Q = \langle 2,2,2\rangle$	0	24	$\mathbb{Z}_2 \times A_4$	$(\mathbb{Z}/2\mathbb{Z})^2$
10	D_5	0		$\mathbb{Z}_2 \times D_6$	$(\mathbb{Z}/2\mathbb{Z})^3$
12	D_6	$\mathbb{Z}/2\mathbb{Z}$		$\mathbb{Z}_3 \times D_4$	$\mathbb{Z}/2\mathbb{Z}$
	A_4	$\mathbb{Z}/2\mathbb{Z}$		$\mathbb{Z}_3 \times Q$	0
	$\langle 2,2,3\rangle$	0		$\mathbb{Z}_4 \times D_3$	$\mathbb{Z}/2\mathbb{Z}$
14	D_7	0		$\mathbb{Z}_2 \times \langle 2,2,3\rangle$	$\mathbb{Z}/2\mathbb{Z}$
16	$\mathbb{Z}_2 \times D_4$	$(\mathbb{Z}/2\mathbb{Z})^3$		D_{12}	$\mathbb{Z}/2\mathbb{Z}$
	$\mathbb{Z}_2 \times Q$	$(\mathbb{Z}/2\mathbb{Z})^2$		S_4	$\mathbb{Z}/2\mathbb{Z}$
	D_8	$\mathbb{Z}/2\mathbb{Z}$		$\langle 2,3,3\rangle$	0
	$\langle -2,4\mid 2\rangle$	0		$(4,6\mid 2,2)$	$\mathbb{Z}/2\mathbb{Z}$
	$\langle 2,2\mid 2\rangle$	0		$\langle -2,2,3\rangle$	0
	$\langle 2,2\mid 4,2\rangle$	$\mathbb{Z}/2\mathbb{Z}$		$\langle 2,2,6\rangle$	0
	$(4,4\mid 2,2)$	$(\mathbb{Z}/2\mathbb{Z})^2$	26	D_{13}	0
	R	$(\mathbb{Z}/2\mathbb{Z})^2$	27	$(3,3\mid 3,3)$	$(\mathbb{Z}/3\mathbb{Z})^2$
	$\langle 2,2,4\rangle$	0		R'''	0
18	$\mathbb{Z}_3 \times D_3$	0	28	D_{14}	$\mathbb{Z}/2\mathbb{Z}$
	D_9	0		$\langle 2,2,7\rangle$	0
	$(3,3,3:2)$	$\mathbb{Z}/3\mathbb{Z}$			
20	D_{10}	$\mathbb{Z}/2\mathbb{Z}$	30	$\mathbb{Z}_3 \times D_5$	0
	R'	0		$\mathbb{Z}_5 \times D_3$	0
	$\langle 2,2,5\rangle$	0		D_{15}	0

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