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Note

Two-layered model of blood flow through stenosed arteries

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Summary. The present investigation deals with a two-layered mathematical model of blood flow through an artery provided with a cosine-shaped constriction. The model consists of a peripheral plasma layer free from red cells and a core region represented by a Casson fluid. The geometry of the interface between the plasma layer and the core region has been determined and compared with that of the constriction along the length of the tube. The theoretical results obtained in this analysis are the expressions for wall shear stress and pressure drop for variable plasma layer thickness. The effect of the variable plasma layer thickness on the flow characteristics has been shown graphically for different parameter values to enable a better understanding of the biomechanical problem.

1 Introduction

The presence of a constriction (medically called stenosis) in the lumen of an artery disturbs the normal blood flow and causes arterial diseases. About fifty percent of the total deaths of people are due to these diseases. The actual reason for formation of stenosis in the artery is not known; but its effect on the flow field in the tube has been studied by several workers. Many researchers have studied blood flow in the artery by considering blood as either Newtonian or non-Newtonian fluids. Since blood is a suspension of red cells in plasma, it behaves as a non-Newtonian fluid at low shear rate and the yield stress is non-zero at that stage. Bugliarello and Sevilla [1] and Scott Blair and Spanner [2] have pointed out that blood exhibits the nature of a Casson fluid in pathological conditions. So it is better to consider blood as Casson fluid in studying the flow field in stenosed arteries.

Two-layered models of blood flow through stenosed arteries have been studied by Shukla et al. [3], [4]. The two-layered models considered in their analysis consist of a cell-free peripheral plasma layer and a core region which is a suspension of red cells in plasma. They have considered blood as being Newtonian or non-Newtonian fluids in both models. However, it has been experimentally verified that blood flowing through an artery consists of a peripheral layer of Newtonian fluid and a core of non-Newtonian fluid. So the models considered in [3], [4] are unrealistic. Chaturani and Samy [5] have taken a two-layered model in which the peripheral plasma layer is Newtonian in character and the central core of red cell suspension in plasma is represented by a Casson fluid.

In the above-mentioned models the thickness of the peripheral plasma layer has been treated as constant. But actually it cannot be constant; it should change along the length of the stenosis. In the present investigation a two-layered model is considered, in which the peripheral plasma layer behaves as Newtonian fluid and the core is represented by a Casson fluid with the supposition that the two fluids are immiscible. The effect of variable thickness of the peripheral layer on the flow characteristics of the problem is investigated for different combinations of the parameter values.

2 Mathematical model

Consider steady, laminar, one-dimensional, axially and fully developed flow of blood through an artery provided with a mild constriction. The constriction develops in the lumen of an artery in an axially symmetric manner and its height depends on the axial distance. The model considered here consists of two regions of which one region is a pheripheral plasma layer free from red cells and the other is a central core which is represented by a Casson fluid. It is also supposed that the fluids in both regions are immiscible.

If u_1 and u_2 are the axial velocities in the two regions, then the equations which govern the fluid motions in the plasma and core regions are respectively

$$\frac{du_1}{dr} = -\frac{Kr}{2\mu_1}, \quad R(x) \geqq r \geqq R_1(x) \tag{1}$$

$$-\frac{du_2}{dr} = \frac{1}{\mu_2}\left[\left(\frac{1}{2}Kr\right)^{1/2} - \tau_0^{1/2}\right]^2, \quad R_1(x) \geqq r \geqq r_0 \tag{2}$$

where

$$K = -\frac{dp}{dx}, \qquad \tau_0 = \frac{1}{2}Kr_0. \tag{3}$$

Here, τ_0 is the yield stress, r_0 is the radius of the plug and $R_1(x)$ is the radius of the interface separating the two media, as shown schematically in Fig. 1. The quantities μ_1 and μ_2 are coefficients of viscosities in the respective regions and p is the fluid pressure.

It should be emphasized that Eqs. (1) and (2) ensure that the shear stress varies linearly with r over the entire crosssection. The continuity of the shear stress across the interface $r = R_1$ is thereby automatically assured. Since the shear stress is less than the yield stress τ_0 in the region $0 \leqq r \leqq r_0$, the fluid in this region is merely carried along by the fluid in the annular region $R_1(x) \geqq r \geqq r_0$ in which the shear stress does exceed the yield stress. Equation (2) is thus valid only in the latter region.

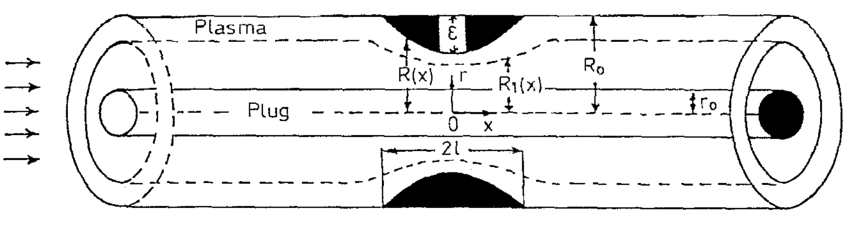

Fig. 1. Geometry of the model

The profile of constriction is described by

$$\frac{R(x)}{R_0} = 1 - \frac{\varepsilon}{2R_0}\left(1 + \cos\frac{\pi x}{l}\right), \qquad -l \leqq x \leqq l \tag{4}$$

where $R(x)$ is the radius of the tube in the constricted region, R_0 is the radius of normal tube, ε is the maximum height of the constriction and $2l$ is the length. The boundary conditions for Eqs. (1) and (2) are the no-slip conditions

$$u_1 = 0 \quad \text{at} \quad r = R(x) \tag{5}$$

$$u_1 = u_2 \quad \text{at} \quad r = R_1(x) \tag{6}$$

at the tube wall and the interface separating the two media, respectively.

3 Solutions of the problem

The solutions of the problem described by Eqs. (1) and (2) subject to the boundary conditions (5) and (6) are obtained simultaneously. They are given by

$$u_1 = \frac{K}{4\mu_1}(R^2 - r^2) \tag{7}$$

and

$$u_2 = \frac{K}{2\mu_2}\left[\frac{4}{3}\sqrt{r_0}\,(r^{3/2} - R_1^{3/2}) - \frac{1}{2}(r^2 - R_1^2) - r_0(r - R_1)\right] + \frac{K}{4\mu_1}(R^2 - R_1^2). \tag{8}$$

Let U be the velocity of the plug. Then the expression for U is obtained from (8) with $r = r_0$ as

$$U = \frac{K}{2\mu_2}\left[\frac{1}{2}R_1^2 + r_0 R_1 - \frac{4}{3}\sqrt{r_0}\,R_1^{3/2} - \frac{1}{6}r_0^2\right] + \frac{K}{4\mu_1}(R^2 - R_1^2). \tag{9}$$

The volumetric flow rates Q_1 and Q_2 in the cell-free plasma layer and core region are given by

$$Q_1 = 2\pi \int_{R_1(x)}^{R(x)} u_1 r\, dr \tag{10}$$

and

$$Q_2 = \pi r_0^2 U + 2\pi \int_{r_0}^{R_1(x)} u_2 r\, dr. \tag{11}$$

Substitutions of u_1 and u_2 from (7) and (8) into (10) and (11), respectively, give, after integration of the expressions for Q_1 and Q_2:

$$Q_1 = \frac{\pi K r_0^4}{8\mu_1}(y^4 - f^2)^2 \tag{12}$$

and

$$Q_2 = \frac{\pi K {r_0}^4}{168\mu_1} [21(\alpha - 2)\, y^8 + 42f^2y^4 - \alpha(48y^7 - 28y^6 + 1)] \tag{13}$$

where

$$\begin{aligned} y^2 &= R_1/r_0 \\ \alpha &= \mu_1/\mu_2 \\ f &= R/r_0 \end{aligned} \tag{14}$$

are dimensionless quantities. Eliminating the pressure gradient $K = -(dp/dx)$ between (12) and (13), one gets

$$\frac{Q_1}{Q_2} = 21(y^4 - f^2)^2/[21(\alpha - 2)\, y^8 + 42f^2y^4 - \alpha(48y^7 - 28y^6 + 1)]. \tag{15}$$

In the case of no stenosis $R = R_0, R_1 = R_0 - h, h$ being the thickness of the plasma layer, and the above relation becomes

$$\frac{Q_1}{Q_2} = 21(\bar{y}^4 - \bar{f}^2)^2/21(\alpha - 2)\, \bar{y}^8 + 42\bar{f}^2\bar{y}^4 - \alpha(48\bar{y}^7 - 28\bar{y}^6 + 1)] \tag{16}$$

where

$$\begin{aligned} \bar{y}^2 &= (1 - \delta)/\beta, \qquad \delta = h/R_0 \\ \beta &= r_0/R_0, \qquad \beta\bar{f} = 1. \end{aligned} \tag{17}$$

Since the flow is steady and there is no transport of fluid from one medium to the other through the interface, therefore, the flow rates Q_1 and Q_2 are constants and hence Q_1/Q_2 is also constant. This ratio maintains the same value in both stenotic and non-stenotic regions of a closed system. Hence we have on equating (15) and (16) that

$$y^8 - \delta_1 y^7 + d_2 y^6 + \delta_3 y^4 - \delta_4 = 0 \tag{18}$$

where

$$\begin{aligned} \delta_1 &= \frac{48\alpha m}{21m(\alpha - 2) - 1}, \qquad \delta_2 = \frac{7}{12}\,\delta_1 \\ \delta_3 &= \frac{(21m + 1)\, f^2}{24\alpha m}\,\delta_1, \qquad \delta_4 = \frac{\alpha m + f^4}{48\alpha m}\,\delta_1 \\ m &= (\bar{y}^4 - f^2)^2/[21(\alpha - 2)\, \bar{y}^8 + 42\bar{f}^2\bar{y}^4 - \alpha(48\bar{y}^7 - 28\bar{y}^6 + 1)]. \end{aligned} \tag{19}$$

4 Wall friction and pressure drop

The shear stress on the surface of the stenosis is given by

$$\tau_1 = -\mu_1 \left(\frac{du_1}{dr}\right)_{r = R(x)}. \tag{20}$$

Substituting u_1 from (7) into (20) we have the expression for τ_1 as

$$\tau_1 = -\frac{1}{2} R \frac{dp}{dx}. \tag{21}$$

By eliminating the pressure gradient dp/dx between (17) and (21) we obtain the following form of wall shear stress:

$$\tau_1 = \frac{4\mu_1 Q_1}{\pi} \cdot \frac{R}{(R^2 - R_1{}^2)^2}. \tag{22}$$

If the thickness h of the plasma layer is assumed to be constant, the relation (22) takes the form

$$\tau_h = \frac{4\mu_1 Q_1}{\pi} \cdot \frac{R}{h^2(2R - h)^2} \tag{23}$$

which in the absence of stenosis becomes

$$\tau_{10} = \frac{4\mu_1 Q_1}{\pi} \cdot \frac{R_0}{h^2(2R_0 - h)^2}. \tag{24}$$

Therefore, the non-dimensional form of shear stress on the surface of the stenosis for variable plasma thickness is

$$\tau = \frac{\tau_1}{\tau_{10}} = \frac{f}{\beta^3} \left[\frac{\delta(2 - \delta)}{f^2 - y^4}\right]^2 \tag{25}$$

where $\delta = h/R_0$.
Again, solving for (dp/dx) we have from (12)

$$-\frac{dp}{dx} = \frac{8\mu_1 Q_1}{\pi} \cdot \frac{1}{(R^2 - R_1{}^2)^2} \tag{26}$$

which, on integration, gives the expression for the pressure drop across the length of the stenosis as

$$\Delta p = p_1 - p_2 = \frac{8\mu_1 Q_1}{\pi} \int_{-l}^{l} \frac{dx}{(R^2 - R_1{}^2)^2} \tag{27}$$

where $p = p_1$ at $x = -l$ and $p = p_2$ at $x = l$.

In the case of constant thickness of the plasma layer relation (27) becomes

$$(\Delta p)_c = \frac{8\mu_1 Q_1}{\pi R_0{}^4 \delta^2} \int_{-l}^{l} \frac{dx}{\left[\frac{2R}{R_0} - \delta\right]^2}. \tag{28}$$

When there is no stenosis, $(R/R_0) = 1$ and the pressure drop across the stenosis length is

$$(\Delta p)_0 = \frac{16\mu_1 Q_1 l}{\pi R_0{}^4 \delta^2 (2 - \delta)^2}\,. \tag{29}$$

Then the dimensionless pressure drop D_1 for variable peripheral layer is

$$D_1 = \frac{(\Delta p)}{(\Delta p_0)} = \frac{\delta^2(2-\delta)^2}{2l\beta^4} \int_{-l}^{l} \frac{dx}{[f^2 - y^4]^2}. \tag{30}$$

5 Numerical results and discussion

The solution provided in Section 3 depends on the dimensionless axial position x/l and the four independent dimensionless parameters $\alpha = \mu_1/\mu_2$, $\beta = r_0/R_0$, $\delta = h/R_0$, and ε/R_0. The auxiliary variable $y^2 = R_1/r_0$, which is governed by the algebraic equation (18), should be confined to the interval from 1.0 to $R/\beta R_0$ in order to assure that the radial extent R_1 of the Casson fluid does not exceed the local radius $R(x)$ of the constriction and yet is greater than the radius r_0 of the plug. The relevant root of Eq. (18) was calculated by means of the "secant method" for 20 equidistant values of x/l in the interval $[-1, 0]$, thereby providing the variation of R_1 and $h = R - R_1$ along the constriction. The integral in Eq. (30) was subsequently obtained by the "trapezoid rule". The viscosity ratio α was 0.6 and the relative thickness δ of the plasma layer upstream of the stenosis was 0.01 in all calculations.

Tables 1, 2 and 3 give numerical results for the position of the interface $R_1(x)$ and the local thickness $h(x) = R(x) - R_1(x)$ of the plasma layer in the range $-1 \leqq x/l \leqq 0$. Due to the symmetry of the problem about $x = 0$ the range $0 \leqq x/l \leqq +1$ was not considered. The tables show that the plasma layer thickness varies along the length of the tube for a particular value of β and attains its minimum value at the throat of the stenosis. It is also seen that h decreases with increasing core radius β for a fixed value of x/l. Moreover, for a given plug width β the plasma layer is substantially thicker for the mild stenosis $\varepsilon/R_0 = 0.1$ than in the case of a severe stenosis $\varepsilon/R_0 = 0.5$.

The variations of the wall shear stress τ and the pressure drop D_1 with stenosis height are shown graphically for different values of β (Figs. 2 and 3) and they are plotted in the same scale. The figures show that there are initially no appreciable changes in the solutions over a small range of ε/R_0. As ε/R_0 increases beyond this range the solutions begin to turn upwards and at higher values of ε/R_0 these solutions increase rapidly. Is is noteworthy that the thinning of the Newtonian plasma layer with increasing height ε of the stenosis enhances the wall friction and the associated pressure drop.

Table 1. ($\beta = 0.1$, $\varepsilon/R_0 = 0.1$)

x/l	R/R_0	R_1/R_0	$R/R_0 - R_1/R_0$
−1.0	1.00000	0.99000	0.01000
−0.9	0.99755	0.98758	0.00997
−0.8	0.99045	0.98057	0.00988
−0.7	0.97939	0.96965	0.00974
−0.6	0.96545	0.95588	0.00957
−0.5	0.95000	0.94063	0.00937
−0.4	0.93455	0.92537	0.00918
−0.3	0.92061	0.91160	0.00901
−0.2	0.90955	0.90068	0.00887
−0.1	0.90245	0.89366	0.00878
0.0	0.90000	0.89125	0.00875

Table 2. ($\beta = 0.5$, $\varepsilon/R_0 = 0.1$)

x/l	R/R_0	R_1/R_0	$R/R_0 - R_1/R_0$
−1.0	1.00000	0.99000	0.01000
−0.9	0.99755	0.98760	0.00995
−0.8	0.99045	0.98064	0.00981
−0.7	0.97939	0.96978	0.00960
−0.6	0.96545	0.95611	0.00934
−0.5	0.95000	0.94095	0.00905
−0.4	0.93455	0.92579	0.00876
−0.3	0.92061	0.91211	0.00850
−0.2	0.90955	0.90125	0.00830
−0.1	0.90245	0.89427	0.00817
0.0	0.90000	0.89187	0.00813

Table 3. ($\beta = 0.1$, $\varepsilon/R_0 = 0.5$)

x/l	R/R_0	R_1/R_0	$R/R_0 - R_1/R_0$
−1.0	1.00000	0.99000	0.01000
−0.9	0.98775	0.97792	0.00985
−0.8	0.95225	0.942 85	0.00940
−0.7	0.89695	0.88823	0.00871
−0.6	0.82725	0.81940	0.00785
−0.5	0.75000	0.74309	0.00691
−0.4	0.67275	0.66677	0.00598
−0.3	0.60305	0.59790	0.00515
−0.2	0.54775	0.54324	0.00450
−0.1	0.51224	0.50814	0.00409
0.0	0.50000	0.49605	0.00395

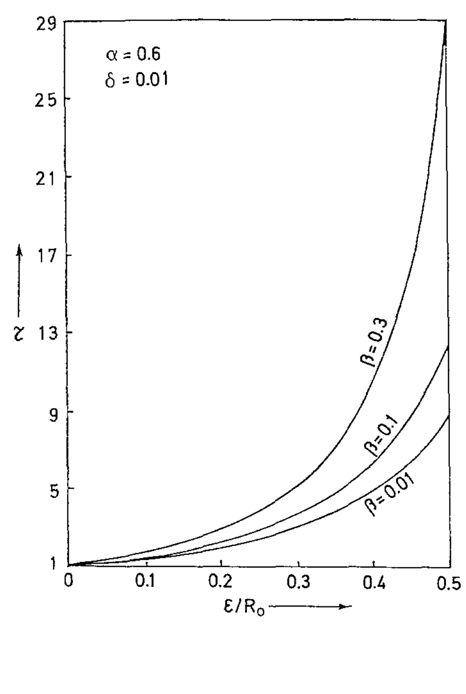

Fig. 2. Variations of τ with ε/R_0 for different values of β

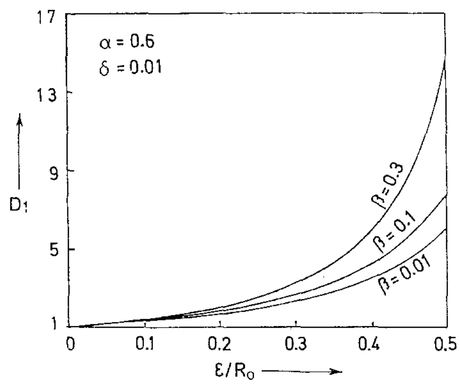

Fig. 3. Variations of D_1 with ε/R_0 for different values of β

It is also observed that for a particular value of ε/R_0 the wall shear stress or pressure drop increases with increasing β and the deviation between results for any two consecutive β-values is significant. It is also interesting to note that the wall shear increases more rapidly than the pressure drop.

From the above discussions the following conclusions can be drawn:

(i) The plasma layer thickness varies along the length of the tube of stenosis and attains its minimum value at the throat of stenosis.

(ii) The wall shear stress and pressure drop increase with increasing plug radius and the former increases more rapidly than the latter.

These conclusions are very important from the physiological point of view.

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