The Perron-Stieltjes Integral.

By

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Considerable attention has been given to the question of integration with respect to a function; various authors have shown how to define an integral which generalises the Stieltjes integral in the same way as Lebesgue generalised the ordinary integral. The Lebesgue-Stieltjes integral, however, is by its very nature restricted to the case in which the integrating function is of bounded variation, while the ordinary Stieltjes integral has no such limitation. An integral which includes both these (Riemann-Stieltjes and Lebesgue-Stieltjes) is therefore not without interest. Such an integral may be obtained by applying the methods of Perron [1]). It is interesting also on account of its close connection with differentiation.

In the first section of this paper I give definitions of differentiation with respect to a function. It is shown that the theorem, due to Denjoy [2]), which governs the possible values of the derivates of a function, can be extended to the case when the base-function is VBG^* on the set of points considered [3]). It seems likely that the class of functions VBG^* on a given set of points is the most general class for which it is profitable to define derivates, and it is noteworthy that the same class of functions proves to be of importance for the theory of the Perron-Stieltjes integral. The second chapter is devoted to the definition and simple properties of the integral. Finally, I consider the case of a base-function which is VBG^*, and give the analogue of the theorem which states that every function ACG^* in an interval is the Perron integral of its derivative, and conversely [4]).

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[1]) Other authors have defined a Perron-Stieltjes integral with respect to functions of bounded variation; see R. L. Jeffery, *Trans. American Math. Soc.* 34 (1932), S. 645; J. Ridder, *Math. Zeitschr.* 40 (1935), S. 127. The definitions employed are not applicable to general base-functions. (The present work was carried out independently of the work of Ridder, and in some places conventions are used which are different from his.)

[2]) A. Denjoy, *Journal de Math.* (1915) S. 105—240, especially S. 190—192.

[3]) S. Saks, *Théorie de l'Intégrale* (Warsaw 1933), S. 158 ff. The definitions are reproduced here for completeness. See Nr. 1.

[4]) Saks, loc. cit. S. 198, 216.

By the kindness of Mr. L. C. Young, I have been able to compare this work with some (unpublished) work of his in which the ideas of the Denjoy integral are applied to Stieltjes integration. It is not surprising that the theorems proved for the 'Perron-Stieltjes' integral are often very similar to those which hold for Young's 'Denjoy-Stieltjes' integral, although the methods of attack are quite different[5]). (In some cases, however, we found that we had carried out similar but independent work.) He suggested that it might be possible to define $(PS)\int(f_1\,d\,\varphi_1 + f_2\,d\,\varphi_2)$ and to prove Theorem 14, which he had proved for Denjoy-Stieltjes integrals. This definition and this theorem have been added in accordance with his suggestions.

1. Unless otherwise stated, all functions mentioned are supposed to be real and finite, and defined for all values of x in a fixed interval $a \leqq x \leqq b$. The 'base-function' with respect to which we differentiate or integrate is usually denoted by $\varphi(x)$ or $\psi(x)$, and other functions by $f(x)$, $g(x)$ and so on. Following Saks, we denote by $\omega(\varphi; E)$ the oscillation of $\varphi(x)$ on a set E. Let now (I_k), $k = 1, 2, \ldots, n$, be any finite system of non-overlapping closed intervals whose end-points lie in E; form the sum $\sum_{k=1}^{n} \omega(\varphi; I_k)$. The upper bound of all such sums is denoted by $V^*(\varphi, E)$: if it is finite we say that $\varphi(x)$ is VB^* on E. Finally, $\varphi(x)$ is said to be VBG^* on E if E can be expressed as a finite or enumerable sum of sets E_n on each of which φ is VB^*.

Given two functions $f(x)$, $\varphi(x)$, we say that $f(x)$ is *AC with respect to* $\varphi(x)$ on a set E, if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, if (I_k) is any finite set of non-overlapping closed intervals whose end-points lie in E, then

$$\sum_k |f(y_k) - f(x_k)| < \varepsilon \quad \text{whenever} \quad \sum_k \omega(\varphi, I_k) < \delta,$$

where x_k, y_k are the end-points of I_k. In the same way, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_k \omega(f, I_k) < \varepsilon \quad \text{whenever} \quad \sum_k \omega(\varphi, I_k) < \delta,$$

the intervals (I_k) fulfilling the same conditions as before, we say that $f(x)$ is *AC^* with respect to* $\varphi(x)$ on E. If a given set E can be expressed as a finite or enumerable sum of sets E_n, on each of which $f(x)$ is AC^* with respect to $\varphi(x)$, we say that $f(x)$ is *ACG^* with respect to* $\varphi(x)$ on E.

[5]) In fact, the final theorems, 12 and 13, are not far from a demonstration that the Perron-Stieltjes integral is equivalent to a Denjoy-Stieltjes integral.

1. 1. Differentiation with respect to a function.

Let $f(x)$, $\varphi(x)$ be any two functions, of which $\varphi(x)$ is defined at all points of the interval $a \leqq x \leqq b$, but $f(x)$ may be undefined at some points. We define the upper right-hand derivate of $f(x)$ with respect to $\varphi(x)$ as

$$D^+(f, x;\ \varphi) = \overline{\lim_{h \to +0}} \frac{f(x+h) - f(x)}{\varphi(x+h) - \varphi(x)}$$

with the conventions that if the quotient takes the form $0/0$ it is not counted in evaluating the limit, and that, if $a > 0$, then $a/0 = +\infty$ and $-a/0 = -\infty$. We define the three other derivates similarly, and if they all exist and are equal we write the common value as $D(f, x;\ \varphi)$. By $df(x)/d\varphi(x)$, however, we denote a rather more general conception, which we may call the Roussel derivative[6]). We say that $f(x)$ is *continuous with respect to* $\varphi(x)$ at the point x, if for some number k

$$\lim_{h \to 0} [f(x+h) - f(x) - k\{\varphi(x+h) - \varphi(x)\}] = 0.$$

If in addition[7]) we have

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - k\{\varphi(x+h) - \varphi(x)\}}{\omega(\varphi;\ x, h)} = 0$$

(where $\omega(\varphi;\ x, h)$ is the oscillation of φ in the closed interval $\langle x, x+h\rangle$ or $\langle x+h, x\rangle$, according to the sign of h), then we write[8])

$$df(x)/d\varphi(x) = k.$$

We take as our starting-point the following theorem, which is a slight generalisation of the Denjoy results.

Theorem 1[9]). *If $\varphi(x)$ is strictly monotone, and $f(x)$ is defined in any set H, then all the points of H fall into one of three sets:*

(i) *a set E_0 where $D^+(f, x;\ \varphi) = D_-(f, x;\ \varphi)$, finite;*

(ii) *a set E_∞ where $D^+(f, x;\ \varphi) = +\infty$, $D_-(f, x, \varphi) = -\infty$;*

(iii) *a set N such that $m\,\varphi(N) = 0$*[10]).

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[6]) Roussel, *Comptes rendus* (Paris), 187, S. 926 (19. Nov. 1928).

[7]) This second condition does not include the continuity condition, as it is possible that $\varphi(x)$ may oscillate infinitely in the neighbourhood of the point x.

[8]) It is easily seen that if φ is bounded in the neighbourhood of x, then the existence of a finite $D(f, x;\ \varphi)$ involves the existence (and equality) of $df(x)/d\varphi(x)$, but not conversely.

[9]) A theorem equivalent to this is proved by Ridder (loc. cit. 134) in the more general case when $\varphi(x)$ is VB [but not when $\varphi(x)$ is VBG^*]. His conventions in the definition of the derivates with respect to $\varphi(x)$ are different from ours; also it should be observed that $m\,\varphi(E)$ is not to be confused with „the φ-measure of E", which cannot in general be defined unless $\varphi(x)$ is VB.

[10]) $\varphi(N)$ denotes the set of values assumed by $\varphi(x)$ for those values of x which lie in N.

To prove this it is only necessary to express $f(x)$ as a function of φ and apply the Denjoy theorems; the arguments involved are trivial. We shall show that the same theorem is true for any $\varphi(x)$ which is VBG^* on H. It will then be shown that if $f(x)$ itself is VBG^*, the set E_∞ can be included in the set N; that is, $m\varphi(E_\infty) = 0$.

1.2. Lemma 1. *If $\varphi(x)$ is VB^* on a set E, it is VB^* on the closed set $E + E'$* [11]).

Lemma 2. *If $\varphi(x)$ is VB^* on a set E, then there exists a strictly increasing function $\chi(x)$ such that all four derivates satisfy $|D^{\pm}_{\pm}(\varphi, x; \chi)| \leqq 1$ at each point of E, except possibly the extreme points.*

By lemma 1, we may suppose E closed; let c, d be its extreme points. Write

$$(1) \qquad \chi(y) = y + V^*(\varphi; [E\langle a, y\rangle + \langle y\rangle]) - V^*(\varphi; [E\langle y, b\rangle + \langle y\rangle]).$$

$\chi(y)$ is obviously strictly increasing, and it is easy to see that for any x of E and any y such that $c \leqq y \leqq d$ we have

$$(2) \qquad |\chi(y) - \chi(x)| \geqq |\varphi(y) - \varphi(x)|.$$

Hence the result follows.

Lemma 3. *If $\chi(x)$ is strictly increasing, and all four derivates of a function $\varphi(x)$ satisfy $|D^{\pm}_{\pm}(\varphi, x; \chi)| \leqq k$ at each point of a set E, then $m_e\varphi(E) \leqq k\, m_e\chi(E)$.*

Given $\varepsilon > 0$, we can find, for any point x of E, a number $h = h(x, \varepsilon)$ so small that

$$(3) \qquad |\varphi(x) - \varphi(\xi)| < (k + \varepsilon)|\chi(x) - \chi(\xi)|$$

whenever $|\chi(x) - \chi(\xi)| < h$ [12]). We can therefore find h_0, depending only on ε, so small that equation (3) is satisfied whenever

$$|\chi(x) - \chi(\xi)| < h_0,$$

if x belongs to a set $E_0 \subset E$ such that $m_e\varphi(E_0) > m_e\varphi(E) - \varepsilon$.

Let U be an open set including $\chi(E)$ and such that $mU < m_e\chi(E) + \varepsilon$; then U can be expressed as a sum of open intervals I_n, possibly overlapping, such that $mI_n < h_0$ for each n, and $\sum_n mI_n < m_e\chi(E) + 2\varepsilon$. Let J_n be the set of points x for which $\chi(x)$ lies in I_n; if x_0, x_1 are any two points of $E_0 J_n$ we clearly have $|\chi(x_0) - \chi(x_1)| < h_0$ and so $|\varphi(x_0) - \varphi(x_1)| < (k + \varepsilon)|\chi(x_0) - \chi(x_1)|$. It follows that $m_e\varphi(E_0 J_n)$

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[11]) Saks, *loc. cit.* 159.

[12]) Since $\chi(x)$ is *strictly* increasing.

$\leqq (k+\varepsilon)\, m\, I_n$. Since E_0 is a sub-set of E, every point of E_0 lies in some set J_n, and therefore

$$\begin{aligned} m_e\,\varphi(E_0) &\leqq \sum_n m_e\,\varphi(E_0 J_n) \\ &\leqq (k+\varepsilon) \sum_n m\, I_n \\ &\leqq (k+\varepsilon)\,\{m_e\,\chi(E) + 2\,\varepsilon\}. \end{aligned}$$

Hence

$$m_e\,\varphi(E) \leqq (k+\varepsilon)\,\{m_e\,\chi(E) + 2\,\varepsilon\} + \varepsilon.$$

Since ε is arbitrarily small, the result follows.

Theorem 2. *If $f(x)$ is defined on a set H, and $\varphi(x)$ is VBG^* on a set $E \subset H$, then all the points of E fall into one of three sets:*

(i) *a set E_0 where $D^+(f, x; \varphi) = D_-(f, x; \varphi)$, finite;*

(ii) *a set E_∞ where $D^+(f, x; \varphi) = +\infty$, $D_-(f, x; \varphi) = -\infty$;*

(iii) *a set N such that $m\,\varphi(N) = 0$.*

Express E as $\sum_{n=1} E_n$ where $\varphi(x)$ is VB^* on each E_n. It is clearly sufficient to prove that the set $N E_n$, which we can express as $E_n - E_\infty - E_0$, satisfies $m\,\varphi\,(N E_n) = 0$ for each n. Write $F_n = E_n + E_n'$ and let c_n, d_n be the extreme points of F_n. By lemmas 1 and 2, we can define $\chi_n(x)$ such that the derivates of φ satisfy $|D^{\pm}_{\pm}(\varphi, x; \chi_n)| \leqq 1$ at each point of F_n except perhaps c_n and d_n. Apply theorem 1 to the functions φ and χ_n; the set can be divided as follows[13]):

A set G_1 where $D(\varphi, x; \chi_n) = 0$;

a set G_2 where $D(\varphi, x; \chi_n)$ exists and is not zero;

a set N_n such that $m\,\chi_n(N_n) = 0$.

The set can, however, also be divided with respect to $f(x)$ and $\chi_n(x)$, as follows:

A set H_1 where $D(f, x; \chi_n)$ exists;

a set H_2 where

$$D^+(f, x; \chi_n) = D_-(f, x; \chi_n), \text{ finite};$$
$$D_+(f, x; \chi_n) = -\infty,\ D^-(f, x; \chi_n) = +\infty;$$

a set H_3 where

$$D_+(f, x; \chi_n) = D^-(f, x; \chi_n), \text{ finite};$$
$$D^+(f, x; \chi_n) = +\infty,\ D_-(f, x; \chi_n) = -\infty;$$

[13]) With each theorem such as theorem 1 there is of course associated a similar theorem relating to the other two derivates, which we assume without further proof. It follows that if all the derivates are finite, they are equal to each other, except on a set satisfying $m\,\chi_n(N_n^-) = 0$.

a set H_4 where

$$D^+(f, x; \chi_n) = D^-(f, x; \chi_n) = +\infty;$$
$$D_+(f, x; \chi_n) = D_-(f, x; \chi_n) = -\infty;$$

a set H_5 such that $m\chi_n(H_5) = 0$.

Now it is clear that all points of $G_2(H_1 + H_2 + H_3 + H_4)$ fall into one of the sets E_0, E_∞. (It may be necessary to consider the sign of $D(\varphi, x; \chi_n)$; for example, for points of the set $G_2 H_2$.) Hence $N E_n \subset H_5 + N_n + G_1$. Apply lemma 3 to the set $H_5 + N_n$ with $k = 1$ and to the set G_1 with $k = 0$; we have $m\varphi(H_5 + N_n + G_1) = 0$ and so $m\varphi(N E_n) = 0$. This proves the theorem.

1.3. Lemma 4. *If $f(x)$ is ACG^* with respect to $\varphi(x)$ on a set E, on which $\varphi(x)$ is VBG^*, then $f(x)$ is VBG^* on E.*

We can split E into at most $\aleph_0$ sets E_n on each of which $\varphi(x)$ is VB^* and $f(x)$ is AC^* with respect to $\varphi(x)$. Define the function $\chi_n(x)$ corresponding to the function $\varphi(x)$ and the set $F_n = E_n + E_n'$, as in lemma 2. Since $f(x)$ is AC^* with respect to $\varphi(x)$ on E_n, we can find $\delta > 0$ such that, for any finite set of intervals with end-points in E_n,

$$\sum_k \omega(f, I_k) < 1 \quad \text{if} \quad \sum_k \omega(\varphi, I_k) < \delta. \tag{4}$$

Consider the sub-set E_{np} of E_n for which $p\delta \leqq \chi_n(x) < (p+1)\delta$. If x_1 and x_2 $(x_1 < x_2)$ are any two points of E_{np}, we have from (1),

$$\omega(\varphi; \langle x_1, x_2\rangle) \leqq V^*(\varphi, \langle x_1, x_2\rangle) \leqq \chi_n(x_2) - \chi_n(x_1).$$

Thus for any set of intervals I_k with end-points in E_{np} we have

$$\Sigma\,\omega(\varphi, I_k) \leqq \Sigma\,\omega(\chi_n, I_k) < \delta,$$

and so from (4), $\Sigma\,\omega(f, I_k) < 1$. That is, $f(x)$ is VB^* on E_{np}; it follows that $f(x)$ is VBG^* on E.

Theorem 3. *If $f(x)$ and $\varphi(x)$ are both VBG^* on a set E, then $D(f, x; \varphi)$ exists, and is finite and equal to $df(x)/d\varphi(x)$, everywhere in E except possibly in a set N such that $m\varphi(N) = 0$.*

Corollary. *If, on E, $\varphi(x)$ is VBG^* and $f(x)$ ACG^* with respect to $\varphi(x)$, then a finite $df(x)/d\varphi(x)$ exists everywhere in E except for a set N such that $m\varphi(N) = 0$.*

We can express E as the sum of sets E_n on each of which both $f(x)$ and $\varphi(x)$ are VB^*. By constructing functions as in lemma 2 for both $f(x)$ and $\varphi(x)$, and adding them together, we obtain a strictly increasing function $\chi_n(x)$ such that

$$|D^\pm_\pm(f, x; \chi_n)| \leqq 1$$

and also

$$|D^\pm_\pm(\varphi, x; \chi_n)| \leqq 1$$

at each point of E_n, except possibly the extreme points. By Theorem 1, both $D(f, x; \chi_n)$ and $D(\varphi, x; \chi_n)$ exist and are finite except in a set N_n such that $m\chi_n(N_n) = 0$. The argument now proceeds as in Theorem 2. We observe that at a point where $D(f, x; \chi_n)$ and $D(\varphi, x; \chi_n)$ both exist and $D(\varphi, x; \chi_n) \neq 0$, both $D(f, x; \varphi)$ and $d f(x)/d\varphi(x)$ must exist (and be equal).

The corollary follows at once from lemma 4.

2. The Perron-Stieltjes Integral.

The Perron integral is defined by means of major and minor functions, which in turn are usually defined by inequalities relating to derivates[14]). There would be obvious difficulties in extending such a definition to the case of integrals with respect to a general function $\varphi(x)$, which may attain the same value at an infinite set of points. For this reason we define the Perron-Stieltjes integral by means of inequalities concerning the increments $M(x+h) - M(x)$, $\varphi(x+h) - \varphi(x)$ *directly*, and not in terms of the derivates of M with respect to φ. The resulting integral is found to include the ordinary Stieltjes integral, whether $\varphi(x)$ is of bounded variation or not. It also includes the Lebesgue-Stieltjes integral with respect to an increasing function; but it does not include the 'Lebesgue-Stieltjes' integral with respect to a function of bounded variation (defined by Mlle N. Bary and Menchoff), which may exist in an interval (a, b) without existing in a smaller interval (a, x)[15]).

2.1. Given any functions $f(x)$, $\varphi(x)$ we say that $M(x)$ is a major function of $f(x)$ with respect to $\varphi(x)$ if $M(a) = 0$, and for any point x of $\langle a, b\rangle$ there exists $\delta(x) > 0$ such that

$$(5) \quad M(\xi) \geqq M(x) + f(x)\{\varphi(\xi) - \varphi(x)\} \quad \text{if} \quad 0 \leqq \xi - x \leqq \delta(x),$$

$$(6) \quad M(\xi) \leqq M(x) + f(x)\{\varphi(\xi) - \varphi(x)\} \quad \text{if} \quad 0 \geqq \xi - x \geqq -\delta(x).$$

(If x is equal to a or b, we consider only the one inequality which is appropriate.)

The upper Perron-Stieltjes integral of $f(x)$ with respect to $\varphi(x)$ is defined as

$$(PS)\int_a^{-b} f(x)\, d\varphi(x) = \underline{\text{bound}}\ M(b) \quad \text{(all major functions)}$$

if finite major functions exist; otherwise we write $+\infty$ for the value of the upper integral.

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[14]) Ridder (loc. cit.) applies the same method in defining his Perron-Stieltjes integral.

[15]) N. Bary and D. Menchoff, Ann. Mat. pura appl. (4) (1928), S. 19—54.

It is easy to see that $M(x) - (PS)\int_a^{\overline{x}} f(t) \cdot d\,\varphi(t)$ is an increasing function of x.

Minor functions and the lower integral are defined in the corresponding way, the inequalities being reversed.

Let $M(x), m(x)$ be any major and minor functions, write

$$\omega(x) = M(x) - m(x).$$

For any x there exists $\delta(x) > 0$ such that

$$\omega(\xi) \geqq \omega(x) \quad \text{if} \quad 0 \leqq \xi - x \leqq \delta(x),$$
$$\omega(\xi) \leqq \omega(x) \quad \text{if} \quad 0 \geqq \xi - x \geqq -\delta(x),$$

(from the definitions). It easily follows that $\omega(x)$ is an increasing function. Since this is true for any $M(x), m(x)$,

$$(PS)\int_a^{\overline{x}} f(t)\,d\,\varphi(t) - (PS)\int_{\underline{a}}^{x} f(t)\,d\,\varphi(t)$$

is a positive increasing function of x. (We do not give the details of this argument, as they are almost exactly the same as for the ordinary Perron integral.

If

$$(PS)\int_a^{\overline{b}} f(x)\,d\,\varphi(x) = (PS)\int_{\underline{a}}^{b} f(x)\,d\,\varphi(x)$$

we write the common value as $(PS)\int_a^b f(x)\,d\,\varphi(x)$; if it is finite we say that $f(x)$ is integrable with respect to $\varphi(x)$ in (a, b). In future we drop the prefix (PS) when there is no ambiguity[16]).

The following properties are easily proved:

(I) If $a < c < b$ and $f(x)$ is integrable in (a, b), then it is integrable in (a, c), (c, b), and

$$\int_a^b f(x)\,d\,\varphi(x) = \int_a^c f(x)\,d\,\varphi(x + \int_c^b f(x)\,d\,\varphi(x).$$

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[16]) The same principles may be applied to form the 'Perron' integral of any differential expression depending on one variable. For example, we may define $(PS)\overline{\int}\{f_1(x)\,d\,\varphi_1(x) + f_2(x)\,d\,\varphi_2(x)\}$ in exactly the same way as we have defined $(PS)\overline{\int} f(x)\,d\,\varphi(x)$, only replacing (5) and (6) by

$M(\xi) - M(x) \geqq f_1(x)\{\varphi_1(\xi) - \varphi_1(x)\} + f_2(x)\{\varphi_2(\xi) - \varphi_2(x)\}$ if $0 \leqq \xi - x \leqq \delta$,

$M(\xi) - M(x) \leqq f_1(x)\{\varphi_1(\xi) - \varphi_1(x)\} + f_2(x)\{\varphi_2(\xi) - \varphi_2(x)\}$ if $0 \geqq \xi - x \geqq -\delta$.

I owe this remark, and Theorem 14 which depends on it, to Mr L. C. Young.

The converse also holds.

(II) $$\overline{\int_a^b} c\, f(x)\, d\,\varphi(x) = c \overline{\int_a^b} f(x)\, d\,\varphi(x) \quad \text{if} \quad c \geqq 0;$$

$$= c \underline{\int_a^b} f(x)\, d\,\varphi(x) \quad \text{if} \quad c \leqq 0.$$

(III) $$\overline{\int_a^b} \{f_1(x) + f_2(x)\}\, d\,\varphi(x) \leqq \overline{\int_a^b} f_1(x)\, d\,\varphi(x) + \overline{\int_a^b} f_2(x)\, d\,\varphi(x),$$

$$\int_a^b (f_1 \pm f_2)\, d\,\varphi(x) = \int_a^b f_1\, d\,\varphi + \int_a^b f_2\, d\,\varphi,$$

$$\overline{\int_a^b} f(x)\, d\,\{\varphi_1(x) + \varphi_2(x)\} \leqq \overline{\int_a^b} f(x)\, d\,\varphi_1(x) + \overline{\int_a^b} f(x)\, d\,\varphi_2(x),$$

$$\int_a^b f(x)\, d\,\{\varphi_1 \pm \varphi_2\} = \int_a^b f(x)\, d\,\varphi_1 \pm \int_a^b f\, d\,\varphi_2,$$

provided that the right-hand side has a meaning, in each case.

It will be noticed that we have not introduced any continuity condition in the definition of the integral. There are, however, continuity properties implicit in the definition, and the integral, if it exists, is always continuous with respect to $\varphi(x)$. For convenience we write

$$F(x) = \int_a^x f(t)\, d\,\varphi(t), \quad \overline{F}(x) = \overline{\int_a^x} f(t)\, d\,\varphi(t), \quad \underline{F}(x) = \underline{\int_a^x} f(t)\, d\,\varphi(t).$$

Given any $\varepsilon > 0$, we can choose a major function with $M(b) < \overline{F}(b) + \varepsilon$. For any x, taking $\delta(x)$ as in the definitions (5) and (6), we then have, since $M(x) - \overline{F}(x)$ is an increasing function,

$$\begin{aligned} \overline{F}(\xi) - \overline{F}(x) &\geqq M(\xi) - M(x) - \varepsilon \\ &\geqq f(x)\,\{\varphi(\xi) - \varphi(x)\} - \varepsilon \text{ if } 0 \leqq \xi - x \leqq \delta(x), \end{aligned}$$

and similarly

$$\overline{F}(\xi) - \overline{F}(x) \leqq f(x)\,\{\varphi(\xi) - \varphi(x)\} + \varepsilon \text{ if } 0 \geqq \xi - x \geqq -\delta(x).$$

Taking into account the corresponding inequalities for $\underline{F}(x)$, we see that if $f(x)$ is integrable,

(7) $$F(\xi) - F(x) - f(x)\,\{\varphi(\xi) - \varphi(x)\} \to 0 \text{ as } \xi - x \to 0.$$

2.2. Relations with other integrals.

In this section we prove first that the PS-integral includes the ordinary Perron integral, and next that it includes the 'modified Stieltjes integral' of S. Pollard[17]), (which is itself a generalisation of the ordinary Stieltjes integral). Finally, we consider its relations with the LS-integral.

Theorem 4. *If* $(P)\int_a^b f(x)\,dx$ *exists, then* $(PS)\int_a^b f(x)\,d\varphi(x)$ (*where* $\varphi(x) = x$) *exists and has the same value.*

Given any $\varepsilon > 0$ we can find a function $M(x)$ with

$$\min.\ [D_+(M, x),\ D_-(M, x)] \geqq f(x)$$

everywhere, and $M(b) < (P)\int_a^b f(x)\,dx + \varepsilon$. Then consider the function $M_1(x) = M(x) + \varepsilon(x-a)/(b-a)$. We have

$$\min\ [D_+(M_1, x),\quad D_-(M_1, x)] > f(x),$$

and hence for sufficiently small $\delta(x)$ the conditions for a PS-major function are satisfied.

Thus $(PS)\overline{\int_a^b} f(x)\,d(x) \leqq M_1(b) < (P)\int_a^b f(x)\,dx + 2\varepsilon$. From this and the corresponding result for the lower integral we have the theorem.

(Note. In the ordinary Perron-Bauer theory $f(x)$ is allowed to take infinite values, which are ruled out here; it is however a standard result that these infinite values may be replaced by zero without altering the integral, so that there is no real restriction in supposing $f(x)$ finite everywhere.)

Theorem 5. *If* $(M)\int_a^b f(x)\,d\varphi(x)$ *exists*[18]), *then* $(PS)\int_a^b f(x)\,d\varphi(x)$ *exists and has the same value.*

Let L be the value of the Pollard integral. For any $\varepsilon > 0$ there exists a set of points $y_1 < y_2 < \ldots y_N$ with the following property: — if $a = x_0 \leqq \xi_1 \leqq x_1 \leqq \xi_2 \leqq \ldots \leqq \xi_n \leqq x_n = b$ is any subdivision of (a, b) such that $y_1, y_2, \ldots, y_N$ occur somewhere among the points $x_1, \ldots, x_n$, then

$$\left| L - \sum_{i=1}^{n} f(\xi_i)\,\{\varphi(x_i) - \varphi(x_{i-1})\} \right| < \varepsilon.$$

Define $M(\alpha, \beta)$ as the upper bound of

$$\sum_{i=1}^{p} f(\xi_i)\,\{\varphi(x_i) - \varphi(x_{i-1})\}$$

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17) S. Pollard, The Stieltjes Integral and its generalisations, Quart. Journ. 49, 193 (1920), S. 87—94.

18) S. Pollard, loc. cit., p. 90.

for all divisions of the type

$$\alpha = x_0 \leqq \xi_1 \leqq x_1 \leqq \ldots \leqq \xi_P \leqq x_P = \beta.$$

Then $M(\alpha, y_1) + M(y_1, y_2) + \ldots + M(y_{N-1}, y_N) + M(y_N, b) \leqq L + \varepsilon.$

Now clearly, if $\alpha < x < \xi$,

$$M(\alpha, \xi) \geqq M(\alpha, x) + f(x)\{\varphi(\xi) - \varphi(x)\};$$

if $\alpha < \xi < x$,

$$M(\alpha, \xi) \leqq M(\alpha, x) + f(x)\{\varphi(\xi) - \varphi(x)\}.$$

For any x, let $n(x)$ be the greatest integer such that $y_{n(x)} \leqq x$.

Write

$$M(x) = M(a, y_1) + \sum_{i=1}^{n(x)-1} M(y_i, y_{i+1}) + M(y_{n(x)}, x).$$

From the inequalities just obtained it is easy to see that $M(x)$ is a major function, and $M(b) \leqq L + \varepsilon$. With the corresponding result for a minor function, this proves the theorem.

In the case when $\varphi(x)$ is monotone increasing, the development of the properties of the PS-integral proceeds almost exactly as for the ordinary P-integral. We show that if $F(x) = (PS)\int_a^x f(t)\, d\varphi(t)$ then $D(F\, x;\, \varphi) = f(x)$ except in a set N, where $m\,\varphi(N) = 0$. We then show that if $(LS)\int_a^b f \cdot d\varphi$ exists, the PS-integral exists and has the same value; that the converse holds if $f(x)$ is always positive. Finally we prove that if $f_n(x) \to f(x)$ and we have always $g(x) \leqq f_n(x) \leqq h(x)$, where g, h are PS-integrable, then if all the $f_n(x)$ are PS-integrable, so is $f(x)$, and

$$\int_a^b f(x)\, d\varphi(x) = \lim_{n \to \infty} \int_a^b f_n(x)\, d\varphi(x).$$

The proofs of these statements follow standard lines, and are omitted.

2. 3. The Stieltjes Transformation.

Theorem 6. *If* $\varphi(x) = \int_a^x g(t)\, d\psi(t)$ *for* $a \leqq x \leqq b$, *and* $f(x)$ *is bounded in* $\langle a, b\rangle$, *then*

$$\int_a^{-b} f(x)\, d\varphi(x) = \int_a^{-b} f(x)\, g(x)\, d\psi(x)$$

and

$$\int_{-a}^b f(x)\, d\varphi(x) = \int_{-a}^b f(x)\, g(x)\, d\psi(x).$$

Suppose that $\overline{\int_a^b} f(x)\, d\,\varphi(x) = L < \infty$, and $|f(x)| \leqq K$. Given $\varepsilon > 0$, choose a major function $M(x)$ of $f(x)$ with respect to $\varphi(x)$, and major and minor functions $N(x)$, $n(x)$ of g with respect to ψ, such that

$$M(b) < L + \varepsilon, \quad \varphi(b) - \varepsilon < n(b) \leqq N(b) < \varphi(b) + \varepsilon.$$

Write $\omega(x) = N(x) - n(x)$, $M_1(x) = M(x) + K\,\omega(x)$; then if $x \leqq \xi$ and $\xi - x$ is sufficiently small we have:

$$n(\xi) - n(x) \leqq \quad \varphi(\xi) - \varphi(x) \quad \leqq N(\xi) - N(x),$$
$$n(\xi) - n(x) \leqq g(x)\,\{\psi(\xi) - \psi(x)\} \leqq N(\xi) - N(x),$$

and so

$$|g(x)\,\{\psi(\xi) - \psi(x)\} - \{\varphi(\xi) - \varphi(x)\}|$$
$$\leqq \{N(\xi) - N(x)\} - \{n(\xi) - n(x)\} = \omega(\xi) - \omega(x).$$

Hence, for sufficiently small positive $\xi - x$,

$$\begin{aligned} M_1(\xi) - M_1(x) &= M(\xi) - M(x) + K\,\{\omega(\xi) - \omega(x)\} \\ &\geqq f(x)\,\{\varphi(\xi) - \varphi(x)\} + K\,\{\omega(\xi) - \omega(x)\} \\ &\geqq f(x)\,\{\varphi(\xi) - \varphi(x)\} + f(x)\,\{g(x)\,[\psi(\xi) - \psi(x)] \\ &\qquad\qquad - [\varphi(\xi) - \varphi(x)]\} \\ &\geqq f(x)\,g(x)\,\{\psi(\xi) - \psi(x)\}. \end{aligned}$$

There is a similar inequality for $\xi \leqq x$ and hence $M_1(x)$ is a major function of $f(x)\,g(x)$ with respect to ψ. Since $M_1(b) < L + (2K + 1)\,\varepsilon$ we see that

$$\overline{\int_a^b} f(x)\,g(x)\,d\,\psi(x) \leqq \overline{\int_a^b} f(x)\,d\,\varphi(x).$$

The reverse inequality is similarly proved; finally, the case of the lower integrals follows by writing $-f(x)$ for $f(x)$.

2. 4. Differential Properties.

We need first a covering theorem (analogous to W. H. Young's lemma).

Lemma 5. *If with each point x of a set E is associated an interval $(x, x + h)$, h varying with x; then given any number A less than $m_e\,\varphi(E)$, we can find a finite non-overlapping set of such intervals $(x_\nu, x_\nu + h_\nu)$, $\nu = 1, 2, \ldots, n$, such that $\Sigma\, m_e\,\varphi\,[E(x_\nu, x_\nu + h_\nu)] > A$.*

We can find p so large that the sub-set E_p, composed of those points of E for which $h > 1/p$, satisfies $m_e\,\varphi(E_p) > A$. Let $\langle c, d\rangle$ be the smallest closed interval containing E_p. We can find a point x_1 of E_p, either coinciding with c or so near to c that $m_e\,\varphi[E_p(x_1, d)] > A$. Let $(x_1, x_1 + h_1)$ be the interval associated with x_1, and let $\langle c_1, d\rangle$ be the smallest closed interval containing the set $E_p \langle x_1 + h_1, d\rangle$. Then

$$m_e\,\varphi\,[E_p\,(x_1, x_1 + h_1)] + m_e\,\varphi\,[E_p\,(c_1, d)] \geqq m_e\,\varphi\,[E_p\,(x_1, d)] > A,$$

so that we can choose a point x_2 of E_p, such that $x_2 \geqq c_1$, lying so near c_1 that

$$m_e\,\varphi\,[E_p\,(x_1, x_1 + h_1)] + m_e\,\varphi\,[E_p\,(x_2, d)] > A.$$

We then take out the interval $(x_2, x_2 + h_2)$, and proceed in the same way; since each value of h is greater than $1/p$ the process must terminate. Finally we obtain a finite set of intervals $(x_1, x_1 + h_1), \ldots, (x_n, x_n + h_n)$ such that

$$\sum_{\nu=1}^{n} m_e\,\varphi\,[E\,(x_\nu, x_\nu + h_\nu)] \geqq \sum_{\nu=1}^{n} m_e\,\varphi\,[E_p\,(x_\nu, x_\nu + h_\nu)] > A.$$

Theorem 7. *If*

$$F(x) = \int_a^x f(t)\,d\,\varphi(t) \quad (a \leqq x \leqq b),$$

then $dF(x)/d\,\varphi(x) = f(x)$ *except possibly at points of a set* N *such that* $m\,\varphi(N) = 0$.

We remark first that if in any interval $\varphi(x)$ is constant, then $F(x)$ also is constant, so that the equation $dF(x)/d\,\varphi(x) = f(x)$ is true in a conventional sense. Consider the set N_1 of points x_0 such that $\varphi(x)$ is not constant in any interval $\langle x_0, x_1\rangle$, $(x_1 > x_0)$, and that

$$\overline{\lim_{x \to x_0 + 0}}\ \frac{F(x) - F(x_0) - f(x_0)\,\{\varphi(x) - \varphi(x_0)\}}{\omega\,(\varphi, \langle x_0, x\rangle)} > 0.$$

We shall show that $m\,\varphi(N_1) = 0$, and a similar argument applied to three other sets defined by analogous inequalities would complete the proof of the theorem, for we have already shown that $F(x)$ is continuous with respect to $\varphi(x)$ at every point.

Suppose on the contrary that $m_e\,\varphi(N_1) > 0$. We can find p such that the set N_p where

$$\overline{\lim_{x \to x_0 + 0}}\ \frac{F(x) - F(x_0) - f(x_0)\,\{\varphi(x) - \varphi(x_0)\}}{\omega\,(\varphi, \langle x_0, x\rangle)} > \frac{1}{p}$$

satisfies $m_e\,\varphi(N_p) > 0$. Take η such that $0 < \eta < m_e\,\varphi(N_p)$ and a minor function $m(x)$ with $F(b) - m(b) < \eta/p$. Since $m(x) - m(x_0) \leqq f(x_0)\,\{\varphi(x) - \varphi(x_0)\}$ for all sufficiently small $x - x_0 \geqq 0$, we have for x_0 in N_p,

$$\overline{\lim_{x \to x_0 + 0}}\ \frac{F(x) - F(x_0) - [m(x) - m(x_0)]}{\omega\,(\varphi, \langle x_0, x\rangle)} > \frac{1}{p}.$$

Applying lemma 5, we can now find a finite non-overlapping set of intervals $(x_\nu, x_\nu + h_\nu)$, $\nu = 1, 2, \ldots, n$, such that

(8) $\quad F(x_\nu + h_\nu) - F(x_\nu) - [m(x_\nu + h_\nu) - m(x_\nu)] > (1/p)\,\omega\,(\varphi, \langle x_\nu, x_\nu + h_\nu\rangle)$

and

$$(9)\qquad \sum_{\nu=1}^{n} \omega(\varphi, \langle x_\nu, x_\nu + h_\nu\rangle) \geqq \sum_{\nu=1}^{n} m_e\, \varphi[N_p(x_\nu, x_\nu + h_\nu)] > \eta.$$

Then since $F(x) - m(x)$ is a positive increasing function, we must have from (8) and (9)

$$F(b) - m(b) > \eta / p,$$

which is a contradiction. Thus $m\,\varphi(N_1) = 0$.

It is also possible to give a differentiation theorem in terms of the ordinary derivates.

Theorem 8. *If* $F(x) = \int_a^x f(t)\, d\,\varphi(t)$, *then* $D^+(F, x) = f(x)\, D^+(\varphi, x)$ *or* $D^+(F, x) = f(x)\, D_+(\varphi, x)$, *according as* $f(x) \geqq 0$ *or* $f(x) \leqq 0$, *almost everywhere in* (a, b), *except possibly where the product takes the form* $0 \times \infty$.

At points x_0 where the statement is untrue, we must have

$$\overline{\lim_{x \to x_0 + 0}} \frac{F(x) - F(x_0) - f(x_0)\,\{\varphi(x) - \varphi(x_0)\}}{x - x_0} > 0$$

or

$$\underline{\lim_{x \to x_0 + 0}} \frac{F(x) - F(x_0) - f(x_0)\,\{\varphi(x) - \varphi(x_0)\}}{x - x_0} < 0.$$

The proof proceeds exactly as in the last theorem, with the substitution of $m_e E$ for $m_e\,\varphi(E)$, and $(x - x_0)$ for $\omega(\varphi, \langle x_0, x\rangle)$.

3. In the preceding sections we have given some properties of the PS-integral, assuming that it exists for the pair of functions considered. We now turn to the questions of the existence and structural properties of the integral. It appears to be necessary to assume that at least one of the functions concerned is VBG^* (on some set considered) in order to obtain any non-trivial results. In fact, we may say that the class of functions VBG^* in $\langle a, b\rangle$ is related to the PS-integral in much the same way as the class of functions of bounded variation is related to the ordinary Stieltjes integral.

After some lemmas dealing with the structure of VBG^* functions[19]), we give a necessary and sufficient condition for the integral to vanish identically. There follows a theorem on integration by parts (which may of course be interpreted as asserting that a certain PS-integral exists whenever the related integral exists). Finally we prove that a PS-integral with respect to a function VBG^* in $\langle a, b\rangle$ is ACG^* with respect to that function, and conversely; subject to the necessary continuity condition.

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[19]) Similar lemmas are used in the work of L. C. Young on Denjoy-Stieltjes integration, but the work here published was carried out independently.

3. 1. Lemma 6. *$\varphi(x)$ is VBG^* on a set E if and only if there exists a strictly increasing function $\chi(x)$ such that all the derivates $D^{\pm}_{\pm}(\varphi, x; \chi)$ are finite at all points of E, except perhaps an enumerable set.*

Suppose the condition satisfied; then the points of E where the derivates are finite can be divided into sets E_n such that, if x lies in E_n,

$$|\varphi(\xi) - \varphi(x)| \leqq n\ |\chi(\xi) - \chi(x)|$$

whenever

$$|\chi(\xi) - \chi(x)| \leqq 1/n.$$

Subdivide each E_n into sub-sets E_{np} defined by the inequality $p/n \leqq \chi(x) < (p+1)/n$. It is clear that $\varphi(x)$ is VB^* on each E_{np} and hence (taking the exceptional points as constituting each a set in itself) $\varphi(x)$ is VBG^* on E.

Conversely, suppose $\varphi(x)$ VBG^* on E, and VB^* on each of a sequence of sets E_n which together make up E. Construct for each E_n the function $\chi_n(x)$ of lemma 2, and write

$$\chi(x) = \sum_n \frac{\chi_n(x) - \chi_n(a)}{2^n\,\{\chi_n(b) - \chi_n(a)\}}.$$

$\chi(x)$ is a strictly increasing function defined in $\langle a, b\rangle$, and it is clear that $|D^{\pm}_{\pm}(\varphi, x; \chi)| \leqq 2^n$ at each point of E_n, except perhaps the extreme points of E_n. Thus the derivates are finite at all points of E except possibly an enumerable set.

Theorem 9. *If $F(x) = \int_a^x f(t)\,d\varphi(t)$ for $a \leqq x \leqq b$, then $F(x)$ is VBG^* on E if and only if E can be expressed as $G + H$, where $\varphi(x)$ is VBG^* on G and $f(x) = 0$ at each point of H.*

Let $M(x)$, $m(x)$ be any major and minor functions of $f(x)$ with respect to $\varphi(x)$, and write $\omega(x) = M(x) - m(x)$. For brevity we write also $\delta\varphi = \delta\varphi(x, h)$ for $\varphi(x+h) - \varphi(x)$, and so on. We observe that, for sufficiently small h, both $f(x)\cdot\delta\varphi(x, h)$ and $\delta F(x, h)$ lie between $\delta m(x, h)$ and $\delta M(x, h)$, so that

$$(10) \qquad |\delta F - f(x)\cdot\delta\varphi| \leqq |\delta\omega|.$$

Suppose now $\varphi(x)$ VBG^* on G, and let $\chi(x)$ be the function of lemma 6 corresponding to the set G. Write $\chi_1(x) = \chi(x) + \omega(x)$. At all points of G except an enumerable set, there exists $K = K(x)$ such that

$$|\delta\varphi| \leqq K(x)\cdot|\delta\chi|$$

for sufficiently small h. Hence, using (10), we have

$$
(11)\qquad \begin{aligned} |\delta F| &\leqq |f(x)\cdot\delta\varphi| + |\delta\omega|,\\ &\leqq K\cdot|f(x)|\cdot|\delta\chi| + |\delta\omega|,\\ &\leqq \{K|f(x)|+1\}\,|\delta\chi_1|, \end{aligned}
$$

since $\chi(x)$ and $\omega(x)$ are both increasing functions. At points of H, where $f(x)=0$, we have immediately

$$
(12)\qquad |\delta F| \leqq |\delta\omega| \leqq |\delta\chi_1|
$$

for sufficiently small h. By lemma 6, it follows from (11) and (12) that $F(x)$ is VBG^* on $G+H$.

Conversely, suppose $F(x)$ VBG^* on $E=G+H$, and $f(x)=0$ on H, but $f(x)\neq 0$ on G.

Let $\chi(x)$ be the function of lemma 6, defined with reference to $F(x)$ and E. At all points of G except an enumerable set, we have, for sufficiently small h,

$$|\delta F| \leqq K(x)\,|\delta\chi|$$

and so

$$|f(x)\cdot\delta\varphi| \leqq K(x)\,|\delta\chi| + |\delta\omega|.$$

Since $f(x)$ does not vanish on G, it follows that

$$|\delta\varphi| \leqq \{K(x)+1\}\,\{|\delta\chi|+|\delta\omega|\}/|f(x)|.$$

Thus $\varphi(x)$ is VBG^* on the set G.

Lemma 7. *If $\chi(x)$ is strictly increasing, and $D(\varphi, x; \chi)$ exists and satisfies $|D(\varphi, x;\chi)| > k > 0$ at each point of a set E such that $m\varphi(E)=0$, then $m\chi(E)=0$.*

For each point x of E we can find $n=n(x)$ so large that

$$|\varphi(\xi)-\varphi(x)| > k|\chi(\xi)-\chi(x)|$$

whenever

$$|\chi(\xi)-\chi(x)| \leqq 1/n.$$

If $m_e\chi(E)>0$, we can find a fixed n_0 such that this condition is satisfied for $n=n_0$ whenever x lies in a subset E_1 of E such that $m_e\chi(E_1)>0$. We may also suppose that $-n_0<\chi(a)<\chi(b)<n_0$. Let now ΣI_p be any open set containing $\varphi(E)$, composed of the non-overlapping intervals I_p. Let E_{pq} be the subset of E_1 composed of points satisfying:

$$\varphi(x) \text{ belongs to } I_p, \text{ and}$$

$$q/n_0 \leqq \chi(x) \leqq (q+1)/n_0.$$

If x, y are any two points of E_{pq} we have

$$mI_p \geqq |\varphi(x)-\varphi(y)| \geqq k|\chi(x)-\chi(y)|.$$

Hence $k \cdot m_e \chi(E_{pq}) \leqq m I_p$, and so, summing over the possible values of q, we have

$$m_e \chi(\sum_q E_{pq}) \leqq 2n^2 (m I_p)/k$$

and so

$$m_e \chi(E_1) \leqq 2n^2 (m \Sigma I_p)/k.$$

Since $m \Sigma I_p$ may be arbitrarily small, we must have $m\chi(E_1) = 0$, contrary to supposition.

Lemma 8. *If $\varphi(x)$ is VBG* on E and continuous at each point of E, and also $m\varphi(E) = 0$, then there exists an increasing function $\psi(x)$ such that $D(\varphi, x; \psi) = 0$ at each point of E.*

Let $\chi(x)$ be the function of lemma 6. By Theorem 1, $D(\varphi, x; \chi)$ exists, and is finite, at the points of E, except possibly for a set N such that $m\chi(N) = 0$. Let E_n be the sub-set of E consisting of points at which $|D(\varphi, x; \chi)| > 1/n$. By lemma 7, $m\chi(E_n) = 0$, and so $m\chi(\Sigma E_n) = 0$. Thus we see that E can be divided into three sets: a set G of points where $D(\varphi, x; \chi) = 0$; a set H, such that $m\chi(H) = 0$, of points at which $\chi(x)$ is continuous and all the derivates of φ with respect to χ are finite; and a set of points (x_n), at most enumerable, where $\chi(x)$ may be discontinuous or one of the derivates of φ not finite.

Let U_n be an open set containing $\chi(H)$ and such that $mU_n < 1/4^n$. Define $\psi_n(x) = m\{U_n[y \leqq \chi(x)]\}$, y being a variable describing the set U_n; $\psi_n(x)$ is clearly an increasing function, and it is easy to see that, since $\chi(x)$ is continuous at points of H, $D(\psi_n; x; \chi) = 1$ at each point of H. We note that $0 \leqq \psi_n(a) \leqq \psi_n(b) < 1/4^n$. If now x_n is one of the enumerable set of points not in G or H, write

$$(13) \qquad \begin{aligned} \beta_n(x) &= 0 && \text{if} \quad x < x_n, \\ \beta_n(x) &= 1/2^{n+1} && \text{if} \quad x = x_n, \\ \beta_n(x) &= 1/2^n && \text{if} \quad x > x_n. \end{aligned}$$

Finally write

$$\psi(x) = \sum_{n=1}^{\infty} 2^n \psi_n(x) + \sum_{x_n} \beta_n(x) + \chi(x).$$

If x belongs to H, let K be the greatest modulus of the four derivates, $|D^{\pm}_{\pm}(\varphi, x; \chi)|$; then

$$\begin{aligned} \overline{\lim_{h\to 0}} \left| \frac{\varphi(x+h) - \varphi(x)}{\psi(x+h) - \psi(x)} \right| &\leqq K \overline{\lim_{h\to 0}} \left| \frac{\chi(x+h) - \chi(x)}{\psi(x+h) - \psi(x)} \right| \\ &\leqq K \overline{\lim_{h\to 0}} \left| \frac{\chi(x+h) - \chi(x)}{2^n[\psi_n(x+h) - \psi_n(x)]} \right| \\ &\leqq K/2^n. \end{aligned}$$

It follows, n being arbitrary, that $D(\varphi, x; \psi) = 0$. The same holds at each point x_n, $\varphi(x)$ being continuous, owing to the presence of the terms $\beta_n(x)$; and at points of G owing to the presence of the term $\chi(x)$ in $\psi(x)$.

Theorem 10. *$\int_a^x f(t)\,d\varphi(t)$ exists and vanishes for all x such that $a \leqq x \leqq b$, if and only if $f(x) = 0$ except in a set E, such that $\varphi(x)$ is VBG^* on E and continuous at each point of E, and $m\varphi(E) = 0$.*

Suppose $\int_a^x f(t)\,d\varphi(t) = 0$ for $a \leqq x \leqq b$. By Theorem 7 the set E on which $f(x) \neq 0$ satisfies $m\varphi(E) = 0$; by the remark at the end of § 2. 1, $\varphi(x)$ must be continuous at each point of E; and by Theorem 9, $\varphi(x)$ is VBG^* on E.

Conversely, suppose the conditions satisfied; let $\psi(x)$ be the function of lemma 8 constructed for the set E. Given any $\varepsilon > 0$ and any x of E, we can find h_0 so small that

$$|f(x)| \cdot |\varphi(x+h) - \varphi(x)| \leqq \varepsilon |\psi(x+h) - \psi(x)| \quad \text{if} \quad |h| \leqq h_0.$$

Thus $\varepsilon\{\psi(x) - \psi(a)\}$ is a major function; similarly $-\varepsilon\{\psi(x) - \psi(a)\}$ is a minor function. It follows that $\int_a^x f(t)\,d\varphi(t) = 0$ for every x.

3. 2. Integration by Parts.

The theorem which we are about to give is in some ways less general than the corresponding result for the ordinary Stieltjes integral, for we restrict one function to be VBG^*. However, it includes the most important case of that result — namely, when $\varphi(x)$ is of bounded variation; for then we know that the Stieltjes integral exists only if $f(x)$ is continuous except at the points of a set N such that $m\varphi(N) = 0$.

Theorem 11. *If $\varphi(x)$ is bounded and VBG^* in $\langle a, b\rangle$, and $f(x)$ is bounded in $\langle a, b\rangle$ and continuous except at the points of a set N, such that $m\varphi(N) = 0$ and $\varphi(x)$ is continuous at each point of N, then*

$$\int_a^{-b} f(x)\,d\varphi(x) + \int_{-a}^{b} \varphi(x)\,df(x) = \varphi(b)f(b) - \varphi(a)f(a)$$

if either integral is finite.

Let $\chi(x)$ be a strictly increasing function such that the derivates of $\varphi(x)$ with respect to $\chi(x)$ are finite except perhaps at an enumerable set of points (lemma 6). Since $\varphi(x)$ is bounded, if we add to $\chi(x)$ a series of functions of the form $\beta_n(x)$, (13), we obtain a function $\chi_1(x)$ such that the derivates of $\varphi(x)$ with respect to χ_1 are finite everywhere.

Suppose now $\int_a^{-b} f(x)\,d\varphi(x) < \infty$ and let $M(x)$ be any major function for this integral. Let $\psi(x)$ be the function of lemma 8 such that $D(\varphi, x; \psi) = 0$ at each point of N. Take any $\varepsilon > 0$ and write

$$m_1(x) = \varphi(x) f(x) - M(x) - \varepsilon\,\{\psi(x) + \chi_1(x)\}.$$

For sufficiently small $h > 0$ we write

$$\begin{aligned} \delta m_1(x, h) &= \delta(\varphi f)\,(x, h) - \delta M(x, h) - \varepsilon\,\{\delta\,\psi(x, h) + \delta\chi_1(x, h)\} \\ (14) \qquad &= \varphi(x)\cdot\delta f + \delta f\cdot\delta\varphi + f(x)\cdot\delta\varphi - \delta M - \varepsilon(\delta\psi + \delta\chi_1) \\ &\leqq \varphi(x)\,\delta f + \delta f\cdot\delta\varphi - \varepsilon\,(\delta\,\psi + \delta\,\chi_1). \end{aligned}$$

If $f(x)$ is continuous at x, we have for some K and sufficiently small $h > 0$, $|\delta\varphi| \leqq K\cdot\delta\chi_1$ and $|\delta f| < \varepsilon/K$; hence from (14)

$$\begin{aligned} \delta m_1(x, h) &\leqq \varphi(x)\cdot\delta f - \varepsilon\cdot\delta\psi \\ &\leqq \varphi(x)\cdot\delta f. \end{aligned}$$

If however x belongs to N, we have for sufficiently small $h > 0$, $|\delta\varphi| < \varepsilon\cdot\delta\psi/2R$, where R is the upper bound of $|f|$. Hence $|\delta f\cdot\delta\varphi| < \varepsilon\cdot\delta\psi$ and so again we have

$$\delta\,m_1(x, h) \leqq \varphi(x)\cdot\delta f(x, h).$$

From this and the corresponding inequality for $h < 0$ we see that $m_1(x) - m_1(a)$ is a minor function for $\int_{-a}^{b} \varphi(x)\,d f(x)$. Thus we have, for any $M(x)$ and any $\varepsilon > 0$,

$$\int_{-a}^{b} \varphi(x)\,d f(x)$$

$$\geqq \varphi(b) f(b) - \varphi(a) f(a) - M(b) - \varepsilon\,\{\psi(b) - \psi(a) + \chi_1(b) - \chi_1(a)\}$$

and so

$$\int_{-a}^{b} \varphi(x)\,d f(x) \geqq \varphi(b) f(b) - \varphi(a) f(a) - \int_a^{-b} f(x)\,d\,\varphi(x).$$

The converse inequality is similarly proved.

It may be remarked that the conditions of the theorem, while far from being necessary, are not so artificial as they might at first appear. In fact, it is possible to prove the following theorem: If $\varphi(x)$ is VBG^* in $\langle a, b\rangle$, and, for $a \leqq x \leqq b$,

$$\int_a^x f(t)\,d\,\varphi(t) + \int_a^x \varphi(t)\,d f(t) = \varphi(x) f(x) - \varphi(a) f(a),$$

then $f(x)$ is continuous except at the points of a set N such that $m\,\varphi(N) = 0$.

3.3. Theorem 12. *If* $F(x) = \int_a^x f(t)\, d\varphi(t)$, *and* $\varphi(x)$ *is* VBG^* *on a set* E, *then* $F(x)$ *is* ACG^* *with respect to* $\varphi(x)$ *on* E.

By Theorem 9, $F(x)$ is VBG^* on E. It follows that we can divide E into an enumerable sequence of sets E_n, on each of which both $\varphi(x)$ and $F(x)$ are VB^*. We shall show that $F(x)$ is AC^* with respect to $\varphi(x)$ on each E_n.

Take any E_n and consider the closed set $E_n + E'_n$. If any of its black intervals (α, β) are such that $\varphi(x)$, and therefore $F(x)$, is constant in the closed interval $\langle\alpha, \beta\rangle$, we add them to $E_n + E'_n$, and thus obtain a closed set H (say), on which, as is easily seen, $F(x)$ and $\varphi(x)$ are VB^*. Given any $\varepsilon > 0$, choose a major function $M(x)$ such that $M(b) < F(b) + \varepsilon$. Since $M(x) - F(x)$ is monotone, $M(x)$ also is VB^* on H. Define $M_1(x)$ as equal to $M(x)$ on H and linear in the black intervals of H; similarly $\varphi_1(x)$ equal to $\varphi(x)$ on H and linear in the intervals. Then $M_1(x)$ and $\varphi_1(x)$ are of bounded variation; let $\Phi(x)$ be the total variation of $\varphi_1(t)$ in the interval $a \leqq t \leqq x$, and $\Psi(x)$ the total variation of $M_1(t)$. Enumerate the black intervals of H as (c_n, d_n), and the discontinuities of $M_1(x)$ as (x_n).

Choose N so large that

$$m_e \Psi[H(|f| < N)] > m\Psi(H) - \varepsilon, \tag{15}$$

$$\sum_{n>N} \{\Psi(d_n) - \Psi(c_n)\} < \varepsilon, \tag{16}$$

$$\sum_{n>N} \{\Psi(x_n + 0) - \Psi(x_n - 0)\} < \varepsilon. \tag{17}$$

Define R as $\max\,[|f(x_1)|, |f(x_2)|, \ldots, |f(x_N)|, 1]$.

Take δ_1 as the least of

$$\omega(\varphi, \langle c_n, d_n\rangle), \text{ for } n \leqq N, \text{ and } \varepsilon/RN\,^{20)}. \tag{18}$$

Divide $\langle a, b\rangle$ by points $a = t_0 < t_1 < \cdots < t_p = b$, such that

$$\sum_{n=1}^{p} |\varphi_1(t_n) - \varphi_1(t_{n-1})| > \Phi(b) - \delta_1. \tag{19}$$

It is clear that we may suppose the points t_n taken as points of H, so that $\varphi_1(t_n) = \varphi(t_n)$; also we may suppose $\omega(\varphi, \langle t_{n-1}, t_n\rangle) > 0$ (otherwise we could simply omit t_n from the set of dividing-points). Take δ as the least of

$$\tfrac{1}{2}\delta_1, \omega(\varphi, \langle t_{n-1}, t_n\rangle) \text{ for } n = 1, 2, \ldots, p. \tag{20}$$

---

[20]) By the construction of H, $\omega(\varphi, \langle c_n, d_n\rangle) > 0$ for each n.

Let (X_i, Y_i) be any finite system of non-overlapping intervals whose end-points lie in H. We shall show that if

(21) $$\Sigma\, \omega(\varphi, \langle X_i, Y_i \rangle) < \delta,$$

then

$$\Sigma\, \{F(Y_i) - F(X_i)\} > -12\,\varepsilon.$$

We see from (20) and (21) that any interval (X_i, Y_i) can contain at most one of the points t_n. If it does contain one, replace it by the two intervals (X_i, t_n) and (t_n, Y_i). Number the whole set of intervals thus obtained as (ξ_i, η_i), $i = 1, 2, \ldots, m$ (say). Since each of the original intervals was divided into at most two parts, we have

(22) $$\begin{aligned}\Sigma\, \omega(\varphi, \langle \xi_i, \eta_i \rangle) &\leqq 2\,\Sigma\, \omega(\varphi, \langle X_i, Y_i \rangle) \\ &< 2\,\delta \\ &< \delta_1.\end{aligned}$$

We see therefore, from (18), that the intervals (ξ_i, η_i) cannot contain any interval (c_n, d_n) for $n \leqq N$.

Let now H_N denote the sub-set of H composed of points where $|f(x)| < N$. We show that

$$\sum_i m_e\, \Psi(H_N \langle \xi_i, \eta_i \rangle) > \sum_i \{\Psi(\eta_i) - \Psi(\xi_i)\} - 8\,\varepsilon.$$

Since $\Psi(x)$ is an increasing function, we have from (15)

(23) $$\sum_i m\, \Psi(H\langle \xi_i, \eta_i \rangle) - \sum_i m_e\, \Psi(H_N \langle \xi_i, \eta_i \rangle) \leqq m\, \Psi(H) - m_e\, \Psi(H_N) < \varepsilon.$$

Again, for each interval $\langle \xi_i, \eta_i \rangle$

(24) $$\begin{aligned} m\, \Psi(H\langle \xi_i, \eta_i \rangle) &= \Psi(\eta_i) - \Psi(\xi_i) - \sum_{\xi_i \leqq x < \eta_i} \{\Psi(x_n + 0) - \Psi(x_n)\} \\ &\quad - \sum_{\xi_i < x_n \leqq \eta_i} \{\Psi(x_n) - \Psi(x_n - 0)\} - \sum_{\xi_i \leqq c_n < d_n \leqq \eta_i} m\, \Psi(c_n, d_n) \\ &= \Psi(\eta_i) - \Psi(\xi_i) - A_i - B_i - C_i, \end{aligned}$$

say. To estimate the last term, we have at once from (16),

(25) $$\begin{aligned}\sum_i C_i &\leqq \sum_{n > N} \{\Psi(d_n) - \Psi(c_n)\} \\ &< \varepsilon.\end{aligned}$$

To estimate the discontinuity terms, write $\omega_+(\varphi, x)$ for

$$\lim_{h \to +0} \omega(\varphi, \langle x, x + h \rangle)$$

(the discontinuity of $\varphi(x)$ on the right). For any set whatever (x_{n_j}) of discontinuities of $M_1(x)$, we have from (7), § 2.1,

$$\begin{aligned}\omega_+(F, x_{n_j}) &= |f(x_{n_j})| \cdot \omega_+(\varphi, x_{n_j}) \\ &\leqq R\, \omega_+(\varphi, x_{n_j}) \text{ if } n_j \leqq N.\end{aligned}$$

Hence, remembering the definitions of $\Psi(x)$ and $M_1(x)$, we have

$$\sum_j \{\Psi(x_{n_j}+0) - \Psi(x_{n_j})\} \leqq \sum_{n_j \leqq N} \omega_+(M_1, x_{n_j}) + \sum_{n>N} \omega_+\{\Psi, x_n\}$$
$$\leqq \sum_{n_j \leqq N} \omega_+(M, x_{n_j}) + \varepsilon$$
$$\leqq \sum_{n_j \leqq N} \omega_+(F, x_{n_j}) + 2\varepsilon$$
$$\leqq \sum_{n_j \leqq N} R\,\omega_+(\varphi, x_{n_j}) + 2\varepsilon.$$

For each point x_n lying in any $\langle\xi_i, \eta_i\rangle$, we have

$$\omega_+(\varphi, x_n) \leqq \omega(\varphi, \langle\xi_i, \eta_i\rangle)$$
$$< \delta_1$$
$$< \varepsilon/(RN).$$

It follows that $\Sigma A_i \leqq N R \varepsilon/(RN) + 2\varepsilon \leqq 3\varepsilon$; similarly $\Sigma B_i \leqq 3\varepsilon$. Hence, combining this result with (24) and (25) we get

$$\sum_i m\,\Psi(H\langle\xi_i, \eta_i\rangle) \geqq \sum_i \{\Psi(\eta_i) - \Psi(\xi_i)\} - 3\varepsilon - 3\varepsilon - \varepsilon,$$

and so from (23)

$$(26) \qquad \sum_i m_e\,\Psi(H_N\langle\xi_i, \eta_i\rangle) > \sum_i \{\Psi(\eta_i) - \Psi(\xi_i)\} - 8\varepsilon.$$

Now with each point x of $H_N\langle\xi_i, \eta_i\rangle$ which is not an end-point of a black interval (c_n, d_n)[21]), we can associate an interval $(x, x+h)$ such that $x+h$ lies in $H\langle\xi_i, \eta_i\rangle$ and also

$$(27) \qquad M_1(x+h) - M_1(x) = M(x+h) - M(x)$$
$$\geqq f(x)\{\varphi(x+h) - \varphi(x)\}$$
$$\geqq -N\,|\varphi(x+h) - \varphi(x)|.$$

By lemma 5, we can find a finite set of such intervals, not overlapping, say (λ_i, μ_i), $i = 1, 2, \ldots, k$, such that

$$(28) \qquad \sum_{i=1}^{k} \{\Psi(\mu_i) - \Psi(\lambda_i)\} \geqq \sum_{i=1}^{k} m_e\,\Psi[H_N(\lambda_i, \mu_i)]$$
$$> \sum_{i=1}^{m} m_e\,\Psi[H_N\langle\xi_i, \eta_i\rangle] - \varepsilon$$
$$> \sum_{i=1}^{m} \{\Psi(\eta_i) - \Psi(\xi_i)\} - 9\varepsilon.$$

Let the complementary intervals, which together with (λ_i, μ_i) make up the intervals (ξ_i, η_i), be enumerated as (l_i, m_i); then clearly from (28)

$$\Sigma\{\Psi(m_i) - \Psi(l_i)\} < 9\varepsilon,$$

[21]) The points which we omit form at most an enumerable set, and so do not affect the measure of $\Psi(H_N)$.

and so, since all the end-points belong to H,

$$\Sigma |M(m_i) - M(l_i)| < 9\varepsilon.$$

Hence, using (27), we have

$$(29) \qquad \sum_{i=1}^{m} \{M(\eta_i) - M(\xi_i)\} > \sum_{i=1}^{k} \{M(\mu_i) - M(\lambda_i)\} - 9\varepsilon$$
$$> -N \sum_{i=1}^{k} |\varphi(\mu_i) - \varphi(\lambda_i)| - 9\varepsilon.$$

Since no interval (ξ_i, η_i) includes, as an interior point, any of the division-points t_n, we see from (19) that

$$\sum_{i=1}^{k} |\varphi_1(\mu_i) - \varphi_1(\lambda_i)| \leqq \sum_{i=1}^{m} |\varphi_1(\eta_i) - \varphi_1(\xi_i)| + \delta_1$$
$$< 2\delta_1 \qquad \text{by (22)}$$
$$< 2\varepsilon/N \qquad \text{by (18).}$$

(All the points concerned belong to H, so that φ and φ_1 are interchangeable.) Thus we have from (29)

$$\sum_{i=1}^{m} \{M(\eta_i) - M(\xi_i)\} > -N(2\varepsilon/N) - 9\varepsilon$$
$$> -11\varepsilon,$$

and so finally

$$\sum_{i=1}^{m} \{F(\eta_i) - F(\xi_i)\} > -12\varepsilon,$$

that is,

$$\Sigma \{F(Y_i) - F(X_i)\} > -12\varepsilon.$$

A similar proof, using a minor function, establishes a similar inequality in the opposite sense (the value of δ may of course be different). Thus there exists $\delta_2 > 0$ such that, if $\Sigma\, \omega(\varphi, \langle X_i, Y_i\rangle) < \delta_2$, then

$$\sum_{F(Y_i) > F(X_i)} \{F(Y_i) - F(X_i)\} < 12\varepsilon,$$

and

$$\sum_{F(Y_i) < F(X_i)} \{F(Y_i) - F(X_i)\} > -12\varepsilon;$$

that is,

$$\Sigma |F(Y_i) - F(X_i)| < 24\varepsilon.$$

Thus $F(x)$ is AC with respect to $\varphi(x)$ on the closed set H. On the other hand, $F(x)$ is VB^* on H, and $\varphi(x)$ is not constant in any of the black intervals, taken closed, $\langle c_n, d_n\rangle$. It easily follows that $F(x)$ is AC^* with respect to $\varphi(x)$ on H, and so also on the smaller set E_n [22]. This completes the proof of the theorem.

[22]) The proof is exactly analogous to that for functions AC with respect to x. Saks, loc. cit. 162—164. Theorem 13.

3.4. Lemma 9. *If $\varphi(x)$ is VB^* on a set E such that $m\,\varphi(E) = 0$, and $F(x)$ is AC^* with respect to $\varphi(x)$ on E, then $m\,F(E) = 0$.*

Corollary. *If $\varphi(x)$ is VBG^* on E, $m\,\varphi(E) = 0$, and $F(x)$ is ACG^* with respect to $\varphi(x)$ on E, then $m\,F(E) = 0$.*

Consider the function $\chi(x)$ of lemma 2, formed with respect to the closed set $E + E'$. We remark that if I is any closed interval whose end-points lie in E, then

$$\omega(\varphi, I) \leqq \omega(\chi, I). \tag{30}$$

As in lemma 8, E may be divided into a set G of points where $D(\varphi, x; \chi) = 0$ and a set H such that $m\,\chi(H) = 0$ We deal with these sets separately, remarking first that, by the omission of an enumerable set of points[23]), we may suppose that no point of G or H is the left-hand end-point of a 'black interval' of E, and that $\chi(x)$ is continuous at each point of H.

Given $\varepsilon > 0$, we can find $\delta > 0$ such that for any set of closed non-overlapping intervals I_k, whose end-points lie in E, $\Sigma\,\omega(F, I_k) < \varepsilon$ whenever $\Sigma\,\omega(\varphi, I_k) < \delta$. With each point x of G, we can associate an interval $(x, x + h)$ such that $x + h$ belongs to E, and

$$\omega(\varphi, \langle x, x + h\rangle) \leqq \eta\,\{\chi(x + h) - \chi(x)\} \tag{31}$$

where $\eta < \delta / \{\chi(b) - \chi(a)\}$. By lemma 5, we can find a finite non-overlapping set of such intervals $(x_\nu, x_\nu + h_\nu)$ such that

$$\begin{aligned} m_e F(G) - \varepsilon &< \sum_\nu m_e F[G(x_\nu, x_\nu + h_\nu)] \\ &< \sum_\nu \omega[F; \langle x_\nu, x_\nu + h_\nu\rangle]. \end{aligned} \tag{32}$$

We have certainly $\sum\limits_\nu \{\chi(x_\nu + h_\nu) - \chi(x_\nu)\} \leqq \chi(b) - \chi(a)$, and so from (31)

$$\begin{aligned} \sum_\nu \omega(\varphi, \langle x_\nu, x_\nu + h_\nu\rangle) &\leqq \eta\,\{\chi(b) - \chi(a)\} \\ &< \delta. \end{aligned}$$

Hence from (32) and the AC^* condition,

$$m_e F(G) - \varepsilon < \varepsilon. \tag{33}$$

Consider now the set H; we can enclose $\chi(H)$ in an open set U of measure less than δ. With each point of H, since we suppose $\chi(x)$ continuous, we can associate an interval $(x, x + h)$ such that $x + h$ belongs to E, and so small that the whole interval $\langle \chi(x), \chi(x + h)\rangle$ lies in U. We can now again pick out a finite set of such intervals such that

$$m_e F(H) - \varepsilon < \sum_\nu \omega[F; \langle x_\nu, x_\nu + h_\nu\rangle].$$

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[23]) This clearlv cannot alter the measure of $F(G)$ or $F(H)$.

By (30), since $\sum_\nu \{\chi(x_\nu + h_\nu) - \chi(x_\nu)\} \leqq m\,U < \delta$, we have $m_e F(H) - \varepsilon < \varepsilon$. From this and (33), since ε is arbitrary, we obtain the desired result.

Theorem 13. *If $\varphi(x)$ is VBG^* in the interval $a \leqq x \leqq b$, and $F(x)$ is, with respect to $\varphi(x)$, both continuous and ACG^* in the interval, then*

$$F(x) - F(a) = \int_a^x f(t)\,d\,\varphi(t), \qquad a \leqq x \leqq b,$$

where $f(x)$ is defined as follows:

(i) *$f(x) = dF(x)/d\,\varphi(x)$ at points where this exists;*

(ii) *At any point x_0 where $\varphi(x)$ is discontinuous*[24]*, $f(x_0)$ is defined so as to make $F(x) - f(x_0)\,\varphi(x)$ continuous at x_0;*

(iii) *At the remaining points, $f(x) = 0$.*

Let N be the set of points at which $\varphi(x)$ is continuous and $dF/d\,\varphi$ does not exist. By Theorem 3, Corollary, $m\,\varphi(N) = 0$, and so, by lemma 9, Corollary, $m\,F(N) = 0$. By lemma 4, $F(x)$ is VBG^* in $\langle a, b\rangle$; we can therefore construct, as in lemma 8, an increasing function $\psi(x)$ such that $D(F, x; \psi) = 0$ at each point of N.

Let $\chi(x)$ be the function of lemma 6, such that the derivates of φ with respect to χ are finite except at an enumerable set of points. Enumerate these exceptional points and the points of discontinuity of $\varphi(x)$, together, as (x_n), $n = 1, 2, \ldots$. Take any $\varepsilon > 0$ and write

$$M(x) = F(x) + \varepsilon\,\{\psi(x) + \chi(x) + \sum_n \beta_n(x)\}$$

where $\beta_n(x)$ is defined as in (13). We say that $M(x) - M(a)$ is a major function of $f(x)$ with respect to $\varphi(x)$.

Consider first a point x where $dF/d\,\varphi$ exists and the derivates of φ with respect to χ are finite. There exists K such that, for sufficiently small h,

$$|\varphi(x+h) - \varphi(x)| \leqq K\,|\chi(x+h) - \chi(x)|.$$

Since $dF/d\,\varphi = f(x)$, for sufficiently small $h > 0$ we have

$$\begin{aligned} |\delta F(x,h) - f(x)\cdot\delta\,\varphi(x,h)| &\leqq \varepsilon\cdot\omega(\varphi, \langle x, x+h\rangle)/(2K) \\ &\leqq \varepsilon\,\{\chi(x+h) - \chi(x)\} \end{aligned}$$

and so

$$\begin{aligned} f(x)\cdot\delta\,\varphi(x,h) &\leqq \delta F(x,h) + \varepsilon\cdot\delta\,\chi(x,h) \\ &\leqq \delta M(x,h). \end{aligned}$$

There is a similar inequality for $h < 0$. In the same way, the conditions for a major function are satisfied at the points of N, where $f(x) = 0$.

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[24]) It is easy to see from lemma 6 that these points of discontinuity are enumerable. The required value of $f(x_0)$ must exist, since $F(x)$ is continuous with respect to $\varphi(x)$.

For since $D(F, x; \psi) = 0$, we have $\delta M \geqq \varepsilon \cdot \delta \psi + \delta F \geqq 0$ for sufficiently small $h > 0$. Finally, at any of those exceptional points x_n which does not lie in N, since $F(x) - f(x_n)\varphi(x)$ is continuous, the conditions are satisfied owing to the presence of the terms $\beta_n(x)$.

Since ε is arbitrarily small, we deduce that, for $a \leqq x \leqq b$,

$$\int_a^{-x} f(t)\, d\varphi(t) \leqq F(x) - F(a),$$

and in a similar way we can show that

$$\int_{-a}^{x} f(t)\, d\varphi(t) \geqq F(x) - F(a).$$

Theorem 14. *If $\varphi(x)$ is VBG* on a set E, and $F(x)$, $h(x)$ are any functions whatever, then*

$$\int_a^{-b} h(x)\, dF(x) = \int_a^{-b} \{h(x) f(x)\, d\varphi(x) + h(x) g(x)\, dF(x)\}$$

if either side is finite; where $f(x)$, $g(x)$ are defined thus: — at any point which belongs to E and where $dF(x)/d\varphi(x)$ is defined, we write $f(x) = dF(x)/d\varphi(x)$, $g(x) = 0$; at all other points, we write $f(x) = 0$, $g(x) = 1$.

Let $\chi(x)$ be an increasing function, as in lemma 6, such that the derivates of φ with respect to χ are finite at all points of E except an enumerable set. Suppose that $\int_a^{-b} h(x)\, dF(x) < \infty$, and let $M(x)$ be a major function for this integral. Let the points of E where the derivates of φ with respect to χ are not finite be enumerated as (x_n); define $\beta_n(x)$ as before and write, for any $\varepsilon > 0$,

$$M_1(x) = M(x) + \varepsilon \{\chi(x) + \sum_n \beta_n(x)\}.$$

Then $M_1(x) - M_1(a)$ is a major function for

$$\int_a^{-b} \{h(x) f(x)\, d\varphi(x) + h(x) g(x)\, dF(x)\}.$$

At points x where $f(x) = dF(x)/d\varphi(x)$ and the derivates of φ with respect to X are finite, we have for sufficiently small $\eta > 0$,

$$|\varphi(x+\eta) - \varphi(x)| \leqq K\{\chi(x+\eta) - \chi(x)\}$$

for some K,

$$\begin{aligned} |h(x)| \cdot |F(x+\eta) - F(x) - f(x)\{\varphi(x+\eta) - \varphi(x)\}| &\leqq \varepsilon \cdot \omega(\varphi, \langle x, x+\eta \rangle)/2K \\ &\leqq \varepsilon\{\chi(x+\eta) - \chi(x)\}, \\ M(x+\eta) - M(x) &\geqq h(x)\{F(x+\eta) - F(x)\}, \end{aligned}$$

and so, since $g(x) = 0$,

$$\begin{aligned} h(x) f(x) \{\varphi(x+\eta) - \varphi(x)\} + h(x) g(x) \{F(x+\eta) - F(x)\} \\ = h(x) f(x) \{\varphi(x+\eta) - \varphi(x)\} \\ \leqq h(x) \{F(x+\eta) - F(x)\} + \varepsilon \{\chi(x+\eta) - \chi(x)\} \\ \leqq M(x+\eta) - M(x) + \varepsilon \{\chi(x+\eta) - \chi(x)\} \\ \leqq M_1(x+\eta) - M_1(x). \end{aligned}$$

There is a similar inequality for $\eta < 0$. At points where $f(x) = 0$ and $g(x) = 1$ the conditions for a major function are obviously satisfied, and the remaining points (namely those points (x_n) where $dF(x)/d\varphi(x)$ exists and therefore $F(x)$ is continuous with respect to $\varphi(x)$) are easily seen to be covered by the terms $\beta_n(x)$. Thus $M_1(x) - M_1(a)$ is a major function. Since $M(x)$ was any major function for $\overline{\int_a^b} h(x)\, dF(x)$ and ε was arbitrary, we have

$$\overline{\int_a^b} \{h(x) f(x)\, d\varphi(x) + h(x) g(x)\, dF(x)\} \leqq \overline{\int_a^b} h(x)\, dF(x).$$

The reverse inequality is similarly proved.

Corollary. *If both $F(x)$ and $\varphi(x)$ are VBG^* in $\langle a, b \rangle$, then there exists a set E such that $m\varphi(E) = 0$, $dF(x)/d\varphi(x)$ exists at every point of comp (E) and at no point of E, and for any $h(x)$*

$$\overline{\int_a^b} h(x)\, dF(x) = \overline{\int_a^b} \{h(x) f(x)\, d\varphi(x) + h(x) g(x)\, dF(x)\},$$

where $f(x) = dF(x)/d\varphi(x)$ where it exists and $f(x) = 0$ on E, $g(x) = 0$ on comp (E) and $g(x) = 1$ on E.

(Eingegangen am 19. März 1936.)