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Generalized "Boolean" theory of universal algebras.

Part II.

Identities and subdirect sums of functionally complete algebras.

By

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1. Introduction. The concept of (general) subdirect sum has been very fruitful in the development of various generalizations of the fundamental structure theorem of Boolean rings. Thus, in particular (see part I of the present paper[1]) for references)

(i) each p-ring is isomorphic with a subdirect sum of F_p (= prime field, characteristic p). Similar structure theorems hold for p^k-rings, Post algebras, and for other classes of algebras.

From a quite different approach it was shown by the author that, beyond the mere structure extensions (i), the algebras in question actually constitute generalizations of the Boolean realm in a very wide sense. For instance the Boolean duality principle extends to

(ii) the theorems and concepts of p-rings (p^k-rings) occur in p-al (p^k-al) sets.

Again, for example,

(iii) each p-function (p^k-function, Post function, etc.) possesses a certain *normal expansion*, which specializes to the familiar normal representation of Boolean functions when $p^k = 2$.

In part I (l. c.) this theory—with emphasis on the property (iii)—was further raised to the level of a rather comprehensive class of universal algebras, designated as *f-algebras*. In the same sense in which F_p and F_{p^k} are the *kernels* of the p- and p^k-level extensions, it was shown in part I that each f-algebra, $\mathfrak{U}$, is the kernel of a corresponding extension. The algebras ($\mathfrak{U}$-algebras) comprising this extension then enjoy—in generalized form—such properties as (i)–(iii). These $\mathfrak{U}$-algebras, on the other hand, were furthermore shown to be completely characterized (up to isomorphisms) by the class of all so-called *normal* subdirect sums of $\mathfrak{U}$,—a certain subclass of all subdirect sums of $\mathfrak{U}$.

It was further shown in part I that particular interest attaches to the case where the kernel $\mathfrak{U}$ is a finite functionally complete f-algebra, as for instance with p- and p^k-rings and also with Post algebras, etc. In this case (of functionally complete kernel) the $\mathfrak{U}$-algebras were found to be additionally characterized by the identities of the kernel.

In the present paper we shall generalize the concepts "f-algebra" and "normal subdirect sum", as given in part I, and shall obtain a number of

[1]) FOSTER, A. L.: Generalized "Boolean" theory of universal algebras, etc. Math. Z. **58**, 306—336 (1953).

results related to the background sketched above. For instance we shall establish

Theorem 9.2 *(Principal theorem for strictly complete kernel).*

Let $\mathfrak{U}$ *be a functionally strictly complete universal algebra of at least two elements* (= *order* $n \geq 2$). *Then the following five classes of algebras are all coextensive, up to isomorphisms:*

(1) *The class* $\{\ldots, \mathbf{U}, \ldots\}$ *of all universal algebras* $\mathbf{U}$ *of the same species as* $\mathfrak{U}$ *and such that* (*a*) *each strict identity of* $\mathfrak{U}$ *is also an identity of* $\mathbf{U}$, *and* (*b*) $\mathbf{U}$ *is of order* $n \geq 2$.

(2) *The class of all subalgebras of direct sums of* $\mathfrak{U}$.

(3) *The class of all subdirect sums of* $\mathfrak{U}$.

(4) *The class of all scalar subdirect sums of* $\mathfrak{U}$.

(5) *The class of all normal subdirect sums of* $\mathfrak{U}$.

From (5) and the fact that all the algebras involved are of class *f*, these algebras of theorem 9.2 then possess (generalizations of) such properties as (i)–(iii) above (see part I).

As an interesting corollary it follows (§ 12) that the identities of such a strictly complete kernel, $\mathfrak{U}$, are equationally closed (saturated).

2. Some preliminaries. Let $\mathfrak{A} = (A, o, \ldots)$ be a universal algebra, with $o = o(\xi, \ldots), \ldots$ as primitive operations in the class $A = \{\ldots, \xi, \ldots\}$. An A-function $f(\xi, n, \ldots)$ is simply a function from $A, A, \ldots$ to A. An $\mathfrak{A}$-*function* is an A-function which is a primitive composition of one or more variables $\xi, \ldots$ over A and a (possibly empty) set of constants (= fixed $\in A$). If no constants are involved the $\mathfrak{A}$-function is called *strict*. An $\mathfrak{A}$-*identity* $f(\xi, \ldots) = g(\xi, \ldots)$ is an identity between $\mathfrak{A}$-functions f, g; if both f and g are strict $\mathfrak{A}$-functions the identity is called *strict*.

$\mathfrak{A}$ is *finite*, of *order n*, if A is a class of n elements.

If $\mathfrak{A}$ is finite and if each A-function may be expressed as some $\mathfrak{A}$-function, —respectively as some strict $\mathfrak{A}$-function—then $\mathfrak{A}$ is *(functionally) complete*, —respectively (functionally) *strictly complete*.

3. Algebras of class *f*; frames. A universal algebra $\mathfrak{U} = (U, o, \ldots)$ is said to be of *class f*, or an *f-algebra*, if there exist elements 0, 1 of $\mathfrak{U}$ ($0 \neq 1$), and $\mathfrak{U}$-functions $\times$ (binary), $^\wedge$, $^\vee$ (unary) such that $^\wedge$, $^\vee$ are permutations of U ($\mathfrak{U}$-permutations), with $^\vee$ the inverse of $^\wedge$ and where

$$0 \times \xi = \xi \times 0 = 0; \qquad 1 \times \xi = \xi \times 1 = \xi \qquad (\xi \in \mathfrak{U}).$$
$$0^\wedge = 1, \qquad 1^\wedge = 0.$$

(It follows that $0^\vee = 1$, $1^\vee = 0$; however in general $\xi^\wedge \neq \xi^\vee$. It is also noted that in general $\xi \times \eta$ is neither associative nor commutative.) We then call the algebra $\mathcal{U} = (U; 0, 1; \times, ^\wedge, ^\vee)$ a *frame* of (or in) $\mathfrak{U}$. An *f*-algebra will in general possess more than one frame. Each $\mathcal{U}$-function is of course a $\mathfrak{U}$-function, though the converse is in general false. We are particularly concerned with the $\mathcal{U}$-function $\times_\wedge$, the $^\wedge$ "transform" of $\times$,

$$\xi \times_\wedge \eta = \text{def} = (\xi^\wedge \times \eta^\wedge)^\vee.$$

We call $\times_\wedge$ and $\times$ the *frame-sum* and *frame-product* of the frame $\mathcal{U}$.

Algebras of class f are very common and widespread. For example every ring $(R, \times, +)$ with identity has $(R; 0, 1; \times, ^*)$ as a frame, where $\xi^* = \xi^\wedge = \xi^\vee = 1 - \xi$; again if α is a unity element, $(R; 0, \alpha; \times_1, ^{*_1})$ is a frame, with $\xi \times_1 \eta = \xi \times \frac{\eta}{\alpha}$, $\xi^{*_1} = \alpha - \xi$. Of the manifold further examples we here mention only the case of functionally complete universal algebras $\mathfrak{U}$ (of order $n \geqq 2$), which are all of class f. Indeed for such $\mathfrak{U}$, if $\alpha_0, \alpha_1 (\alpha_0 \neq \alpha_1)$ are any distinct $\in \mathfrak{U}$, there exists at least one frame $\mathcal{U} = (U; \alpha_0, \alpha_1; \times, ^\wedge, ^\vee)$. In particular all finite fields, and also the Post algebra kernel (of order n) are functionally complete and therefore of class f. The same holds for the cyclic algebra of order n, $\mathcal{C}_n = (C_n, \times, ^*)$, where $(C_n, ^*)$ is the cyclic group of order $n-1$, augmented by a null, $0: 0 \times \xi = \xi \times 0 = 0$;

$$\xi^* = (0\,1) = \begin{cases} 1 & \text{if} \quad \xi = 0 \\ 0 & \text{if} \quad \xi = 1 \\ \xi & \text{if} \quad \xi \neq 0, \neq 1. \end{cases}$$

Again $(C_n, \times, ^\frown)$ is complete, where $^\frown$ is taken as an arbitrary but fixed cyclic permutation of the n elements comprising C_n. (Compare with part I.)

4. Normal- and scalar-subdirect sums. Let $\mathfrak{U} = (U, o, \ldots)$ be a universal algebra of class f, and let $\mathcal{U} = (U; 0, 1; \times, ^\wedge, ^\vee)$ be a frame thereof. Let $\mathfrak{U}^{(\mathfrak{N})}$ be a direct sum (= direct power) of $\mathfrak{U}$, where $\mathfrak{N}$ is of arbitrary (finite or transfinite) cardinality. If $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{\mathfrak{N}}) \in \mathfrak{U}^{(\mathfrak{N})}$ and if $\mu \in \mathfrak{U}$, then by $P_\mu(\boldsymbol{\alpha})$, called the μ^{th}-*projection* of $\boldsymbol{\alpha}$ (in, or relative to the frame $\mathcal{U}$), we mean:

$$P_\mu(\boldsymbol{\alpha}) = (\alpha'_1, \alpha'_2, \ldots, \alpha'_{\mathfrak{N}}), \quad \text{where} \quad \alpha'_i = \begin{cases} 1 & \text{if} \quad \alpha_i = \mu \\ 0 & \text{if} \quad \alpha_i \neq \mu. \end{cases}$$

Let $\boldsymbol{\mathfrak{U}} = (\boldsymbol{U}, o, \ldots)$ be a subalgebra of $\mathfrak{U}^{(\mathfrak{N})}$. If $\boldsymbol{\mathfrak{U}}$ satisfies

(1°) all elements (= 'scalars') $(\mu, \mu, \mu, \ldots, \mu)$ are $\in \boldsymbol{\mathfrak{U}}$ $(\mu \in \mathfrak{U})$, we call $\boldsymbol{\mathfrak{U}}$ a *scalar-subdirect sum* of $\mathfrak{U}$. If in addition $\boldsymbol{\mathfrak{U}}$ also satisfies

(2°) for each $\boldsymbol{\alpha} \in \boldsymbol{\mathfrak{U}}$ and for each $\mu \in \mathfrak{U}$, $P_\mu(\boldsymbol{\alpha}) \in \boldsymbol{\mathfrak{U}}$, we call $\boldsymbol{\mathfrak{U}}$ a *normal-subdirect* sum of $\mathfrak{U}$[2]).

The independence of the concept of normality from the particular frame chosen is shown by

Theorem 4.1. *If $\boldsymbol{\mathfrak{U}}$ is a normal subdirect sum of $\mathfrak{U}$ relative to some frame $\mathcal{U}$, then it is a normal subdirect sum relative to any frame $\mathcal{U}'$.*

Proof. Let P' denote projection relative to the frame $\mathcal{U}' = (U; 0', 1'; \times', ^{\wedge'}, ^{\vee'})$. One easily verifies the formula

(4.1) $$P'_\mu(\boldsymbol{\alpha}) = \{(1', 1', 1', \ldots) \times P_\mu(\boldsymbol{\alpha})\} \times_\wedge \{(0', 0', 0', \ldots) \times (P_\mu(\boldsymbol{\alpha}))^\wedge\}.$$

Since the right of (4.1) is $\in \boldsymbol{\mathfrak{U}}$ the theorem is proved.

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[2]) That $\boldsymbol{\mathfrak{U}}$ is a subdirect sum (see Birkhoff [*1*]) follows from (1°).

5. Normal representation; kernel, core. As in part I we shall agree to identify the *kernel*, $\mathfrak{U}$, with its isomorph $\mathfrak{U}_1$ = subalgebra of all scalars

$$(\mu, \mu, \mu, \ldots, \mu), \qquad (\mu \in \mathfrak{U}):$$

$$\mathfrak{U} \cong \mathfrak{U}_1, \quad \mathfrak{U} = \mathfrak{U}_1: \ \mu \longleftrightarrow (\mu, \mu, \ldots, \mu), \ \ \mu = (\mu, \mu, \ldots, \mu); \ \ \mathfrak{U} \subseteqq \boldsymbol{\mathfrak{U}}.$$

With this understanding the identity (4.1), for instance, is written simply

$$P'_\mu(\boldsymbol{\alpha}) = 1' P_\mu(\boldsymbol{\alpha}) \times_\wedge 0'(P_\mu(\boldsymbol{\alpha}))^\wedge.$$

The kernel $\mathfrak{U} (= \mathfrak{U}_1)$ of $\boldsymbol{\mathfrak{U}}$ is obviously the same for all frames in $\boldsymbol{\mathfrak{U}}$.

By the *core*, J, of $\boldsymbol{\mathfrak{U}}$ — relative to a given frame $\mathcal{U}$ — we mean the set of all elements $P_\mu(\boldsymbol{\alpha})$; $(\mu \in \mathfrak{U}, \boldsymbol{\alpha} \in \boldsymbol{\mathfrak{U}})$. As in part I $(J, \times, {}^\wedge, {}^\vee)$ is shown to be a Boolean algebra with $a \times b$ as Boolean intersection, $a^\wedge = a^\vee = a^* =$ Boolean complement $(a, b \in J)$, and 0, 1 as null and universe. With only very minor modifications of the corresponding proofs in part I we have

Theorem 5.1 (*Normal representation theorem*). *Let $f(\boldsymbol{\xi}, \boldsymbol{\eta}, \ldots)$ be any $\boldsymbol{\mathfrak{U}}$-function, where $\boldsymbol{\mathfrak{U}}$ is a normal subdirect sum of a universal algebra $\mathfrak{U}$ of class f. Then for each fixed frame in $\boldsymbol{\mathfrak{U}}$ we have a normal decomposition*[3])

$$f(\boldsymbol{\xi}, \boldsymbol{\eta}, \ldots) = \sum_{\alpha, \beta, \ldots \in \mathfrak{U} = \text{kernel of } \boldsymbol{\mathfrak{U}}}^{\times_\wedge} f(\alpha, \beta, \ldots)\, P_\alpha(\boldsymbol{\xi})\, P_\beta(\boldsymbol{\eta}) \ldots .$$

Theorem 5.1′ (*Alternate formulation*). *In the notation of theorem 5.1, each $\boldsymbol{\xi} \in \boldsymbol{\mathfrak{U}}$ may be expressed in one and only one way in the normal form*

$$\boldsymbol{\xi} = \sum_{\mu \in \mathfrak{U}}^{\times_\wedge} \mu\, a_\mu$$

in which the a_μ are pairwise disjoint core $(= J)$ elements which cover J $\left(i.e., \sum\limits_{\mu \in \mathfrak{U}}^{\times_\wedge} a_\mu = 1\right)$. Then

$$f(\boldsymbol{\xi}, \boldsymbol{\eta}, \ldots) = \sum_{\alpha, \beta, \ldots \in \mathfrak{U}}^{\times_\wedge} f(\alpha, \beta, \ldots)\, a_\alpha\, b_\beta \ldots,$$

where $\{a_\mu\}$, $\{b_\mu\}$, ... are the core components of $\boldsymbol{\xi}, \boldsymbol{\eta}, \ldots$.

6. Functionally complete kernel. Let $\boldsymbol{\mathfrak{U}}$ be a universal algebra and $\mathfrak{U}$ a subalgebra. A $\mathfrak{U}$-identity $f(\xi, \ldots) = g(\xi, \ldots)$ is said to *extend* to $\boldsymbol{\mathfrak{U}}$ or to be *satisfied* by $\boldsymbol{\mathfrak{U}}$ if $f(\boldsymbol{\xi}, \ldots) = g(\boldsymbol{\xi}, \ldots)$ is a $\boldsymbol{\mathfrak{U}}$-identity; here the $\mathfrak{U}$-variables $\xi, \ldots$ but not the constants (if any) appearing in $f(\xi, \ldots)$, $g(\xi, \ldots)$ are replaced by $\boldsymbol{\mathfrak{U}}$-variables $\boldsymbol{\xi}, \ldots$.

Theorem 6.1. *Let $\boldsymbol{\mathfrak{U}}$ be a universal algebra possessing a subalgebra, $\mathfrak{U}$, where*

(1°) *$\mathfrak{U}$ is finite and functionally complete;*

(2°) *each $\mathfrak{U}$-identity extends to $\boldsymbol{\mathfrak{U}}$;*

(3°) *$\mathfrak{U}$ contains at least two elements.*

Then $\boldsymbol{\mathfrak{U}}$ is isomorphic with a normal subdirect sum of $\mathfrak{U}$.

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[3]) For the limited associative and commutative properties which make parentheses in $\sum\limits^{\times_\wedge}$ unnecessary see FOSTER, part I, l. c.

Proof. Let 0, 1 be any distinct elements of $\mathfrak{U}$. Then $\mathfrak{U}$ is of class f and possesses a frame $\mathfrak{U}=(U; 0, 1; \times, {}^{\wedge}, {}^{\vee})$. That $\mathbf{\mathfrak{U}}$ is a normal subdirect sum of $\mathfrak{U}$—relative to the frame $\mathfrak{U}$—follows with only minor modifications (required because of our more general formulation of the concepts 'of class f' and 'normal subdirect sum') from the proof of a similar theorem of part I. Theorem 4.1 then shows that $\mathbf{\mathfrak{U}}$ is independent of the particular frame chosen.

The following more general converse of theorem 6.1 is an immediate consequence of the definitions involved:

Theorem 6.2. *If $\mathbf{\mathfrak{U}}$ is a scalar-subdirect sum of a universal algebra, $\mathfrak{U}$, then the scalars of $\mathbf{\mathfrak{U}}$ form a subalgebra isomorphic (and to be identified) with $\mathfrak{U}$, and every $\mathfrak{U}$-identity extends to $\mathbf{\mathfrak{U}}$.*

By forward reference to lemma 8 and from a joint consideration of theorems 6.1 and 6.2 we have

Theorem 6.3 *(Principal theorem for complete kernel). Let $\mathfrak{U}$ be a functionally complete universal algebra of order $n \geq 2$. Then the following three classes of algebras are all coextensive, up to isomorphisms:*

(1) *The class $\{\ldots, \mathbf{\mathfrak{U}}, \ldots\}$ of all overalgebras $\mathbf{\mathfrak{U}}$ of $\mathfrak{U}$ such that each $\mathfrak{U}$-identity extends to $\mathbf{\mathfrak{U}}$.*

(2) *The class of all scalar-subdirect sums of $\mathfrak{U}$.*

(3) *The class of all normal-subdirect sums of $\mathfrak{U}$.*

That theorem 6.1 cannot be proved if the condition (1°) thereof is deleted is seen by the following example. Let $\mathfrak{A}=(A, \times)=\{\alpha, \alpha^2, \alpha^3, \ldots\}$, with $\alpha^m \times \alpha^n = \alpha^{m+n}$. Also let $\mathfrak{A}'=(A', \times')$ be any group, and let $\mathbf{\mathfrak{A}}=\mathfrak{A}\times\mathfrak{A}'$ (direct sum). Since $\mathfrak{A}$ is the free algebra (of one generator) of the identities $\xi\times\eta=\eta\times\xi$, $\xi\times(\eta\times\xi)=(\xi\times\eta)\times\xi$, it is obvious that $\mathfrak{A}'$, and hence also $\mathbf{\mathfrak{A}}$ satisfies all identities satisfied by $\mathfrak{A}$. Moreover $\mathfrak{A}$ is (isomorphic with) a subalgebra of $\mathbf{\mathfrak{A}}$. Thus, with the exception of (1°) the conditions of theorem 6.1 are satisfied. However the conclusion is now false, in fact a simple argument, which we omit, shows that $\mathbf{\mathfrak{A}}$ is not isomorphic with any subalgebra of a direct sum of $\mathfrak{A}$.

7. Strictly complete kernel. The finite fields $\mathcal{F}_p=(F_p, \times, +)$ and $\mathcal{F}_{p^k}=(F_{p^k}, \times, +)$ are complete but not strictly complete, e.g., the constant function, 1, cannot be expressed as a strict $\mathcal{F}_p$- or an $\mathcal{F}_{p^k}$-function. However (see part I) expressed in the form $(F_p, \times, ')$ and $(F_{p^k}, \times, {}^{\frown})$ these fields are strictly complete. Here

$$\xi' = \xi + 1 \pmod{p}; \quad \xi^{\frown} = \begin{Bmatrix} \lambda\xi & \text{for} & \xi \neq 0, \; \neq \lambda^{p^k-2} \\ 1 & \text{for} & \xi = 0 \\ 0 & \text{for} & \xi = \lambda^{p^k-2} \end{Bmatrix},$$

$$(\xi \in F_{p^k};\ \lambda = \text{multip. generator of } F_{p^k}).$$

Again, the algebra $\mathcal{C}_n=(C_n, \times, {}^{\frown})$ previously noted (§ 3) is strictly complete. [For $n=p^k=$ prime power this algebra is identical with the Galois field $(F_{p^k}, \times, {}^{\frown})$.] Further, the Post kernel $(P_n, \times, ')$ of order n is strictly complete. Other examples, and criterea for completeness (ordinary and strict) may be found in part I.

The 'simplicity' of the concept of strict completeness is shown by

Theorem 7.1. *If* $\mathfrak{U}$ *is strictly complete,*

(1°) $\mathfrak{U}$ *contains no proper subalgebra;*

(2°) $\mathfrak{U}$ *contains no proper ideals, i.e., if* $\mathfrak{U} \to \mathfrak{U}'$ *is a homomorphism, then* $\mathfrak{U} \cong \mathfrak{U}'$ *(isomorphism), or* $\mathfrak{U}'$ *is the one-element algebra.*

Proof. Suppose $\mathfrak{B}$ were a proper subalgebra, and suppose $\alpha \in \mathfrak{U}$, $\alpha \notin \mathfrak{B}$. There exists a strict $\mathfrak{U}$-function $f(\xi)$ such that $f(\xi) \equiv \alpha$. Then in particular for $\eta \in \mathfrak{B}$, $f(\eta) = \alpha$. But this contradicts the assumption that $\mathfrak{B}$ is a subalgebra, and (1°) is proved.

Consider (2°). Suppose $\mathfrak{U}$ possesses a proper ideal (= partition), $\mathfrak{q}$. Then there exist at least two residue classes Q, Q' of $\mathfrak{q}$ neither of which is empty and one of which contains at least two elements. Say $\alpha_1, \alpha_2 \in Q$, $\alpha' \in Q'$. There exists (at least one) strict $\mathfrak{U}$-function $h(\xi)$ such that $h(\alpha_1) = \alpha_2$, $h(\alpha_2) = \alpha'$. This is in contradiction with the assumption that $\mathfrak{q}$ is an ideal, and (2°) is proved. From (1°) we have the immediate

Corollary. *If* β *is any element of* $\mathfrak{U}$ *then* β *generates* $\mathfrak{U}$, $\mathfrak{U} = (\beta)$.

8. Complete kernel (continued). If $\mathfrak{U}$ is a universal algebra of class f, simple counterexamples show that (a) a subalgebra of a direct sum of $\mathfrak{U}$ is not in genaral a subdirect sum of $\mathfrak{U}$, and further that (b) a subdirect sum of $\mathfrak{U}$ is in general not normal, nor even scalar. [In fact (a) and (b) are still true even when $\mathfrak{U}$ is (merely) complete.] We have however the

Lemma 8. (1) *If* $\mathfrak{U}$ *is a functionally strictly complete universal algebra of order* $n \geq 2$, *then every subalgebra of a direct sum of* $\mathfrak{U}$ *is a normal-subdirect sum of* $\mathfrak{U}$.

(2) *If* $\mathfrak{U}$ *is a functionally complete universal algebra of order* $n \geq 2$, *then every scalar-subdirect sum of* $\mathfrak{U}$ *is normal.*

Proof of (1). Let $\mathcal{U}$ be a frame in $\mathfrak{U}$, let $\mu \in \mathfrak{U}$ and let the constant $\mathfrak{U}$-function $f(\xi) \equiv \mu$ and the characteristic function $\delta_\mu(\xi)$, (relative to $\mathcal{U}$), be expressed as strict $\mathfrak{U}$-functions. If $\boldsymbol{\mathfrak{U}}$ is a subalgebra of a direct sum of $\mathfrak{U}$ and if $\boldsymbol{\alpha} = (\alpha_1, \ldots) \in \boldsymbol{\mathfrak{U}}$, then

$$f_\mu(\boldsymbol{\alpha}) = (\mu, \mu, \mu, \ldots) = \in \boldsymbol{\mathfrak{U}}$$
$$\delta_\mu(\boldsymbol{\alpha}) = P_\mu(\boldsymbol{\alpha}) \doteq \in \boldsymbol{\mathfrak{U}}.$$

Hence all scalars are $\in \boldsymbol{\mathfrak{U}}$, and with $\boldsymbol{\alpha}$ also all projections of $\boldsymbol{\alpha}$ are $\in \boldsymbol{\mathfrak{U}}$, which proves (1). The proof of (2) is similar and will be omitted.

9. Proof of theorem 9.2 (stated in § 1), etc.

Theorem 9.1. *Let* $\mathfrak{U}$, $\boldsymbol{\mathfrak{U}}$ *be any universal algebras of the same species and each containing at least two elements, and where*

(1°) $\mathfrak{U}$ *is finite and functionally strictly complete.*

(2°) *Each strict identity of* $\mathfrak{U}$ *is also an identity of* $\boldsymbol{\mathfrak{U}}$.

Then

(i) $\boldsymbol{\mathfrak{U}}$ *is isomorphic with a normal subdirect sum of* $\mathfrak{U}$.

(ii) *Each strict identity of* $\boldsymbol{\mathfrak{U}}$ *is an identity of* $\mathfrak{U}$.

Proof. Let $\boldsymbol{\mathfrak{U}}$, $\mathfrak{U}$ satisfy (1°) and (2°) above, and let $\alpha \in \mathfrak{U}$, $f_\alpha(\xi) \equiv \alpha$, where $f_\alpha(\xi)$ is expressed as a strict $\mathfrak{U}$-function. Then $f_\alpha(\boldsymbol{\xi})$ is a strict $\boldsymbol{\mathfrak{U}}$-function. Applying (2°) to the $\mathfrak{U}$-identity $f_\alpha(\xi) = f_\alpha(\eta)$, we have the strict $\boldsymbol{\mathfrak{U}}$-identity $f_\alpha(\boldsymbol{\xi}) = f_\alpha(\boldsymbol{\eta})$. That is, for given $\alpha \in \mathfrak{U}$, $f_\alpha(\boldsymbol{\xi})$ is a constant $\boldsymbol{\mathfrak{U}}$-function, call it α': $f_\alpha(\boldsymbol{\xi}) \equiv \alpha' = \in \boldsymbol{\mathfrak{U}}$.

Consider the mapping

$$(9.1) \qquad \alpha \to \alpha' = f_\alpha(\boldsymbol{\xi}), \qquad (\alpha \in \mathfrak{U}).$$

Let $\alpha \to \alpha'$, $\beta \to \beta'$, ... let $o(\xi, \eta, \ldots)$ be a primitive operation and let $o(\alpha, \beta, \ldots) = \gamma$. From (2°) and the strict $\mathfrak{U}$-identities

$$o\big(f_\alpha(\xi), f_\beta(\xi), \ldots\big) = f_\gamma(\eta)$$

we obtain the strict $\boldsymbol{\mathfrak{U}}$-identities

$$o\big(f_\alpha(\boldsymbol{\xi}), f_\beta(\boldsymbol{\xi}), \ldots\big) = f_\gamma(\boldsymbol{\eta}),$$

that is

$$o(\alpha', \beta', \ldots) = \gamma'.$$

We have thus shown that $\mathfrak{U}'$, the class of all images $\alpha', \ldots$ under (9.1), is a subalgebra of $\boldsymbol{\mathfrak{U}}$, and that (9.1) is a homomorphism of $\mathfrak{U}$ onto $\mathfrak{U}'$. By theorem 7.1 either $\mathfrak{U} \cong \mathfrak{U}'$ or else $\mathfrak{U}' = \tau'$ (= one element algebra). The latter is impossible, for if so, from (2°) of theorem 9.1 and obvious corresponding $\mathfrak{U}$-identies we have: for any $\boldsymbol{\xi} \in \boldsymbol{\mathfrak{U}}$,

$$\boldsymbol{\xi} = \sum_{\mu \in \mathfrak{U}}^{\times\wedge} f_\mu(\boldsymbol{\xi})\, \delta_\mu(\boldsymbol{\xi}) = \sum_{\mu \in \mathfrak{U}}^{\times\wedge} \tau'\, \delta_\mu(\boldsymbol{\xi}) = \tau'.$$

This contradicts our assumption that $\boldsymbol{\mathfrak{U}}$ contains at least two elements. Hence $\mathfrak{U} \cong \mathfrak{U}'$. The assertion (i) of theorem 9.1 then follows from theorem 6.1. Finally (ii) is a consequence of the fact that $\mathfrak{U} (= \mathfrak{U}')$ is a subalgebra of $\boldsymbol{\mathfrak{U}}$.

The proof of theorem 9.2 (see § 1) now follows at one from lemma 8 and theorem 9.1.

Employing corollary 22 of part I we have the special

Theorem 9.3. *Let* $\mathfrak{U}$, $\boldsymbol{\mathfrak{U}}$ *satisfy either the hypotheses of theorem* 9.1 *or those of theorem* 6.1. *Then if* $\boldsymbol{\mathfrak{U}}$ *is a finite algebra,* $\boldsymbol{\mathfrak{U}}$ *is isomorphic with a direct sum of* $\mathfrak{U}$, *and the order of* $\boldsymbol{\mathfrak{U}}$ *is a power of the order of* $\mathfrak{U}$.

10. Core, normal expansion, for complete kernel. The considerations of §§ 4, 5, holding for arbitrary f-algebra kernels, are perforce synthetic in nature. For complete kernels, however, they may be given an internal formulation. We shall merely sketch this section since the arguments are essentially those of §§ 19, 14 of part I, where however the less general definition of f-algebra and normal subdirect sum were used.

Let $\mathfrak{U}$ be functionally complete, $\boldsymbol{\mathfrak{U}}$ a normal subdirect sum of $\mathfrak{U}$ and $\mathcal{U} = (U; 0, 1; \times, {}^\wedge, {}^\vee)$ a frame in $\mathfrak{U}$. Then the μ^{th} *characteristic* function $(\mu \in \mathfrak{U})$,

$$\delta_\mu(\xi) = \begin{cases} 1 & \text{if } \xi = \mu \\ 0 & \text{if } \xi \neq \mu \end{cases} \qquad (\xi \in \mathfrak{U})$$

may—and we assume it to—be expressed as a $\mathfrak{U}$-function, or, in case $\mathfrak{U}$ is strictly complete, as a strict $\mathfrak{U}$-function.

The μ^{th} projection $P_\mu(\boldsymbol{\xi})$, $(\boldsymbol{\xi} \in \mathfrak{U})$ is then given by

$$P_\mu(\boldsymbol{\xi}) = \delta_\mu(\boldsymbol{\xi}),$$

where $\delta_\mu(\boldsymbol{\xi})$ is the extension to $\mathfrak{U}$ of the characteristic function $\delta_\mu(\xi)$, that is, $\delta_\mu(\boldsymbol{\xi})$ is obtained from $\delta_\mu(\xi)$ by replacing the $\mathfrak{U}$-variable ξ by the $\mathfrak{U}$-variable $\boldsymbol{\xi}$. [If $\mathfrak{U}$ is merely simply, but not strictly complete, any constants occurring in the $\mathfrak{U}$-function $\delta_\mu(\xi)$ remain unaltered in passing to $\delta_\mu(\boldsymbol{\xi})$.] Thus $P_\mu(\boldsymbol{\xi})$ is 'internally' expressed, as a $\mathfrak{U}$-function, or even—in case $\mathfrak{U}$ is strictly complete—as a strict $\mathfrak{U}$-function.

From earlier sections we then readily have

Theorem 10.1. *Let $\mathfrak{U}$ be a universal algebra possessing a functionally complete subalgebra (of at least two elements) such that every $\mathfrak{U}$-identity extends to $\mathfrak{U}$. Let $\mathcal{U} = (U; 0, 1; \times, ^\wedge, ^\vee)$ be any frame in $\mathfrak{U}$ and let $\delta_\mu(\boldsymbol{\xi})$ be the extension to $\mathfrak{U}$ of the characteristic function $\delta_\mu(\xi)$ of $\mathfrak{U}$. Then each $\mathfrak{U}$-function $f(\boldsymbol{\xi}, \boldsymbol{\eta}, \ldots)$ possesses the normal decomposition*

$$f(\boldsymbol{\xi}, \boldsymbol{\eta}, \ldots) = \sum_{\alpha, \beta, \ldots \in \mathfrak{U}}^{\times\wedge} f(\alpha, \beta, \ldots)\, \delta_\alpha(\boldsymbol{\xi})\, \delta_\beta(\boldsymbol{\eta}) \ldots.$$

Theorem 10.2. *Let $\mathfrak{U}$ be a functionally strictly complete algebra (of at least two elements) and let $\mathfrak{U}$ be any algebra that satisfies all the strict identities of $\mathfrak{U}$. If $f_\alpha(\xi) \equiv \alpha$, $(\alpha \in \mathfrak{U})$ is the constant function, α, expressed as a strict $\mathfrak{U}$-function, the class of corresponding (strict) $\mathfrak{U}$-functions, $\mathfrak{U}' = \{f_\alpha(\boldsymbol{\xi})\}$, $(\alpha \in \mathfrak{U}$, $\boldsymbol{\xi} \in \mathfrak{U})$ form an isomorphic subalgebra of $\mathfrak{U}$, $\mathfrak{U} \cong \mathfrak{U}'$ and, in the notation of theorem 10.1, each (not necessarily strict-) $\mathfrak{U}$-function possesses the normal expansion of theorem 10.1, with $\sum\limits_{\alpha, \beta, \ldots \in \mathfrak{U}}$ replaced by $\sum\limits_{\alpha, \beta, \ldots \in \mathfrak{U}'}$.*

Core. For $\mathfrak{U}$, $\mathfrak{U}$ as in theorem 10.1 or theorem 10.2, the core J of $\mathfrak{U}$—relative to the frame $\mathcal{U}$ in $\mathfrak{U}$—is internally expressed by (see part I)

$$(10.1)\quad \begin{aligned} &J = \{\delta_\mu(\boldsymbol{\xi})\}, \text{ where } \mu \text{ runs through } \mathfrak{U} \text{ and } \boldsymbol{\xi} \text{ runs through } \mathfrak{U}. \\ &J = \{\delta_\alpha(\boldsymbol{\xi}), \text{ where } \alpha = \text{fixed} \in \mathfrak{U} \text{ and } \boldsymbol{\xi} \text{ runs through } \mathfrak{U}. \end{aligned}$$

In particular, $J = \{\delta(\boldsymbol{\xi})\}$, where $\delta(\xi) = \delta_0(\xi)$.

If 0, 1 are any $\in \mathfrak{U}$, since $\mathfrak{U}$ is complete we may always choose a *special* frame $\mathcal{U}' = (U; 0, 1; \times, ^\wedge, ^\vee)$ in which 0 and 1 are the only idempotent elements (relative to $\times$). A comparison of the synthetic and internal aspects of normal subdirect sums then yields

Theorem 10.3. *If $\mathfrak{U}$ is a normal subdirect sum of a functionally complete algebra, $\mathfrak{U}$, and if $\mathcal{U}'$ is a special frame in $\mathfrak{U}$, then, relative to $\mathcal{U}'$, the core of $\mathfrak{U}$ consists of the totality of idempotent elements of $\mathfrak{U}$, $(\boldsymbol{\xi} \times \boldsymbol{\xi} = \boldsymbol{\xi})$.*

Such special frames, with corresponding easily computed cores, are useful in connection with the normal decomposition of $\mathfrak{U}$-functions given by theorem 5.1'.

Thus, in the case of the kernel F_p of a p-ring, and F_{p^k} of a p^k-ring, the 'natural' frames $(F_p, \times, *)$, $(F_{p^k}, \times, *)$ are special in the above sense; correspondingly the (natural) core of a p- or of a p^k-ring consists of its totality of idempotent elements. A similar remark applies to the $\mathcal{C}_n$-algebras [satisfying all identities of $(C_n, \times, *)$, or all strict identities of $(C_n, \times, ^\frown)$,—see

§ 3]. However any 'natural' frame in the Post kernel $(P_n, \times, ')$, that is any frame utilizing $\times$ as frame product, is not special (— every element of P_n is here idempotent). With respect to such frames the core of Post algebras is determined by the general formulas (10.1).

11. Universal fundamental set of identities. A set $\mathfrak{I} = \{\ldots, I_i, \ldots\}$ of identities of an algebra is called a *fundamental* set if every identity of the algebra is a logical consequence thereof.

Let $\mathfrak{U}$ be functionally complete, let $\mathcal{U}$ be a frame thereof, and let $\delta(\xi) = \delta_0(\xi)$ be its 0^{th} characteristic function and $\xi^\frown$ any transitive $0 \to 1$ $\mathfrak{U}$-permutation. From the considerations of § 19 of part I one may, in terms of these, write down a certain finite *universal* fundamental set, $\mathfrak{I}^*$

$$\mathfrak{I}^* = \begin{cases} \delta(\xi) \times \delta(\xi) = \delta(\xi) \\ \vdots \end{cases}$$

which is 'universal' in the following sense: if $\mathfrak{U}$ is any functionally complete algebra with a frame chosen therein, and if $\delta(\xi)$, $\xi^\frown$ are expressed as $\mathfrak{U}$-functions, the expressions $\mathfrak{I}^*$ form a fundamental set of identities for $\mathfrak{U}$. Thus, for instance, for $(F_p, \times, +)$, using the natural frame $(F_p, \times, {}^*)$, the first expression of $\mathfrak{I}^*$ becomes the identity $(1-\xi^{p-1})\,(1-\xi^{p-1}) = (1-\xi^{p-1})$, etc.

Of course for a given complete algebra $\mathfrak{U}$ the universal fundamental set will generally not be as short or simple as one which is geared to the particular peculiarities of structure of $\mathfrak{U}$. The universal set may however serve as a standard for testing a candidate set of identities for 'fundamentalness'.

12. Equational closure and functional completeness. Let $o, \ldots$ be a class of operations and let $\mathcal{E} = \{\ldots, E_i, \ldots\}$ be a set of (strict) equations in these operations. Every such set $\mathcal{E}$ has the one-element model, called the *trivial* model. If $\mathcal{E}$ possesses only the trivial model, $\mathcal{E}$ is called *inconsistent*; otherwise $\mathcal{E}$ is *consistent*.

Let E' be a strict equation in the operations $o, \ldots$ and let $\mathcal{E}' = \mathcal{E}(E')$ denote the class $\mathcal{E}$ augmented by E'. If E' is a logical consequence of the identities $\mathcal{E}$ we write $\mathcal{E}$ *equiv.* $\mathcal{E}'$.

A set $\mathcal{E}$ of equations is called *equationally closed* (or *saturated*) if, for every E', either $\mathcal{E}$ equiv. $\mathcal{E}'$ or else $\mathcal{E}'$ is inconsistent.

Theorem 12.1. *Let $\mathfrak{U}$ be a strictly complete (finite) algebra of order $n \geq 2$, and let $\mathfrak{I} = \{\ldots, I_i, \ldots\}$ be a fundamental set of (or else the set of all) strict identities of $\mathfrak{U}$. Then $\mathfrak{I}$ is equationally closed.*

Proof. Let I' be any strict equation in the primitive operations $o, \ldots$ of $\mathfrak{U}$. Now by theorem 9.2 all models of $\mathfrak{I}$, and hence also all models of $\mathfrak{I}' = \mathfrak{I}(I')$ if $\mathfrak{I}'$ is consistent, are normal subdirect sums of $\mathfrak{U}$. But, by theorem 9.1, each such model satisfies those and *only* those strict identities satisfied by $\mathfrak{U}$. Hence if $\mathfrak{I}'$ is consistent it follows that I' is a logical consequence of $\mathfrak{I}$, and the theorem is proved.

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