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Micromechanics and effective moduli of elastic composites containing randomly dispersed ellipsoidal inhomogeneities

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Summary. A micromechanical framework is proposed to investigate effective mechanical properties of elastic multiphase composites containing many randomly dispersed ellipsoidal inhomogeneities. Within the context of the representative volume element (RVE), four governing micromechanical ensemble-volume averaged field equations are presented to relate ensemble-volume averaged stresses, strains, volume fractions, eigenstrains, particle shapes and orientations, and elastic properties of constituent phases of a linear elastic particulate composite. A renormalization procedure is employed to render absolutely convergent integrals. Therefore, the micromechanical equations and effective elastic properties of a statistically homogeneous composite are independent of the shape of the RVE. Various micromechanical models can be developed based on the proposed ensemble-volume averaged constitutive equations. As a special class of models, inter-particle interactions are completely ignored. It is shown that the classical Hashin-Shtrikman bounds, Walpole's bounds, and Willi's bounds for isotropic or anisotropic elastic multiphase composites are related to the "noninteracting" solutions. Further, it is demonstrated that the Mori-Tanaka method *coincides* with the Hashin-Shtrikman bounds and the "noninteracting" micromechanical model in some cases. Specialization to unidirectionally aligned penny-shaped microcracks is also presented. An accurate, higher order (in particle concentration), probabilistic pairwise particle interaction formulation coupled with the proposed ensemble-volume averaged equations will be presented in a companion paper.

1 Introduction

The prediction and estimation of effective (overall) mechanical and conductive properties of random heterogeneous multiphase materials are of great interest to researchers and engineers in many science and engineering disciplines. The so-called "effective" properties of a heterogeneous composite are obtained by some volume- and ensemble-averaging processes over a "representative volume element" (RVE) featuring a "mesoscopic" length scale which is much larger than the characteristic length scale of particles (inhomogeneities) but smaller than the characteristic length scale of a macroscopic specimen. In this paper, all particles are assumed to be *non-intersecting* (impenetrable) and embedded firmly into a homogeneous matrix material; i.e., perfect interfacial bonding is assumed. Further, we assume that *statistical homogeneity* holds. Therefore, effective (averaged) material properties remain the same for arbitrary averaging domains inside a composite medium. As a consequence, heterogeneous composites can be represented by equivalent homogeneous continuum media with appropriately defined effective properties. Mathematically, this procedure is related to the homogenization method. Examples of heterogeneous particles or fibrous composites are abundant, such as graphite/epoxy composites, ceramic matrix composites, porous and cracked media, concrete and cementitious composites, polymer-blended soils, and rocks, etc.

There are many theoretical methods in the literature to tackle this class of problems (see Hashin [1]). The first school, stemming from the pioneering work of Hashin and Shtrikman [2]–[4], employs variational principles to obtain mathematical lower and upper bounds for effective properties of multiphase particulate composites. See, for example, Walpole [5]–[8], Beran and Molyneux [9], and Willis [10]. "Improved" higher-order mathematical bounds which depend on statistical microstructural informations of random heterogeneous composites have been proposed by, e.g., McCoy [11], Silnutzer [12], Milton [13]–[15], Milton and Phan-Thien [16], Torquato and Lado [17], and Sen, Lado and Torquato [18]. We refer to Torquato [19] for a state-of-the-art review on the improved higher-order statistical bounds of two-phase linear composites. Recently, novel *nonlinear* variational bounds for isotropic elastic and viscoelastic particulate composites were proposed by Ponte Castaneda and Willis [20], Ponte Castaneda [21], and Willis [22]. The second school for micromechanical estimation of effective properties of heterogeneous composites is known as the effective medium approach. This school includes the self-consistent method (Hill [23], Budiansky [24], Budiansky and O'Connell [25]), the differential scheme (Roscoe [26], [27], McLaughlin [28], Hashin [29]), the Mori-Tanaka method (Mori and Tanaka [30], Taya and Chou [31], Taya and Mura [32], Taya [33], Weng, [34], Benveniste [35], Zhao et al. [36], Weng [37], Qiu and Weng [38]), and the generalized self-consistent method (Christensen and Lo [39]), etc. However, effective medium methods do *not* depend on particle locations or their relative configurations. See also Hashin [1], Laws and Dvorak [40], Nemat-Nasser and Hori [41], and Christensen [42] for some comparisons and assessments of these methods.

The third school aims at *direct* determination of effective properties of composites with *randomly located* particles by introducing some approximations or by assuming certain special configurations for particles (inhomogeneities) dispersing in matrix materials. For example, effective elastic properties of two-phase composites can be evaluated by a power series in particle volume fraction ϕ, with each ϕ^n-term corresponding to averaged n-particle interaction effects. At *dilute* particle concentrations, some results were obtained by Dewey [43], Kerner [44], Eshelby [45] and Hashin [46] to accommodate the $O(\phi)$-term by considering only effects due to *single* particles (i.e., no inter-particle interactions). On the other hand, Walpole [8], Batchelor and Green [47], Willis and Acton [48], and Chen and Acrivos [49], [50] determined the $O(\phi^2)$ contributions by bringing in the effects of pairwise *inter*-particle interactions. These methods are suitable for low particle densities; e.g., for ϕ around $15 \sim 25\%$. It is noted that the n-th order corrections to effective properties require micromechanical solutions of the complicated n-particle interaction problem. This difficult problem still remains mostly intractable for n greater than two. Therefore, there has been no further analytical work in deriving the higher-order interactions and corrections. Recently, nonetheless, a numerical approach to compute many-particle interactions was proposed by Rodin and Hwang [51] for *deterministic* (not statistical) particle arrays. Their results, contrary to all existing results, concluded that particle interactions and thus higher-order corrections at very high particle concentrations could be completely ignored.

Further, assuming that composites contain particles of *periodic arrays*, several analytical results were obtained in recent years. See, for instance, Nemat-Nasser and Taya [52], [53], Iwakuma and Nemat-Nasser [54], Nunan and Keller [55] and Sangani and Lu [56]. Although the assumption of periodic arrays of particles makes it possible to solve the many-particle interaction problem exactly, it is, however, unrealistic for most composites containing randomly dispersed particles. In addition, it is noteworthy that an *exact* expression for the effective isotropic *bulk modulus* (not shear modulus) was obtained by the "composite sphere assemblage" (Hashin [57]) if there is a continuous distribution of particle sizes (starting from infinitesimally small). Namely, the model requires that composite spheres fill all spaces in a composite body.

In this paper, we attempt to establish governing micromechanical field equations for an RVE and to improve theoretical predictions (not mathematical bounds) of effective elastic moduli of composites containing many *randomly dispersed* particles embedded in homogeneous matrix materials. An outline of this paper is as follows. In Section 2, we consider a linear elastic multiphase composite containing randomly located *ellipsoidal* inhomogeneities. Following the "eigenstrain concept" introduced by Eshelby [45], four governing micromechanical ensemble-volume averaged field equations for an RVE are *explicitly* constructed to relate macroscopic (ensemble-volume averaged) stresses, strains, volume fractions, particle shapes and orientations, average strain-perturbations due to inhomogeneities, and elastic properties of constituent phases. Effective properties of heterogeneous composites can then be derived based on the proposed general framework. A probabilistic "renormalization procedure" similar to that of Willis [10] and Sen and Torquato [58] is employed to ensure the *absolute convergence* of relevant integrals; i.e., the governing micromechanical equations and overall effective properties are independent of the shape of the RVE used. In Section 3, as a special class of models, *inter*-particle interactions are completely ignored. It is shown that the classical Hashin-Shtrikman [2]–[4] variational bounds, Walpole's bounds [5]–[7], and Willi's bounds [10] can be recovered as special cases. Further, in Section 4, it is demonstrated that the Mori-Tanaka method *coincides* with the Hashin-Shtrikman bounds and the "noninteracting" micromechanical model in some cases. The effects of unidirectionally aligned penny-shaped microcracks on stiffness reduction of microcracked brittle solids will be examined in Section 5. A novel, accurate, probabilistic pairwise particle interaction formulation for two-phase composites with *randomly located* isotropic spherical particles will be presented in a companion paper (Ju and Chen [59]) within the proposed general framework. In particular, overall elastic properties are analytically derived in terms of *fractional expressions* of ϕ (not a power series of ϕ).

2 Ensemble-volume averaged micromechanical field equations

To obtain "effective" constitutive equations and properties of random heterogeneous composites, one typically performs the ensemble-volume averaging process (homogenization) within a mesoscopic representative volume element (RVE). To avoid the truncation errors of *Green's functions* outside the domain of an RVE, an ellipsoidal RVE itself is embedded in an *infinite* (and identical) matrix material within our framework; see Fig. 1. The entire assembly is subjected to specified far-field stresses or strains. Furthermore, all particles are assumed to be non-intersecting (impenetrable).

The volume-averaged stress tensor is defined as

$$\bar{\boldsymbol{\sigma}} \equiv \frac{1}{V}\int_V \boldsymbol{\sigma}(\mathbf{x})\, d\mathbf{x} = \frac{1}{V}\left[\int_{V_m} \boldsymbol{\sigma}(\mathbf{x})\, d\mathbf{x} + \sum_{r=1}^{n}\int_{V_r} \boldsymbol{\sigma}(\mathbf{x})\, d\mathbf{x}\right], \tag{1}$$

where V is the volume of an RVE, V_m is the volume of the matrix, V_r is the volume of the rth-phase particles (inhomogeneities), and n denotes the number of particulate phases of different material properties (excluding the matrix). Similarly, the volume-averaged strain tensor is defined as

$$\bar{\boldsymbol{\varepsilon}} \equiv \frac{1}{V}\int_V \boldsymbol{\varepsilon}(\mathbf{x})\, d\mathbf{x} = \frac{1}{V}\left[\int_{V_m} \boldsymbol{\varepsilon}(\mathbf{x})\, d\mathbf{x} + \sum_{r=1}^{n}\int_{V_r} \boldsymbol{\varepsilon}(\mathbf{x})\, d\mathbf{x}\right] \equiv \frac{1}{V}\left[V_m\bar{\boldsymbol{\varepsilon}}_m + \sum_{r=1}^{n} V_r\bar{\boldsymbol{\varepsilon}}_r\right]. \tag{2}$$

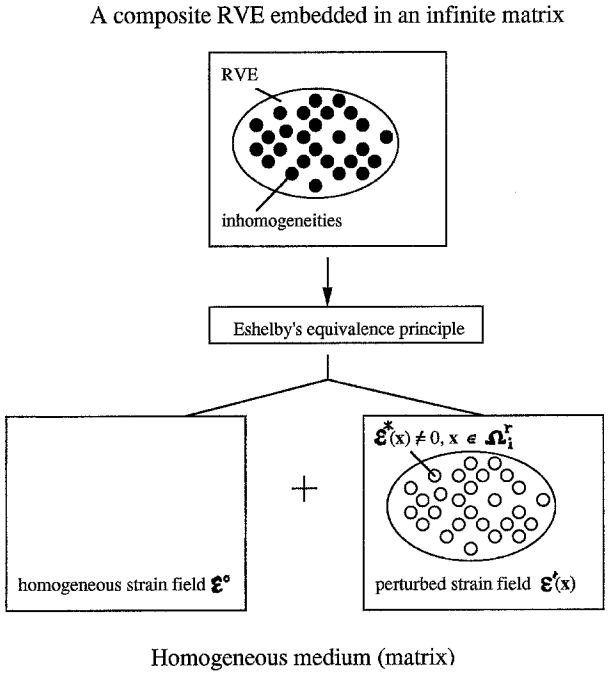

Fig. 1. The RVE representation for a composite medium and linear superposition treatment involving ε^0 and distributed eigenstains $\varepsilon^*(\mathbf{x})$

Moreover, the effective elastic stiffness tensor $\mathbf{C}_*$ of the composite is defined through

$$\bar{\boldsymbol{\sigma}} \equiv \mathbf{C}_* : \bar{\boldsymbol{\varepsilon}} \tag{3}$$

where ":" signifies the tensor contraction.

According to Eshelby's equivalence principle (Eshelby [45], [60]), the perturbed strain field $\varepsilon'(\mathbf{x})$ induced by inhomogeneities (particles with properties different from those of the homogeneous matrix) can be related to specified eigenstrains $\varepsilon^*(\mathbf{x})$ by replacing the inhomogeneities with the matrix material (or vice versa). That is, for the domain of the rth-phase particles with elastic stiffness tensor $\mathbf{C}_r$, we have

$$\mathbf{C}_r : [\boldsymbol{\varepsilon}^0 + \boldsymbol{\varepsilon}'(\mathbf{x})] = \mathbf{C}_0 : [\boldsymbol{\varepsilon}^0 + \boldsymbol{\varepsilon}'(\mathbf{x}) - \boldsymbol{\varepsilon}^*(\mathbf{x})], \tag{4}$$

where $\mathbf{C}_0$ is the stiffness tensor of the matrix and ε^0 is the uniform strain field induced by far-field loads for a homogeneous matrix material only; see Fig. 1. $\mathbf{C}_0$ and $\mathbf{C}_r$ could be isotropic or anisotropic if the eigenstrain field $\varepsilon^*(\mathbf{x})$ is uniform in V. If $\varepsilon^*(\mathbf{x})$ is nonuniform in V, it is assumed that, for simplicity, $\mathbf{C}_0$ and $\mathbf{C}_r$ are isotropic tensors. In Fig. 1, the strain at any point within an RVE is decomposed into two parts: (a) the uniform strain ε^0 (without inhomogeneities), and (b) the perturbed strain $\varepsilon'(\mathbf{x})$ due to distributed eigenstrains $\varepsilon^*(\mathbf{x})$. It is emphasized that the eigenstrain $\varepsilon^*(\mathbf{x})$ is *nonzero* in the particle domain and *zero* in the matrix domain, respectively.

In particular, in accordance with Eshelby [45], [60], the perturbed strain field induced by distributed eigenstrains ε^* can be expressed as

$$\boldsymbol{\varepsilon}'(\mathbf{x}) = \int_V \mathbf{G}(\mathbf{x} - \mathbf{x}') : \boldsymbol{\varepsilon}^*(\mathbf{x}')\, d\mathbf{x}', \tag{5}$$

where $\mathbf{x}, \mathbf{x}' \in V$. In addition, $\mathbf{G}$ is the (second derivative of the) Green's function in a linear elastic homogeneous matrix. For example, for a linear elastic *isotropic* matrix, we have

$$G_{ijkl} = \frac{1}{8\pi(1-\nu_0)\, r^3}\,[(1-2\nu_0)\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}) + 3\nu_0(\delta_{ik}n_jn_l + \delta_{il}n_jn_k + \delta_{jk}n_in_l + \delta_{jl}n_in_k) + 3\delta_{ij}n_kn_l + 3(1-2\nu_0)\,\delta_{kl}n_in_j - 15n_in_jn_kn_l], \tag{6}$$

where $\mathbf{r} = \mathbf{x} - \mathbf{x}'$, $r = \|\mathbf{x} - \mathbf{x}'\|$ and $\mathbf{n} = \mathbf{r}/r$. Further, summation convention applies, δ_{ij} denotes the Kronecker delta, and ν_0 is Poisson's ratio of the homogeneous matrix. From Eqs. (4) and (5), we arrive at

$$-\mathbf{A}_r : \boldsymbol{\varepsilon}^*(\mathbf{x}) = \boldsymbol{\varepsilon}^0 + \int_V \mathbf{G}(\mathbf{x}-\mathbf{x}') : \boldsymbol{\varepsilon}^*(\mathbf{x}')\, d\mathbf{x}' \tag{7}$$

for $\mathbf{x} \in V$, where

$$\mathbf{A}_r \equiv (\mathbf{C}_r - \mathbf{C}_0)^{-1} \cdot \mathbf{C}_0. \tag{8}$$

Furthermore, the total *local* strain field $\boldsymbol{\varepsilon}(\mathbf{x})$ can be expressed as

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \boldsymbol{\varepsilon}^0 + \boldsymbol{\varepsilon}'(\mathbf{x}) = \boldsymbol{\varepsilon}^0 + \int_V \mathbf{G}(\mathbf{x}-\mathbf{x}') : \boldsymbol{\varepsilon}^*(\mathbf{x}')\, d\mathbf{x}'. \tag{9}$$

Therefore, the volume-averaged strain tensor is given by

$$\bar{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}^0 + \frac{1}{V}\int_V \int_V \mathbf{G}(\mathbf{x}-\mathbf{x}') : \boldsymbol{\varepsilon}^*(\mathbf{x}')\, d\mathbf{x}'\, d\mathbf{x} = \boldsymbol{\varepsilon}^0 + \frac{1}{V}\int_V \left[\int_V \mathbf{G}(\mathbf{x}-\mathbf{x}')\, d\mathbf{x}\right] : \boldsymbol{\varepsilon}^*(\mathbf{x}')\, d\mathbf{x}'. \tag{10}$$

It is well known that integrals such as $\left[\int_V \mathbf{G}(\mathbf{x}\,\mathbf{x}')\, d\mathbf{x}\right]$ are conditionally convergent since $\mathbf{G}$ is of the order r^{-3}; i.e., they depend on the shape of the RVE. Nonetheless, effective material properties certainly cannot depend on such shape dependence. Therefore, "renormalization procedures" were proposed in the literature to avoid this conditional convergence problem. We refer to, for instance, Batchelor and Green [47], Willis [10], Chen and Acrivos [50], and Sen and Torquato [58], etc. In particular, the "renormalization schemes" proposed by Willis [10] and Sen and Torquato [58] are particularly attractive since the former involves a two-point probability function $\chi_{rs}(r, s =$ phase numbering indices) and the latter involves n-point probability functions $S_n^{(i)}$ ($n = 1, 2, 3, 4,$ etc. and $i =$ phase numbering index). More specifically, Willis' [10] renormalization scheme involves the two-point correlation functions χ_{rs} and integrals such as

$$\int_V \mathbf{G}(\mathbf{x}-\mathbf{x}')\,[\chi_{rs}(\mathbf{x}-\mathbf{x}') - \phi_r\phi_s]\, d\mathbf{x}', \tag{11}$$

where ϕ_r is the volume fraction of the rth phase inclusions. The integral in (11) absolutely converges because $\chi_{rs}(\mathbf{x}-\mathbf{x}') \to \phi_r\phi_s$ as $\|\mathbf{x}-\mathbf{x}'\| \to \infty$; see Willis [10, p. 188]. However, in Willis [10, Eqs. (3.18)–(3.20)] only unidirectionally aligned and similarly shaped ellipsoidal inclusions are explicity treated, which depend on the microstructure of a composite via the *shape* of the inclusions and volume fractions only. On the other hand, emanating from a perturbation expansion scheme, Sen and Torquato [58] explicitly derive the nth-order tensorial coefficients $\mathbf{A}_n^{(i)}$ (termed n-point microstructural parameters) for *two-phase* linear composite media based on

n-point probability functions $S_n^{(i)}(\mathbf{r})$ and volume fractions ϕ_i. Sen and Torquato [58, p. 4513] show that Willis' **P** tensor (defined by Willis [10, Eq. (3.19)]) is trivially related to the well-known "depolarization factor" tensor $\mathbf{A}_2^*$ for an ellipsoid at the two-point level. If inclusions are not aligned or not similarly shaped, knowledge of the two-point probability functions $S_2^{(i)}(\mathbf{r})$ enables one to evaluate $\mathbf{A}_2^{(i)}$ in (2.11a) and $\mathbf{A}_2^*$ in (3.14) of Sen and Torquato [58] for two-phase linear composites. Moreover, third-order bounds and fourth-order bounds of effective conductivities for two-phase linear composite media are explicitly derived in Sen and Torquato [58]. However, Torquato and his coworkers have never presented any explicit examples demonstrating two-phase elastic composites with non-aligned and/or dissimilar ellipsoidal inclusions. Further, generalizations of Sen and Torquato's [58] work to accommodate *multiphase* (more than two-phase) elastic composites containing non-aligned and/or dissimilar inclusions are apparently needed in the future.

Following the concepts of Willis [10] and Sen and Torquato [58], a similar ensemble-average "renormalization" procedure is applied to Eq. (10) at the two-point level. It follows that the following "renormalized" integrals are involved:

$$\int_V \mathbf{G}(\mathbf{x}-\mathbf{x}') \frac{[\chi_{rs}(\mathbf{x}-\mathbf{x}') - \phi_r\phi_s]}{\left[\dfrac{\chi_{rs}(\mathbf{0})}{\phi_r\phi_s} - 1\right]} \, d\mathbf{x}. \tag{12}$$

These integrals are absolutely convergent because the integrand $[\chi_{rs}(\mathbf{x}-\mathbf{x}') - \phi_r\phi_s] \to 0$ as $\|\mathbf{x}-\mathbf{x}'\| \to \infty$. For simplicity, let us assume that all particles are unidirectionally aligned and similarly shaped. In addition, we assume that the two-point probability functions are isotropic (i.e. $\chi_{rs}(\mathbf{x}-\mathbf{x}')$ depends on $\|\mathbf{x}-\mathbf{x}'\|$ only). Therefore, according to Willis [10], Eq. (12) becomes

$$\int_V \mathbf{G}(\mathbf{x}-\mathbf{x}') \frac{[\chi_{rs}(\mathbf{x}-\mathbf{x}') - \phi_r\phi_s]}{\left[\dfrac{\chi_{rs}(\mathbf{0})}{\phi_r\phi_s} - 1\right]} \, d\mathbf{x} = \mathbf{S} \frac{[\chi_{rs}(\mathbf{0}) - \phi_r\phi_s]}{[\chi_{rs}(\mathbf{0}) - \phi_r\phi_s]/[\phi_r\phi_s]} = \mathbf{S}\phi_r\phi_s, \tag{13}$$

where

$$\mathbf{S} \equiv \int_{\bar{\varrho}<b} \mathbf{G}(\mathbf{x}) \, d\mathbf{x} = \text{a constant tensor}. \tag{14}$$

Here, $\bar{\varrho} \equiv (\mathbf{x}^T\bar{\mathbf{A}}^T\bar{\mathbf{A}}\mathbf{x})^{1/2}$ is determined by the ellipsoidal inclusion aspect ratios ($\bar{\mathbf{A}}$) (see Willis [10, Eq. (3.18)]) and $b > 0$ is any real positive value. As a consequence, Eq. (10) can be rephrased as

$$\langle\bar{\boldsymbol{\varepsilon}}\rangle = \boldsymbol{\varepsilon}^0 + \sum_{r=0}^{n}\sum_{s=0}^{n} \mathbf{1}\cdot[\phi_r\phi_s\mathbf{S} : \langle\bar{\boldsymbol{\varepsilon}}_r^*\rangle] = \boldsymbol{\varepsilon}^0 + \mathbf{S} : \left[\sum_{r=1}^{n} \phi_r\langle\bar{\boldsymbol{\varepsilon}}_r^*\rangle\right], \tag{15}$$

where the angled brackets signifiy the ensemble-average quantities and

$$\langle\bar{\boldsymbol{\varepsilon}}_r^*\rangle \equiv \frac{1}{V_r}\int_{V_r} \langle\boldsymbol{\varepsilon}^*(\mathbf{x}')\rangle \, d\mathbf{x}'. \tag{16}$$

Therefore, we can relate our **S** to Willis' **P** as follows: $\mathbf{S} \equiv \mathbf{P}\cdot\mathbf{C}_0$. For two-phase linear composites, we can further write $\mathbf{S} \equiv \mathbf{A}_2^*$ (cf. Sen and Torquato [58]). For example, if the linear elastic matrix material is isotropic and all inclusions are spherical, then the **S** tensor takes the form

$$S_{ijkl} = \frac{1}{15(1-\nu_0)}[(5\nu_0 - 1)\,\delta_{ij}\delta_{kl} + (4 - 5\nu_0)\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]. \tag{17}$$

On the other hand, for linear composites containing arbitrarily non-aligned and/or dissimilar ellipsoidal inclusions, we can define the integral in Eq. (12) as $\mathbf{S}_2{}^{rs}$ and characterize $\mathbf{S}^{rs}$ by $\mathbf{S}^{rs} = \mathbf{S}_2{}^{rs}/(\phi_r\phi_s)$. In such cases Eq. (15) can be recast as

$$\langle\bar{\boldsymbol{\varepsilon}}\rangle = \boldsymbol{\varepsilon}^0 + \sum_{r=0}^{n}\sum_{s=0}^{n} \phi_r\phi_s\mathbf{S}^{rs} : \langle\bar{\boldsymbol{\varepsilon}}_r{}^*\rangle. \tag{18}$$

Similarly, we can define

$$\mathbf{P}_2{}^{rs} \equiv \int_V \boldsymbol{\Gamma}^{\infty}(\mathbf{x}-\mathbf{x}')\,[\chi_{rs}(\mathbf{x}-\mathbf{x}') - \phi_r\phi_s]\,d\mathbf{x}' \tag{19}$$

in Willis [10, Eq. (3.20)] and characterize $\mathbf{P}^{rs}$ by $\mathbf{P}^{rs} = \mathbf{P}_2{}^{rs}/[\chi_{rs}(\mathbf{0}) - \phi_r\phi_s]$ for linear composites with arbitrary microgeometries. Therefore, we still have $\mathbf{S}^{rs} = \mathbf{P}^{rs}\cdot\mathbf{C}_0$. Since all field quantities in this paper only occur when ensemble averaged, the ensemble averaged brackets will be *dropped* for simplicity in what follows.

Similarly, the ensemble-volume averaged stress field can be recast as [cf. Eqs. (1), (2), (4)]:

$$\begin{aligned}\bar{\boldsymbol{\sigma}} &= \frac{1}{V}\left[\int_{V_m}\mathbf{C}_0 : \boldsymbol{\varepsilon}(\mathbf{x})\,d\mathbf{x} + \sum_{r=1}^{n}\int_{V_r}\mathbf{C}_0 : [\boldsymbol{\varepsilon}(\mathbf{x}) - \boldsymbol{\varepsilon}^*(\mathbf{x})]\,d\mathbf{x}\right]\\ &= \frac{1}{V}\left[V_m\mathbf{C}_0 : \bar{\boldsymbol{\varepsilon}}_m + \sum_{r=1}^{n} V_r\mathbf{C}_0 : (\bar{\boldsymbol{\varepsilon}}_r - \bar{\boldsymbol{\varepsilon}}_r{}^*)\right] = \mathbf{C}_0 : \left[\bar{\boldsymbol{\varepsilon}} - \sum_{r=1}^{n}\phi_r\bar{\boldsymbol{\varepsilon}}_r{}^*\right].\end{aligned} \tag{20}$$

The effective elastic moduli (Eq. (3)) can be obtained, in principle, from Eqs. (7), (15) (or (18)) and (20) since the variables are $\bar{\boldsymbol{\sigma}}$, $\bar{\boldsymbol{\varepsilon}}$, $\boldsymbol{\varepsilon}^0$ and $\bar{\boldsymbol{\varepsilon}}_r{}^*$. In essence, one needs to solve the relation between $\bar{\boldsymbol{\varepsilon}}$ and $\bar{\boldsymbol{\varepsilon}}_r{}^*$ (or between $\boldsymbol{\varepsilon}^0$ and $\bar{\boldsymbol{\varepsilon}}_r{}^*$), which involves the solution of the integral equation (7). However, $\bar{\boldsymbol{\varepsilon}}_r{}^*$ depends on inter-particle interactions, particle-matrix interactions and microstructure (i.e. particle sizes, orientations, shapes, volume fractions, locations, configurations, and probability functions) of a composite system. Therefore, for *randomly dispersed* particles, one needs to obtain the *ensemble-volume averaged* relation between $\bar{\boldsymbol{\varepsilon}}$ and $\bar{\boldsymbol{\varepsilon}}_r{}^*$ by averaging *all possible* solutions of the integral equation (7) for any particle configurations generated according to specified probability functions.

By taking the ensemble-volume average of Eq. (7) (Eshelby's equivalence principle) over all rth-phase particles, we obtain

$$-\mathbf{A}_r : \bar{\boldsymbol{\varepsilon}}_r{}^* = \boldsymbol{\varepsilon}^0 + \bar{\boldsymbol{\varepsilon}}_r{}', \tag{21}$$

where

$$\begin{aligned}\bar{\boldsymbol{\varepsilon}}_r{}' &= \frac{1}{V_r}\int_{V_r}\int_{V}\mathbf{G}(\mathbf{x}-\mathbf{x}') : \boldsymbol{\varepsilon}^*(\mathbf{x}')\,d\mathbf{x}'\,d\mathbf{x}\\ &= \frac{1}{V_r}\sum_{i=1}^{N_r}\int_{\Omega_r{}^i}\left[\int_{V-\Omega_r{}^i}\mathbf{G}(\mathbf{x}-\mathbf{x}') : \boldsymbol{\varepsilon}^*(\mathbf{x}')\,d\mathbf{x}'\right]d\mathbf{x} + \frac{1}{V_r}\sum_{i=1}^{N_r}\int_{\Omega_r{}^i}\left[\int_{\Omega_r{}^i}\mathbf{G}(\mathbf{x}-\mathbf{x}')\,d\mathbf{x}\right] : \boldsymbol{\varepsilon}^*(\mathbf{x}')\,d\mathbf{x}',\end{aligned} \tag{22}$$

Ω_r^i = the domain of the ith particle in the rth phase domain V_r, and N_r = the number of the phase r particles dispersed in V. For convenience, we shall define

$$\bar{\boldsymbol{\varepsilon}}_r'^p \equiv \frac{1}{V_r} \sum_{i=1}^{N_r} \int_{\Omega_r^i} \left[\int_{V-\Omega_r^i} \mathbf{G}(\mathbf{x}-\mathbf{x}') : \boldsymbol{\varepsilon}^*(\mathbf{x}')\, d\mathbf{x}' \right] d\mathbf{x} \tag{23}$$

which represents the *inter-particle interaction* effects. In a companion paper, it will be shown that the leading order of $\bar{\boldsymbol{\varepsilon}}_r'^p$ is of $O(a/L)^3$ where L denotes the average spacing between the centers of two neighboring particles and a is the average radius of particles. Therefore, when particle spacings are very large relative to sizes of particles, $\bar{\boldsymbol{\varepsilon}}_r'^p$ becomes negligible. Further, let us define

$$\mathbf{s}_r^i \equiv \int_{\Omega_r^i} \mathbf{G}(\mathbf{x}-\mathbf{x}')\, d\mathbf{x}, \qquad \mathbf{x} \text{ and } \mathbf{x}' \in \Omega_r^i \tag{24}$$

as the *Eshelby tensor* associated with the ith particle in the rth-phase inhomogeneity. The components of the Eshelby tensor $\mathbf{s}_r^i$ depend on elastic moduli of the matrix material and the shape and orientation of the ellipsoidal particle Ω_r^i. See Mura [61] and Zhu and Weng [62] for details.

If all particles in the rth-phase have the *same* ellipsoidal shape and orientation (although shapes and orientations could be *different* from phases to phases), then $\mathbf{s}_r^i$ becomes $\mathbf{s}_r$ which is a *constant* tensor within the rth phase only. Accordingly, Eq. (22) can be recast as

$$\bar{\boldsymbol{\varepsilon}}_r' = \bar{\boldsymbol{\varepsilon}}_r'^p + \mathbf{s}_r : \left[\frac{1}{V_r} \int_{V_r} \boldsymbol{\varepsilon}^*(\mathbf{x}')\, d\mathbf{x}' \right] \equiv \bar{\boldsymbol{\varepsilon}}_r'^p + \mathbf{s}_r : \bar{\boldsymbol{\varepsilon}}_r^*. \tag{25}$$

From Eqs. (21) and (25), we arrive at

$$(-\mathbf{A}_r - \mathbf{s}_r) : \bar{\boldsymbol{\varepsilon}}_r^* = \boldsymbol{\varepsilon}^0 + \bar{\boldsymbol{\varepsilon}}_r'^p \tag{26}$$

for the rth-phase particles with *aligned* and identically *shaped* ellipsoidal inhomogeneities. Since different phases could have different shapes and orientations, the Eshelby tensors $\mathbf{s}_r$ are *distinct* for phases $r = 1$ to n.

In addition, the ensemble-volume average of the perturbed strain takes the form

$$\bar{\boldsymbol{\varepsilon}}' = \frac{1}{V} \int_V \boldsymbol{\varepsilon}'(\mathbf{x})\, d\mathbf{x} = \frac{1}{V} \left[\frac{V_m}{V_m} \int_{V_m} \boldsymbol{\varepsilon}'(\mathbf{x})\, d\mathbf{x} + \sum_{r=1}^{n} \frac{V_r}{V_r} \int_{V_r} \boldsymbol{\varepsilon}'(\mathbf{x})\, d\mathbf{x} \right]. \tag{27}$$

For compactness, let us define

$$\bar{\boldsymbol{\varepsilon}}'^m \equiv \frac{1}{V_m} \int_{V_m} \boldsymbol{\varepsilon}'(\mathbf{x})\, d\mathbf{x}. \tag{28}$$

Clearly, $\bar{\boldsymbol{\varepsilon}}'^m$ corresponds to the ensemble-volume averaged perturbed strain in the *matrix* due to the existence of all particles. As a result of Eqs. (25) and (28), Eq. (27) can be rephrased as

$$\bar{\boldsymbol{\varepsilon}}' = \frac{V_m}{V} \bar{\boldsymbol{\varepsilon}}_r'^m + \sum_{r=1}^{n} \frac{V_r}{V} \bar{\boldsymbol{\varepsilon}}_r' = \left(1 - \sum_{r=1}^{n} \phi_r \right) \bar{\boldsymbol{\varepsilon}}'^m + \sum_{r=1}^{n} \phi_r (\mathbf{s}_r : \bar{\boldsymbol{\varepsilon}}_r^* + \bar{\boldsymbol{\varepsilon}}_r'^p). \tag{29}$$

For linear composites with unidirectionally aligned and similarly shaped particles, Eqs. (15) and (29) then lead to the relationship between $\bar{\boldsymbol{\varepsilon}}'^m$ and $\bar{\boldsymbol{\varepsilon}}_r'^p$:

$$\left(1 - \sum_{r=1}^{n} \phi_r\right) \bar{\boldsymbol{\varepsilon}}'^m + \sum_{r=1}^{n} \phi_r \bar{\boldsymbol{\varepsilon}}_r'^p = \sum_{r=1}^{n} \phi_r (\mathbf{S} - \mathbf{s}) : \bar{\boldsymbol{\varepsilon}}_r^* = \mathbf{0}. \tag{30}$$

In contrast, Eqs. (18) and (29) should be used to derive the counterpart of (30) for different $\mathbf{s}_r$.

In summary, the four governing micromechanical ensemble-volume averaged field equations are recapitulated as follows (dropping the ensemble angled brackets):

$$\bar{\boldsymbol{\sigma}} = \mathbf{C}_0 : \left(\bar{\boldsymbol{\varepsilon}} - \sum_{r=1}^{n} \phi_r \bar{\boldsymbol{\varepsilon}}_r^*\right), \tag{31}$$

$$\bar{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}^0 + \sum_{r=1}^{n} \phi_r \mathbf{S} : \bar{\boldsymbol{\varepsilon}}_r^*, \tag{32}$$

$$(-\mathbf{A}_r - \mathbf{s}) : \bar{\boldsymbol{\varepsilon}}_r^* = \boldsymbol{\varepsilon}^0 + \bar{\boldsymbol{\varepsilon}}_r'^p, \tag{33}$$

$$\left(1 - \sum_{r=1}^{n} \phi_r\right) \bar{\boldsymbol{\varepsilon}}'^m + \sum_{r=1}^{n} \phi_r \bar{\boldsymbol{\varepsilon}}_r'^p = \sum_{r=1}^{n} \phi_r (\mathbf{S} - \mathbf{s}) : \bar{\boldsymbol{\varepsilon}}_r^* = \mathbf{0}. \tag{34}$$

It is noted that Eqs. (32), (33) and (34) are valid when all particles are unidirectionally aligned and identically shaped (with $\mathbf{S} = \mathbf{s}$). If the rth-phase particles are not even unidirectionally aligned and identically shaped within the rth-phase, then $\mathbf{s}_r$ is undefined and one must employ $\mathbf{s}_r^i$ defined in Eq. (24) for *each particle*. For example, if all particles are *randomly oriented*, then $\mathbf{s}_r^i$ is not a common constant tensor for all rth-phase particles in (22). Equations (33) and (34) must be modified by using the "*orientational average*" procedure proposed by Tandon and Weng [63], and Zhao, Tandon and Weng [36]. Basically, one first evaluates the product $\mathbf{s}_r^i : \boldsymbol{\varepsilon}^*(\mathbf{x}')$ in (22) in the *local* inclusion axes for each particle, then one performs coordinate transformation to the *global* material axes and computes the global "orientational average" of the product. We refer to Weng [37, p. 1117] for a similar treatment.

To actually solve Eqs. (31)–(34) and obtain effective moduli of composites, as previously commented, it is essential to express the rth-phase average eigenstrain $\bar{\boldsymbol{\varepsilon}}_r^*$ in terms of the average strain $\bar{\boldsymbol{\varepsilon}}$ (or $\boldsymbol{\varepsilon}^0$). Namely, one has to solve the integral equation (7) exactly for each phase, which involves details of random microstructure. Unfortunately, in most situations, the exact solution for this (many-particle interaction) integral equation does not exist. In a companion paper (Ju and Chen [59]), however, some reasonably accurate approximations will be introduced to estimate effects of inter-particle intractions. Further, in Section 3, we will present a "noninteracting" solution for (7) and compare the resulting effective elastic moduli with some existing variational bounds and micromechanical models.

3 Noninteracting approximation and its relation to classical variational bounds

As mentioned in the previous Section, the exact solution of the many-particle-interaction integral equation (7) or (33) appears to be an insurmountable task for arbitrary and random configurations of particles. However, as the first attempt, "noninteracting solutions" can be easily obtained in this Section by *neglecting* the inter-particle interaction effects (i.e. the averaged perturbations in a particle due to other surrounding particles). Furthermore, for the first time, it

will be *explicitly* shown that effective elastic moduli due to classical variational bounds correspond *precisely* to the "noninteracting" micromechanical solutions with $\mathbf{S} = \mathbf{s}$.

From a mathematical viewpoint, classical variational principles (Hashin and Shtrikman [2], [3], [4]) for obtaining bounds on effective properties amount to finding *stationary* conditions and values of a given scalar function (such as the total elastic energy function of a composite medium) under specified macroscopic information of the system (e.g. material properties of constituent phases, volume fractions and boundary strain field). By using the classical variational principle (without introducing the Green's function), Hashin and Shtrikman [2]–[4] derived upper and lower bounds for linear elastic *isotropic* particulate composites. Walpole [5]–[7] generalized the variational principle to *anisotropic* elastic composites. By introducing the Green's function, Willis [10] derived generalized Hashin-Shtrikman bounds for multiphase anisotropic composites, which contain bounds of Hashin and Shtrikman [2], [3], Hashin [64], and Walpole [5]–[7] as special cases. The use of Green's function is a significant advancement from a micromechanics standpoint. As indicated by Weng [37, p. 1116], the approximate fields of Hashin and Shtrikman [4] derived after carrying out the integration of Green's function (similar to Willis' [10] procedures) are known to correspond to those associated with a *single sphere.*

3.1 Noninteracting approximation of effective moduli for multiphase composites

If we neglect the *inter-particle* interaction effects, then the ensemble-volume averaged perturbed strain $\bar{\boldsymbol{\varepsilon}}_r'^p$ in (26) can be dropped. The resulting "noninteracting approximation" for the rth-phase particles becomes

$$\bar{\boldsymbol{\varepsilon}}_r^* = -(\mathbf{A}_r + \mathbf{s}_r)^{-1} : \boldsymbol{\varepsilon}^0. \tag{35}$$

It is emphasized that the above equation is only valid for unidirectionally aligned and identically shaped ellipsoids within the rth-phase. Equation (35) implies that any material point within a particle is only influenced by other material points within the *same* particle. In other words, Eq. (35) is simply the solution of averaged eigenstrain for the *single* ellipsoidal inclusion problem (Eshelby [45]).

If all particles are unidirectionally aligned and similarly shaped, substitution of (35) into (32) then leads to (with $\mathbf{s} = \mathbf{S}$)

$$\bar{\boldsymbol{\varepsilon}}_r^* = -(\mathbf{A}_r + \mathbf{s})^{-1} \left[\mathbf{I} - \sum_{r=1}^{n} \phi_r \mathbf{S}(\mathbf{A}_r + \mathbf{s})^{-1} \right]^{-1} : \bar{\boldsymbol{\varepsilon}}. \tag{36}$$

Equation (36) together with (31) then renders the following effective stiffness tensor for a multiphase composite medium:

$$\mathbf{C}_* = \mathbf{C}_0 \cdot [\mathbf{I} + \mathbf{B}(\mathbf{I} - \mathbf{SB})^{-1}], \tag{37}$$

where

$$\mathbf{B} \equiv \sum_{r=1}^{n} \phi_r (\mathbf{s} + \mathbf{A}_r)^{-1}. \tag{38}$$

The matrix and the inclusions can be isotropic or anisotropic.

3.2 Some analytical examples

A number of analytical examples is presented for two-phase and three-phase composites by using the aforementioned "noninteracting" approximate solutions.

Category I: two-phase composites. Since $\mathbf{S} = \mathbf{s}_1$, Eq. (37) reduces to

$$\mathbf{C}_* = \mathbf{C}_0\{\mathbf{I} + \phi_1[(\mathbf{C}_1 - \mathbf{C}_0)^{-1}\,\mathbf{C}_0 + (1 - \phi_1)\,\mathbf{s}_1]^{-1}\}. \tag{39}$$

Equation (39) is valid for any linearly elastic isotropic or anisotropic $\mathbf{C}_0$ and $\mathbf{C}_1$. Clearly, the tensorial expression for effective moduli given in Eq. (39) is *identical* to that given by Willis [10], which is Walpole's [5], [6] generalization of Hashin and Shtrikman's [4] result.

Case I.1: All particles are spherical and both matrix and particles are isotropic elastic. The effective bulk and shear moduli are [cf. Eq. (17)]

$$\varkappa_* = \varkappa_0\left\{1 + \frac{3(1 - \nu_0)\,(\varkappa_1 - \varkappa_0)\,\phi_1}{3(1 - \nu_0)\,\varkappa_0 + (1 - \phi_1)\,(1 + \nu_0)\,(\varkappa_1 - \varkappa_0)}\right\}, \tag{40}$$

$$\mu_* = \mu_0\left\{1 + \frac{15(1 - \nu_0)\,(\mu_1 - \mu_0)\,\phi_1}{15(1 - \nu_0)\,\mu_0 + (1 - \phi_1)\,(8 - 10\nu_0)\,(\mu_1 - \mu_0)}\right\}. \tag{41}$$

It is noted that Eqs. (40) and (41) are entirely *identical* to the lower (or upper) bounds, if the matrix is the softer (or harder) phase, derived by Hashin and Shtrikman [4]. In addition, (40) and (41) also coincide with the corresponding results obtained by using the Mori-Tanaka method (see Weng [34] and [37]).

Case I.2: Similar to Case I.1, but all particles are *rigid spheres.* The effective bulk and shear moduli are

$$\varkappa_* = \varkappa_0\left\{1 + \frac{3(1 - \nu_0)\,\phi_1}{(1 - \phi_1)\,(1 + \nu_0)}\right\}, \tag{42}$$

$$\mu_* = \mu_0\left\{1 + \frac{15(1 - \nu_0)\,\phi_1}{(1 - \phi_1)\,(8 - 10\nu_0)}\right\}. \tag{43}$$

These are the lower-bound solutions of the classical variational principle.

Case I.3: Similar to Case I.1, but all particles are now *spherical voids.* The effective bulk and shear moduli are

$$\varkappa_* = \varkappa_0\left\{1 - \frac{3(1 - \nu_0)\,\phi_1}{3(1 - \nu_0) - (1 - \phi_1)\,(1 + \nu_0)}\right\}, \tag{44}$$

$$\mu_* = \mu_0\left\{1 - \frac{15(1 - \nu_0)\,\phi_1}{15(1 - \nu_0) - (1 - \phi_1)\,(8 - 10\nu_0)}\right\}. \tag{45}$$

These are the upper-bound solutions of the classical variational principle; see also Zhao et al. [36, p. 111]. Further, if the matrix material becomes *incompressible*, we have

$$\varkappa_* = \mu_0\left\{\frac{4(1 - \phi_1)}{3\phi_1}\right\}, \tag{46}$$

$$\mu_* = \mu_0\left\{1 - \frac{2.5\phi_1}{1.5 + \phi_1}\right\}. \tag{47}$$

Category II: three-phase composites. If all inclusions are aligned and similarly shaped ($\mathbf{S} = \mathbf{s}_1 = \mathbf{s}_2$), our "noninteracting" solutions are again identical to the classical variational bounds.

Case II.1: If the first and the second phases contain *spherical rigid* particles and *spherical voids*, respectively, then $\mathbf{A}_1 = \mathbf{0}$ and $\mathbf{A}_2 = -\mathbf{I}$. After some lengthy but straightforward derivations, the effective bulk and shear moduli are

$$\varkappa_* = \varkappa_0 \left\{1 + \frac{3(1-\nu_0)}{(1+\nu_0)} \cdot \frac{2(1-2\nu_0)\,\phi_1 - (1+\nu_0)\,\phi_2}{2(1-2\nu_0)\,(1-\phi_1) + (1+\nu_0)\,\phi_2}\right\}, \tag{48}$$

$$\mu_* = \mu_0 \left\{1 + \frac{15(1-\nu_0)}{(8-10\nu_0)} \cdot \frac{(7-5\nu_0)\,\phi_1 - (8-10\nu_0)\,\phi_2}{(7-5\nu_0)\,(1-\phi_1) + (8-10\nu_0)\,\phi_2}\right\}. \tag{49}$$

On the other hand, it is interesting to examine Walpole's [8] solutions for a two-phase composite with randomly distributed *spherical* particles. Walpole [8] employed an "image" strain field to arrive at the second-order expressions in stiffness moduli, which are identical to the Taylor's series expansion of the aforementioned "noninteracting" solutions (or Hashin and Shtrikman's bounds [4]) to the second order. Therefore, the second-order moduli derived by Walpole [8] do *not* include the inter-particle interaction effects. This observation was also noted by Chen and Acrivos [50, p. 350–351]. In particular, the effective bulk modulus takes the form

$$\frac{\varkappa_*}{\varkappa_0} = 1 + \frac{3(1-\nu_0)\,(\varkappa_1 - \varkappa_0)}{2(1-2\nu_0)\,\varkappa_0 + (1+\nu_0)\,\varkappa_1}\,\phi_1 + \frac{3(1-\nu_0^2)\,(\varkappa_1-\varkappa_0)^2}{[2(1-2\nu_0)\,\varkappa_0 + (1+\nu_0)\,\varkappa_1]^2}\,\phi_1^2. \tag{50}$$

It is observed that the second order term in Eq. (50) is always *positive*. In the event that spherical particles become spherical *voids* ($\varkappa_1 = 0$), Eq. (50) yields

$$\frac{\varkappa_*}{\varkappa_0} = 1 - \frac{3(1-\nu_0)}{2(1-2\nu_0)}\,\phi_1 + \frac{3(1-\nu_0^2)}{4(1-2\nu_0)^2}\,\phi_1^2. \tag{51}$$

Equation (51) implies that the effective bulk modulus of a composite could *increase* due to the existence of randomly located spherical voids if the matrix material is nearly incompressible. For instance, when $\nu_0 = 0.45$, the effective bulk modulus increases when the void volume fraction ϕ_1 is greater than 0.138. Similarly, it can be shown that the second-order models of Willis and Acton [48] and Chen and Acrivos [50], which include *inter*-particle interactions, predict an increase in $\varkappa_*$ when the void volume ratio ϕ_1 is greater than 0.142 and 0.143, respectively. Therefore, although a second-order *stiffness* theory is well suited for elastic particles, it is inappropriate for composites containing voids.

4 The connection with the Mori-Tanaka method

The connection between the "noninteracting" solutions and the Mori-Tanaka method will be examined in this Section. Section 3 together with this Section then reveals the relationships among the classical variational bounds, the "noninteracting" approximation, and the Mori-Tanaka method.

In the original paper by Mori and Tanaka [30], the difference ($\bar{\boldsymbol{\varepsilon}}_r^{pt}$) between the average strain of the rth-phase inclusions and the matrix is assumed to be (see Eq. (1) on page 572 in Mori and Tanaka [30])

$$\bar{\boldsymbol{\varepsilon}}_r^{pt} = \mathbf{s}_r : \bar{\boldsymbol{\varepsilon}}_r^* \tag{52}$$

(see also Zhao et al. [36, Eq. (2.5)], and Weng [37, Eq. (2.3)]). Furthermore, the RVE shape is assumed to be *similar* to that of the inclusions in Eqs. (3) and (5) of Mori and Tanaka [30].

Accordingly, we shall assume that all particles are unidirectionally aligned and similarly shaped in what follows. If an RVE domain is removed from the surrounding infinite matrix, the ensemble-volume averaged strain perturbation $\bar{\boldsymbol{\varepsilon}}'$ can be expressed as [cf. Eq. (32)]

$$\bar{\boldsymbol{\varepsilon}}' = \mathbf{S}^{-1} : \left(\sum_{r=1}^{n} \phi_r \mathbf{S} : \bar{\boldsymbol{\varepsilon}}_r^* \right) = \sum_{r=1}^{n} \phi_r \bar{\boldsymbol{\varepsilon}}_r^* \tag{53}$$

(see also Mori and Tanaka [30, Eq. (2)] and Zhao et al. [36, Eq. (2.6)]). As a consequence, the ensemble-volume averaged field equations (31)–(34) can be modified as follows:

$$\bar{\boldsymbol{\sigma}} = \mathbf{C}_0 : \boldsymbol{\varepsilon}^0, \tag{54}$$

$$\bar{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}^0 + \sum_{r=1}^{n} \phi_r \bar{\boldsymbol{\varepsilon}}_r^*, \tag{55}$$

$$(-\mathbf{A}_r - \mathbf{s}) : \bar{\boldsymbol{\varepsilon}}_r^* = \boldsymbol{\varepsilon}^0 + \bar{\boldsymbol{\varepsilon}}_r'^p, \tag{56}$$

$$\left(1 - \sum_{r=1}^{n} \phi_r\right) \bar{\boldsymbol{\varepsilon}}'^m + \sum_{r=1}^{n} \phi_r \bar{\boldsymbol{\varepsilon}}_r'^p = \sum_{r=1}^{n} \phi_r (\mathbf{I} - \mathbf{s}) : \bar{\boldsymbol{\varepsilon}}_r^*. \tag{57}$$

The average strain perturbation in the rth-phase is given by (25), and the average strain perturbation in the matrix is denoted by $\bar{\boldsymbol{\varepsilon}}'^m$. Since the difference $\bar{\boldsymbol{\varepsilon}}_r^{pt}$ is assumed to follow (52), we conclude that (from (25) and (52))

$$\bar{\boldsymbol{\varepsilon}}'^m = \bar{\boldsymbol{\varepsilon}}_r'^p \tag{58}$$

is assumed for all phases in the Mori-Tanaka method. In addition, Eqs. (57) and (58) lead to

$$\bar{\boldsymbol{\varepsilon}}'^m = \bar{\boldsymbol{\varepsilon}}_r'^p = \sum_{r=1}^{n} \phi_r (\mathbf{I} - \mathbf{s}) : \bar{\boldsymbol{\varepsilon}}_r^*. \tag{59}$$

Therefore, explicit solutions for effective moduli of multiphase composites are possible by using the micromechanical Mori-Tanaka method. Indeed, Mori-Tanaka method emerges as a popular micromechanical method to derive effective moduli of composites; see, for instance, Taya and Chou [31], Taya and Mura [32], Weng [34], [37], Benveniste [35], [65], Norris [66], Zhao et al. [36], and Qiu and Weng [38]. In what follows, let us consider some analytical examples.

Case 1: A two-phase composite. From (55), (56) and (59), we arrive at (dropping the subscript 1 for inclusions)

$$\bar{\boldsymbol{\varepsilon}}^* = -(\mathbf{A} - \mathbf{s} + \phi \mathbf{s})^{-1} : \bar{\boldsymbol{\varepsilon}}. \tag{60}$$

Substitution of (60) into (54) and (55) then renders the effective stiffness tensor (see also Zhao, Tandon and Weng [36, p. 107])

$$\mathbf{C}_* = \mathbf{C}_0 \{\mathbf{I} + \phi [\mathbf{A} + (1 - \phi)\, \mathbf{s}]^{-1}\}. \tag{61}$$

Comparison between Eq. (61) and (39) immediately reveals that the Mori-Tanaka method, the classical variational bounds, and the "noninteracting" approximation yield *identical* results in this case. See also Zhao et al. [36, pp. 108–111] and Weng [37, pp. 1116–1117].

Case 2: A multi-phase composite with same s. Since all particles are identically shaped and unidirectionally aligned in all phases, we have $\mathbf{S} = \mathbf{s}$. This analysis has been done in a paper by

Weng [37, pp. 1112–1114, p. 1116]. Here, we intend to recast the problem and compare the results with the "noninteracting" solutions. From (52) (or (58)) and (54)–(57), we obtain the averaged eigenstrains

$$\sum_{r=1}^{n} \phi_r \bar{\boldsymbol{\varepsilon}}_r^* = -\left[\mathbf{I} - \left(\sum_{r=1}^{n} \phi_r (\mathbf{A}_r + \mathbf{s})^{-1}\right)\mathbf{s}\right]^{-1}\left[\sum_{r=1}^{n} \phi_r (\mathbf{A}_r + \mathbf{s})^{-1}\right] : \bar{\boldsymbol{\varepsilon}}. \tag{62}$$

Furthermore, the anisotropic effective moduli can be easily derived as follows:

$$\mathbf{C}_* = \mathbf{C}_0[\mathbf{I} + (\mathbf{I} - \mathbf{Bs})^{-1}\,\mathbf{B}], \tag{63}$$

where $\mathbf{B}$ has been previously defined in (38). It is clear that Eq. (63) coincides with (37) with $\mathbf{S} = \mathbf{s}$. Namely, the anisotropic effective moduli derived from the Mori-Tanaka method are *identical* to those derived from the "noninteracting" approximation and classical variational bounds (Walpole [5], [6], Willis [10]).

5 Elastic solids with unidirectionally aligned penny-shaped microcracks

In this Section, we consider the effective properties of an *isotropic* elastic matrix material containing unidirectionally aligned *penny-shaped* microcracks. In particular, the aforementioned *noninteracting approximation* will be employed to estimate increases in effective compliances (or reductions in effective stiffnesses) of brittle solids with microcracks. This class of microcrack problems has been studied extensively in the micromechanics literature by using Taylor's model (e.g., Krajcinovic and Fanella [67], Ju [68]), the self-consistent model (e.g., Budiansky and O'Connell [25], Horii and Nemat-Nasser [69], Laws et al. [70], Laws and Dvorak [40], Laws and Brockenbrough [71], Sumarac and Krajcinovic [72], [73], Krajcinovic and Sumarac [74], Ju [75], Ju and Lee [76], Lee and Ju [77]), the differential scheme (e.g., Roscoe [26], [27], McLaughlin [28], Laws and Dvorak [40], Hashin [29]), the variational bounds (e.g., Willis [10]), the Mori-Tanaka method (e.g., Taya [33], Zhao et al. [36]), the deterministic microcrack interaction model (e.g., Kachanov [78]), and the statistical microcrack interaction model (e.g., Ju and Chen [79], [80], Ju and Tseng [81]).

Unidirectionally aligned penny-shaped microcracks can be regarded as the limiting case of unidirectionally aligned *spheroidal voids* with the aspect ratio $\varrho \to 0$; see Fig. 2 for a schematic diagram. That is, one can collapse one axis of a spheroidal microvoid to recover a penny-shaped microcrack. In this event, the *noninteracting approximation* (39) becomes

$$\mathbf{C}_* = \mathbf{C}_0 + \frac{4\pi}{3}\omega\mathbf{C}_0 \cdot \left[\frac{1}{\varrho}(\mathbf{s} - \mathbf{I}) - \frac{4\pi}{3}\omega\mathbf{s}\right]^{-1}_{\varrho \to 0}, \tag{64}$$

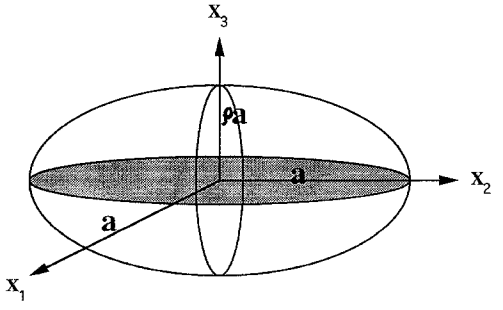

Fig. 2. The aspect ratio of a spheroidal domain

where $\mathbf{A} = -\mathbf{I}$, the volume ratio $\phi = \frac{4\pi}{3}\varrho\omega$ with $\omega \equiv na^3$ denoting the microcrack density, and n, a = the number density (per unit volume) and radius of the penny-shaped microcrack, respectively; see also Zhao, Tandon and Weng [36, p. 121]. For simplicity, all microcracks are assumed to be open and of the same radius a. The components of Eshelby's tensors $\mathbf{s}$ for a *spheroid* can be found in, for instance, Mura [61]. By substituting the components of $\mathbf{s}$ into (64) and taking the limit $\varrho \to 0$ for the *inclusion*, one can obtain the (transversely isotropic) effective compliance tensor $\mathbf{M}^*(\equiv \mathbf{C}_*^{-1})$ of a microcracked solid. The nontrivial effective compliance components are

$$M^*_{3333} = \frac{1 - \nu_0 - 2\alpha\nu_0^2}{E_0(1-\nu_0)(1-\alpha)}, \tag{65}$$

$$M^*_{1313} = M^*_{2323} = M^*_{3131} = M^*_{3232} = \frac{1}{4\mu_0(1-\beta)}, \tag{66}$$

and the remaining components of $\mathbf{M}^*$ are *identical* to those of the isotropic matrix compliance $\mathbf{M}_0$ $(\equiv \mathbf{C}_0^{-1})$. Here, the parameters α and β take the form:

$$\alpha = 1 - \frac{3(1-2\nu_0)}{3(1-2\nu_0) + 16(1-\nu_0)^2\,\omega}, \tag{67}$$

$$\beta = 1 - \frac{3(2-\nu_0)}{3(2-\nu_0) + 16(1-\nu_0)\,\omega}. \tag{68}$$

Substitution of (67) and (68) into (65) and (66) then renders nontrivial components of the effective compliance tensor,

$$E_0 M^*_{3333} = 1 + \frac{16}{3}(1-\nu_0^2)\,\omega, \tag{69}$$

$$4\mu_0 M^*_{1313} = 1 + \frac{16(1-\nu_0)}{3(2-\nu_0)}\,\omega = 4\mu_0 M^*_{2323} = 4\mu_0 M^*_{3131} = 4\mu_0 M^*_{3232}. \tag{70}$$

As expected, the nontrivial compliance components M^*_{3333} and M^*_{1313} are found to *coincide* exactly with those given by the Mori-Tanaka method (see Zhao, Tandon and Weng [36, Eqs. (7.3), (7.5)]). Further, they *coincide* exactly with Willis' [10] variational bounds if thermal conductivities are replaced by elasticities, see Willis [10, p. 197] and Laws and Dvorak [40, p. 1274].

6 Conclusion

Emanating from the "eigenstrain concept" introduced by Eshelby [45] and the RVE representation, ensemble-volume averaged governing micromechanical constitutive equations are derived for multiphase elastic composites to relate macroscopic stresses, macroscopic strains and average eigenstrains. It is demonstrated that the shapes and orientations of inclusions have a direct impact on the averaged constitutive equations. A "renormalization" procedure inspired by Willis [10] and Sen and Torquato [58]) is employed to obtain absolutely convergent integrals. Although micromechanical ensemble-volume averaged field equations can be constructed, exact solutions of pertinent integral equations are not available for arbitrary and random configura-

tions. Therefore, exact expressions for effective elastic moduli of multiphase composites are generally not available.

Nevertheless, "noninteracting" solutions are constructed for multiphase composites in Section 3 by neglecting *inter*-particle interaction effects. Moreover, in Sections 3, 4 and 5, it is explicitly shown that Hashin-Shtrikman-Walpole bounds, Willis' bounds, and the Mori-Tanaka estimates *coincide* with the "noninteracting" solutions if all inclusions are aligned and similarly shaped.

In a companion paper (Ju and Chen [59]), effects due to *inter-particle interactions* will be included for *two-phase* elastic composites with randomly located *spherical* particles. Specifically, both matrix and particles will be taken as isotropic. It will be shown that a higher-order, accurate, probabilistic particle interaction model and resulting effective moduli can actually be constructed micromechanically. By mathematical analogy, effective shear viscosities of hard-sphere dispersions under high shear rates can also be obtained in a companion paper from the results of incompressible matrix with rigid spheres.

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