Acta Mechanica 103, 1–15 (1994)

ACTA MECHANICA

Flutter analysis using transversality theory

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(Received October 23, 1992; revised December 17, 1992)

Summary. A new method of calculating the flutter boundaries of undamped aeroelastic "typical section" models is presented. The method is an application of the weak transversality theorem used in catastrophe theory. In the first instance, the flutter problem is cast in matrix form using a frequency domain method, leading to an eigenvalue matrix. The characteristic polynomial resulting from this matrix usually has a smooth dependence on the system's parameters. As these parameters change with operating conditions, certain critical values are reached at which flutter sets in. Our approach is to use the transversality theorem in locating such flutter boundaries using this criterion: *at a flutter boundary, the characteristic polynomial does not intersect the axis of the abscissa transversally.* Formulas for computing the flutter boundaries of structures with two degrees of freedom are presented, and extension to multi degree of freedom systems is indicated. The formulas have obvious applications in, for instance, problems of panel flutter at supersonic Mach numbers. Substantial savings in computation resources are possible when this non-iterative method is used, compared to existing frequency domain methods which are essentially iterative.

1 Introduction

Flutter prevention is a very important consideration in the design and development of various engineering structures and components for aeronautics and space applications. In recent years, a research program for the development of advanced propulsion engines and their components has been on-going at NASA Lewis Research Center. A significant part of the development effort is devoted to flutter considerations. The work reported here is part of that development.

In this work, a computationally efficient method is developed for calculating the flutter boundaries of an engineering structure with two degrees of freedom, based on the typical section model. The typical section of an airfoil is a simple but very effective concept for modeling the aeroelastic behavior of structures such as fixed airplane wings (Bairstow [1]; Frazer and Duncan [2]; Theodorsen [3]), and the rotary wings of helicopters (see, e.g., Johnson [4]). By a simple extension, the model has been used in the analysis of a cascade of turbomachine blades in various regimes of flow (Whitehead [5], [6]; Kaza and Kielb [7]; Dugundji and Bundas [8]; Bahkle et al. [9]). Such cascades are used in a variety of mathematical models of engine components such as propfans, compressor fans, and turbine bladed-disk assemblies.

1.1 Qualitative approach

The conventional approach in theoretical flutter analysis is basically *quantitative*, in which computationally intensive codes are developed for calculating flutter boundaries. However, the ultimate consideration in a flutter analysis is, essentially, a *qualitative* one: will flutter occur in the designed system under its normal operating conditions or not? The qualitative nature of the problem to be solved is, in some cases, masked by quantitative computational strategies.

An innovative aspect of our method is that it enables the solution of the qualitative flutter problem by means of a qualitative method of mathematical analysis well known in catastrophe theory or singularity theory. It is based on the concept of "structural stability" of mathematical objects such as matrices, smooth functions or differential equations; see, for instance, Poincaré [10, Lemma IV, p. LXI], Andronov and Pontryagin [11], Thom [12], and Arnol'd [13], [14], among others.

1.2 Structural stability

The concept of "structural stability" as used in mathematical texts is quite different from the notion of structural stability as used in engineering. In order to avoid any confusion here, the stability of structures in the aeroelastic sense will explicitly be called "*elastic* stability", or "*aeroelastic* stability", while stability in the mathematical sense will be referred to as "structural stability", in those situations where the intended meaning is not obvious from the context.

1.3 Parametric dependence

The problem of flutter analysis may be formulated as a problem of matrices depending on parameters. The transversality theorem was used by Arnol'd [15] in arriving at his versal deformation theorem for matrices depending on parameters. In this paper, we draw motivation from Arnol'd's work, but do not apply the transversality theorem to *matrices* directly, as he did. Instead, we apply the transversality theorem to the *characteristic polynomials* of matrices depending on parameters. In this way, we obtain a computationally efficient method for calculating flutter boundaries.

The format of this paper is as follows. In Section 2, a brief review of pertinent definitions and concepts from matrix theory and algebraic geometry is presented. In Section 3, we show that, in an undamped vibrating system, the condition of a non-transversal intersection of the characteristic polynomial with the axis of the abscissa may be used to detect the onset of flutter. The material in Section 4 is more or less standard aeroelasticity, but the flavor of our presentation is new; it is included here for continuity. Our main results are in Section 5, where various formulas for calculating flutter boundaries are presented. These formulas are applied to the computation of flutter boundaries in the remainder of the paper (Section 6).

2 Preliminaries

In this Section, we recall pertinent definitions from matrix theory, and illustrate certain concepts from other branches of mathematics not normally encountered in engineering analysis or aeroelasticity, but which are used in the ensuing discussion.

2.1 Matrix bundles

A *matrix bundle* (see, for instance, Arnol'd [15]) is a set of Jordan, scalar or ordinary diagonal matrices in which all the matrices in the set have the same structure, but the corresponding elements on the diagonals of the matrices are not necessarily equal. For example, all scalar and diagonal matrices of order n are in the same bundle.

By a *Jordan matrix* in the foregoing, we mean a square, block diagonal matrix in which each block is either in the Jordan normal form or is purely diagonal. In a Jordan matrix of order n, all the n elements on the main diagonal are equal to the same scalar.

2.2 Objects in general position

Various objects of a category (such as matrices, smooth functions, or polynomials depending on parameters) may be partitioned into two types: those that are in "general position", and those that are not. Objects in general position are also called "generic", while those that are not generic are said to be in "exceptional position", and called "degenerate".

A generic object has structural stability, and does not change its qualitative properties or behavior under small, arbitrary perturbations. A degenerate object, on the other hand, is structurally unstable. An arbitrarily small perturbation will cause it to bifurcate into two (or more) generic objects. As a result of this instability, degenerate objects are unobservable, and are "almost always" not encountered in engineering practice.

2.3 Transversality

The weak transversality theorem is one of the foundations of catastrophe theory, Thom and Levin [16]. It arises in the context of intersections of manifolds, a discussion of which has been given by, for instance, Abraham and Robbin [17]. Its significance in algebraic geometry has been outlined by Brieskorn and Knörrer [18] and Zeeman [19], among others.

The theorem asserts that if two manifolds intersect in such a way that the intersection is not in general position, then an arbitrarily small perturbation will lead to its bifurcation, and place the resulting intersections in general position.

In Fig. 1 a, the two intersections between the horizontal line and the curve are in general position, and are called *transversal.* At each intersection, the local tangent to the curve is different from that to the line, and the set of local tangents spans the ambient space.

On the other hand, the intersection shown in Fig. 1 b is *non-transversal.* The tangent to the curve at its only intersection with the line cannot be distinguished from the tangent to the line at that point. At this non-transversal intersection, the local tangents do not span the two-dimensional ambient space.

In Fig. 1 c there are no real intersections between the curve and the line, but there is an imaginary or complex intersection. The imaginary or complex intersection in Fig. 1 c is just as transversal as the real intersections in Fig. 1 a. For a further discussion of these ideas see, for instance, Poston and Stewart [20].

Transversality, being a property of objects in general position, implies structural stability. On the other hand, loss of transversality implies loss of structural stability.

3 Non-transversality implies flutter boundary

Many flutter problems may be analyzed as vibrating systems with two or more degrees of freedom; see, for instance, Bisplinghoff and Ashley [21], Dowell et al. [22], Dugundji and Bundas [8], and other references cited earlier. Often, the typical section model is used, in which there exists a coupling between two coordinates of vibration such as torsion and bending. In what follows, we consider such a coupled two degree-of-freedom system in order to illustrate how a non-transversal intersection of its characteristic polynomial with the axis of the abscissa indicates the onset of flutter.

3.1 Undamped system

If a coupled vibrating system with two degrees of freedom has no damping, then its characteristic equation may be written as a quadratic polynomial in λ,

$$p(\lambda) = \lambda^2 + a\lambda + b = 0; \quad \lambda, a, b \in \mathbb{R}. \tag{1}$$

The eigenvalues $\lambda = \omega^2$ must be real and positive in order for the structure to have elastic stability. A complex value of λ in a coupled undamped system implies flutter instability, while a real but negative value of λ implies divergence instability.

Since the coefficients a and b are real, the graph of $p(\lambda)$ in $\mathbb{R}^2$ is a *real algebraic curve*, Brieskorn and Knörrer [18]. The zero level set of this graph comes from the intersections of the polynomial with the axis of the abscissa, and are the eigenvalues of the coupled vibrating system. From a corollary of the fundamental theorem of algebra, it follows that there are at most two roots of (1), counting multiplicities. If the magnitudes of the roots are distinct, then the roots must be real; if the magnitudes are equal, the roots are either real and degenerate, or are complex conjugates.

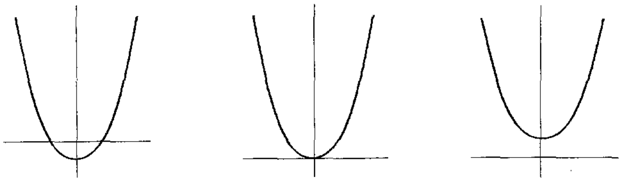
Now, the coefficients a and b have parametric dependence on system variables, such as the air speed in an aeroelastic system. As these system variables change with operating conditions, a and b also vary, and the graph of (1) becomes a family of curves in the plane. There are exactly three *qualitatively* different types of intersections with the axis of the abscissa, with regard to the number and nature of the roots in this family. All three are illustrated in Figs. 1 a to 1 c.

In Fig. 1 a, there are two distinct real roots; two real but coincident roots in Fig. 1 b; and no real roots at all in Fig. 1 c. The only case where transversal intersections do not occur is Fig. 1 b. We shall now show how the loss of transversality, as in Fig. 1 b, marks the flutter boundary in a coupled two degree-of-freedom system without damping.

Coupled vibrating systems with two degrees of freedom having the graph in Fig. 1 a *cannot* flutter because the eigenvalues $\lambda = \omega^2$, being the two roots of the polynomial in (1), are always real and distinct.

Coupled two degree-of-freedom vibrating systems having the graph in Fig. 1 c *must flutter.* The system flutters because the eigenvalues $\lambda_{1,2}$, which should always be real and positive if flutter is to be avoided, have now become complex.

Intermediate between the two cases considered above is that of Fig. 1 b, with a non transversal intersection. The following points of view may now be taken:



a **b** **c**

Fig. 1. Transversal and non-transversal intersections. **a** Transversal, **b** non-transversal, **c** transversal

(i) From the mathematical point of view, the intersection of Fig. 1 b is not non-transversal, is not in general position, is structurally unstable, and should "almost always" not occur.

(ii) From the engineering point of view, the degenerate characteristic polynomial of Fig. 1 b is inadmissive in a coupled two degree of freedom system, because the only way a coupled two degree of freedom system can have degenerate eigenvalues is when it disintegrates into two identical, *uncoupled* sub-systems. This contradicts the assumption that the sub-systems constitute a single *coupled* structure. Therefore, the characteristic polynomial depicted in Fig. 1 b should, physically, "almost always" not occur.

If flutter occurs at any time in an initially stable system (1) as its parameters a and b are varied, then the graph of the characteristic polynomial must have changed from that of Fig. 1 a to that of Fig. 1 c. There is *only one route* for passing from Fig. 1 a to Fig. 1 c, and that is through Fig. 1 b. Therefore, the case of Fig. 1 b constitutes a *flutter boundary.*

From what has been said above, we come to the following result:

The flutter boundaries of a coupled two degree-of-freedom system without damping may be obtained simply by inspecting its characteristic polynomial, and noting the critical parameters at which a non-transversal intersection with the axis of the abscissa, such as in Fig. 1 b, occurs.

Although we reached the above result by considering the transversal intersection of a real curve with a real axis of the abscissa, similarly useful results could be obtained by considerations of the transversality of complex algebraic curves intersecting with a complex axis of the abscissa, using the appropriate singularity theory for complex polynomial germs; see, for instance, Milnor [23] or Arnol'd et al. [24].

3.2 *Effect of damping*

Characteristic polynomials with complex coefficients and complex roots appear in vibrating systems the moment we account for damping, irrespective of the status of its elastic stability. Thus, if we have a damped two degree of freedom system, we have to work with a complex polynomial intersecting with a complex axis of the abscissa in a complex 2-dimensional space $\mathbb{C}^2$ or, by realification, a real polynomial of the fourth degree in a real 4-dimensional space, $\mathbb{R}^4$. Thus, the characteristic polynomial of a two degree of freedom system with damping may be written as a quartic equation in s,

$$p(s) = s^4 + as^3 + bs^2 + cs + d = 0; \qquad a, b, c, d \in \mathbb{R}, \qquad s \in \mathbb{C} \tag{2}$$

or, if $s = i\omega$ is assumed as is the usual custom in vibration analysis, then one has $p(\omega) = \omega^4 - ia\omega^3 - b\omega^2 + ic\omega + d = 0$. The presence of odd powers of ω is indicative of damping. The characteristic polynomial of an undamped system with n degrees of freedom is an even function of ω of order $2n$. Hence, if damping is removed, then one gets a biquadratic in ω for the two degree of freedom system,

$$p(\omega) = \omega^4 - b\omega^2 + d = 0, \tag{3}$$

which is equivalent to (1), upon setting $\lambda = \omega^2$.

3.3 Computational aspects and degeneracies

Computationally, the loss of transversality of the characteristic polynomial with the axis of the abscissa is indicated by the occurrence of *degenerate eigenvalues*. Degenerate eigenvalues, like all degenerate mathematical objects, are not in general position. Therefore, they are structurally unstable, and should "almost always" not be encountered in realistic engineering analysis. If they are encountered in the mathematical model of a physical system, it is only because one has made a theoretical *assumption* which is not *qualitatively* valid in the actual physical problem. For example, one might have assumed perfect symmetry when, in fact, there is a small but non-vanishing amount of imperfection, leading to a coupling between, say, two modes of vibration. Although the imperfection may be *quantitatively* small, the dynamic behavior of the coupled systems could be dramatically different from that predicted by ignoring the small imperfection altogether.

There are also other kinds of ambiguities associated with the eigenvalue degeneracy. For example, to which form of (4) below does the system eigenmatrix corresponding to Fig. 1b reduce under a similarity transformation: a diagonal matrix $\mathbf{D}_2$ of order 2, or a Jordan matrix $\mathbf{J}_2$ of order 2?

$$\mathbf{D}_2 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad \mathbf{J}_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}. \tag{4}$$

Computationally, in order to resolve whether or not a coupled two degree-of-freedom system will flutter, one has to calculate the eigenvalues in the first instance. If degenerate eigenvalues are encountered, then it means that the characteristic polynomial is not transversal to the axis of the abscissa. We then must inspect the corresponding *eigenvectors* or, equivalently, the eigenvalue matrix at the point where transversality is lost. If the eigenvectors are not linearly independent or, equivalently, the eigenvalue matrix is a Jordan matrix, then flutter *must* occur.

3.4 A note on divergence

The graph of (1) loses transversality with the axis of its abscissa in only one way, as in Fig. 1b. In contrast, the graph of (3) intersects the axis of its abscissa non-transversally in two ways as in Fig. 2b or Fig. 2c. Now, (1) is a quadratic in λ, whereas (3) is a biquadratic in ω, and both describe the same system. The loss of transversality depicted in Fig. 2b signifies a flutter boundary, whereas that in Fig. 2c indicates a divergence condition. In order to use the transversality criterion for divergence, therefore, one has to use $p(\omega)$ rather than $p(\lambda)$.

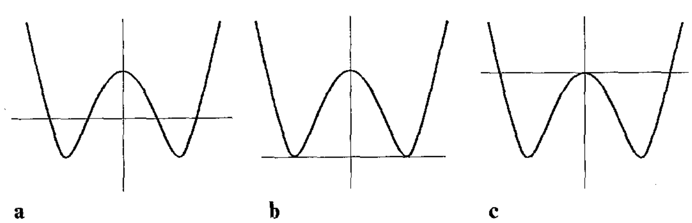


Fig. 2. Symmetric unfolding of the cusp catastrophe germ showing transversal and non-transversal intersections of the characteristic polynomial of a vibrating system with two degrees of freedom. **a** Elastic stability, **b** flutter boundary, **c** divergence boundary

4 Equations of motion

Many problems of "dynamic instability" encountered in engineering are actually various versions of the two aeroelastic phenomena known as "flutter" and "divergence". They include instability problems of: (i) rotating shafts, plates, or shells having a cyclic or polygonal cross section in which the cyclic or dihedral symmetry of the structure is slightly destroyed by a variety of mechanisms; (ii) elastic systems with convective forces induced by moving loads, moving fluids, or thermal transport; (iii) elastic systems coupled to magnetic forces; (iv) elastic systems subject to thermal convection; (v) various other physical systems in which spatial or temporal *direction sensitivity* has a qualitative significance; and so on, and so forth.

These kinds of problems are all characterized by the fact that system matrices are no longer symmetric, i.e. Maxwell's reciprocal theorem is not obeyed by such systems. Consequently, such systems are not governed by a potential, and are often called *non-conservative systems;* see, for instance, Bolotin [25]. Before the onset of flutter, it is admissible to assume that motion is harmonic, with small amplitudes in the neighborhood of equilibrium. This is the essence of linear stability analysis. Large amplitudes of vibration arise only *after* the flutter boundary has been passed. If a prediction of post-flutter behavior is not required, then a linear analysis is sufficiently accurate for many engineering applications. Therefore, many flutter problems are analyzed as linear systems.

The powerful techniques of catastrophe theory may, at first glance, seem to be inapplicable to the solution of the physical problems outlined above since, in the first instance, such problems are *linear* and, secondly, they are not gradient dynamic systems, or systems governed by potentials. However, if we use matrix techniques such as the receptance method, we may apply catastrophe theoretic ideas to gain insight into the stability of such systems, simply by studying the transversality of the *characteristic polynomial* of the system's matrix to the axis of the abscissa.

The technical term "receptance" as proposed by Duncan, Biot, Johnson and Bishop [26] relates to a concept initially called mechanical admittance; see, for instance, Duncan [27], or Bisplinghoff and Ashley [21, p. 204]. It is a powerful technique that enables one to make a frequency domain analysis of a complex engineering structure. A detailed account of this technique has been provided by Bishop and Johnson [28]. Similar ideas are also used in the static analysis of engineering structures, where receptances are called "displacement influence coefficients".

The basic concept of receptance is to relate generalized forces to generalized displacements in a multi degree of freedom system vibrating at a frequency ω using matrix methods. If $\mathbf{f}$ and $\mathbf{x}$ represent the generalized force vector and generalized displacement vector, respectively, then the relationship between the two may be expressed as

$$\mathbf{f} = \mathbf{K}(\omega)\,\mathbf{x}, \quad \mathbf{x} = \mathbf{A}(\omega)\,\mathbf{f}, \quad \mathbf{AK} = \mathbf{KA} = \mathbf{I} \in \mathbb{C}^{n\times n}, \quad \mathbf{x}, \mathbf{f} \in \mathbb{C}^{n}, \tag{5}$$

where $\mathbf{A}(\omega)$ is the receptance matrix and its inverse, $\mathbf{K}(\omega)$, is the "dynamic stiffness matrix" in the frequency domain. If $\mathbf{f}$ is due to aerodynamic forces, then $\mathbf{K}(\omega)$ may be termed the "aerodynamic stiffness matrix".

The flutter problem, being essentially a problem of mechanical vibration analysis, may be treated by the method of receptance. This means that the equation of motion of an aeroelastic system undergoing small displacements in the neighborhood of equilibrium may be written in the standard notation of mechanical vibration as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}. \tag{6}$$

For harmonic vibrations at the circular frequency ω one may write

$$\mathbf{D}(\omega)\,\mathbf{x} = \mathbf{f}, \qquad \mathbf{D}(\omega) = (\mathbf{K} - \omega^2\mathbf{M}) + i\omega\mathbf{C}, \tag{7}$$

where $\mathbf{D}$ is the dynamic stiffness matrix of the multi degree of freedom vibrating system. For sinusoidal motion of an airfoil in an air stream, the forcing vector $\mathbf{f}$ in (6) may be written in matrix form

$$\mathbf{f} = \omega^2\mathbf{L}(\omega)\,\mathbf{x}, \tag{8}$$

where $\mathbf{L}(\omega)$ is an "aerodynamic stiffness matrix". It may be noted that $\mathbf{L}$ generally has a smooth, nonlinear dependence on the vibration frequency ω.

From (6) and (8), one gets the equation of motion, when $\mathbf{C} = 0$,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \omega^2\mathbf{L}(\omega)\,\mathbf{x}. \tag{9}$$

Under harmonic vibrations at small amplitudes, $\ddot{\mathbf{x}} = -\omega^2\mathbf{x}$, and the above becomes

$$\mathbf{K}\mathbf{x} = \omega^2[\mathbf{M} + \mathbf{L}(\omega)]\,\mathbf{x}, \tag{10}$$

which may be written in the *eigenvalue problem form* as

$$\mathbf{A}(\lambda)\,\mathbf{u} = \lambda\mathbf{u}, \qquad \mathbf{A}(\lambda) = \mathbf{K}^{-1}[\mathbf{M} + \mathbf{L}(\lambda)]. \tag{11}$$

The stability of $\mathbf{A}$ in the above may be investigated by using the techniques published by Arnol'd [15] on matrices depending on parameters. However, our approach here is to map $\mathbf{A}$ from the space of matrices to the space of polynomials, and treat it there as a problem of smooth functions depending on parameters. In this way we apply the transversality theorem in a more efficient way to suit the problem under consideration.

5 Determination of flutter boundaries

Equation (11) is a nonlinear eigenvalue problem, which is traditionally solved iteratively, in a computationally intensive procedure, in order to determine the flutter boundaries. In this Section, we shall outline a new and computationally more efficient procedure for finding the flutter boundaries, based on applications of the weak transversality theorem of catastrophe theory.

First, we fix λ in (11) at some nominal value λ_0 to get

$$\mathbf{A}(\lambda_0)\,\mathbf{u} = \lambda\mathbf{I}\mathbf{u}, \tag{12}$$

which may be written as

$$[\mathbf{A}(\lambda_0) - \lambda\mathbf{I}]\,\mathbf{u} = 0, \tag{13}$$

from which one obtains the well known *flutter determinant*,

$$|\mathbf{A}(\lambda_0) - \lambda\mathbf{I}| = 0. \tag{14}$$

Expanding the flutter determinant yields the characteristic polynomial, $p(\lambda)$. To do this, one may write $\mathbf{A}(\lambda_0)$ in (13) as

$$\mathbf{A}(\lambda_0) = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}, \quad a_{ij} \in \mathbb{C}, \tag{15}$$

and, since $\mathbf{A}(\lambda_0)$ is not a symmetric matrix, it may be decomposed into its symmetric and skew-symmetric parts,

$$\mathbf{A}(\lambda_0) = \begin{bmatrix} a_{11} & \frac{1}{2}(a_{12} + a_{21}) \\ \frac{1}{2}(a_{12} + a_{21}) & a_{22} \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2}(a_{12} - a_{21}) \\ \frac{1}{2}(a_{12} - a_{21}) & 0 \end{bmatrix}, \tag{16}$$

in which all the matrix elements a_{ij} are functions of λ_0. In what follows, we consider the special case in which a_{ij} are independent of frequency. With the substitutions

$$\begin{aligned} a_0 &= \frac{1}{2}(a_{11} + a_{22}), \quad b_0 = \frac{1}{2}(a_{12} + a_{21}), \\ c_0 &= \frac{1}{2}(a_{12} - a_{21}), \quad d_0 = \frac{1}{2}(a_{11} - a_{22}), \end{aligned} \tag{17}$$

one obtains from (16)

$$\mathbf{A}(\lambda_0) = \begin{bmatrix} a_0 - d_0 & b_0 \\ b_0 & a_0 + d_0 \end{bmatrix} + \begin{bmatrix} 0 & -c_0 \\ c_0 & 0 \end{bmatrix}. \tag{18}$$

The characteristic polynomial of (18) is

$$p(\lambda) = \lambda^2 - 2a_0\lambda + (a_0^2 - b_0^2 + c_0^2 - d_0^2) = 0; \quad a_0, b_0, d_0 \in \mathbb{R}, \tag{19}$$

from which one obtains the following discriminant of p:

$$\Delta = b_0^2 + d_0^2 - c_0^2. \tag{20}$$

When p is not a quadratic, its discriminant may be computed by means of Sylvester's eliminant; see, for instance, Afolabi [29].

The condition for a non-transversal intersection of (19) with the axis of the abscissa is equivalent to the vanishing of the discriminant of the polynomial. The vanishing of the discriminants of polynomials is very significant in catastrophe theory, Zeeman [19], where the projection of singular surfaces to the parameter space is called the *bifurcation set*. Perhaps the most well known discriminant surface is that associated with the polynomial germ $y = x^4$ in the real 2-dimensional parameter space, and called the *cusp catastrophe*. The geometry of discriminant surfaces of algebraic varieties in a more general context is discussed in the work of Brieskorn and Knörrer [18]. In the specific case of our two degree-of-freedom typical section model the non-transversality condition, of the vanishing of the discriminant of the characteristic polynomial, is also the same as eigenvalue degeneracy.

If we calculate the eigenvalues and corresponding eigenvectors of (18), we get

$$\lambda_1 = a_0 - \sqrt{\Delta}, \quad \lambda_2 = a_0 + \sqrt{\Delta}, \quad \mathbf{u}_1 = \begin{Bmatrix} 1 \\ -d_0 + \sqrt{\Delta} \end{Bmatrix}, \quad \mathbf{u}_2 = \begin{Bmatrix} 1 \\ -d_0 - \sqrt{\Delta} \end{Bmatrix}. \tag{21}$$

When the discriminant vanishes, $\Delta = 0$ in (21), and one obtains the following degenerate eigenvalues, the corresponding *eigenvectors* of which *are also degenerate:*

$$\lambda_1 = \lambda_2 = a_0, \quad \mathbf{u}_1 = \mathbf{u}_2 = \begin{Bmatrix} 1 \\ -d_0 \end{Bmatrix}. \tag{22}$$

Thus, at the non-transversal condition signified by the vanishing of the discriminant, the eigenvalues are degenerate. It is precisely this kind of eigenvalue degeneracy, usually noted in undamped models of coupled bending-torsion vibrations, that gives rise to the well known terms, *coupled mode flutter* and *coalescence flutter.*

It is now pertinent to make the following remarks:

(i) A *flutter boundary* corresponds to the parameter values where a simultaneous degeneracy of the eigenvalues and eigenvectors occurs.

(ii) The *degeneracy of eigenvectors necessarily implies flutter,* because the existence of degenerate eigenvectors at the flutter boundary implies that the system's eigenmatrix cannot be diagonalized; it is only reducible to a Jordan matrix.

(iii) Just before and just after a flutter boundary, the *system's eigenmatrix is catastrophically switched from one matrix bundle to another.*

A summary of the foregoing is this. The flutter boundaries obtained from the transversality criterion, as determined by the vanishing of the discriminant, also correspond to the conditions of simultaneous degeneracy of the eigenvalues and their corresponding eigenvectors.

Two types of flutter information may be deduced from the characteristic polynomial of an aeroelastic system. In the first place, one tests if flutter will occur at all. If flutter is to occur, the discriminant of the polynomial of the undamped system must vanish. If the occurrence of flutter has been determined, the second thing is to compute the flutter boundaries. The flutter boundaries are obtained simply by setting the discriminant to zero when solving for the roots of the characteristic polynomial. The formula for computing the discriminant of a quadratic equation is very well known in engineering, but not so for a polynomial of arbitrary order. A general algorithm for computing the discriminant of a polynomial of arbitrary order by means of Sylvester's resultant, or eliminant, is well known in the theory of equations, Turnbull [30]; its applications for vibrating systems have been described by Afolabi [29].

The following conditions may be used to test if a given aeroelastic system, whose characteristic polynomial is written in the form of (19), will flutter or not:

$$\text{if} \quad b_0 = 0, \quad \text{flutter occurs when} \quad c_0 = \pm d_0, \tag{23}$$

$$\text{if} \quad d_0 = 0, \quad \text{flutter occurs when} \quad c_0 = \pm b_0, \tag{24}$$

$$\text{if} \quad d_0 = 0, \quad \text{and} \quad b_0 = 0, \quad \text{flutter occurs for all} \quad c_0 \in \mathbb{R}, \tag{25}$$

$$\text{if} \quad d_0 \neq 0, \quad \text{and} \quad b_0 \neq 0, \quad \text{flutter occurs when} \quad c_0 = \pm\sqrt{b_0^2 + d_0^2}. \tag{26}$$

The variables $a_0 \dots d_0$ in the foregoing are functions of λ_0, and are defined in (17). Although all of the above Eqs. (23)–(26) are theoretically equivalent in that they all give the same flutter boundaries, there are instances when it may be advantageous to use a particular form, rather than another. For example, if $b_0 = 0$ in a model, then it is computationally more efficient to use (23). Similarly, if $d_0 = 0$ in some mathematical model, then flutter boundaries are easier to predict for such a model using (24). If $b_0 = 0$ and $d_0 = 0$, then flutter *always* occurs, as seen from (25). In the most general case, (26) applies and the flutter boundaries may be obtained from

$$c_0^2 = b_0^2 + d_0^2. \tag{27}$$

If it has been definitely determined that flutter will occur, e.g. by using any of (23)–(26), then the flutter boundaries, or flutter frequencies, may be computed by means of the formula

$$\lambda_F = \omega_F = a_0, \tag{28}$$

which follows upon substituting (26) in (19). The formula (28) for calculating flutter boundaries is especially efficient because a_0 is simply the semi-trace of the eigenvalue matrix; the off-diagonal terms in the matrix contribute nothing. Thus, we arrive at the remarkable result:

The off-diagonal terms or coupling terms in the eigenmatrix (18) have no influence whatsoever on the flutter boundaries; the flutter speed is determined simply by averaging up the diagonal terms in (18) and equating the sum thus obtained to the eigenvalue parameter, λ, as in (28).

The formula (28) is also easy to obtain from the monic form of the characteristic polynomial: it is, quite simply, the coefficient of the linear term divided by 2.

6 Flutter boundaries in steady aerodynamics

Steady aerodynamics applies, approximately, to an aeroplane at cruise speed, or to various aeroelastic structures such as flat plates or panels at supersonic Mach numbers. This flutter problem has been treated in several texts; see, for instance, Dowell et al. [22, Section 3, Eqs. (3.3.48) et seq.] from where one gets the lift and moment coefficients

$$L_h = qS\,\frac{\partial C_L}{\partial \alpha}\,\alpha \tag{29}$$

$$M_\alpha = eqS\,\frac{\partial C_L}{\partial \alpha}\,\alpha. \tag{30}$$

Writing the equations of motion in terms of the aerodynamic stiffness matrix, by using the above lift and moment expressions, one gets

$$\begin{Bmatrix} L_h \\ M_\alpha \end{Bmatrix} = \frac{\varrho_\infty U^2}{2M} \begin{bmatrix} 0 & qS\,\dfrac{\partial C_L}{\partial \alpha} \\ 0 & eqS\,\dfrac{\partial C_L}{\partial \alpha} \end{bmatrix} \begin{Bmatrix} h \\ \alpha \end{Bmatrix}. \tag{31}$$

The remainder of the equations of motion are

$$\begin{bmatrix} m & S_\alpha \\ S_\alpha & I_\alpha \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix} + \begin{bmatrix} K_h & 0 \\ 0 & K_\alpha \end{bmatrix} \begin{Bmatrix} h \\ \alpha \end{Bmatrix} = \begin{Bmatrix} -L_h \\ M_\alpha \end{Bmatrix}. \tag{32}$$

From (31) and (32), one gets $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{L}\mathbf{x}$. The inverse mass matrix is

$$\mathbf{M}^{-1} = \frac{1}{d}\begin{bmatrix} I_\alpha & -S_\alpha \\ -S_\alpha & m \end{bmatrix}, \quad d = mI_\alpha - S_\alpha^2. \tag{33}$$

One may use the above to compute an eigenvalue matrix, $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, where $\mathbf{A} = \mathbf{M}^{-1}(\mathbf{K} + \mathbf{L})$, $\lambda = \omega^2$.

6.1 Flutter boundaries from the flutter determinant

The flutter determinant obtainable from the eigenvalue matrix formed from (31) and (32) above is

$$|\mathbf{A} - \lambda \mathbf{I}| = \frac{1}{d} \begin{vmatrix} K_h I_\alpha - \lambda d & \dfrac{\varrho_\infty U^2}{2M} qS \dfrac{\partial C_L}{\partial \alpha} (I_\alpha + eS_\alpha) - K_\alpha S_\alpha \\ -S_\alpha & -\dfrac{\varrho_\infty U^2}{2M} qS \dfrac{\partial C_L}{\partial \alpha} (S_\alpha + em) + mK_\alpha - \lambda d \end{vmatrix}, \tag{34}$$

where d is defined in (33). Summing up the diagonal elements of (34) and equating the sum thus obtained to zero yields the equation for the flutter boundaries,

$$\lambda_F = \omega_F{}^2 = \frac{k^2 U^2}{b^2} = \frac{mK_\alpha + I_\alpha K_h - \dfrac{\varrho_\infty U^2}{2M} qS(S_\alpha + em) \dfrac{\partial C_L}{\partial \alpha}}{2(mI_\alpha - S_\alpha{}^2)}, \tag{35}$$

where k is the reduced frequency. It has been assumed that, in the above, C_L is not a function of k. Before using (35), therefore, one must specify the appropriate C_L for the given aeroelastic problem. Certain problems lend themselves to simplified expressions of C_L. For instance, for a flat plate in two-dimensional flow, one gets

$$\frac{\partial C_L}{\partial \alpha} = 2\pi, \tag{36}$$

in which case the flutter boundary expression becomes, in terms of the reduced frequency,

$$k = \frac{b}{U} \sqrt{\frac{mK_\alpha + I_\alpha K_h - \pi \varrho_\infty U^2 qS(me + S_\alpha)/M}{2(mI_\alpha - S_\alpha{}^2)}}. \tag{37}$$

For a numerical example of how the formulas obtained by using catastrophe theoretic methods in the preceding Sections may be implemented, we consider a case previously treated by Dowell et al. [22] by the classical methods. The governing equations of motion may be cast in matrix form

$$-\omega^2 ml \begin{bmatrix} \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} + \frac{\varrho_\infty U^2}{2M} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = 0, \tag{38}$$

or, $-\omega^2 \mathbf{M}\ddot{\mathbf{x}} + \mathbf{Kx} + \mathbf{Lx} = 0$. Premultiplying (38) by $\mathbf{M}^{-1}$ leads to the following flutter determinant, where $\lambda = \omega^2$:

$$|\mathbf{A} - \lambda \mathbf{I}| = \frac{1}{5ml} \begin{vmatrix} 8k + \dfrac{\varrho_\infty U^2}{M} - \lambda_m & -2k - 4\varrho_\infty U^2 M \\ -2k + 4\varrho_\infty U^2 M & 8k - \dfrac{\varrho_\infty U^2}{M} - \lambda_m \end{vmatrix}, \tag{39}$$

and $\lambda_m = (5ml)\,\lambda$. Summing up the diagonal terms in the above flutter determinant and equating the sum thus obtained to zero yields the flutter boundary

$$\lambda_m = 8k, \qquad \Rightarrow \lambda = \omega^2 = \frac{8k}{5ml}, \qquad \Rightarrow \omega = \pm\sqrt{\frac{8k}{5ml}}, \tag{40}$$

which agrees with results previously published by Dowell et al.

6.2 *Flutter boundaries from the characteristic polynomial*

The following characteristic equation is obtained from the flutter determinant (34):

$$p(\lambda) = a\lambda^2 + b\lambda + c = 0, \tag{41}$$

where

$$a = mI_\alpha - S_\alpha{}^2$$

$$b = qS(me + S_\alpha)\,\frac{\partial C_L}{\partial \alpha} - mK_\alpha - I_\alpha K_h$$

$$c = K_h K_\alpha - qK_h eS\,\frac{\partial C_L}{\partial \alpha}.$$

In the first instance, one tests if the system will flutter at all. If flutter is to occur, then the discriminant of the characteristic polynomial must vanish at a flutter boundary. Secondly, one uses the transversality theorem to obtain a geometric criterion for the location of the flutter boundaries. Computationally, this may be achieved by means of Sylvester's eliminant, as described by Afolabi [29].

The discriminant of (33) is $\Delta = b^2 - 4ac$. If flutter is to occur, this discriminant must vanish. Thus, setting $\Delta = b^2 - 4ac = 0$ in (33) gives the parameter values which guarantee the occurrence of flutter. When this occurs at a non-transversal intersection of $p(\lambda)$ with the axis of the abscissa, we get the equation of the flutter boundary as

$$\lambda = \omega^2 = -\frac{b}{2a}. \tag{42}$$

The above may be written in terms of the reduced frequency, k, and the system's elastic and aerodynamic parameters as

$$k = \frac{b}{U}\sqrt{\frac{mK_\alpha + I_\alpha K_h - \frac{1}{2}\varrho U^2 S(me + S_\alpha)\,\frac{\partial C_L}{\partial \alpha}}{K_h\left(K_\alpha - \frac{1}{2}\varrho U^2 eS\,\frac{\partial C_L}{\partial \alpha}\right)}}, \tag{43}$$

which is identical to results obtained from the flutter determinant (37).

Using the numerical example in Dowell et al. [22], one obtains the following characteristic polynomial:

$$5m^2l^2\lambda - 16mlk\lambda + 12\left(k^2 + \frac{\varrho_\infty U^2}{2M}\right) = 0. \tag{44}$$

When the determinant of (44) vanishes, one obtains the flutter boundaries as

$$\lambda_F = \frac{8k}{5ml}, \quad \Rightarrow \omega_F = \pm\sqrt{\frac{8k}{5ml}}, \tag{45}$$

which agrees with the result of the previous Section.

7 Conclusions

With the aid of the weak transversality theorem from catastrophe theory, simple formulas have been outlined for computing the flutter boundaries of vibrating systems representable as aeroelastic "typical sections", and which are characterized by *asymmetric* system matrices. The procedures developed here provide the flutter boundaries much more quickly and with much less effort when compared with existing iterative methods. Two different techniques have been outlined, both of which are equivalent. In the first case, one simply sums up the diagonal elements of the "flutter determinant" (14), and equates the sum thus obtained to zero, yielding Eq. (28). In the second case, one writes the characteristic polynomial in the form $p = a\lambda_2 + b\lambda + c = 0$, so that the equation of the flutter boundaries becomes $\lambda_F = -b/2a$. In either case, whether flutter takes place or not depends on the nature of the intersection of the characteristic polynomial with the axis of the abscissa. We test for transversality of the intersections by computing the discriminant of the polynomial, $\Delta = b^2 - 4ac$ if the polynomial is a quadratic function. For a polynomial of arbitrary order, one may compute the discriminant, Δ, by means of Sylvester's eliminant. If $\Delta = 0$, then a non-transversal intersection exists, and coupled mode flutter will occur. If $\Delta \neq 0$, then the intersections are always transversal, and coupled mode flutter cannot occur.

Acknowledgements

Part of this work was carried out at NASA Lewis Research Center, Cleveland, Ohio. I am grateful to O. Mehmed for sharing with me his experimental insights on flutter, and to Dr. T. S. R. Reddy for his stimulating discussions on the theory of flutter. Funding was provided through the Institute for Computational Mechanics in Propulsion, ICOMP.

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