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Rings for which every module is a direct sum of cyclic modules.

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Let R be a principal ideal ring with the descending chain condition. Then it is known that any R-module[1]) is a direct sum of cyclic modules. This was apparently first proved by KÖTHE [3], though the fundamental ideas go back to PRÜFER [5]. In the commutative case, KÖTHE also proved a converse: if a commutative ring with unit satisfies the descending chain condition, and has the property that all its modules are direct sums of cyclic modules, then it is necessarily a principal ideal ring. The purpose of this note is to show that the assumption of the descending chain condition is actually redundant.

Theorem. *Let R be a commutative ring with unit, and suppose every R-module is a direct sum of cyclic modules. Then R is a principal ideal ring satisfying the descending chain condition.*

The corresponding problem in the non-commutative case remains open, even with the descending chain condition assumed in advance. NAKAYAMA [4] has noted that the ring need not be a principal ideal ring, so that the obvious conjecture fails.

Before proceeding to the theorem itself, we prove several lemmas. We first consider a ring A which is a complete direct sum of an infinite set of fields $\{K_i\}$. (We use the term "complete" to distinguish this from the usual weak or discrete direct sum, in which only a finite number of non-zero components are allowed.) There are some obvious maximal ideals in A: for any index i, we have the maximal ideal M_i consisting of all elements with vanishing K_i-component. There also exist some "less accessible" maximal ideals in A. To see this, let I be the weak direct sum of $\{K_i\}$; I is an ideal in A. By ZORN's lemma, I can be expanded to a maximal ideal, necessarily different from any M_i. Following the terminology of HEWITT [2], we shall refer to the latter type as a *free* maximal ideal.

Lemma 1. *Let A be the complete direct sum of an infinite number of fields, and let M be a free maximal ideal in A. Then M is not a direct sum of principal ideals.*

[1]) All rings under discussion will have a unit element, which is assumed to act as unit operator on any module.

Proof. We remark that any principal ideal in A can be generated by an idempotent: given an element x, take e to be 0 where x is 0 and 1 where x is non-zero; then $xA = eA$. A direct sum of a finite number of principal ideals is again principal, being generated by the sum of the generators.

Suppose then that M is a direct sum of principal ideals $\{e_j A\}$, where the e_j's are (necessarily orthogonal) idempotents. The number of summands must be infinite, for otherwise M would be principal, and it is clear that a free maximal ideal cannot be principal. Split $\{e_j\}$ into two infinite subsets, and let f and g be the sums of the respective subsets (these "infinite sums" having an obvious meaning). Then $fg = 0$ is in M, and hence either f or g is in M; say f for definiteness. But this means that f is a linear combination of a finite number of e_j's, a contradiction.

Lemma 2. *Let R be a commutative ring with unit and no nilpotent elements. Suppose $\{x_i\}$ are orthogonal non-zero elements in R, and define S to be the complete direct sum of $\{x_i R\}$; S is in a natural way a ring (coordinate-wise multiplication) and an R-module. Then any decomposition of S (as an R-module) into a direct sum of cyclic modules can have only a finite number of (non-zero) summands, and moreover the generators of the cyclic summands are orthogonal.*

Proof. Suppose S is the direct sum of cyclic modules generated by y_j. We may write, formally

$$(1) \qquad y_j = \sum_m a_{jm} x_m \qquad (a_{jm} \in R).$$

Then (for $j \neq k$),

$$(2) \qquad (a_{km} x_m) y_j = a_{km} a_{jm} x_m^2 = (a_{jm} x_m) y_k .$$

Since the sum is direct, it must be the case that (2) is zero. Hence $y_j y_k = 0$ for $j \neq k$, and we have verified the last statement of the lemma.

Now suppose the number of y's to be infinite. Let z be the element

$$z = x_1 + x_2 + \cdots .$$

Then z is in S, and so is a linear combination of a finite number of y's. This means that z is orthogonal to the remaining y's. But

$$z y_j = \sum_m a_{jm} x_m^2 .$$

Hence $z y_j = 0$ entails $a_{jm} x_m^2 = 0$, $(a_{jm} x_m)^2 = 0$, $a_{jm} x_m = 0$, $y_j = 0$, a contradiction.

For convenience in the remainder of the discussion, let us call R a D-ring if it is a commutative ring with unit with the property that every R-module is a direct sum of cyclic modules. It is clear that a homomorphic image of a D-ring is again a D-ring. Also, if R is a

D-ring without divisors of zero, then R must be a field; for otherwise the quotient field of R, as an R-module, could not be a direct sum of cyclic modules. We have proved:

Lemma 3. *In a D-ring all prime ideals are maximal.*

We shall now dispose of the semi-simple case of our theorem.

Lemma 4. *Let R be a D-ring which is semi-simple in the sense that the intersection of the maximal ideals is* 0. *Then R is the direct sum of a finite number of fields.*

Proof. Our task is to prove that R satisfies the descending chain condition. It follows from Lemma 3 and [1, Th. 1] that it will suffice to prove the ascending chain condition. If the latter fails, then there exists in R an ideal I which is not finitely generated. The expression of I as a direct sum of principal ideals $\{x_i R\}$ must therefore have an infinite number of summands. The x's are of course necessarily orthogonal. We now follow the notation of Lemma 2 and form S, the complete direct sum of $\{x_i R\}$; S is the direct sum of cyclic modules generated by $y_1, \ldots, y_k$. For at least one y_j it must be the case that the expression (1) has an infinite number of non-zero terms. Suppose this happens for $j = 1$, and let us write T for the set of indices m for which $a_{1m} x_m \neq 0$. For each $t \in T$, choose a maximal ideal M_t in R with the property that $a_{1t} x_t$ is not in M_t; since the x's are orthogonal, the M_t's are surely distinct.

For each $t \in T$, let an arbitrary element $c_t \in R$ be prescribed. We claim that there exists in R an element which agrees with c_t modulo M_t for each t. To see this, we form the element u in S given formally by

$$u = \sum_t c_t a_{1t} x_t .$$

We note

$$(a_{1t} x_t) u = c_t a_{1t}^2 x_t^2. \tag{3}$$

Suppose $u = d_1 y_1 + \cdots + d_k y_k$. Since the y's are orthogonal (Lemma 2), we have

$$(a_{1t} x_t) u = (a_{1t} x_t) d_1 y_1 = d_1 a_{1t}^2 x_t^2. \tag{4}$$

From (3) and (4) we obtain

$$d_1 \equiv c_t \pmod{M_t},$$

and so d_1 is the desired element.

Write J for the intersection of the M_t's. Then the preceding paragraph can be summarized as follows: R/J is the complete direct sum of the fields R/M_t. But R/J, along with R, is a D-ring. This contradicts Lemma 1.

For completeness, we include the well known proof of the following lemma.

Lemma 5. *Let R be a commutative ring with unit, and suppose that every prime ideal in R is maximal. Then there exists a set of primary ideals, one for each maximal ideal, with intersection* 0.

Proof. Given a maximal ideal M, define Q to be the set of all elements x of R for which the annihilator (the set of all a with $ax = 0$) is not contained in M. One easily verifies that Q is an ideal contained in M, and that $yz \in Q$ and $y \bar{\in} Q$ imply $z \in M$. To complete the proof that Q is primary for M, we take t in M and have to prove that some power of t is in Q. If not, consider the set S of all $t^k c$, for $k = 0, 1, 2, \ldots$ and $c \bar{\in} M$; S is closed under multiplication and does not contain 0. So we may form an ideal N which is maximal with respect to disjointness from S; N is a prime and hence a maximal ideal. It is clear that N is contained in M. On the other hand, M contains t which is not in N. Thus N is properly contained in M, a contradiction.

Now let $\{M_i\}$ be the maximal ideals of R, form the corresponding primary ideals $\{Q_i\}$ as above, and suppose $r \in \bigcap Q_i$. Consider the ideal I annihilating r. If $I \neq R$, then I is contained in some M_i. But $r \in Q_i$ means that there exists an element $c \bar{\in} M_i$ with $cr = 0$, i. e. with $c \in I$. This contradicts $I < M_i$. Hence $I = R$ and $r = 0$.

Proof of the theorem. Let R be a D-ring. If N is the intersection of the maximal ideals, R/N is also a D-ring. Lemma 4 then shows that R has only a finite number of maximal ideals, say $M_1, \ldots M_n$. By Lemma 5,

$$0 = Q_1 \cap \ldots \cap Q_n,$$

where Q_i is a primary ideal with M_i as radical. Since the Q_i are pairwise comaximal, R is the direct sum of rings isomorphic to R/Q_i. So it is sufficient to show that R/Q_i is a principal ideal ring, and for this it suffices to show that its maximal ideal is principal.

Hence assume now that the D-ring R has only one maximal ideal M, $M \neq 0$. By hypothesis,

$$M = a_1 R \oplus a_2 R \oplus \cdots \tag{5}$$

where the a_i are orthogonal non-zero elements. Likewise

$$M^2 = a_1^2 R \oplus a_2^2 R \oplus \cdots.$$

Then M/M^2 is the direct sum of the non-zero modules $a_i R/a_i^2 R$. Since $a_i M = a_i^2 R$, $a_i R/a_i^2 R$ is a vector space over R/M. So if we can show that M/M^2 is one-dimensional over R/M, then there is only one a_i, and M is principal, as desired. Now M/M^2 is the maximal ideal of R/M^2, and $(R/M^2)/(M/M^2)$ is isomorphic to R/M, so we can make the further assumption that $M^2 = 0$.

We wish to show that M has length 1, as an R-module. Every principal ideal contained in M has length 1, since it is cyclic over

R/M. If M has length greater than 1, we can, in (5) above, pass to the residue class ring modulo $a_3 R \oplus \cdots$; so we may assume that M has the form $M = aR \oplus bR$. Thus M has length 2 and R has length 3.

At this point we could quote KÖTHE's result, but for completeness we give the remainder of the proof. Let G be the R-module generated by u and v, subject to the relation $au + bv = 0$; G is a module of length 5. We claim that any cyclic submodule Rw of G has length 1 or 3. This is clear if $Mw = 0$, so we suppose the contrary. Write $w = cu + dv$. Since $Mw \neq 0$, either c or d is a unit in R, so we may assume $c = 1$. Let r be an element of R annihilating w. Then $ru + rdv = 0$, so there exists an element s with $r = sa$, $rd = sb$. From the equation $sb = sad$, and the fact that b is not a multiple of a, we see that s is not a unit, $s \in M$, $r = sa = 0$. Hence the mapping $r \to rw$ is an isomorphism of R onto Rw, and the latter must have length 3.

Now G is a direct sum of cyclic modules, not all of length 1, since $MG \neq 0$. Hence

$$G = Rw \oplus Rx \oplus Ry,$$

with Rw of length 3, Rx and Ry of length 1. Thus $MG = Mw$,

$$G/MG \cong (Rw/Mw) \oplus Rx \oplus Ry,$$

giving G/MG length 3. But G/MG is isomorphic to $R/M \oplus R/M$, and consequently has length 2. This contradiction completes the proof of the theorem.

Literature.

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