Int J Adv Manuf Technol (1996) 12:349–355

The International Journal of Advanced Manufacturing Technology

Optimising Tolerance Allocation for Mechanical Components Correlated by Selective Assembly

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Selective assembly can enlarge the tolerances of mechanical components for easier manufacturing. However, the non-independent dimensions of correlated components make it difficult to optimise tolerance allocation for an assembly. This paper proposes a solution for this constrained optimisation problem consisting of tolerances and non-independent dimensions as design variables. The approach is to develop a simplified algorithm applying a Lagrange multiplier method to evaluate the optimal tolerances efficiently. The solution is shown to be a global optimum at the given correlation coefficients. The correlation coefficients are key elements in determining the optimal solution, which is demonstrated in the given examples. The results are helpful in designing tolerances for selective assembly.

Keywords: Kuhn–Tucker theorem; Lagrange multiplier method; Nonlinear programming; Optimal tolerance allocation; Selective assembly; Statistical model

1. Introduction

Selective assembly is generally used to achieve easier manufacturing for mass-produced mechanical components, especially for high precision components. Compared with random assembly, selective assembly maintains the same assembly quality but at less manufacturing cost since it loosens the tolerances of component dimensions selected to be assembled in pairs. Thus, an efficient tolerance allocation is of importance at the design stage. However, for selective assembly the tolerance allocation thus far discussed is just for simple linear assembly, such as the assembly of a shaft in hole with a clearance or an interference. In practice, a general mechanical assembly will have complicated and multiple dimensional relationships between components and their assemblies, which is difficult to solve by traditional methods. Besides, it would be better to allocate the tolerances with optimisation techniques rather than with other approaches. This requires an optimisation design of tolerance allocation on which linear or nonlinear constraints are imposed.

For random assembly, optimising tolerance allocation to the dimensions, based on minimising the manufacturing cost, subject to single or multiple dimensional constraints, has been studied by applying various optimisation techniques in the last two decades. Some of these techniques can be found in [1–10]. These studies used a variety of cost models proposed in past years [4,11], and used either a statistical model or worst-case model of tolerance to formulate the constraints. The statistical model employed by most researchers is superior to the worst-case model because the former can transform the constraint functions into a well-defined form and has well-known economic benefits compared to the latter. However, past studies have applied the statistical model only for random assembly, which always assumes that the component dimensions are independent variables. For selective assembly, all the dimensions cannot be so defined because some of them may be selected to be correlated in pairs and so are non-independent. Parkinson [12] studied cost optimisation of dimensional tolerances involving a covariance matrix using the Hasofer–Lind index (reliability index) to develop an applicable algorithm, but indicated that it is not efficient to optimise tolerance allocation by this method because it takes a very long computer run-time and does not guarantee convergence to a minimum solution. At present, there is little or no research to derive the algorithm using non-independent dimensional variables and to solve such an optimisation problem of tolerance allocation efficiently.

In view of this, the study presented in this paper is to propose a simplified algorithm for solving such nonlinear constrained optimisation problems of selective assembly. The work begins with transforming the constraint functions into a quadratic form to formulate the optimisation problem. Next, a simplified algorithm is developed, and is later used to evaluate efficiently the optimal dimensional tolerances. The numerical algorithm is coded into a Fortran program that can be run on a personal computer. The designers give the values of assembly tolerances and correlation coefficients *a priori* to determine the optimal component tolerances, as illustrated in the numerical examples.

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2. Optimisation Problem for Selective Assembly

For a mechanical assembly, the optimisation problem of tolerance allocation usually minimises a cost function subject to a set of linear or nonlinear constraints formed from the dimensional relationships between components and their constituent assemblies. As selective assembly is used, the component dimensions in the constraint functions are correlated in pairs. This study transforms these constraint functions into a quadratic form by taking standard deviations as new variables. In the following parts, the constraint functions are derived, followed by the formulation of the optimisation problem.

2.1 Dimensional Constraint Functions

A schematic example of simple stack-up assembly is shown in Fig. 1, where the clearance Z is derived from the component dimensions as follows:

$$Z = x_1 - x_2 - x_3 \quad (1)$$

If the clearance value is given as 0.2 mm, the constraint function is:

$$F = x_1 - x_2 - x_3 - 0.2 = 0 \quad (2)$$

As the assembly line uses the dimensions x_3 and $(x_1 - x_2)$ as an order index to assemble components 3 and 4, then the dimensional relationship between x_1 and x_3, and x_2 and x_3 are non-independent in the optimisation problem. In this case, the variance of Z is expressed in the following form:

$$\sigma_Z^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + \sigma_{x_3}^2 - 2\rho_{x_1x_3}\sigma_{x_1}\sigma_{x_3} + 2\rho_{x_2x_3}\sigma_{x_2}\sigma_{x_3} \quad (3)$$

where $\rho_{x_ix_k}$ is the correlation coefficient of x_i and x_k, and $-1 \leq \rho_{x_ix_k} \leq 1$.

Supposing the assembly policy is that a large x_1 and a small x_2 are assembled with a large x_3, and a small x_1 and a large x_2 are assembled with a small x_3 (i.e. $\rho_{x_1x_3} > 0$ and $\rho_{x_2x_3} < 0$), then the system can increase the variances of x_1, x_2 and x_3. This assembly condition means that larger component tolerances are allowed, and indicates that a particular mechanical system can use selective assembly to widen tolerances while keeping the same quality of product.

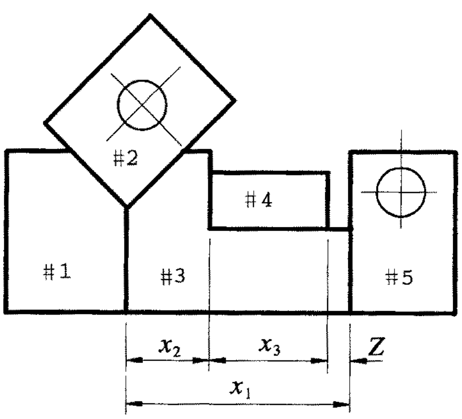

Fig. 1. Schematic example of stack-up assembly.

For generalisation, let x_i and Z_j denote the dimensions of components and their constituent assemblies, respectively, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, then the constraint equations for the optimisation tolerance problem can be expressed with dimensional variables as:

$$F_j(\mathbf{x}) = Z_j(\mathbf{x}) - \delta_j = 0 \qquad (j = 1, 2, \ldots, p) \quad (4)$$

$$F_j(\mathbf{x}) = \begin{cases} Z_j(\mathbf{x}) - \delta_j \geq 0 & \text{if } Z_j(\mathbf{x}) \geq \delta_j \\ \delta_j - Z_j(\mathbf{x}) \geq 0 & \text{if } Z_j(\mathbf{x}) \leq \delta_j \end{cases} \qquad (j = p+1, p+2, \ldots, m) \quad (5)$$

where $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ and δ_j is the given design value of assembly dimension.

The constraint function $F_j(\mathbf{x})$ may be in either equality or inequality, or both, and may be linear or nonlinear. For example, in Fig. 1 the dimensional relationship between two holes forms a nonlinear equality constraint, and the distance between the left-hand side of component 1 and the righthand side of component 5 has a linear equality relationship. The adjacent components 4 and 5 produce a linear inequality constraint because of a clearance $(Z_j(\mathbf{x}) \geq \delta_j)$ or an interference $(Z_j(\mathbf{x}) \leq \delta_j)$ between the two components.

2.2 Optimisation Formulation with Quadratic Constraints

A statistical model is used to introduce standard deviations and to transform the form of constraint function. Assume each dimension follows a normal distribution with symmetry at the midpoint. γ_i and β_j represent the confidence coefficients for the component and assembly dimension, respectively, then their tolerances are expressed as $T_{x_i} = 2\gamma_i\sigma_{x_i}$ and $T_{Z_j} = 2\beta_j\sigma_{Z_j}$. The variance of assembly dimension is of the following form:

$$V(Z_j) = \sigma_{Z_j}^2 = \sum_{i=1}^{n} \left(\frac{\partial Z_j}{\partial x_i}\right)_{\dot{x}_i}^2 \sigma_{x_i}^2 + 2\sum_{k=1}^{n-1}\sum_{i=k+1}^{n} \left(\frac{\partial Z_j}{\partial x_i}\right)_{\dot{x}_i} \left(\frac{\partial Z_j}{\partial x_k}\right)_{\dot{x}_k} \rho_{x_ix_k}\sigma_{x_i}\sigma_{x_k} \quad (6)$$

where $\dot{x}_i$ is the midpoint value of the component dimension.

The correlation coefficients are given *a priori*, which may be based on existing or sampling data in the manufacturing processes. Therefore, the new form of constraint function takes only the standard deviations as design variables as follows. For the equality constraints in equation (4), the transformed expression is:

$$f_j(\boldsymbol{\sigma}_\mathbf{x}) = \sum_{i=1}^{n} \left(\frac{\partial Z_j}{\partial x_i}\right)_{\dot{x}_i}^2 \sigma_{x_i}^2 + 2\sum_{k=1}^{n-1}\sum_{i=k+1}^{n} \left(\frac{\partial Z_j}{\partial x_i}\right)_{\dot{x}_i} \left(\frac{\partial Z_j}{\partial x_k}\right)_{\dot{x}_k} \rho_{x_ix_k}\sigma_{x_i}\sigma_{x_k} - \sigma_{Z_j}^2 = 0 \qquad (j = 1, 2, \ldots, p) \quad (7)$$

where $\sigma_{Z_j} = T_{Z_j}/2\beta_j$ and $\boldsymbol{\sigma}_\mathbf{x} = [\sigma_{x_1}, \sigma_{x_2}, \ldots, \sigma_{x_n}]^T$. For the inequality constraints in equation (5), the given design value of δ_j must at least be at the lower limit of $Z_j(\mathbf{x})$ if $Z_j(\mathbf{x}) \geq \delta_j$. The difference of midpoint $Z_j(\dot{\mathbf{x}})$ and half of the assembly tolerance must at least be equal to δ_j, so that this work can consider half of assembly tolerance as $\frac{1}{2}T_{Z_j} = Z_j(\dot{\mathbf{x}}) - \delta_j$, where $\dot{\mathbf{x}} = [\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n]^T$. Similarly, in the case of $Z_j(\mathbf{x}) \leq \delta_j$, δ_j must at most be at the upper limit of $Z_j(\mathbf{x})$, so that we have $\frac{1}{2}T_{Z_j} = \delta_j - Z_j(\dot{\mathbf{x}})$. Therefore, the transformed expression is:

$$f_j(\boldsymbol{\sigma}_\mathbf{x}) = \sum_{i=1}^{n} \left(\frac{\partial Z_j}{\partial x_i}\right)^2_{\dot{x}_i} \sigma^2_{x_i} + 2 \sum_{k=1}^{n-1} \sum_{i=k+1}^{n} \left(\frac{\partial Z_j}{\partial x_i}\right)_{\dot{x}_i} \left(\frac{\partial Z_j}{\partial x_k}\right)_{\dot{x}_k} \rho_{x_i x_k} \sigma_{x_i} \sigma_{x_k} - \sigma^2_{Z_j} = 0 \quad (j = p+1, p+2, \ldots, m) \tag{8}$$

where

$$\sigma_{Z_j} = \begin{cases} [Z_j(\dot{\mathbf{x}}) - \delta_j]/\beta_j & \text{if } Z_j(\mathbf{x}) \geq \delta_j \\ [\delta_j - Z_j(\dot{\mathbf{x}})]/\beta_j & \text{if } Z_j(\mathbf{x}) \leq \delta_j \end{cases}$$

The objective function of this problem is the total manufacturing cost denoted as $C(\mathbf{T}) = \sum_{i=1}^{n} C_i(T_{x_i})$ or $C(\sigma_\mathbf{x}) = \sum_{i=1}^{n} C_i(\sigma_{x_i})$, where C_i is the individual manufacturing cost of component dimension and $\mathbf{T} = [T_{x_1}, T_{x_2}, \ldots, T_{x_n}]^T$. Optimising tolerance allocation usually considers monotonously decreasing cost models, which means that the first and second derivatives of the continuous cost functions are negative and positive, respectively, with respect to component tolerances. This is rarely violated in general manufacturing processes. A variety of cost models proposed in past years [4,11] are available. Thus, based on the transformation, the optimisation problem of allocating tolerances for selective assembly is presented as follows:

$$\min \quad C(\boldsymbol{\sigma}_\mathbf{x}) \tag{9}$$

$$\text{subject to } f_j(\boldsymbol{\sigma}_\mathbf{x}) = 0 \quad (j = 1, 2, \ldots, m \ (m < n)) \tag{10}$$

$$\sigma_{x_i} > 0 \quad (i = 1, 2, \ldots, n) \tag{11}$$

3. Algorithm for the Optimum Solution

Each constraint function $f_j(\boldsymbol{\sigma}_\mathbf{x})$ in equation (10) is an equality instead of both the equality and inequality functions in equations (4) and (5), and represents a quadratic form. Such a quadratic function always forms a well-defined figure of either convex or concave shape. Although equations (9)–(11) may be solved by the other nonlinear algorithms, the equality constraints are convenient for formulating an optimisation function by the Lagrange multiplier method. The Lagrange multipliers represent the sensitivity indices when carrying out a sensitivity analysis for the given assembly tolerances and correlation coefficients. With this point of view, this study considers a Newton-based algorithm to derive the optimal solution in equations (9)–(11).

3.1 Simplified Algorithm for Selective Assembly

The constraint equation can be expressed in an explicit quadratic form as:

$$f_j(\boldsymbol{\sigma}_\mathbf{x}) = \boldsymbol{\sigma}_\mathbf{x}^T \mathbf{A}_j \boldsymbol{\sigma}_\mathbf{x} - \sigma^2_{Z_j} = 0 \tag{12}$$

where coefficient matrix $\mathbf{A}_j$ is:

$$\mathbf{A}_j = \begin{bmatrix} \left(\frac{\partial Z_j}{\partial x_1}\right)^2_{\dot{x}_1} & \left(\frac{\partial Z_j}{\partial x_1}\right)_{\dot{x}_1}\left(\frac{\partial Z_j}{\partial x_2}\right)_{\dot{x}_2}\rho_{x_1x_2} & \cdots & \left(\frac{\partial Z_j}{\partial x_1}\right)_{\dot{x}_1}\left(\frac{\partial Z_j}{\partial x_n}\right)_{\dot{x}_n}\rho_{x_1x_n} \\ \left(\frac{\partial Z_j}{\partial x_2}\right)_{\dot{x}_2}\left(\frac{\partial Z_j}{\partial x_1}\right)_{\dot{x}_1}\rho_{x_2x_1} & \left(\frac{\partial Z_j}{\partial x_2}\right)^2_{\dot{x}_2} & \cdots & \left(\frac{\partial Z_j}{\partial x_2}\right)_{\dot{x}_2}\left(\frac{\partial Z_j}{\partial x_n}\right)_{\dot{x}_n}\rho_{x_2x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \left(\frac{\partial Z_j}{\partial x_n}\right)_{\dot{x}_n}\left(\frac{\partial Z_j}{\partial x_1}\right)_{\dot{x}_1}\rho_{x_nx_1} & \left(\frac{\partial Z_j}{\partial x_n}\right)_{\dot{x}_n}\left(\frac{\partial Z_j}{\partial x_2}\right)_{\dot{x}_2}\rho_{x_nx_2} & \cdots & \left(\frac{\partial Z_j}{\partial x_n}\right)^2_{\dot{x}_n} \end{bmatrix}$$

Chen [10] proposed a simplified algorithm applying the Lagrange multiplier method to evaluate efficiently the optimal solution for the same class of problem with independent random variables. This study develops a similar algorithm to solve the optimisation problem in equations (9)–(10). The additional concern in the problem is the term $\rho_{x_ix_k}$ creating non-zero off-diagonal elements of the coefficient matrix $\mathbf{A}_j$ in which the coefficient values are given *a priori* in the numerical algorithm.

On the basis of Kuhn–Tucker theorem suitable for general inequality constraints and the facts that the constraint $f_j(\boldsymbol{\sigma}_\mathbf{x})$ is an equality and the standard deviation σ_{x_i} is larger than zero, the simplified necessary conditions for the optimisation problem presented in equations (9)–(11) are written as:

$$\nabla_{\boldsymbol{\sigma}_\mathbf{x}} L(\boldsymbol{\sigma}^*_\mathbf{x}, \boldsymbol{\lambda}^*) = \mathbf{0} \tag{13}$$

$$\mathbf{f}(\boldsymbol{\sigma}^*_\mathbf{x}) = \mathbf{0} \tag{14}$$

$$\boldsymbol{\sigma}^*_\mathbf{x} > \mathbf{0} \tag{15}$$

where $\mathbf{0} = [0, 0, \ldots, 0]^T$, $\mathbf{f} = [f_1, f_2, \ldots, f_m]^T$, $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_m]^T$ is a Lagrange multiplier vector, $(\boldsymbol{\sigma}^*_\mathbf{x}, \boldsymbol{\lambda}^*)$ denotes the feasible point satisfying the solution, and the Lagrangian $L(\boldsymbol{\sigma}_\mathbf{x}, \boldsymbol{\lambda}) = C(\boldsymbol{\sigma}_\mathbf{x}) + (\boldsymbol{\lambda})^T \mathbf{f}(\boldsymbol{\sigma}_\mathbf{x})$. In this algorithm equation (15) is excluded, so that equations (13) and (14) become the necessary conditions and then can be easily solved by the Newton method. After obtaining the solution, it is necessary to check if $\boldsymbol{\sigma}^*_\mathbf{x} > \mathbf{0}$ is satisfied, otherwise it is necessary to make adjustments until a satisfactory solution is reached. The simplified algorithm developed for selective assembly should be as efficient and accurate as that for random assembly in [10].

3.2 Existence of Optimum Solution

For the problem with non-independent dimensions, the Lagrange multipliers exist to guarantee the existence of an optimum solution, and the solution represents a global minimum valid for a set of correlation coefficients given *a priori*. Briefly, it is proved as follows.

Suppose some candidate point $\boldsymbol{\sigma}^0_\mathbf{x}$ satisfying the constraints $\mathbf{f}(\boldsymbol{\sigma}^0_\mathbf{x}) = \mathbf{0}$ is found. If the point is the minimum, based on the necessary conditions $\nabla_{\boldsymbol{\sigma}_\mathbf{x}} L(\boldsymbol{\sigma}_\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$, some $\boldsymbol{\lambda}$ must exist for:

$$\left[\frac{\partial \mathbf{f}}{\partial \boldsymbol{\sigma}_\mathbf{x}}\right]_{\boldsymbol{\sigma}^0_\mathbf{x}} \boldsymbol{\lambda} = -\nabla_{\boldsymbol{\sigma}^0_\mathbf{x}} C(\boldsymbol{\sigma}^0_\mathbf{x}) \tag{16}$$

Where the Jacobian $[\partial \mathbf{f}/\partial \boldsymbol{\sigma}_\mathbf{x}]_{\boldsymbol{\sigma}^0_x}$ is an $n \times m$ matrix with the element J_{kj} as follows:

$$J_{kj} = 2 \sum_{i=1}^{n} \left(\frac{\partial Z_j}{\partial x_i}\right)_{\dot{x}_i} \left(\frac{\partial Z_j}{\partial x_k}\right)_{\dot{x}_k} \rho_{x_i x_k} \sigma^0_{x_i} \tag{17}$$

$$j = 1, 2, \ldots, m; \ k = 1, 2, \ldots, n$$

Thus the Jacobian matrix has to be at least of rank m at the minimum $\boldsymbol{\sigma}_{\mathbf{x}}^{0}$ so that Lagrange multipliers exist. It is impossible that the Jacobian matrix is not at least of rank m because σ_{x_i} is larger than zero and the constraint equations are linearly independent, which implies that the solution of equation (16) exists. In addition, in numerical computation this work also tested various proper values of $\rho_{x_ix_k}$ and easily obtained the solutions at all times. These tests show that the optimal solution of equation (16) exists by using the Lagrange multiplier method.

Furthermore, the Hessian of the constraint function in equation (12) is calculated by:

$$\mathbf{H}_j = 2\mathbf{A}_j \tag{18}$$

Apparently, it can be proved that the Hessian matrix is positive definite because the following positive property can be derived from equation (12) for any standard deviation vector as:

$$\tfrac{1}{2}\boldsymbol{\sigma}_{\mathbf{x}}^{T}\mathbf{H}_j\boldsymbol{\sigma}_{\mathbf{x}} = \boldsymbol{\sigma}_{\mathbf{x}}^{T}\mathbf{A}_j\boldsymbol{\sigma}_{\mathbf{x}} = \sigma_{Z_j}^{2} > 0 \tag{19}$$

This result shows that the constraint functions are convex. Moreover, the cost function $C_i(\sigma_{x_i})$ represents a monotonously decreasing function, which proves that the objective function of this optimisation problem is convex. Based on these points, it is established that the optimal solution evaluated by this simplified algorithm is also a global minimum at the given correlation coefficients.

3.3 Sensitivity Analysis

From equation (9), it is intuitively clear that the optimal solution $(\boldsymbol{\sigma}_{\mathbf{x}}^{*},\boldsymbol{\lambda}^{*})$ will change as the assembly variance $\sigma_{Z_j}^{2}$ (or tolerance T_{Z_j}) and the correlation coefficient $\rho_{x_ix_k}$ vary, which will affect the total manufacturing cost. The special feature of applying the Lagrange multiplier method is that the designers can use Lagrange multipliers as sensitivity indices to analyse the effect of varying $\sigma_{Z_j}^{2}$ and $\rho_{x_ix_k}$ on the total manufacturing cost. Derived from the gradient of Lagrangian and the necessary conditions in equations (13) and (14) yield:

$$\frac{\partial C(\boldsymbol{\sigma}_{\mathbf{x}}^{*})}{\partial \sigma_{Z_j}} = -2\lambda_j^{*}\sigma_{Z_j} \tag{20}$$

$$\frac{\partial C(\boldsymbol{\sigma}_{\mathbf{x}}^{*})}{\partial \rho_{x_ix_k}} = 2\sum_{j=1}^{m} \lambda_j^{*}\left(\frac{\partial Z_j}{\partial x_i}\right)_{\bar{x}_i}\left(\frac{\partial Z_j}{\partial x_k}\right)_{\bar{x}_k}\sigma_{x_i}^{*}\sigma_{x_k}^{*} \tag{21}$$

The gradient in equations (20) and (21) is the so-called sensitivity coefficient showing that the Lagrange multipliers represent sensitivity indices of the total manufacturing cost versus assembly tolerances (or standard deviations) and correlation coefficients, respectively. Apparently, λ_j^{*} has to be positive for optimising tolerance allocation, and $\text{sign}[(\partial Z_j/\partial x_i)_{\bar{x}_i}(\partial Z_j/\partial x_k)_{\bar{x}_k}]$ has to be equal to $\text{sign}(\rho_{x_ix_k})$ for the same purpose according to equation (21). Based on these conditions, the designer can carry out sensitivity analysis in the design stage to evaluate the reasonableness and feasible range of individual assembly tolerance and correlation coefficients given *a priori*. For details of evaluating assembly tolerances, refer to the random assembly case in [10]. Example 2 in Section 4 evaluates the given correlation coefficients.

3.4 Numerical Solution

For either random assembly or selective assembly, the Newton method is suitable for integrating the solution since it can evaluate the steepest direction for the objective function, where the constraint function is in a quadratic form. For details of the procedures for the numerical algorithm, refer to [10].

This study considers that the designers are to allocate optimal component tolerances for satisfying the expected assembly tolerances and correlation coefficients. Alternatively, they may use a trial-and-error test in the design stage to search for a set of coefficients' values. Based on these values, the optimal tolerances determined by applying the simplified algorithm are believed to be close to the global optimum of this optimisation problem (see example 2). In manufacturing, the values of correlation coefficients may depend on the strategy of dividing a batch of correlative components into several groups.

4. Examples

This study presents two numerical examples to illustrate the optimisation problem. The numerical algorithm is implemented in a Fortran code, TOLKIT2, and is run on a personal computer. The first example involving single linear constraint focuses on discussing the effect of correlation coefficients on manufacturing cost and the optimal solution of tolerances. The second example shows the determination of the optimal tolerances of a nonlinear constrained optimisation problem with various assembly cases. Both examples use the inverse power cost function proposed by Lee and Woo [6].

Example 1 Figure 2 represents the geometry of a simple selective assembly having four relevant dimensions. The dimensional relation in the assembly constructs one linear constraint equation:

$$F(\mathbf{x}) = x_1 + x_2 - x_3 - x_4 = 0 \tag{22}$$

The cost function is expressed in terms of standard deviation as:

$$C_i(\sigma_{x_i}) = B_i(2\gamma_i\sigma_{x_i})^{-2} \tag{23}$$

Suppose the dimensional pairs for selective assembly are x_1 and x_2, x_2 and x_3, x_3 and x_4, and the following data are given for this example:

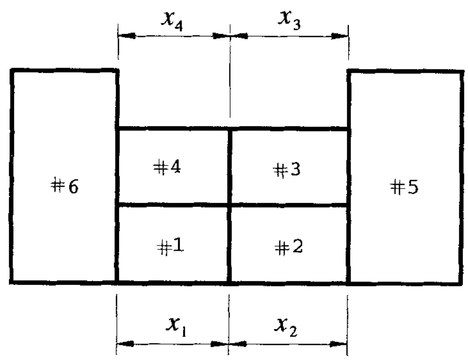

Fig. 2. Selective assembly with one linear constraint.

$B_i = 0.002, \quad \gamma_i = 3, \quad \dot{x}_i = 50, \quad (i = 1, 2, 3, 4);$
$\frac{1}{2}T_Z = 0.01, \quad \beta = 3$

where all dimensions are measured in mm.

The given correlation coefficients will affect the optimum solution. For comparing the effect, this example tests the following six cases of coefficients:

1. $\rho_{x_1x_2} \neq 0, \quad \rho_{x_2x_3} = \rho_{x_3x_4} = 0$ or $\rho_{x_3x_4} \neq 0, \quad \rho_{x_1x_2} = \rho_{x_2x_3} = 0$
2. $\rho_{x_2x_3} \neq 0, \quad \rho_{x_1x_2} = \rho_{x_3x_4} = 0$
3. $\rho_{x_1x_2} = \rho_{x_3x_4} \neq 0, \quad \rho_{x_2x_3} = 0$
4. $\rho_{x_1x_2} = -\rho_{x_3x_4} \neq 0, \quad \rho_{x_2x_3} = 0$
5. $\rho_{x_1x_2} = \rho_{x_3x_4} = \rho_{x_2x_3} \neq 0$
6. $\rho_{x_1x_2} = \rho_{x_3x_4} = -\rho_{x_2x_3} \neq 0$

By running TOLKIT2, the results of manufacturing cost versus different sets of coefficients are shown in Fig. 3. Each case has its own lowest cost, for instance, $\min C(\mathbf{T}) = 4.41$ for $\rho_{x_1x_2} = \rho_{x_3x_4} = -1.0$ for case 3, and the optimal tolerances are $T_{x_1} = T_{x_3} = 0.051331$ and $T_{x_2} = T_{x_4} = 0.037188$. Compared with random assembly (min $C(\mathbf{T}) = 80.00$ and $T_{x_1} = T_{x_2} = T_{x_3} = T_{x_4} = 0.01$), Fig. 3 indicates that the manufacturing cost is considerably reduced as the components are properly correlated. On the other hand, it increases when components correlate improperly such as $\rho_{x_1x_2} = \rho_{x_3x_4} = -\rho_{x_2x_3} > 0$ for case 6. In the six cases, cases 6 and 3 produce "lower" lowest manufacturing cost, and case 6 has the lowest cost. However, since there is no constraint for the distance between components 5 and 6 in Fig. 2, we should notice that the cost will approach or be equal to zero because the tolerances will be unreasonably large as $\rho_{x_1x_2} = \rho_{x_3x_4} = -\rho_{x_2x_3} < -0.65$ for case 6 shown in Fig. 3. In other words, a set of improper correlation coefficients will result in unexpected and insignificant tolerances for this case. Figure 3 also shows that the solution diverges as $\rho_{x_1x_2} = \rho_{x_3x_4} > 0.96$ for case 3 and $\rho_{x_1x_2} = \rho_{x_3x_4} = -\rho_{x_2x_3} > 0.97$ for case 6, which implies that improper coefficients may cause a divergent solution for this optimisation problem. Those results in this case study indicate that the effect of correlation coefficients on manufacturing cost and optimum tolerances is clear.

Example 2 Suppose the dimensional relationships of components considered to be correlated in pairs in Fig. 4 form four

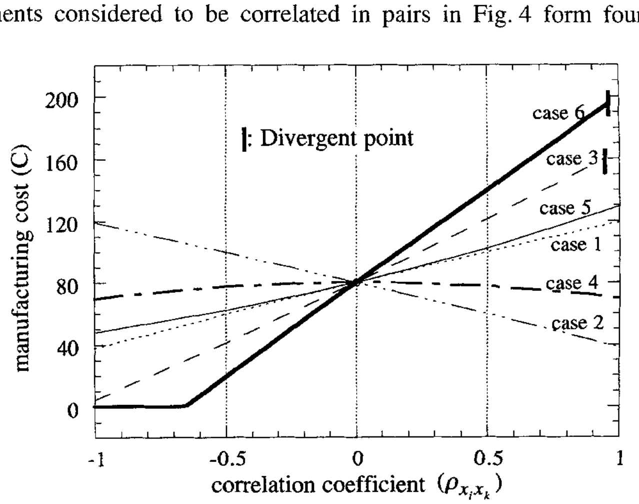

Fig. 3. Effect of correlation coefficients on manufacturing cost in example 1.

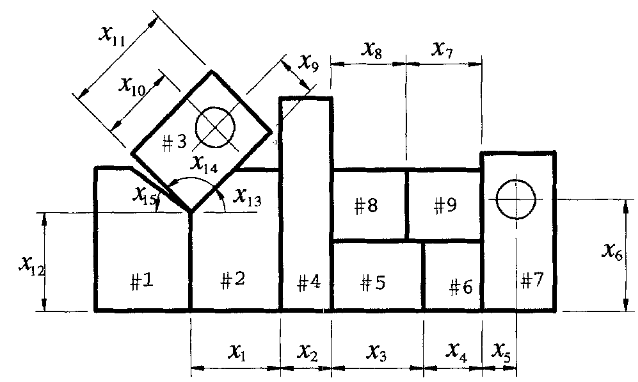

Fig. 4. Selective assembly with four nonlinear constraints.

constraints of the optimisation problem, where F_1 controls the central distance between the two holes, F_2 and F_3 represent the dimensional constraints to assemble components 4–9 and 1–3, respectively, and F_4 constrains the clearance between components 3 and 4. These constraint equations are written with the given data of assembly dimensions as follows:

$$F_1(\mathbf{x}) = \{[x_1 + x_2 + x_3 + x_4 + x_5 - (x_{10} \cos x_{13} - x_9 \sin x_{13})]^2 + [-x_6 + x_{12} + (x_{10} \sin x_{13} + x_9 \cos x_{13})]^2\}^{1/2} - 162.5428 = 0 \quad (24)$$

$$F_2(\mathbf{x}) = x_3 + x_4 - x_7 - x_8 = 0 \quad (25)$$

$$F_3(\mathbf{x}) = 180° - x_{13} - x_{14} - x_{15} \geq 0 \quad (26)$$

$$F_4(\mathbf{x}) = x_1 - x_{11} \cos x_{13} \geq 0 \quad (27)$$

where all the dimensions are measured in mm and angles x_{13}, x_{14} and x_{15} are in degrees. This example uses the cost function shown in equation (23), and gives the following data:

$\dot{\mathbf{x}} = [57\ 26\ 50\ 30\ 20\ 75\ 40\ 40\ 25\ 60\ 80.3\ 52\ 45°\ 44.9°\ 90°]^T$
$B_1 = B_2 = B_9 = B_{10} = 2 \times 10^{-5}, \quad B_3 = B_4 = 0.02, \quad B_5 = B_6 = 5 \times 10^{-5}$
$B_7 = B_8 = 0.2, \quad B_{11} = 3 \times 10^{-5}, \quad B_{12} = 1 \times 10^{-4}, \quad B_{13} = B_{14} = B_{15} = 2 \times 10^{-4}$
$\gamma_i = 3 \quad (i = 1, 2, \ldots, 15), \quad \beta_j = 3 \quad (j = 1, 2, 3, 4)$
$\frac{1}{2}T_{Z_1} = \frac{1}{2}T_{Z_2} = 0.01, \quad \frac{1}{2}T_{Z_3} = 0.1°, \quad \frac{1}{2}T_{Z_4} = 0.2193$

After evaluating the assembly conditions and manufacturing cost of each dimension, this work selects five pairs of correlated dimensions preliminarily as follows:

$$\{x_3 \text{ and } x_4,\ x_7 \text{ and } x_8,\ x_3 \text{ and } x_8,\ x_{13} \text{ and } x_{14},\ x_{13} \text{ and } x_{15}\}$$

By running TOLKIT2, the sensitivity coefficients and total manufacturing costs with respect to the correlation coefficients are presented in Fig. 5 indicating that $\rho_{x_7x_8}$ has the most effect on the total cost. The sensitivity coefficients are 2752.4 at $\rho_{x_7x_8} = -0.48$ and 2435.1 at $\rho_{x_7x_8} = -0.1$, which proves they should be deterministic. On the other hand, $\rho_{x_{13}x_{14}}$ and $\rho_{x_{13}x_{15}}$ are not sensitive (sensitivity coefficients are only between 7.4 and 8.7). Therefore, the last two pairs shown above can be excluded from the selective assembly. Figure 5 also shows the feasible range for each coefficient that has its own lowest cost.

For optimising tolerance allocation, suppose the designer assigns the coefficients by referencing to the existing or sampling data as $\rho_{x_3x_4} = -0.35$, $\rho_{x_7x_8} = -0.25$, and $\rho_{x_3x_8} = 0.4$, then the low-

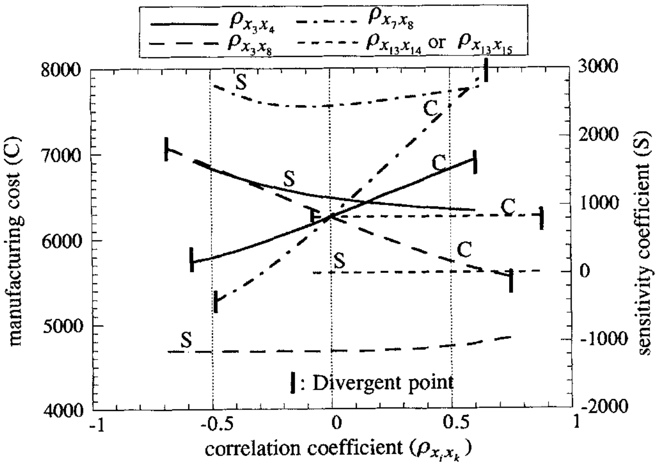

Fig. 5. Sensitivity and feasible range of correlation coefficients in example 2.

est cost at these coefficient values is immediately obtained as min$C(\mathbf{T})$ = 5113.94. On the other hand, based on the belief that a trial-and-error test can search for a set of correlation coefficients producing a lowest cost near the global minimum for this problem, a procedure for this test starting with small coefficients by running TOLKIT2 is shown in Table 1, and the result is min$C(\mathbf{T})$ = 4843.89 ($\rho_{x_3x_4} = -0.38$, $\rho_{x_7x_8} = -0.51$ and $\rho_{x_3x_8} = 0.37$). Table 1 also shows several improper sets of coefficients resulting in divergent solutions. The lowest cost for the above first case is higher than that for the second case by 5.6%. Furthermore, assigning only one pair of correlated dimensions x_7 and x_8 for selective assembly (say, $\rho_{x_7x_8} = -0.48$) increases the lowest cost by only 9.1% compared to the second case. However, the benefit is that this case uses only one pair, which can reduce the extra costs arising from selective assembly such as selecting and inspecting.

The optimal tolerances for these cases are listed in Table 2 which also presents the results for the random assembly case.

Table 1. Trial-and-error test for selective assembly.

Test no.	$\rho_{x_7x_8}$	$\rho_{x_3x_8}$	$\rho_{x_3x_4}$	Lowest cost
1	–0.1	0.1	–0.1	5807.28
2	–0.15	0.2	–0.2	5511.05
3	–0.2	0.2	–0.2	divergent*
4	–0.23	0.26	–0.27	5260.16
5	–0.3	0.3	–0.3	divergent
6	–0.28	0.35	–0.3	divergent
7	–0.35	0.3	–0.4	4988.99
8	–0.4	0.4	–0.4	4922.82
9	–0.45	0.4	–0.35	4890.49
10	–0.45	0	–0.55	4878.06
11	–0.48	0.38	–0.35	4875.51
12	–0.49	0.38	–0.35	divergent
13	–0.49	0.37	–0.36	4864.32
14	–0.5	0.37	–0.37	4853.89
15	–0.51	0.37	–0.38	4843.89
16	–0.52	0.37	–0.38	divergent
17	–0.51	0.38	–0.38	divergent
18	–0.51	0.37	–0.39	divergent

*This set of coefficients results in a divergent solution.

Table 2. Optimal tolerances for selective and random assembly.

Tolerances	Selective assembly			Random assembly
	Trial-and-error test	Existing or sampling data	Single pair ($\rho_{x_7x_8}$ = –0.48)	
T_{x_1}	0.001223	0.001219	0.001263	0.001274
T_{x_2}	0.001223	0.001219	0.001263	0.001274
T_{x_3}	0.010529	0.010409	0.006289	0.006036
T_{x_4}	0.006309	0.006291	0.006289	0.006036
T_{x_5}	0.001538	0.001533	0.001588	0.001602
T_{x_6}	0.003177	0.003167	0.003279	0.003309
T_{x_7}	0.023480	0.018213	0.019690	0.012789
T_{x_8}	0.015260	0.015211	0.014197	0.012789
T_{x_9}	0.001646	0.001641	0.001699	0.001714
$T_{x_{10}}$	0.002091	0.002084	0.002158	0.002177
$T_{x_{11}}$	0.619958	0.619961	0.619934	0.619927
$T_{x_{12}}$	0.003778	0.003767	0.003899	0.003935
$T_{x_{13}}$	0.015357°	0.015312°	0.015852°	0.015996°
$T_{x_{14}}$	0.141000°	0.141003°	0.140973°	0.140965°
$T_{x_{15}}$	0.141000°	0.141005°	0.140973°	0.140965°
Relative cost (Actual cost)	1 (4843.89)	1.056 (5113.94)	1.091 (5282.37)	1.292 (6258.10)

The lowest cost for this last case is higher than that for the second case by 29.2%, however, the optimal tolerances are not much different except T_{x_3}, T_{x_7} and T_{x_8} because the manufacturing costs of these dimenions are much higher than the others. Hence, for selective assembly it is clear that the designers should employ those manufactured more expensively as correlated dimensions in pairs since they cause a deterministic effect on the total manufacturing cost.

5. Conclusion

This study has presented an optimisation technique associated with a statistical method to optimise tolerance allocation for a mechanical assembly whose component dimensions are correlated by using selective assembly. By giving the values of correlation coefficients *a priori*, a simplified algorithm applying a Lagrange multiplier method has been developed to evaluate efficiently the solution for this class of optimisation problem. In the case of a monotonously decreasing cost model, the solution can be determined immediately and demonstrated to be the global optimum valid for the given correlation coefficients.

It is a special feature, when applying the Lagrange multiplier method, that Lagrange multipliers represent sensitivity indices in carrying out sensitivity analysis for the given assembly tolerances and correlation coefficients. Furthermore, as shown in the example, designers may use a trial-and-error test to search for a set of correlation coefficients determining a solution believed to be close to the global optimum of the problem.

The designers should select the dimensions manufactured more expensively as correlated dimensions in pairs for a lower manufacturing cost, and use the least number of pairs possible to reduce the extra costs arising from selective assembly. The effect of correlation coefficients is clear. From the examples provided, it can be seen that a set of improper coefficients may cause a divergent solution, larger cost, or unreasonable tolerances.

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Notation

$\mathbf{A}_j$	coefficient matrix of f_j
B_i	coefficient of cost function
C	total manufacturing cost function
C_i	manufacturing cost function for x_i
F_j	the jth dimensional constraint function
f_j	the jth quadratic constraint function
$\mathbf{f}$	quadratic constraint vector
$\mathbf{H}_j$	the jth Hessian matrix
J_{kj}	element of $n \times m$ Jacobian matrix
L	Lagrangian
m	number of assembly dimensions
n	number of component dimensions
p	number of equality dimensional constraints
$\mathbf{T}$	tolerance vector of component dimensions [mm] or [°]
T_{x_i}	tolerance of x_i [mm] or [°]
T_{Z_j}	tolerance of Z_j [mm] or [°]
$\mathbf{x}$	component dimension vector
$\dot{\mathbf{x}}$	midpoint vector
x_i	component dimension [mm] or [°]
$\dot{x}_i$	midpoint of x_i [mm] or [°]
Z_j	assembly dimension [mm] or [°]
β_j	confidence coefficient for Z_j
γ_i	confidence coefficient for x_i
δ_j	given design value of Z_j [mm] or [°]
$\boldsymbol{\lambda}$	Lagrange multiplier vector
λ_j	the jth Lagrange multiplier
$\boldsymbol{\lambda}^*$	Lagrange multiplier vector at the optimum solution
$\rho_{x_i x_k}$	correlation coefficient for x_i and x_k
$\boldsymbol{\sigma}_{\mathbf{x}}$	standard deviation vector
$\boldsymbol{\sigma}_{\mathbf{x}}^*$	standard deviation vector at the optimum solution
$\boldsymbol{\sigma}_{\mathbf{x}}^0$	candidate point satisfying the constraints $\mathbf{f}(\boldsymbol{\sigma}_{\mathbf{x}}^*) = \mathbf{0}$
σ_{x_i}	standard deviation of x_i