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Infinitely Generated Subgroups of Finitely Presented Groups. I

Gilbert Baumslag[1], Frank B. Cannonito[2], and Charles F. Miller III[3]

[1] Graduate Center of the City University of New York, 33 West 42nd Street, New York, New York 10036, USA

[2] Department of Mathematics, University of California, Irvine, California 92717, USA

[3] Department of Mathematics, University of Melbourne, Parkville Victoria 3052, Australia

§ 1. Introduction

1.1. The object of this paper is to throw some light on the nature of the infinitely generated subgroups of finitely presented groups. In order to help put our results into perspective we term a group *recursively presentable* if it can be generated by a recursively enumerable set of generators and completely defined in terms of these generators by a recursively enumerable set of relators.

In 1961 Graham Higman [10] proved the following remarkable theorem: *A recursively presentable group can be embedded in a finitely presented group.* Higman's theorem provides a simple characterization of the finitely generated subgroups of finitely presented groups, viz.: *A finitely generated group is a subgroup of a finitely presented group if and only if it is recursively presentable* (Higman [10]).

No such characterization exists, however, for the infinitely generated subgroups of finitely presented groups. Of course, a countable subgroup of a finitely presented group is locally embeddable in a finitely presented group (where, as usual, a group is said to have a property locally if all its finitely generated subgroups have this property). On the other hand, Higman has constructed a countable group H which is locally embeddable in a finitely presented group but which is not itself so embeddable. This serves to underline the difficulty in obtaining satisfactory conditions which ensure that a countable group can be embedded in a finitely presented group. This paper is mainly concerned with obtaining such conditions and, in this, we have been fairly successful.

1.2. Our main theorem revolves around a technique embodied in Theorem 2.1, in §2, for realizing a countable ascending union in terms of HNN-constructions. Theorem 2.1 is a little too complicated to formulate here. We prefer, instead, to record at this point two extremely interesting applications of this theorem. First we have the

Theorem 3.1. *A countable locally free group can be embedded in a finitely presented group (with solvable word problem).*

Next we have the

Theorem 4.8. *Every countable locally polycyclic-by-finite group can be embedded in a finitely presented group.*

Theorem 4.8 yields, in particular, the embeddability of countable locally finite groups, as well as countable abelian groups, in finitely presented groups (Higman [10]). However, its scope is obviously far greater than these two results of Higman, which together with the theorems of Higman cited at the outset, constituted all that was known about the subgroups of finitely presented groups until now.

Corollary 2.5 will be utilized in the forthcoming part II of this work to give embeddability results for certain locally solvable groups. For the present we note that Theorem 2.1 can be utilized in other ways. In particular it provides us with a necessary and sufficient condition for a countable group to be embedded in a finitely presented group, detailed in Theorem 2.4 in §2.

1.3. We turn our attention next in §5 to automorphism groups where we prove the pleasing

Theorem 5.1. *The automorphism group* Aut(G) *of any finitely presented group* G *is recursively presentable. If* G *has solvable word problem then* Aut(G) *has a presentation for which the word problem is solvable.*

Actually, Theorem 5.1 can be formulated in very much more general terms, for universal algebras (see Theorem 5.3).

Putting Theorem 5.1 and Higman's embedding theorem together we obtain our second necessary and sufficient condition for a countable group to be embeddable in a finitely presented group.

Theorem 5.3. *A group* H *is embeddable in a finitely presented group if and only if it can be faithfully represented as a group of automorphisms of a finitely presented group.*

1.4. In §6 we turn our attention to linear groups, i.e., groups isomorphic to matrix groups over commutative fields. It is easy to prove, by combining a well known theorem of Mal'cev [14] with work of Rabin [15] the

Theorem 6.1. *A countable group, locally linear of bounded degree, can be embedded in a finitely presented group with solvable word problem.*

In particular Theorem 6.1 yields again the embeddability of countable locally free groups in finitely presented groups, since a countable free group has a faithful representation as a linear group of degree two. It is worth noting that we have been unable to determine whether *every* countable locally linear group can be embedded in a finitely presented group.

1.5. We conclude, in §7, with some examples of countable groups which are not embeddable in finitely presented groups.

First of all we establish a necessary and sufficient condition, Theorem 7.1, for the wreath product of two finitely generated groups to be recursively presentable. This provides us with a host of finitely generated groups which are not embeddable in finitely presented groups (see §5.1).

Next, we concoct, in §7.3, a simple example of an extension E of two finitely generated solvable groups A and B such that E is not embeddable in a finitely presented group although A and B are.

Higman's example, cited in §1.1, of a countable group H which is locally embeddable in a finitely presented group but which is itself not embeddable contains a finitely generated subgroup with unsolvable word problem. In §7.3 we construct another countable group K which is like Higman's group H in that it is locally embeddable but not itself embeddable in a finitely presented group. However, unlike H, all the finitely generated subgroups of K have solvable word problems.

Our final example is one of Macintyre's [12] countable algebraically closed groups which is locally embeddable in a finitely presented group but which, as we show, is not itself embeddable (see §7.4).

§2. Local Systems

2.1. Let H be a group which is an ascending union of its subgroups

$$H_1 \leqq H_2 \leqq \cdots \leqq H = \bigcup_{n \geqq 1} H_n.$$

Furthermore, let W be another group such that each $H_n (n \geqq 1)$ is embeddable in W. If the group A is embeddable in group B we denote this by the symbol $A \hookrightarrow B$. Now H itself need not be embeddable in W. However, as we shall show, H can be embedded in a suitable HNN-construction over W.

To this end, we need to introduce some terminology. A *system of monomorphisms in* W is a family of monomorphisms $\psi_{\alpha,\beta}\colon U_\alpha \to U_\beta$ from various subgroups U_α $(\alpha \in I)$ of W into various others. We allow the possibility that several of the U_α's may coincide. The system of monomorphisms $\{\psi_{\alpha,\beta}\}$ need not contain a monomorphism for each pair of indices, but when it does we say $\psi_{\alpha,\beta}$ is *defined.* An *immersion of the sequence* $\{H_n\}$ *in the system* $\{\psi_{\alpha,\beta}\}$ of *monomorphisms* is a function γ from the natural numbers to the index set I for the U_α's together with a sequence of embeddings $f_n\colon H_n \to U_{\gamma(n)}$ such that, for each n, $\psi_{\gamma(n),\gamma(n+1)}$ is defined and the diagram

$$\begin{array}{ccc} H_n & \xrightarrow{\text{inclusion}} & H_{n+1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ U_{\gamma(n)} & \xrightarrow{\psi_{\gamma(n),\gamma(n+1)}} & U_{\gamma(n+1)} \end{array}$$

commutes. Note that the embeddings f_n need not be onto the $U_{\gamma(n)}$. Notice that if the H_n's are all embedded in W, then there always exists a system of monomorphisms in W in which $\{H_n\}$ can be immersed – for example, the family of all monomorphisms between all the subgroups of W. Roughly speaking, the sequence $\{H_n\}$ can be immersed in the system $\{\psi_{\alpha,\beta}\}$ if the H_n's can all be embedded in W and there are enough monomorphisms $\psi_{\alpha,\beta}$ to realize the embeddings $H_n \hookrightarrow H_{n+1}$.

The HNN-*construction G associated to the system* $\{\psi_{\alpha,\beta}\}$ of monomorphisms in W is the HNN-construction obtained from W by adding for each $\psi_{\alpha,\beta}$ which is defined a new generator $s_{\alpha,\beta}$ together with defining relations $s_{\alpha,\beta} u s_{\alpha,\beta}^{-1} = \psi_{\alpha,\beta}(u)$ for all $u \in U_\alpha$. A well known result of G. Higman, B.H. Neumann and Hanna Neumann asserts that W is naturally embedded in G. We shall adopt this notation in the statement and proof of the following

Theorem 2.1. *Let H be a group which is an ascending union of its subgroups*

$$H_1 \leqq H_2 \leqq \cdots \leqq H = \bigcup_{n \geqq 1} H_n.$$

Let $\{\psi_{\alpha,\beta}\}$ *be a system of monomorphisms between a family of subgroups of W. If the sequence* $\{H_n\}$ *can be immersed in the system* $\{\psi_{\alpha,\beta}\}$, *then H can be embedded in the HNN-construction G associated to the* $\{\psi_{\alpha,\beta}\}$.

Proof. For each n, put $t_n = s_{\gamma(n),\gamma(n+1)}$. Then, for each $h \in H_n$, $t_n f_n(h) t_n^{-1} = f_{n+1}(h)$ by the definition of an immersion. Define a map $\theta\colon H \to G$ as follows: $\theta | H_1 = f_1$. For $n \geqq 1$ and $x \in H_{n+1}$, put

$$\theta | H_{n+1}(x) = t_1^{-1} t_2^{-1} \dots t_n^{-1} f_{n+1}(x) t_n \dots t_2 t_1.$$

Note that if $x \in H_n$, then

$$\begin{aligned}\theta | H_{n+1}(x) &= t_1^{-1} t_2^{-1} \dots t_n^{-1} f_{n+1}(x) t_n \dots t_2 t_1\\ &= t_1^{-1} \dots t_{n-1}^{-1} f_n(x) t_{n-1} \dots t_1\\ &= \theta | H_n(x).\end{aligned}$$

Hence, $\theta | H_{n+1}$ extends $\theta | H_n$ for each n. Also $\theta | H_n$ is an embedding for each n since f_n is an embedding. Hence $\theta\colon H \to G$ is an embedding.

By the *full system of finitely generated monomorphisms* in W we mean the system of all monomorphisms from one finitely generated subgroup of W into another.

Corollary 2.2. *Let H be a countable group which is locally embeddable in W. Then H can be embedded in the HNN-construction G associated with the full system of finitely generated monomorphisms in W.*

Proof. Let $\{h_1, h_2, \dots\}$ be a list of the elements of H, and let H_n be the subgroup generated by $\{h_1, \dots, h_n\}$. Then by hypothesis H_n is isomorphic to a finitely generated subgroup, say, to $U_{\gamma(n)}$, of W, via an isomorphism $f_n\colon H_n \to U_{\gamma(n)}$. But then $f_{n+1} \circ i_n \circ f_n^{-1}$, where i_n is the inclusion of H_n into H_{n+1}, is a monomorphism of $U_{\gamma(n)}$ into $U_{\gamma(n+1)}$, and so belongs to the full system of finitely generated monomorphisms in W. Thus, the sequence $\{H_n\}$ is immersed in the full system, and the result follows from Theorem 1.

2.2. To produce embeddings into finitely presented groups, we need some recursion theoretic conditions. Suppose that W is recursively presented. We say that the system $\{\psi_{\alpha,\beta}\}$ is a *recursively enumerable system of finitely generated* *monomorphisms* if the U_α's are all finitely generated and if the collection of pairs

$$(\{x_{\alpha,1}, \ldots, x_{\alpha,n(\alpha)}\}, \{\psi_{\alpha,\beta}(x_{\alpha,1}), \ldots, \psi_{\alpha,\beta}(x_{\alpha,n(\alpha)})\})$$

of finite sets of words specifying the subgroups U_α and the monomorphisms $\psi_{\alpha,\beta}$ which are defined is recursively enumerable. From the definition and Higman's theorem, we have the following

Lemma 2.3. *Let W be recursively presented. If $\{\psi_{\alpha,\beta}\}$ is a recursively enumerable system of finitely generated monomorphisms, then the HNN-construction G associated to the system can be recursively presented. Hence, G can be embedded in a finitely presented group.*

We are now in a position to give a characterization of the subgroups of finitely presented groups.

Theorem 2.4. *Let H be a countable group. Then H can be embedded in a finitely presented group if and only if H is an ascending union of subgroups*

$$H_1 \leqq H_2 \leqq \cdots \leqq H = \bigcup_{n \geqq 1} H_n$$

such that the sequence $\{H_n\}$ can be immersed in a recursively enumerable system of finitely generated monomorphisms $\{\psi_{\alpha,\beta}\}$ in a recursively presented group W.

Proof. The sufficiency is clear from Theorem 2.1 and Lemma 2.3. For the necessity, suppose H can be embedded in the finitely presented group K. Take $W = K$ and let the system consist of the identity map on W. Then, in fact, any representation of H as an ascending union can be immersed in this system.

This result is of use mainly for proving local embedding results because of the following:

Corollary 2.5. *Let W be a recursively presented group. Suppose that the collection of pairs $(\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_n\})$ of finite sets of words of W such that $x_i \mapsto y_i$ defines an isomorphism of finitely generated subgroups is a recursively enumerable set of pairs. If H is a countable group which is locally embeddable in W, then H can be embedded in a finitely presented group.*

Proof. The desired result follows immediately from Corollary 2.2 and Lemma 2.3.

2.3. We will now give several embeddability results which are consequences of Corollary 2.5. In each case, the main task is to verify the hypothesis of Corollary 2.5 concerning the recursive enumerability of finitely generated subgroups.

§ 3. Locally Free Groups

3.1. We now give our first embeddability result.

Theorem 3.1. *A countable locally free group can be embedded in a finitely presented group.*

Proof. Let H be a countable locally free group, and let W be a free group of rank 2. Then H is locally a subgroup of W. From an enumeration of all pairs $(\{x_u, \ldots, x_n\}, \{y_1, \ldots, y_n\})$ of finite sets of words of W we select those pairs such that $x_i \mapsto y_i$ $(i=1, \ldots, n)$ defines an isomorphism. This can be accomplished effectively as follows. By successively applying Nielsen transformations to $\{x_1, \ldots, x_n\}$ (see Nielsen [17] or Magnus-Karrass-Solitar [13]) we can transform this set into an equipotent set $\{u_1, \ldots, u_m, 1, \ldots, 1\}$ where $u_1, \ldots, u_m$ freely generate the subgroup of W generated by $x_1, \ldots, x_n$. Suppose that the same sequence of Nielsen transformations applied now to $\{y_n, \ldots, y_n\}$ yields $\{v_1, \ldots, v_n\}$. Then the map $x_i \mapsto y_i$ $(i=1, \ldots, n)$ is an isomorphism if and only if $v_{m+1} = \cdots = v_n = 1$ and $\mathrm{gp}(v_1, \ldots, v_m)$ is a free group of rank m. Since the rank of a given finitely generated subgroup of a free group can be effectively computed (Nielsen [17]), it follows that Corollary 2.5 can be applied here. This completes the proof.

§ 4. The Embeddability of Locally Polycyclic-by-Finite Groups

4.1. A *polycyclic group* is, by definition, a group with a normal series with cyclic factors, i.e., a series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G = G \tag{1}$$

where G_i/G_{i-1} is cyclic $(i=1, \ldots, k)$. We term k the *length* of the polycyclic series (1). Choose $a_i \in G_i$ so that $G_i = \mathrm{gp}(a_i, G_{i-1})$. We term $a_1, \ldots, a_k$ a polycyclic *basis* for G. If a_i is of order $e_i > 1$ (possibly $e_i = \infty$) modulo G_{i-1} then every element $g \in G$ can be written uniquely in the *standard form*

$$g = a_1^{\gamma_1} \ldots a_k^{\gamma_k}, \qquad \text{where } 0 \leqq \gamma_i < e_i \text{ if } e_i < \infty. \tag{2}$$

Notice that we do *not* allow $e_i = 1$; i.e., $G_{i-1} = G_i$. Given the above information, we can form a presentation π_k for G as follows:

generators of π_k: $a_1, \ldots, a_k$
relations of π_k: (Type I) $a_i^{e_i} = u_i(a_1, \ldots, a_{i-1})$
if $1 < e_i < \infty$ where $u_i(a_1, \ldots, a_{i-1})$ is a word on $a_1, \ldots, a_{i-1}$
in the form of (2); if $e_i = \infty$ Type I relations do not occur. (3)
(Type II) $a_j^{-1} a_i a_j = v_{ij}(a_1, \ldots, a_{j-1})$
where $1 \leqq i < j \leqq k$ and $v_{ij}(a_1, \ldots, a_{j-1})$ is a word on $a_1, \ldots, a_{j-1}$
in the form of (2).

Then $G = \mathrm{gp}(\pi_k)$, the group presented by π_k. Also notice that, for any G_i in the polycyclic series (1) for G, we can similarly form π_i and $\pi_i \subseteq \pi_{i+1}$ in the obvious sense. We term k the length of the presentation π_k and note that we have a natural order

$$\varnothing = \pi_0 \subseteq \pi_1 \subseteq \pi_2 \subseteq \cdots \subseteq \pi_k$$

on subpresentations of (3) corresponding to the series (1). A presentation of the form (3) will be called a *polycyclic presentation.*

Conversely, any presentation of form (3) has a naturally ordered collection of subpresentations. Such a polycyclic presentation will be termed *honest* if for each $1 \leqq i \leqq k$, the inclusion $\pi_{i-1} \subseteq \pi_i$ induces an embedding $\mathrm{gp}(\pi_{i-1}) \hookrightarrow \mathrm{gp}(\pi_i)$. (Then the order of a_i mod $\mathrm{gp}(\pi_{i-1})$ is e_i.) Notice that even if π_{k-1} is honest π_k may fail to be honest because (for instance) the type II relations specifying the action of a_k on $\mathrm{gp}(\pi_{k-1})$ may not define an automorphism. Obviously, every polycyclic group has an honest polycyclic presentation. Below we shall give an algorithm to determine of an arbitrary polycyclic presentation where or not it is honest.

4.2. Two facts concerning polycyclic groups beyond the definition will be needed: (i) polycyclic groups satisfy the maximum condition for subgroups (since cyclic groups satisfy the maximum condition this follows easily by induction on the length of a polycyclic series); and (ii) polycyclic groups are hopfian, i.e., every surjective endomorphism is an automorphism (see, for example, H. Neumann [16] Lemma 32.1 and Corollary 41.44).

We also observe the following fact which is easily established by induction on length.

Lemma 4.1. *If π_k is an honest polycyclic presentation as in* (3), *and if $g \in \mathrm{gp}(\pi_k)$ has finite order, then $|g|$ divides* $\prod_{\substack{1 \leqq i \leqq k \\ e_i < \infty}} e_i$.

We also observe the following:

Lemma 4.2. *If π_k is an honest polycyclic presentation as in* (3), *then there is a recursive procedure which, when applied to any word W in the generators of π_k, gives the unique word in standard form equal to W. In particular, this algorithm solves the word problem for* $\mathrm{gp}(\pi_k)$.

Proof. Use type II relations to push higher subscripted generators to right and type I relations to reduce mod $e_i < \infty$. Eventually, standard form will be arrived at which is unique since π_k is honest. This concludes the proof.

Theorem 4.3. *There is an algorithm to determine of an arbitrary polycyclic presentation π_k as in* (3) *whether or not π_k is honest. Moreover, if π_k is honest, there are algorithms (uniform in π_k) for solving the following problems:*

(i) *decide for arbitrary finite sets of words $w_1, \ldots, w_n$ in π's and arbitrary $g \in \pi_k$ whether or not $g \in \mathrm{gp}(w_1, \ldots, w_n)$ (called the* generalized word problem, *membership problem, occurrence problem, etc.),*

(ii) *for an arbitrary finite set of words $w_1, \ldots, w_n$ in π_k find a finite presentation for $\mathrm{gp}(w_1, \ldots, w_n)$ (such a presentation exists since the subgroup is polycyclic), where the generators in the presentation naturally correspond to $w_1, \ldots, w_n$.*

Proof. By induction on the length k of the polycyclic presentation π_k. For $k=1$, all assertions are easily checked. Suppose the theorem has been established for polycyclic presentations of length $<k$. Consider π_k. Decide inductively if π_{k-1} is honest. If not, π_k is not honest and we are done. So suppose π_{k-1} is honest.

Consider the type II relations

$$a_k^{-1} a_i a_k = v_{ik}(a_1, \ldots, a_{k-1}) = v_{ik} (1 \leqq i < k).$$

Now π_k is honest if and only if the map $a_i \mapsto v_{ik}$ $(1 \leqq i < k)$ defines an automorphism of $\mathrm{gp}(\pi_{k-1})$ and if the e_k-th power of this automorphism is conjugation by u_k (cf type I relations) when $e_k < \infty$. Using the solution to the word problem in $\mathrm{gp}(\pi_{k-1})$ (Lemma 4.2) we may check to see whether the map defines a homomorphism. Using the solution to the generalized word problem (part (i)) for π_{k-1}, we may decide whether

$$a_i \in \mathrm{gp}(v_{1,k}, \ldots, v_{k-1,k}) \quad \text{for all } 1 \leqq i < k.$$

Thus we may effectively decide whether $a_i \mapsto v_{ik}$ defines a surjective endomorphism of $\mathrm{gp}(\pi_{k-1})$ and, hence, an automorphism, by hopficity of polycyclic groups. Moreover, using the solution to the word problem in π_k we may check whether the e_k-th power of this automorphism is conjugation by u_k. Thus, we may effectively decide whether π_k is honest.

Assume π_k is honest. We must produce algorithms for solving problems (i) and (ii). By induction hypothesis, we already have algorithms for solving these problems for π_{k-1}. Let $w_1, \ldots, w_n$ be an arbitrary finite set of words in π_k and let $S = \mathrm{gp}(w_1, \ldots, w_n)$ be the subgroup they generate. By Lemma 4.2, we may assume each w_i is in standard form (2). Let $\gamma_{k,1}, \ldots, \gamma_{k,n}$ be the powers of a_k appearing in their standard forms, and let α be the greatest common divisor of

$$\gamma_{k,1}, \ldots, \gamma_{k,n}, e_k \quad (\text{if } e_k < \infty)$$

and take integers β_i so that

$$\alpha = \beta_1 \gamma_{k,1} + \cdots + \beta_n \gamma_{k,n} + \beta_{n+1} e_k.$$

Then, by applying the relations of types I and II, the standard form of $r = w_1^{\beta_1} \ldots w_n^{\beta_n}$ has a_k to the power α appearing. Moreover, for each i, w_i multiplied by a suitable power $\delta(i)$ of r has $w_i r^{\delta(i)} \in \mathrm{gp}(\pi_{k-1})$. So

$$S = \mathrm{gp}(r, w_1 r^{\delta(1)}, \ldots, w_n r^{\delta(n)}).$$

Now, clearly this new set of generators for S can effectively be found since only the Euclidean algorithm is involved. (Note, we have tactily assumed that all the w_i's involve a non-zero power of a_k; if some w_i's have zero power of a_k apply the above to the remaining w_i's. However, if all w_i's have zero power of a_k, then S and all the w_i's lie in $\mathrm{gp}(\pi_{k-1})$ and the desired result follows easily by the induction assumption. Thus, we assume hereafter that some w_i has a non-zero power of a_k and so $1 \leqq \alpha < e_k$.)

Let us relabel the generators for S as $r, x_1, \ldots, x_n$ all in standard form, where $x_i \in \mathrm{gp}(\pi_{k-1})$ and a_k appears in r with power $\alpha (1 \leqq \alpha < e_k)$. If $e_k < \infty$, let δ denote the order of r mod $\mathrm{gp}(\pi_{k-1})$ and add the element $r^s \in \mathrm{gp}(\pi_{k-1})$ to the generators of S. Re-label the generators of S $r, y_1, \ldots, y_m$. Decide, using (i) for π_{k-1}, whether $r^{-1} y_j r \in \mathrm{gp}(y_1, \ldots, y_m)$ for $j = 1, \ldots, m$. If not, add these elements $r^{-1} y_j r$ to the generators of S, re-label the generators, and repeat. By the maximum condition, this process must terminate. Thus, we can effectively find a set of generators

$r, t_1, \ldots, t_q$ for S with $t_i \in \mathrm{gp}(\pi_{k-1})$ *and* $r^{-1} t_i r \in \mathrm{gp}(t_1, \ldots, t_q)$ *and* $r^\delta \in \mathrm{gp}(t_1, \ldots, t_q)$ in case $e_k < \infty$ and δ = order of $r \bmod \mathrm{gp}(\pi_{k-1})$.

In terms of the generators $r, t_1, \ldots, t_q$ we can easily obtain a presentation for S. By the induction hypothesis we can effectively find a presentation Δ for $\mathrm{gp}(t_1, \ldots, t_q) \leqq \mathrm{gp}(\pi_{k-1})$ on generators $\bar{t}_1, \ldots, \bar{t}_q$ corresponding to the t_i's. A presentation for S can be obtained by adding a new generator $\bar{r}$ together with relations $\bar{r}^{-1} \bar{t}_i \bar{r} = \bar{v}_i(\bar{t}_1, \ldots, \bar{t}_q)$ and $\bar{r}^\delta = \bar{u}(\bar{t}_1, \ldots, \bar{t}_q)$ where the words $\bar{v}_i$, $\bar{u}$ can be effectively found by the properties of the generators for S. Moreover, since $r, t_1, \ldots, t_q$ were effectively constructed out of $w_1, \ldots, w_n$ we can reverse the process and by Tietze transformation obtain a presentation for S on generators corresponding to $w_1, \ldots, w_n$. This completes the required algorithm for (ii).

Let g be an arbitrary element of $\mathrm{gp}(\pi_k)$. We want to decide whether $g \in S$. We may suppose g is in standard for (2). If a_k appears in g with power β, and α does not divide $\beta \bmod e_k$ then $g \notin S$. This may be effectively tested. Suppose $\alpha \cdot n \equiv \beta \bmod e$. Let $h = r^{-n} g$ so $h \in S$ if and only if $g \in S$ and $h \in \mathrm{gp}(\pi_{k-1})$. By properties of the constructed set of generators for S, $h \in S$ if and only if $h \in \mathrm{gp}(t_1, \ldots, t_q)$ which is decidable by induction hypothesis. This solves the generalized word problem (i) for π_k and completes the proof.

4.3. Remark: That polycyclic groups have solvable generalized word problems was known. Indeed, Remmeslenikov and Toh have (independently) shown polycyclic groups are subgroup separable. However, we obtained the solubility result for only a little extra work on the way to proving (ii).

Corollary 4.4. *There is an effective procedure to tell of two arbitrary finite sets of words* $\{w_1, \ldots, w_n\}$ *and* $\{z_1, \ldots, z_n\}$ *in an honest polycyclic presentation* π *whether or not the map* $w_i \mapsto z_i$ *defines and isomorphism between the subgroups of* π *which they generate.*

Proof. Since π is honest, by Theorem 4.3, we may effectively find finite presentations for $W = \mathrm{gp}(w_1, \ldots, w_n)$ and $Z = \mathrm{gp}(z_1, \ldots, z_n)$ on the given generators. Let $\varphi\colon W \to Z$ be the map defined by $w_i \mapsto z_i$ and $\psi\colon Z \to W$ the map defined by $z_i \mapsto w_i$. Using the solution to the word problem for π we can decide whether φ and ψ determine homomorphisms by checking to see that they preserve defining relations. If so, they are obviously isomorphisms as they are inverse to each other. This completes the proof.

4.4. Observe that if π_k and π'_l are honest polycyclic presentations of lengths k and l then there is an obvious honest polycyclic presentation Δ_{k+l} of length $k+l$ for $\pi_k \times \pi'_l$ (direct product) obtained from π_k by successively adding generators of π'_l acting trivially on generators of π_k. Thus if $\pi, \pi', \pi'', \ldots$ is a sequence of honest polycyclic presentations we may form a sequence

$$\varnothing \subseteq \Delta_1 \subseteq \cdots \subseteq \Delta_i \subseteq \cdots$$

of honest polycyclic presentations such that π, $\pi \times \pi'$, $\pi \times \pi' \times \pi''$, ... etc., are all honestly presented at some state Δ_i. By Theorem 4.3 the sequence of all polycyclic presentations is recursive; let Δ_i be the corresponding ordered sequence of honest polycyclic presentations and put $\bar{\Delta}=\mathrm{gp}(\bigcup_{i\geq 1}\Delta_i)$. Then $\bar{\Delta}$ contains an isomorphic copy of every polycyclic group and Δ is locally polycyclic. By construction, $\bar{\Delta}$ is recursively presented; by Corollary 4.4 and the fact that the Δ_i's are honest, the set of pairs of finite sets of words $\{w_1, \ldots, w_n\}$ and $\{z_1, \ldots, z_n\}$ such that $w_i \mapsto z_1$ is an isomorphism of subgroups is recursive. Thus by Corollary 2.5 we have

Theorem 4.5. *Every countable locally polycyclic group can be embedded in a finitely presented group.*

4.5. We now begin the study of polycyclic-by-finite groups. We proceed as follows. We recursively enumerate all wreath products of polycyclic groups by all finite groups, giving us the groups

$$W_1, W_2, W_3, \ldots,$$

say, together with their presentations, where the polycyclic component has been presented by means of an honest presentation (c.f. 4.3). We need then the

Lemma 4.6. *Let $W=G \wr F$ where G is polycyclic and F is finite and let $J=\mathrm{gp}(w_1, \ldots, w_n)$ be a finitely generated subgroup of W. Then we can effectively find a finite presentation for J.*

Proof. Note that W is a split extension of B by F where, as usual,

$$B=\mathrm{gp}_W(G)=\prod_{f\in F} f^{-1}Gf.$$

Our first objective is to obtain a set of generators for $J \cap B$. To this end let us write each w_i in the form

$$w_i=f_i b_i \quad (f_i\in F,\ b_i \in B).$$

We may assume that if $f_i f_j=f_k$ then $f_k b_k$ is included among the generators of J. Consider now the elements

$$f_i b_i f_j b_j (f_k b_k)^{-1}=c_{i,j},$$

where $f_k b_k$ is any one of the generators of J for which $f_i f_j=f_k$. We claim that

$$J \cap B=\mathrm{gp}(c_{ij}) \quad (i,j=1, \ldots, n).$$

For, suppose,

$$w_{i_1} w_{i_2} \ldots w_{i_r} \in B.$$

(Note that we need only consider "positive" words by adjoining all inverses of the $c_{i,j}$ to the given set of generators.) Then, assuming $f_{i_1} f_{i_2}=f_{j_1}$, we find

$$\begin{aligned} w_{i_1} w_{i_2} \ldots w_{i_r} &=(f_{i_1} b_{i_1} f_{i_2} b_{i_2} \ldots f_{i_r} b_{i_r}) \\ &=(f_{i_1} b_{i_1} f_{i_2} b_{i_2} (f_{j_1} b_{j_1})^{-1})(f_{j_1} b_{j_1} \ldots f_{i_r} b_{i_r}) \\ &=c_{i_1, j_1}(f_{j_1} b_{j_1} \ldots f_{i_r} b_{i_r}) \\ &=c_{i_1, j_1} \cdot c_{i'_2 j'_2} \ldots c_{i'_h, j'_r} f b, \end{aligned}$$

for some $f \in F$, $b \in B$ and an appropriate choice of the subscripts. It follows that $f=1$. Notice that $fb=b$ arises from a product

$$(f'b'f''b''(f'f''b''')^{-1})f'f''b''',$$

i.e., as $f'f''b'''=b$; in other words, b is simply one of the original $c_{i,j}$'s or an inverse of such. Thus, we "have our hands" on $J \cap B$, an implicitly given finitely generated subgroup of a polycyclic group B which has an obvious *basis* made up from a basis for each of the conjugates $f^{-1}Gf$ of the given polycyclic group G. By Theorems 4.3 and 4.4 we can effectively find an honest presentation for $J \cap B$. This leads to a presentation for J itself since we have a natural presentation for $J/J \cap B$ and we can easily find the action of the elements w_i on the generators of $J \cap B$ since W is a group with an explicitly given multiplication table.

Corollary 4.7. *Let $W_i = G_i \wr F_i$ $(i=1,2)$ be the wreath product of the polycyclic groups G_i by the finite groups F_i, where the G_i are explicitly given in terms of a polycyclic basis. Let θ be the mapping of a finite subset $\{w_1, \ldots, w_n\}$ of W_1 into the subset $\{w'_1, \ldots, w'_n\}$ of W_2 given by $w_i \mapsto w'_i$ $(i=i, \ldots, n)$. Then one can determine effectively whether θ is an isomorphism.*

Proof. Put $J_1 = \mathrm{gp}(w_1, \ldots, w_n)$, $J_2 = \mathrm{gp}(w'_1, \ldots, w'_n)$. We effectively obtain finite presentations for J_1 and J_2 by the lemma above. Thus, we can check whether the maps

$$\varphi: w_i \mapsto w'_i$$

and

$$\psi: w'_i \mapsto w_i$$

define homomorphisms since W_1 and W_2 have solvable word problems. Then θ is an isomorphism if and only if the composed maps $\varphi\psi$ and $\psi\varphi$ are the identify which can be effectively verified. This concludes the proof.

4.6. We are now in a position to apply Corollary 2.5. We enumerate all the wreath products W_i, enumerate all the finite pairs of objects (X, Y) of equipotent sets of generators $X \subseteq W_i$, $Y \subseteq W_j$ where $X = \{x_1, \ldots, x_r\}$, $Y = \{y_1, \ldots, y_r\}$ *and* $x_i \mapsto y_i$ defining an isomorphism. The resultant HNN construction then gives the following

Theorem 4.8. *A countable locally polycyclic-by-finite group can be embedded in a finitely presented group.*

§ 5. Automorphism Groups

5.1. In this section we prove the

Theorem 5.1. *The automorphism group* Aut(G) *of any finitely presented group G is recursively presentable. If G has solvable word problem then* Aut(G) *has a presentation for which the word problem is solvable.*

Proof. The argument is extremely simple. Instead of trying to enumerate the automorphisms of G alone, we enumerate an automorphism and its inverse at the same time. Let

$$G=\langle x_1, \ldots, x_n;\; R_1=1, \ldots, R_m=1\rangle.$$

Each $R_i=R_i(x_1, \ldots, x_n)$ is a word on the x_i's – we will sometimes use vector notation $\vec{x}=(x_1, \ldots, x_n)$ for brevity. Suppose $\alpha\colon G\to G$ is an automorphism. Then, for suitable words $W_i(\vec{x})$, $\alpha(x_i)=W_i(\vec{x})$ in G for $1\leqq i\leqq n$. Since α is a homomorphism, the equations

$$R_j(W_1(\vec{x}), \ldots, W_n(\vec{x}))=1 \quad \text{where } 1\leqq j\leqq m \text{ hold in } G. \tag{1}$$

Suppose that $\beta\colon G\to G$ is the inverse of α. Then $\beta(x_i)=U_i(\vec{x})$ for suitable words U_i in the x_j's. Moreover, the following equations hold in G:

$$R_j(U_1(\vec{x}), \ldots, U_n(\vec{x}))=1 \quad \text{where } 1\leqq j\leqq m. \tag{2}$$

In addition, because α and β are inverses of each other, the following equations hold in G:

$$\begin{aligned} &U_k(W_1(\vec{x}), \ldots, W_n(\vec{x}))=x_k \quad \text{for } 1\leqq k\leqq n\\ &W_j(U_1(\vec{x}), \ldots, U_n(\vec{x}))=x_j \quad \text{for } 1\leqq j\leqq n. \end{aligned} \tag{3}$$

Conversely, if $(W_1, \ldots, W_n)$ and $(U_1, \ldots, U_n)$ are pairs of n-tuples of words in the x_i's satisfying the systems of Equations (1), (2), and (3), then the maps defined by $\alpha\colon x_i\mapsto W_i$ and $\beta\colon x_i\mapsto U_i$ for $1\leqq i\leqq n$ are automorphisms of G which are inverse to one another.

Since G is finitely presented, the set of pairs $(W_1, \ldots, W_n)$ and $(U_1, \ldots, U_n)$ of words in the x_i's satisfying the finite systems of Equations (1), (2), and (3) is recursively enumerable. Note that if G has solvable word problem, then this set of pairs is even recursive. This enumeration clearly gives a recursively enumerable set of generators for $\mathrm{Aut}(G)$.

Words in the above set of generators for $\mathrm{Aut}(G)$ correspond to the automorphism (which will again appear among the generators) gotten by composing the actions on the x_i's. Thus to enumerate a set of relations in the above set of generators it suffices to enumerate those pairs representing the identity automorphism. But $\alpha\colon x_i\mapsto W_i(\vec{x})$ for $1\leqq i\leqq n$ represents the identity if and only if $W_i(\vec{x})=x_i$ for all $i=1, \ldots, n$. Since G is finitely presented, the pairs satisfying this condition are recursively enumerable, and so $\mathrm{Aut}(G)$ is recursively presentable. If, in addition, G has solvable word problem, then this condition is recursive and so $\mathrm{Aut}(G)$ has solvable word problem in the given set of generators. This completes the proof.

Corollary 5.2. *The automorphism group* $\mathrm{Aut}(G)$ *of a finitely presented group G is embeddedable in a finitely presented group H. If G has solvable word problem we may choose H to have solvable word problem.*

Proof. By Theorem 5.1 and Higman [10] the first half of Corollary 3.2 follows. The second half follows by Cannonito-Gatterdam [5] and either Clapham [6] or Gatterdam [8].

Theorem 5.3. *A countable group G is embeddable in a finitely presented group H if and only if it has a faithful representation as a group of automorphisms of a finitely presented group.*

Proof. "If" follows from Corollary 5.2. To prove only if, we assume G is embedded in the nontrivial finitely presented group H. Then if K is the free product of H with an infinite cyclic group, K is finitely presented and has no center. Thus, K is isomorphic to $\mathrm{Inn}(K)$, the group of inner automorphisms, and so G is embeddable in $\mathrm{Aut}(K)$.

5.2. An examination of the proof of Theorem 5.1 yields the somewhat more general (see Cohn [7], p. 153 f.).

Theorem 5.4. *Let $\mathscr{V}$ be a variety of universal algebras in which the finitely generated free algebras have solvable word problem and let A be a finitely presented algebra in $\mathscr{V}$. Then* $\mathrm{Aut}(A)$ *is recursively presentable; moreover, if A has solvable word problem there is a presentation of* $\mathrm{Aut}(A)$ *for which the word problem is also solvable.*

We would like to draw attention to three special cases of Theorem 5.3 (besides Theorem 5.1).

Corollary 5.5. *The automorphism group of a finitely presented semi-group is recursively presentable.*

Corollary 5.6. *The automorphism group of a finitely generated metabelian group is recursively presentable and has solvable word problem in some presentation.*

Remark. It is not hard to show that the word problem in a finitely generated metabelian lie algebra is solvable (see, e.g. [2]) and so we have the

Corollary 5.7. *The automorphism group of a finitely generated metabelian lie algebra is recursively presentable and has solvable word problem in some presentation.*

Corollary 5.6 and Corollary 5.7 depend on the fact that finitely generated metabelian groups and finitely generated metabelian lie algebras satisfy the maximal condition for normal subgroups and ideals (P. Hall [9], Amayo and Stewart [1]). As a consequence they are finitely presented if observed as algebras in their respective metabelian varieties.

Finally, let us remark that the automorphism group of a finitely presented group need not be finitely generated (see Lewin [11]).

§ 6. Linear Groups

6.1. We begin by giving an embedding theorem for linear groups (over any field) which is of independent interest and which gives an alternative proof for the embeddability of countable locally free groups.

Let Φ be the countable algebraically closed field of transcendence degree $\aleph_0$ of characteristic p, $p \geqq 0$. If $GL(n, \Phi)$ is the group of $n \times n$ invertible matrices over Φ, we regard it as embedded in the "upper left corner" of $GL(n+1, \Phi)$. Denote by $GL(\Phi)$ the direct limit of the system

$$GL(1, \Phi) < GL(2, \Phi) < \cdots < GL(n, \Phi) < \cdots .$$

Then $GL(\Phi)$ may be thought of as the group of $\omega \times \omega$ matrices which differ from the identity on only finitely many places and which have nonzero determinant (in the obvious sense).

Next, we need the concept of a *computable field*, introduced by Rabin in [18]. Intuitively, a countable field is computable if it can be mapped injectively onto a subset S of the natural numbers in such a way that (CF1) there is an algorithm which computes the characteristic function of S, and (CF2) the field operations induce functions $S \times S \to S$ which are recursive or algorithmically computable. Thus, a computable field has a recursive representation in the natural numbers. It is easily seen that all prime fields are computable. Further, a pure transcendental extension of degree $\leqq \aleph_0$ of a computable field is again computable. Finally, Rabin [18] proves that the algebraic closure of a computable field is computable. Thus, our field Φ can be viewed as a universal countable computable field of characteristic p.

Now we note that $GL(\Phi)$ is recursively presentable. This can be seen by, for example, providing $GL(\Phi)$ with its multiplication table presentation, taking as generators all $\omega \times \omega$ matrices of the kind described above and as defining relators all words ABC^{-1} where A, B and C are matrices and $AB = C$. That this system of generators is recursively enumerable follows from an easily conceived procedure that, again, is clumsy to write out. It suffices to note merely that each matrix in $GL(\Phi)$ is completely specified by a finite number of field elements and their coordinates; that is, by a finite sequence of elements of S. It is easily seen that we can effectively list all finite sequences of elements of S and so to effectively list the elements of $GL(\Phi)$ we merely use the computable field operations to cast out those sequences which do not correspond to elements by computing a determinant. In a similar manner we effectively obtain all defining relators of the form ABC^{-1}.

We note that when the multiplication table presentation of a group is recursive the group has a solvable word problem for this presentation. Thus, in particular, $GL(\Phi)$ has a solvable word problem for the presentation given above. Hence, by Cannonito-Gatterdam [5] and Clapham [6] or Gatterdam [8], we have the following:

Theorem 6.1. *The group $GL(\Phi)$ can be embedded in a finitely presented group with solvable word problem.*

This has several remarkable consequences:

Corollary 6.2. *A countable linear group G over any field is embeddable in a finitely presented group with solvable word problem.*

Proof. Since G is countable, the field elements which appear as entries in the elements of G all lie in some countable field, hence, in some subfield of Φ (where Φ is as in 4.1). Thus, G is a subgroup of $GL(n, \Phi)$ and, hence, of $GL(\Phi)$, so the result follows from 4.1.

This immediately gives the following:

Corollary 6.3. *A countable group G which is locally linear of bounded degree is embeddable in a finitely presented group with solvable word problem.*

Proof. For, by Mal'cev's local theorem [14], G is linear. Hence, we may apply 6.2.

As an application of 6.3, we have

Corollary 6.4. *A countable locally free group G can be embedded in a finitely presented group with solvable word problem.*

Proof. $GL(2, \mathbb{Q})$ contains finitely generated free groups of arbitrary rank (see [19]). Thus, G is locally faithfully representable in $GL(2, \mathbb{Q})$ and we may apply Corollary 6.3.

Remark. Since finitely generated linear groups are countable, and finitely generated subgroups of finitely generated groups with solvable word problem have, themselves, solvable word problem, Corollary 6.2 gives a proof of the theorem announced in Rabin [18], viz.: a finitely generated linear group has solvable word problem.

§ 7. Some Nonembeddable Groups

7.1. In this section we consider several examples of groups which are not embeddable in finitely presented groups.

7.2. We begin by first proving the

Theorem 7.1. *The wreath product $W = U \wr T$ of two finitely generated groups U and T is recursively presentable if and only if both U and T are recursively presentable and either U is abelian or T has solvable word problem.*

Proof. We recall that W is defined by the following properties:

$$W = \mathrm{gp}(U, T); \tag{1}$$

(2) The normal subgroup B of W generated by U is the (restricted) direct product of its conjugates $t^{-1} U t$ $(t \in T)$. It follows that $W = B] T$ is a split extension of B by T with T acting on B essentially by the right regular representation of T on itself.

Now suppose that W is recursively presentable. Then it follows from Higman's theorem [10] that both U and T are recursively presentable. Let

$$U=\langle u_1,\ldots,u_m;\, r_1,r_2,\ldots\rangle$$
$$T=\langle t_1,\ldots,t_n;\, s_1,s_2,\ldots\rangle \tag{3}$$

be presentations for U and T on finite sets of generators and recursively enumerable sets of defining relators. If U is nonabelian, we can find u_i, u_j among the generators $u_1,\ldots,u_m$ such that the commutator $[u_i,u_j]\neq 1$. Since the defining relators of W are recursively enumerable, the relators of the form

$$[u_i,\, w(t_1,\ldots,t_n)^{-1}u_j w(t_1,\ldots,t_n)] \tag{4}$$

are also recursively enumerable, where here $w(t_1,\ldots,t_n)$ is a word in the generators $t_1,\ldots,t_n$ of T. But (4) is a relator if and only if $W\neq 1$. Hence, if W is recursively presented and U is nonabelian, then T has solvable word problem, for we can recursively enumerate both the set of words $w(\underline{t})$ such that $w=1$ in T and the set of words $w(\underline{t})$ such that $w\neq 1$ in T.

Conversely, suppose U and T are recursively presentable, with recursive presentations (3). If T has a solvable word problem then W can be presented in the form

$$\begin{aligned}W=\langle u_1,\ldots,u_m,t_1,\ldots,t_n;\, &r_1,r_2,\ldots,s_1,s_2,\ldots\\ &[u_i,w(\underline{t})^{-1}u_jw(\underline{t})]\ (i,j=1,\ldots,m,\ w(\underline{t})\\ &\text{an arbitrary word on } t_1,\ldots,t_1 \text{ such that } w(\underline{t})\neq 1 \text{ in } T\rangle\end{aligned} \tag{5}$$

and so W is recursively presentable. On the other hand, if U is abelian, no restriction need be placed on the words $w(\underline{t})$ in the presentation (5) and so again W is recursively presentable.

As a consequence of Theorem 7.1, note that the wreath product $W=U\wr T$ of a finite nonabelian group U by a finitely presented group T with unsolvable word problem is not recursively presentable. In particular, the wreath product of two finitely presented groups need not be embeddable in a finitely presented group.

7.3. Let p be a prime. In [9], P. Hall constructed a recursively presented, 2-generator center-by-metabelian group A, having center C, which is a direct sum of cyclic groups of order p. Let c_i $(i\geqq 1)$ be the generators of these cyclic groups (from [9] one sees they can be written explicitly in terms of the generators of A). Let K be a recursively enumerable, non-recursive set of natural numbers. Let N be the normal closure in A of $\{c_i \mid i\notin K\}$. Put $B=A/N$. Finally, let G be the direct product of A and B with central amalgamations $c_i=c_iN$ for $i\in K$.

Now G is a finitely generated, center-by-metabelian group with the following properties:

(1) A is normal in G, and $G/A\cong A/C$ is a 2-generator metabelian group. In fact, one can easily solve the word problem for A, so G is an extension of one finitely generated group with solvable word problem by another.

(2) C is normal in A and C is normal in G (as a subgroup of A) and $G/C\cong A/C\oplus A/C$ which is metabelian. Thus, G is locally finite-by-metabelian.

(3) G is not embeddable in a finitely presented group. For in terms of generators for two copies A and $\bar{A}$ of the group A, in G we have $c_i = \bar{c}_i \Leftrightarrow i \notin K$. Hence, G cannot be recursively presented, and thus is not embeddable in a finitely presented group.

Remark. Properties (1) and (3) contradict a claim made in [15] Lemma 1 p. 4, that "finitely generated with solvable word problem" is a poly-property. The property is only preserved by recursively presented extensions (and, in particular, for finitely presented groups).

7.4. Let $H_1, H_2, \ldots$ be a list of all finitely presented groups with solvable word problem (by Boone-Rogers [4] this list is not recursively enumerable). According to results of Boone-Higman [3] combined with Clapham [6], a finitely generated group with solvable word problem can be embedded in a simple subgroup of a finitely presented group with solvable word problem. Put $A_1 = H_1$ and let A_{n+1} be the result of applying Boone-Higman-Clapham to the free product $A_n * H_{n+1}$ for $n \geqq 1$. Put $G = \bigcup_{n \geqq 1} A_n$. Then G is simple (being a union of simple groups) and locally has a solvable word problem (since each A_n has solvable word problem). However, G cannot be embedded in a finitely presented group. For if G could be so embedded there would be a uniform partial algorithm for solving the word problem for the H_i's (see Boone-Higman [3] Theorem III) which would in turn contradict Boone-Rogers [4].

7.5. In [12] MacIntyre constructs a countable algebraically closed group G whose finitely generated subgroups are exactly the finitely generated recursively presented groups. Thus, G is locally embeddable in a finitely presented group. However, G itself not embeddable in a finitely presented group. For since G is simple and has finitely presented subgroups with unsolvable word problem, by Boone-Higman [3] G cannot be so embedded.

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