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Existence and uniqueness of the integrable-exactly hypoelastic equation $\overset{\circ}{\boldsymbol{\tau}}^* = \lambda(\mathrm{tr}\boldsymbol{D})\boldsymbol{I} + 2\mu\boldsymbol{D}$ and its significance to finite inelasticity

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Summary. In Eulerian rate type finite inelasticity models postulating the additive decomposition of the stretching $\boldsymbol{D}$, such as finite deformation elastoplasticity models, the simple rate equation indicated in the above title is widely used to characterize the elastic response with $\boldsymbol{D}$ replaced by its "elastic" part. In 1984 Simo and Pister (Compt. Meth. Appl. Mech. Engng. **46,** 201–215) proved that none of such rate equations with several commonly-known stress rates is exactly integrable to deliver an elastic relation, and thus any of them is incompatible with the notion of elasticity. Such incompatibility implies that Eulerian rate type inelasticity theory based on any commonly-known stress rate is self-inconsistent, and thus it is hardly surprising that some aberrant, spurious phenomena such as the so-called shear oscillatory response etc., may be resulted in. Then arises the questions: *Whether or not is there a stress rate* $\overset{\circ}{\boldsymbol{\tau}}^*$ *such that the hypoelastic equation of grade zero with this stress rate is exactly integrable to really define an elastic relation? If the answer is yes, what is or are such stress rate(s)?* The answer for these questions is crucial to achieving rational, self-consistent Eulerian rate type formulations of finite inelasticity models. It seems that there has been no complete, natural and convincing treatment for the foregoing questions until now. It is the main goal of this article to prove the fact: among all possible (infinitely many) objective corotational stress rates and other well-known objective stress rates $\overset{\circ}{\boldsymbol{\tau}}^*$, there is one and only one such that the hypoelastic equation of grade zero with this stress rate is exactly integrable to define a hyperelastic relation, and this stress rate is just the newly discovered *logarithmic stress rate* by these authors and others. This result, which provides a complete answer for the aforementioned questions, indicates that in Eulerian rate type formulations of inelasticity models, the logarithmic stress rate is the only choice in the sense of compatibility of the hypoelastic equation of grade zero that is used to represent the elastic response with the notion of elasticity.

1 Introduction

In Eulerian rate type formulation of continuum models characterizing inelastic behaviours of isotropic materials, such as finite deformation elastoplasticity, an additive decomposition of the stretching $\boldsymbol{D}$ into the "elastic" part $\boldsymbol{D}^e$ and "inelastic" part(s) is introduced. Then, the inelastic response is formulated by certain inelastic laws, e.g., associated or nonassociated flow law and hardening rule etc., while the elastic response is widely assumed to be characterized by a hypoelastic equation of grade zero (see Truesdell and Noll [47] for a comprehensive account of hypoelasticity), i.e.,

$$\overset{\circ}{\boldsymbol{\tau}}^* = \boldsymbol{H}_0 : \boldsymbol{D}^e = \lambda(\mathrm{tr}\boldsymbol{D}^e)\,\boldsymbol{I} + 2\mu\boldsymbol{D}^e. \tag{1}$$

In the above, $\overset{\circ}{\boldsymbol{\tau}}^*$ is an objective stress rate, and $\boldsymbol{H}_0$ is the constant isotropic linear elasticity tensor

$$\boldsymbol{H}_0 = \lambda\boldsymbol{I}\otimes\boldsymbol{I} + 2\mu\boldsymbol{\Upsilon}, \tag{2}$$

where λ and μ are the Lamé elastic constants, and

$$I_{ij} = \delta_{ij}; \qquad \Upsilon_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{3}$$

Throughout, δ_{rs} is used to denote the Kronecker delta.

How to choose the stress rate $\overset{\circ}{\boldsymbol{\tau}}{}^*$ in (1) has been one of the crucial points. Although use of various stress rates makes no difference in the structure of this rate equation, fundamental difference in modelling material behaviours may occur. Lehmann [24]–[25], Dienes [8], and Nagtegaal and de Jong [32] disclosed that hypoelasticity models and elastoplasticity models with the Zaremba-Jaumann-Noll stress rate predict aberrant, spurious responses at finite simple shear deformation, now known as *shear oscillatory phenomena.* Prompted by these discoveries, several alternative stress rates have been suggested, examined and compared for simple shear responses and some other responses. For this aspect, we refer to, e.g., Truesdell [44]–[46], Lehmann [24]–[25], Dienes [8], Nagtegaal and de Jong [32], Dafalias [5]–[7], Lee, Mallett and Wertheimer [23], Loret [27], Reed and Atluri [36]–[37], Atluri [1], Johnson and Bammann [18], Key [19], Moss [31], Sowerby and Chu [42], Paulun and Pechersky [35], Haupt and Tsakmakis [13], Dubey [9], Metzger and Dubey [30], Zbib and Aifantis [56]–[57], Szabó and Balla [43], Yang, Cheng and Hwang [54], Xia and Ellyin [48], et al. A detailed account of this aspect can be found in Khan and Huang [20]. However, it seems that a complete, natural and convincing treatment has not been attained yet.

It seems that the decisive conclusion for a model can not be drawn only from some reasonable particular responses such as simple shear responses etc., and, moreover, for each model it is impossible to calculate all possible responses. One of the decisive criteria should be: when $\boldsymbol{D} = \boldsymbol{D}^e$, *the rate equation (1) that is used to represent elastic behaviour must be exactly integrable to really define an elastic relation.* If the just-stated criterion is violated, the rate equation (1) will be incompatible with the definition of any elastic behaviour, and therefore fails to represent any elastic behaviour faithfully and undistortedly in nonlinear range. In such a case, it is hardly surprising that some aberrant, spurious phenomena inappropriate for any elastic behaviour may be resulted in, such as hysteretic energy dissipation (see Bernstein [2], and Truesdell and Noll [47]) and residual stress for a closed strain path (see, e.g., Kleiber [21], Kojic and Bathe [22], and Roy et al. [39]), etc.

Simo and Pister [41] have proved that with several commonly-known stress rates $\overset{\circ}{\boldsymbol{\tau}}{}^*$, none of the corresponding hypoelastic equations of grade zero is integrable to yield an elastic relation, and any of them is thus incompatible with the definition of elasticity, in particular, hyperelasticity. Such incompatibility implies that any existing Eulerian rate type finite inelasticity theory with (1) is not self-consistent in the sense of characterizing the elastic response. To further pursue Simo and Pister's study from a general point of view, we may ask: whether or not is there a stress rate such that the hypoelastic equation of grade zero with this stress rate is exactly integrable to define an elastic relation? If the answer is yes, what is or are such stress rate(s)? The answer for these questions is crucial to achieving a rational self-consistent Eulerian rate type finite inelasticity theory with (1).

It is the objective of this article to find the complete answer for the above questions. We shall prove the fact: There is one and only one corotational stress rate $\overset{\circ}{\boldsymbol{\tau}}{}^*$ such that the hypoelastic equation of grade zero with this corotational stress rate is exactly integrable to define an elastic relation, and this corotational stress rate is just the newly discovered *logarithmic stress* rate by these authors [50]–[53] and other researchers (see Lehmann et al. [26], and Reinhardt and Dubey [38]). This fact implies that, to achieve the positive answer for the aforementioned questions, the logarithmic stress rate is the only choice among infinitely many objective corotational stress rates (see [52] and [53]) and other well-known objective stress rates including Oldroyd rate [34], Cotter-Rivlin rate [4] and Truesdell rate [44]–[45]. As a result, the new finite deformation elastoplasticity models based upon the logarithmic stress rate, suggested by these authors in a succeeding article [3], are not only the first, but also unique, self-consistent ones of their kinds, in the sense of compatibility of the hypoelastic equation of grade zero used to represent elastic response with the definition of elasticity.

We conclude this introduction with some facts that will be used. Let $\mathcal{T}_2$, Sym and Skw be the sets of all second-order tensors, all symmetric and all antisymmetric second-order tensors over a real three-dimensional inner-product space $\mathcal{V}$, respectively. As usual, each second-order tensor over $\mathcal{V}$ is regarded as a linear transformation from $\mathcal{V}$ to itself. Furthermore, each fourth-order tensor $\boldsymbol{H}$ over $\mathcal{V}$ with the index symmetry properties

$$H_{ijkl} = H_{ijlk} = H_{jikl}$$

is identified with a linear transformation on Sym by virtue of the third expression below. In so doing, we aim to have the benefit of utilizing the powerful spectral theory for real symmetric linear transformations, which will prove to be essential to our subsequent account.

In what follows, $\boldsymbol{A}$, $\boldsymbol{B} \in \mathcal{T}_2$, and $\boldsymbol{H}$, $\boldsymbol{H}'$, $\boldsymbol{\Lambda}$ are two fourth-order tensors and a sixth-order tensor over $\mathcal{V}$, respectively. Throughout, we shall use the notations $\boldsymbol{A} : \boldsymbol{B}$, $\boldsymbol{AB}$, $\boldsymbol{H} : \boldsymbol{A}$, $\boldsymbol{A} : \boldsymbol{H}$, $\boldsymbol{H} : \boldsymbol{H}'$, $\boldsymbol{\Lambda} : \boldsymbol{A}$ and $\boldsymbol{\Lambda} : \boldsymbol{H}$ to designate the scalar, the second-order tensors, the fourth-order tensors and the sixth-order tensor defined by:

$$\boldsymbol{A} : \boldsymbol{B} = A_{ij}B_{ij},$$

$$(\boldsymbol{AB})_{ij} = A_{ik}B_{kj},$$

$$(\boldsymbol{H} : \boldsymbol{A})_{ij} = H_{ijkl}A_{kl},$$

$$(\boldsymbol{A} : \boldsymbol{H})_{ij} = A_{kl}H_{klij},$$

$$(\boldsymbol{H} : \boldsymbol{H}')_{ijkl} = H_{ijrs}H'_{rskl},$$

$$(\boldsymbol{\Lambda} : \boldsymbol{A})_{ijkl} = \Lambda_{ijklrs}A_{rs},$$

$$(\boldsymbol{\Lambda} : \boldsymbol{H})_{ijklrs} = \Lambda_{ijklpq}H_{pqrs}.$$

Moreover, let $\boldsymbol{Q} \in \mathcal{T}_2$ be an orthogonal tensor over $\mathcal{V}$, i.e.,

$$Q_{ik}Q_{jk} = \delta_{ij}.$$

Then we shall use $\boldsymbol{Q} \star \boldsymbol{A}$ and $\boldsymbol{Q} \star \boldsymbol{H}$ to represent the second-order and fourth-order tensors given by

$$(\boldsymbol{Q} \star \boldsymbol{A})_{ij} = Q_{ik}Q_{jl}A_{kl}, \qquad \text{i.e.,} \qquad \boldsymbol{Q} \star \boldsymbol{A} = \boldsymbol{QAQ}^{\mathrm{T}},$$

$$(\boldsymbol{Q} \star \boldsymbol{H})_{ijkl} = Q_{ip}Q_{jp}Q_{kr}Q_{ls}H_{pqrs}.$$

Throughout, the symbol $\boldsymbol{A}^{\mathrm{T}}$ is used to represent the transpose of the tensor $\boldsymbol{A} \in \mathcal{T}_2$. The following identities will be useful:

$$\boldsymbol{Q} \star (\boldsymbol{H} : \boldsymbol{A}) = (\boldsymbol{Q} \star \boldsymbol{H}) : (\boldsymbol{Q} \star \boldsymbol{A}), \tag{4}$$

$$\boldsymbol{Q} \star (\boldsymbol{R} \star \boldsymbol{A}) = (\boldsymbol{QR}) \star \boldsymbol{A}, \tag{5}$$

$$\boldsymbol{Q} \star (\boldsymbol{R} \star \boldsymbol{H}) = (\boldsymbol{QR}) \star \boldsymbol{H}, \tag{6}$$

for any orthogonal tensors $\boldsymbol{Q}$ and $\boldsymbol{R}$.

Finally, we shall use the notation $\boldsymbol{p} \vee \boldsymbol{q}$, where $\boldsymbol{p}$ and $\boldsymbol{q}$ are any given two vectors, to signify a symmetric second-order tensor over $\mathcal{V}$ given by

$$\boldsymbol{p} \vee \boldsymbol{q} = \boldsymbol{p} \otimes \boldsymbol{q} + \boldsymbol{q} \otimes \boldsymbol{p}. \tag{7}$$

2 Logarithmic strain and objective corotational rates

2.1 Logarithmic strain

In this subsection, we recapitulate some related basic facts about kinematics of finite deformations. A detailed account of this aspect can be found in, e.g., Truesdell and Noll [47], and Marsden and Hughes [28].

Consider a continuous body $\mathcal{B}$ experiencing continuing deformation. We identify each particle of the body $\mathcal{B}$ with a position vector $\boldsymbol{X}$ in a referential configuration, e.g., an initial configuration. The current position vector of a particle $\boldsymbol{X}$ is denoted by $x = \bar{x}(\boldsymbol{X}, t)$, and hence the velocity vector of a particle $\boldsymbol{X}$ is given by

$$v = \dot{x}.$$

Throughout, the superimposed dot is used to designate the material time derivative of a scalar or tensor field.

The local deformation at a particle $\boldsymbol{X}$ is described by the *deformation gradient*

$$\boldsymbol{F} = \frac{\partial x}{\partial \boldsymbol{X}},$$

while the rate of change of deformation at a particle $\boldsymbol{X}$ is characterized by the *velocity gradient*

$$\boldsymbol{L} = \frac{\partial v}{\partial x} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}.$$

The following polar decomposition formula and additive decomposition formula are well-known:

$$\boldsymbol{F} = \boldsymbol{V}\boldsymbol{R}, \qquad \boldsymbol{V}^2 = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}, \tag{8}$$

$$\begin{cases} \boldsymbol{L} = \boldsymbol{W} + \boldsymbol{D}, \\ \boldsymbol{W} = \dfrac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^{\mathrm{T}}), \quad \boldsymbol{D} = \dfrac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}}). \end{cases} \tag{9}$$

In the above, the symmetric positive definitive tensor $\boldsymbol{V} \in Sym$ and the proper orthogonal tensor $\boldsymbol{R}$ are known as the *left stretch tensor* and the *rotation tensor*, and the symmetric and antisymmetric tensors $\boldsymbol{D} \in Sym$ and $\boldsymbol{W} \in Skw$ are called the *stretching* and the *vorticity tensor*.

Let $\lambda_1 > 0, \cdots, \lambda_m > 0$ be the distinct eigenvalues of $\boldsymbol{V}$ and $\boldsymbol{P}_1, \cdots, \boldsymbol{P}_m$ the corresponding subordinate eigenprojections of $\boldsymbol{V}$. Then we have the simple manipulation formula

$$\boldsymbol{P}_\sigma \boldsymbol{P}_\tau = \delta_{\sigma\tau} \boldsymbol{P}_\sigma \quad \text{(no summation)}, \tag{10}$$

$$\boldsymbol{P}_1 + \cdots + \boldsymbol{P}_m = \boldsymbol{I}. \tag{11}$$

Following Hill [15], we define a class of strain measures by

$$e = f(\boldsymbol{V}) = \sum_{\sigma=1}^{m} f(\lambda_\sigma)\boldsymbol{P}_\sigma, \tag{12}$$

where the *scale function* $f(\lambda)$ is a smooth strictly increasing function with the property $f(0) = f'(0) - 1 = 0$. In particular, the logarithmic scale function $f(\lambda) = \ln \lambda$ yields the Hencky's *logarithmic strain measure* (see, e.g., Hencky [14], Hill [15], and Fitzjerald [10])

$$\boldsymbol{h} = \ln \boldsymbol{V} = \sum_{\sigma=1}^{m} (\ln \lambda_\sigma) \boldsymbol{P}_\sigma, \tag{13}$$

which here will be of particular interest. Some important results on the logarithmic strain and its rates can be found in Fitzgerald [10], Gurtin and Spear [12], and Hoger [16], et al.

2.2 *Objective corotational rates*

Let $\boldsymbol{\Omega}^* \in Skw$ be a time-dependent spin tensor. In a rotating frame with the spin $\boldsymbol{\Omega}^*$, an objective Eulerian symmetric second-order tensor $\boldsymbol{S} \in Sym$ in a fixed background frame, becomes $\boldsymbol{QSQ}^{\mathrm{T}}$, and hence its time rate in this rotating frame is given by

$$\begin{aligned}
\overline{(\boldsymbol{Q} \star \boldsymbol{S})}^{\cdot} &= \overline{(\boldsymbol{QSQ}^{\mathrm{T}})}^{\cdot}, \\
&= \boldsymbol{Q}\dot{\boldsymbol{S}}\boldsymbol{Q}^{\mathrm{T}} + \dot{\boldsymbol{Q}}\boldsymbol{S}\boldsymbol{Q}^{\mathrm{T}} + \boldsymbol{QS}\dot{\boldsymbol{Q}}^{\mathrm{T}}, \\
&= \boldsymbol{Q}\overset{\circ}{\boldsymbol{S}}{}^{*}\boldsymbol{Q}^{\mathrm{T}} = \boldsymbol{Q} \star \overset{\circ}{\boldsymbol{S}}{}^{*}.
\end{aligned} \tag{14}$$

In the above, $\boldsymbol{Q}$ is a proper orthogonal tensor defining the spin $\boldsymbol{\Omega}^*$, i.e.,

$$\boldsymbol{\Omega}^* = \dot{\boldsymbol{Q}}^{\mathrm{T}}\boldsymbol{Q} = -\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{Q}}, \tag{15}$$

and moreover

$$\overset{\circ}{\boldsymbol{S}}{}^{*} = \dot{\boldsymbol{S}} + \boldsymbol{S}\boldsymbol{\Omega}^* - \boldsymbol{\Omega}^*\boldsymbol{S}. \tag{16}$$

The latter, i.e., $\overset{\circ}{\boldsymbol{S}}{}^{*}$, is called the *corotational rate* of the tensor $\boldsymbol{S}$ defined by the spin $\boldsymbol{\Omega}^*$. It is evident that there are infinitely many kinds of corotational rates. Not all of them, however, are objective. The objectivity of a corotational rate depends on its defining spin tensor, and the latter must be associated with the rotation and deformation of the deforming body under consideration in a suitable manner, as shown by several known examples, e.g., $\boldsymbol{\Omega}^* = \boldsymbol{W}$ (Zaremba-Jaumann-Noll rate; see, e.g., Zaremba [55], Jaumann [17], Noll [33]) and $\boldsymbol{\Omega}^* = \dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}}$ (Green-Naghdi-Dienes rate; see, e.g., Green and Naghdi [11], and Dienes [8]), etc.

Since the deformation gradient $\boldsymbol{F}$ and the velocity gradient $\boldsymbol{L}$ respectively characterize the local deformation state and the rate-of-change of local deformation state at a generic material particle, the most general form of spins $\boldsymbol{\Omega}^*$ defining objective corotational rates may be assumed to be of the form:

$$\boldsymbol{\Omega}^* = \omega(\boldsymbol{F}, \boldsymbol{L}),$$

where $\omega(\boldsymbol{F}, \boldsymbol{L})$ is an antisymmetric second-order tensor-valued function of the deformation gradient $\boldsymbol{F}$ and the velocity gradient $\boldsymbol{L}$. To make the corotational rate defined by the above spin $\boldsymbol{\Omega}^*$ to be a reasonable objective rate measure, certain necessary requirements must be imposed on the defining spin $\boldsymbol{\Omega}^*$. From the following requirements (see the second footnote in [52] and the three conditions listed at the start of Sect. 3 in [52]. In the following, the third one of the latter is dropped, and hence more general case is treated here):

(i) any superimposed constant rigid rotation has no effect on $\boldsymbol{\Omega}^*$,

(ii) the corotational rate of an Eulerian tensor defined by the spin $\boldsymbol{\Omega}^*$ depends linearly on the change of time scale,

(iii) the corotational rate of each time-differentiable objective Eulerian symmetric second-order tensor field defined by the spin $\boldsymbol{\Omega}^*$ is objective, and

(iv) the tensor function $\omega(\boldsymbol{F}, \boldsymbol{L})$ is continuously differentiable at $\boldsymbol{L} = \boldsymbol{O}$,

these authors [52]–[53] have derived a general form of spin $\boldsymbol{\Omega}^*$ as follows (see the formula preceding the condition (34) given in [52]. The former, i.e., the formula given below, is more general than that given by Eq. (26) in Theorem 2 in [51] due to the less restrictive requirements adopted here, as mentioned before):

$$\boldsymbol{\Omega}^* = \boldsymbol{W} + \sum_{\sigma,\tau=1}^{m} h(\lambda_\sigma, \lambda_\tau, I)\, \boldsymbol{P}_\sigma \boldsymbol{D} \boldsymbol{P}_\tau, \tag{17}$$

where $I = \boldsymbol{V} : \boldsymbol{I} = \mathrm{tr}\,\boldsymbol{V}$ is the trace, i.e., the first principal invariant, of the left stretch tensor $\boldsymbol{V}$, and the function $h(x, y, z)$ from $R^+ \times R^+ \times R^+$ to R, which defines the spin tensor $\boldsymbol{\Omega}^*$ and is hence called the *spin function,* is antisymmetric with respect to its first two variables, i.e.,

$$h(x, y, z) = -h(y, x, z). \tag{18}$$

The general formula (17)–(18) incorporates several commonly-known spin tensors as particular cases [52]–[53]. For example, according to (17), the following four particular forms of spin functions

$$h(x,y,z) = 0, \qquad \frac{1 - \dfrac{x}{y}}{1 + \dfrac{x}{y}}, \qquad \frac{1 + \left(\dfrac{x}{y}\right)^2}{1 - \left(\dfrac{x}{y}\right)^2}, \qquad \frac{2\dfrac{x}{y}}{1 - \left(\dfrac{x}{y}\right)^2},$$

define the vorticity tensor $\boldsymbol{W}$, the spin tensor $\dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}}$ and the twirl tensors $\boldsymbol{\Omega}^{\mathrm{E}}$ and $\boldsymbol{\Omega}^{\mathrm{L}}$ of the Eulerian and Lagrangean triads $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)$ and $(\boldsymbol{R}^{\mathrm{T}}\boldsymbol{n}_1, \boldsymbol{R}^{\mathrm{T}}\boldsymbol{n}_2, \boldsymbol{R}^{\mathrm{T}}\boldsymbol{n}_3)$, here $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)$ being three orthonormal eigenvectors of the left stretch tensor $\boldsymbol{V}$. A detailed account of these spin tensors can be found, e.g., in Hill [15], and Mehrabadi and Nemat-Nasser [29]. A spin tensor that will prove to be essential to our purpose will be introduced slightly later.

Now we are in a position to establish the relationship between the stretching $\boldsymbol{D}$ and the general objective corotational rate $\overset{\circ}{\boldsymbol{h}}{}^*$ of the logarithmic strain $\boldsymbol{h}$ (see (13)) defined by the spin $\boldsymbol{\Omega}^*$ of the form (17), which will be needed in the proof of the main result of this article. It seems that Fitzgerald [10], Gurtin and Spear [12], and Hoger [16] were the first to obtain some rigorous and complete results on this aspect for several well-known objective corotational rates.

First, we have the equality (see Eq. (40) in Xiao et al. [50])

$$\dot{\boldsymbol{V}} = \sum_{\sigma,\tau=1}^{m} \left(\frac{\lambda_\sigma{}^2 + \lambda_\tau{}^2}{\lambda_\sigma + \lambda_\tau} \boldsymbol{P}_\sigma \boldsymbol{D} \boldsymbol{P}_\tau - (\lambda_\sigma - \lambda_\tau)\, \boldsymbol{P}_\sigma \boldsymbol{W} \boldsymbol{P}_\tau \right), \tag{19}$$

and the gradient formula (see Eqs. (30)–(31 a, b) in [49]; note that $\dot{\boldsymbol{E}} = \partial \boldsymbol{E}/\partial \boldsymbol{U} : \dot{\boldsymbol{U}}$ therein, and that $\boldsymbol{E}$ and $\boldsymbol{U}$ are replaced by $\boldsymbol{h}$ and $\boldsymbol{V}$ here; see also [53])

$$\dot{\boldsymbol{h}} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{V}} : \dot{\boldsymbol{V}} = \sum_{\sigma,\tau=1}^{m} \frac{\ln \lambda_\sigma - \ln \lambda_\tau}{\lambda_\sigma - \lambda_\tau} \boldsymbol{P}_\sigma \dot{\boldsymbol{V}} \boldsymbol{P}_\tau, \tag{20}$$

with the limiting process $\lim_{\lambda_\sigma \to \lambda_\tau} (\ln \lambda_\sigma - \ln \lambda_\tau)/(\lambda_\sigma - \lambda_\tau) = \lambda_\sigma{}^{-1}$ understood for $\sigma = \tau$. Then, substituting the above two expressions and (13) and (17) into the second equality below and then using the simple manipulation formula (10) for the eigenprojections, we deduce (see

Xiao et al. [53])

$$
\begin{aligned}
\overset{\circ}{\boldsymbol{h}}{}^{*} &= \dot{\boldsymbol{h}} + \boldsymbol{h}\boldsymbol{\Omega}^{*} - \boldsymbol{\Omega}^{*}\boldsymbol{h} \\
&= \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{V}} : \dot{\boldsymbol{V}} + \boldsymbol{h}\boldsymbol{\Omega}^{*} - \boldsymbol{\Omega}^{*}\boldsymbol{h} \\
&= \sum_{\sigma,\tau=1}^{m} \frac{\ln \lambda_\sigma - \ln \lambda_\tau}{\lambda_\sigma - \lambda_\tau} \left(\frac{{\lambda_\sigma}^2 + {\lambda_\tau}^2}{\lambda_\sigma + \lambda_\tau} \boldsymbol{P}_\sigma \boldsymbol{D} \boldsymbol{P}_\tau - (\lambda_\sigma - \lambda_\tau)\, \boldsymbol{P}_\sigma \boldsymbol{W} \boldsymbol{P}_\tau \right) \\
&\quad + \sum_{\sigma,\tau=1}^{m} (\ln \lambda_\sigma - \ln \lambda_\tau) \left(\boldsymbol{P}_\sigma \boldsymbol{W} \boldsymbol{P}_\tau + h(\lambda_\sigma, \lambda_\tau, I)\, \boldsymbol{P}_\sigma \boldsymbol{D} \boldsymbol{P}_\tau \right) \\
&= \sum_{\sigma,\tau=1}^{m} (\ln \lambda_\sigma - \ln \lambda_\tau) \left(\frac{{\lambda_\sigma}^2 + {\lambda_\tau}^2}{\lambda_\sigma^2 - \lambda_\tau^2} + h(\lambda_\sigma, \lambda_\tau, I) \right) \boldsymbol{P}_\sigma \boldsymbol{D} \boldsymbol{P}_\tau .
\end{aligned} \tag{21}
$$

Hence, we have

$$
\overset{\circ}{\boldsymbol{h}}{}^{*} = \boldsymbol{\Gamma} : \boldsymbol{D}, \tag{22}
$$

where $\boldsymbol{\Gamma} = \bar{\boldsymbol{\Gamma}}(\boldsymbol{V})$ is a fourth-order tensor depending on the left stretch tensor $\boldsymbol{V}$.

The spin function

$$
h(x, y, z) = h^{\log}\left(\frac{x}{y}\right) = \frac{1 + \left(\frac{x}{y}\right)^2}{1 - \left(\frac{x}{y}\right)^2} + \frac{2}{\ln \left(\frac{x}{y}\right)^2} \tag{23}
$$

defines a particular spin tensor $\boldsymbol{\Omega}^{\log}$, called the *logarithmic spin* [50]–[53]. Accordingly, the corotational rate of an objective Eulerian tensor $\boldsymbol{S}$ defined by the spin $\boldsymbol{\Omega}^{\log}$, i.e.,

$$
\overset{\circ}{\boldsymbol{S}}{}^{\log} = \dot{\boldsymbol{S}} + \boldsymbol{S}\boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log}\boldsymbol{S}, \tag{24}
$$

is called the *logarithmic rate* of $\boldsymbol{S}$. Eqs. (23), (21) and (11) yield the defining tensor equation for the logarithmic spin $\boldsymbol{\Omega}^{\log}$ as follows:

$$
\overset{\circ}{\boldsymbol{h}}{}^{\log} = \dot{\boldsymbol{h}} + \boldsymbol{h}\boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log}\boldsymbol{h} = \boldsymbol{D}, \tag{25}
$$

which indicates that the logarithmic rate of the logarithmic strain $\boldsymbol{h}$ is identical with the stretching $\boldsymbol{D}$. Further properties and results for the logarithmic spin and the logarithmic rate can be found in Xiao et al. [50]–[53]. In Xiao et al. [50], the introduction of the logarithmic spin was motivated by examining the just-stated fact and inspired by the work by Gurtin and Spear [12], the latter authors being the first to indicate the fact that several well-known corotational rates of the Eulerian logarithmic strain can equal the stretching under certain conditions (see also Hoger [16]).

Finally, we provide the expressions for the two fourth-order tensors $\partial \boldsymbol{h}/\partial \boldsymbol{V}$ and $\boldsymbol{\Gamma}$, both depending on $\boldsymbol{V}$. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{T}_2$. We define a fourth-order tensor $\boldsymbol{A} \circ \boldsymbol{B}$ over $\mathcal{V}$ by

$$
(\boldsymbol{A} \circ \boldsymbol{B})_{ijkl} = A_{ik} B_{jl}. \tag{26}
$$

Hence, we have

$$
(\boldsymbol{A} \circ \boldsymbol{B}) : \boldsymbol{X} = \boldsymbol{A}\boldsymbol{X}\boldsymbol{B}^{\mathrm{T}}
$$

for each second-order tensor $\boldsymbol{X}$ over $\mathcal{V}$. Further, we introduce the following fourth-order tensor

$$
[\boldsymbol{A} \circ \boldsymbol{B}] = \boldsymbol{\Upsilon} : (\boldsymbol{A} \circ \boldsymbol{B}) : \boldsymbol{\Upsilon}, \tag{27}
$$

where $\boldsymbol{\Upsilon}$ is the identity transformation on Sym given by (3. 2). We have

$$[\boldsymbol{A} \circ \boldsymbol{B}] = [\boldsymbol{B} \circ \boldsymbol{A}]$$

for any $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{T}_2$. Evidently, as a map from $\mathcal{T}_2 \times \mathcal{T}_2$ to $\mathcal{T}_4$, $[\boldsymbol{A} \circ \boldsymbol{B}]$ is bilinear. Moreover, the fourth-order tensor $[\boldsymbol{A} \circ \boldsymbol{B}]$ is a linear transformation on Sym. In particular, we have

$$[(\boldsymbol{v}_1 \otimes \boldsymbol{v}_2) \circ (\boldsymbol{v}_3 \otimes \boldsymbol{v}_4)] = \frac{1}{4}(\boldsymbol{v}_1 \vee \boldsymbol{v}_3) \otimes (\boldsymbol{v}_2 \vee \boldsymbol{v}_4), \tag{28}$$

for any four vectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_4$. Then, from the above definition and (20)–(22) we obtain

$$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{V}} = \sum_{\sigma,\tau=1}^{m} \frac{\ln \lambda_\sigma - \ln \lambda_\tau}{\lambda_\sigma - \lambda_\tau} [\boldsymbol{P}_\sigma \circ \boldsymbol{P}_\tau], \tag{29}$$

$$\boldsymbol{\Gamma} = \sum_{\sigma,\tau=1}^{m} (\ln \lambda_\sigma - \ln \lambda_\tau) \left(\frac{\lambda_\sigma^2 + \lambda_\tau^2}{\lambda_\sigma^2 - \lambda_\tau^2} + h(\lambda_\sigma, \lambda_\tau, I) \right) [\boldsymbol{P}_\sigma \circ \boldsymbol{P}_\tau]. \tag{30}$$

Both $\partial \boldsymbol{h}/\partial \boldsymbol{V}$ and $\boldsymbol{\Gamma}$ are symmetric linear transformations on Sym. An important property of (29) and (30) is: either of them is a *spectral expression.* In fact, by using (26)–(27) we have

$$([\boldsymbol{P}_\alpha \circ \boldsymbol{P}_\beta] : [\boldsymbol{P}_\sigma \circ \boldsymbol{P}_\tau]) : \boldsymbol{X} = [\boldsymbol{P}_\alpha \circ \boldsymbol{P}_\beta] : ([\boldsymbol{P}_\sigma \circ \boldsymbol{P}_\tau] : \boldsymbol{X}),$$
$$= \frac{1}{2} \boldsymbol{\Upsilon} : (\boldsymbol{P}_\alpha \boldsymbol{P}_\sigma \boldsymbol{X} \boldsymbol{P}_\tau \boldsymbol{P}_\beta + \boldsymbol{P}_\alpha \boldsymbol{P}_\tau \boldsymbol{X} \boldsymbol{P}_\sigma \boldsymbol{P}_\beta)$$

for each $\boldsymbol{X} \in Sym$. Then, by Eq. (10) we infer

$$2[\boldsymbol{P}_\alpha \circ \boldsymbol{P}_\beta] : 2[\boldsymbol{P}_\sigma \circ \boldsymbol{P}_\tau] = \begin{cases} 4[\boldsymbol{P}_\tau \circ \boldsymbol{P}_\tau], & \alpha = \beta = \sigma = \tau, \\ 2[\boldsymbol{P}_\sigma \circ \boldsymbol{P}_\tau], & (\alpha, \beta) = (\sigma, \tau) \quad \text{or} \quad (\tau, \sigma), \sigma \neq \tau, \\ \boldsymbol{O}, \quad \text{otherwise.} \end{cases}$$

Moreover, by Eq. (11) we deduce

$$\sum_{\sigma,\tau=1}^{m} [\boldsymbol{P}_\sigma \circ \boldsymbol{P}_\tau] = \boldsymbol{\Upsilon} : (\boldsymbol{I} \circ \boldsymbol{I}) : \boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}. \tag{31}$$

Hence, we conclude that the $m(m+1)/2$ fourth-order tensors

$$\begin{cases} [\boldsymbol{P}_\sigma \circ \boldsymbol{P}_\sigma], & \sigma = 1, \cdots, m; \\ 2[\boldsymbol{P}_\sigma \circ \boldsymbol{P}_\tau], & \tau > \sigma = 1, \cdots, m-1, \end{cases}$$

are the eigenprojections of either of the symmetric linear transformations $\partial \boldsymbol{h}/\partial \boldsymbol{V}$ and $\boldsymbol{\Gamma}$ on Sym, and

$$\frac{\ln \lambda_\sigma - \ln \lambda_\tau}{\lambda_\sigma - \lambda_\tau}$$

and

$$(\ln \lambda_\sigma - \ln \lambda_\tau) \left(\frac{\lambda_\sigma^2 + \lambda_\tau^2}{\lambda_\sigma^2 - \lambda_\tau^2} + h(\lambda_\sigma, \lambda_\tau, I) \right)$$

are the corresponding eigenvalues of the foregoing two symmetric linear transformations on Sym, respectively.

Thanks to the spectral property indicated above, it becomes tractable to deal with the existence of and the expressions for the inverses of the two fourth-order tensors mentioned before, which will be needed. Now we can assert that the gradient $\partial \boldsymbol{h}/\partial \boldsymbol{V}$ is invertable, since every eigenvalue of it is nonvanishing. We have

$$\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{h}} = \left(\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{V}}\right)^{-1} = \sum_{\sigma,\tau=1}^{m} \frac{\lambda_\sigma - \lambda_\tau}{\ln \lambda_\sigma - \ln \lambda_\tau} [\boldsymbol{P}_\sigma \circ \boldsymbol{P}_\tau], \tag{32}$$

where for the gradient $\partial \boldsymbol{V}/\partial \boldsymbol{h}$ the left stretch tensor $\boldsymbol{V}$ is regarded as being determined by a tensor function of the logarithmic strain $\boldsymbol{h}$, i.e., $\boldsymbol{V} = e^{\boldsymbol{h}}$. Moreover, we assert that $\boldsymbol{\Gamma}$ is invertable if

$$h(x, y, z) \neq \frac{1 + \left(\frac{y}{x}\right)^2}{1 - \left(\frac{y}{x}\right)^2}. \tag{33}$$

When the above condition is fulfilled, we have

$$\begin{aligned}\boldsymbol{\Gamma}^{-1} &= \sum_{\sigma,\tau=1}^{m} (\ln \lambda_\sigma - \ln \lambda_\tau)^{-1} \left(\frac{\lambda_\sigma^2 + \lambda_\tau^2}{\lambda_\sigma^2 - \lambda_\tau^2} + h(\lambda_\sigma, \lambda_\tau, I)\right)^{-1} [\boldsymbol{P}_\sigma \circ \boldsymbol{P}_\tau] \\ &= \boldsymbol{\Upsilon} + \sum_{\sigma \neq \tau}^{m} \left((\ln \lambda_\sigma - \ln \lambda_\tau)^{-1} \left(\frac{\lambda_\sigma^2 + \lambda_\tau^2}{\lambda_\sigma^2 - \lambda_\tau^2} + h(\lambda_\sigma, \lambda_\tau, I)\right)^{-1} - 1 \right) [\boldsymbol{P}_\sigma \circ \boldsymbol{P}_\tau].\end{aligned} \tag{34}$$

In deriving the latter, (31) is used. Besides, we use the symbol $\sum_{\sigma \neq \tau}^{m}$ to mean the summation for all $\sigma, \tau = 1, \cdots, m$ and $\sigma \neq \tau$. When $m = 1$, this summation is assumed to vanish.

2.3 A chain rule for corotational derivatives of a second-order tensor-valued isotropic function

In this subsection, we supply a useful result for corotational derivatives of second-order tensor-valued isotropic functions of a symmetric second-order tensor.

A symmetric second-order tensor-valued fuction ϕ of a symmetric second-order tensor $\boldsymbol{A} \in Sym$, i.e.,

$$\boldsymbol{\tau} = \phi(\boldsymbol{A}) \in Sym,$$

is said to be isotropic if

$$\phi(\boldsymbol{Q} \star \boldsymbol{A}) = \boldsymbol{Q} \star (\phi(\boldsymbol{A})),$$

i.e.,

$$\boldsymbol{Q} \star \boldsymbol{\tau} = \phi(\boldsymbol{Q} \star \boldsymbol{A}) \tag{35}$$

for each $\boldsymbol{A} \in Sym$ and for each orthogonal tensor $\boldsymbol{Q} \in \mathcal{T}_2$. Let $\phi(\cdot)$ be differentiable and moreover let the tensor argument $\boldsymbol{A}$ be time-dependent. Then the following chain rule holds:

$$\dot{\boldsymbol{\tau}} = \frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{A}} : \dot{\boldsymbol{A}}. \tag{36}$$

We intend to prove that a corresponding chain rule holds for any corotational rate of $\boldsymbol{\tau}$, i.e., for any given time-dependent spin $\boldsymbol{\Omega}^* \in Skw$, we have

$$\overset{\circ}{\boldsymbol{\tau}}{}^* = \frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{A}} : \overset{\circ}{\boldsymbol{A}}{}^*, \tag{37}$$

where $\overset{\circ}{\boldsymbol{\tau}}{}^*$ and $\overset{\circ}{\boldsymbol{A}}{}^*$ are the corotational rates of the tensors $\boldsymbol{\tau}$ and $\boldsymbol{A}$ defined by the spin $\boldsymbol{\Omega}^*$, obtained by replacing $\boldsymbol{S}$ with $\boldsymbol{\tau}$ and $\boldsymbol{A}$, respectively, in the defining formula (16).

In fact, by differentiating the two sides of (35) and utilizing the chain rule (36) we deduce

$$\overline{(\boldsymbol{Q}\star\boldsymbol{\tau})}^{\cdot} = \left(\left.\frac{\partial\boldsymbol{\tau}}{\partial\boldsymbol{A}}\right|_{\boldsymbol{Q}\star\boldsymbol{A}}\right) : \overline{(\boldsymbol{Q}\star\boldsymbol{A})}^{\cdot}.$$

Then, letting the $\boldsymbol{Q}$ above be an orthogonal tensor defining the spin $\boldsymbol{\Omega}^*$ (see (15)) and applying (14), we infer

$$\boldsymbol{Q}\star\overset{\circ}{\boldsymbol{\tau}}{}^* = \left.\frac{\partial\boldsymbol{\tau}}{\partial\boldsymbol{A}}\right|_{\boldsymbol{Q}\star\boldsymbol{A}} : (\boldsymbol{Q}\star\overset{\circ}{\boldsymbol{A}}{}^*).$$

Hence, by acting $\boldsymbol{Q}^{\mathrm{T}}$ on the two sides of the above equality and then using (4)–(5) we obtain

$$\overset{\circ}{\boldsymbol{\tau}}{}^* = \left(\boldsymbol{Q}^{\mathrm{T}}\star\left(\left.\frac{\partial\boldsymbol{\tau}}{\partial\boldsymbol{A}}\right|_{\boldsymbol{Q}\star\boldsymbol{A}}\right)\right) : \overset{\circ}{\boldsymbol{A}}{}^*. \tag{38}$$

Here and below, we use the notation $\Theta|_\xi$ to designate the value of the function $\Theta = \bar{\Theta}(\omega)$ at $\omega = \xi$, i.e.,

$$\Theta|_\xi = \bar{\Theta}(\xi).$$

On the other hand, by the definition of gradient of a tensor function (see, e.g., Truesdell and Noll [47], and Marsden and Hughes [28]) and the isotropy of ϕ, as well as (4)–(5) we derive

$$\begin{aligned}
\left.\frac{\partial\boldsymbol{\tau}}{\partial\boldsymbol{A}}\right|_{\boldsymbol{Q}\star\boldsymbol{A}} : \boldsymbol{X} &= \lim_{\alpha\to 0}\frac{\phi(\boldsymbol{Q}\star\boldsymbol{A}+\alpha\boldsymbol{X}) - \phi(\boldsymbol{Q}\star\boldsymbol{A})}{\alpha}\\
&= \lim_{\alpha\to 0}\frac{\boldsymbol{Q}\star\phi(\boldsymbol{A}+\alpha\boldsymbol{Q}^{\mathrm{T}}\star\boldsymbol{X}) - \boldsymbol{Q}\star\phi(\boldsymbol{A})}{\alpha}\\
&= \boldsymbol{Q}\star\left(\lim_{\alpha\to 0}\frac{\phi(\boldsymbol{A}+\alpha\boldsymbol{Q}^{\mathrm{T}}\star\boldsymbol{X}) - \phi(\boldsymbol{A})}{\alpha}\right)\\
&= \boldsymbol{Q}\star\left(\frac{\partial\boldsymbol{\tau}}{\partial\boldsymbol{A}} : (\boldsymbol{Q}^{\mathrm{T}}\star\boldsymbol{X})\right)\\
&= \left(\boldsymbol{Q}\star\frac{\partial\boldsymbol{\tau}}{\partial\boldsymbol{A}}\right) : \boldsymbol{X}
\end{aligned}$$

for any $\boldsymbol{X}\in Sym$. The latter yields

$$\left.\frac{\partial\boldsymbol{\tau}}{\partial\boldsymbol{A}}\right|_{\boldsymbol{Q}\star\boldsymbol{A}} = \boldsymbol{Q}\star\frac{\partial\boldsymbol{\tau}}{\partial\boldsymbol{A}}. \tag{39}$$

Thus, from Eqs. (6), (38) and (39) we conclude that Eq. (37) holds.

3 Gradient of a fourth-order tensor-valued function

To prove the main result of this article, the main effort will be to evaluate the gradient of the fourth-order tensor-valued function $\boldsymbol{\Gamma}^{-1}$ of $\boldsymbol{V}$ (see (34)). In this section we shall attack this tough problem by virtue of the powerful eigenprojection method.

For our purpose, it is sufficient to deal with the case when $\boldsymbol{V}$ has three distinct eigenvalues. Let $\lambda_1, \cdots, \lambda_3$ be the three distinct eigenvalues of $\boldsymbol{V}$ and $\boldsymbol{n}_1, \cdots, \boldsymbol{n}_3$ three corresponding orthonormal eigenvectors of V. Hence, the three eigenprojections of V are given by

$$P_1 = n_1 \otimes n_1, \qquad P_2 = n_2 \otimes n_2, \qquad P_3 = n_3 \otimes n_3. \tag{40}$$

The fourth-order tensor-valued function considered here is of the form:

$$\tilde{\Gamma} = \Phi(V) = \sum_{i \neq j}^{3} \tilde{g}(\lambda_i, \lambda_j, I)[P_i \circ P_j], \tag{41}$$

where the function

$$g = \tilde{g}(x, y, z)$$

is assumed to be differentiable with respect to each variable of it and symmetric with respect to its first two variables, i.e.,

$$\tilde{g}(x, y, z) = \tilde{g}(y, x, z). \tag{42}$$

By the definition of gradient we have

$$\frac{\partial \tilde{\Gamma}}{\partial V} : X = \lim_{\alpha \to 0} \frac{\Phi(V + \alpha X) - \Phi(V)}{\alpha}, \qquad \forall X \in Sym. \tag{43}$$

To evaluate the gradient $\partial\tilde{\Gamma}/\partial V$, it suffices to calculate the above limits for

$$X = n_i \vee n_j, \qquad i, j = 1, 2, 3,$$

each of which is a fourth-order tensor. First, let $X = P_i = n_i \otimes n_i$ and (ijk) a permutation of (123). Then, the eigenvalues of the symmetric second-order tensor $V + \alpha P_i$ are given by $\lambda_i + \alpha$, λ_j and λ_k, and the corresponding eigenprojections by P_i, P_j and P_k. Hence we have

$$\begin{aligned} \frac{\partial \tilde{\Gamma}}{\partial V} : P_i &= \lim_{\alpha \to 0} \frac{\Phi(V + \alpha P_i) - \Phi(V)}{\alpha} \\ &= 2\alpha_{ij}[P_i \circ P_j] + 2\alpha_{ik}[P_i \circ P_k] + 2\alpha_{jk}[P_j \circ P_k], \end{aligned}$$

where

$$\begin{aligned} \alpha_{ij} &= \lim_{\alpha \to 0} \frac{\tilde{g}(\lambda_i + \alpha, \lambda_j, I + \alpha) - \tilde{g}(\lambda_i, \lambda_j, I)}{\alpha} = \left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\right)\bigg|_{(\lambda_i, \lambda_j, I)}, \\ \alpha_{ik} &= \lim_{\alpha \to 0} \frac{\tilde{g}(\lambda_i + \alpha, \lambda_k, I + \alpha) - \tilde{g}(\lambda_i, \lambda_k, I)}{\alpha} = \left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\right)\bigg|_{(\lambda_i, \lambda_k, I)}, \\ \alpha_{jk} &= \lim_{\alpha \to 0} \frac{\tilde{g}(\lambda_j, \lambda_k, I + \alpha) - \tilde{g}(\lambda_j, \lambda_k, I)}{\alpha} = \frac{\partial g}{\partial z}\bigg|_{(\lambda_j, \lambda_k, I)}. \end{aligned}$$

In deriving the above results, the symmetry condition (42) has been used. Then we obtain

$$\begin{aligned} \frac{1}{2}\frac{\partial \tilde{\Gamma}}{\partial V} : P_i = {}& \left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\right)\bigg|_{(\lambda_i, \lambda_j, I)} [P_i \circ P_j] + \left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\right)\bigg|_{(\lambda_i, \lambda_k, I)} [P_i \circ P_k] \\ & + \frac{\partial g}{\partial z}\bigg|_{(\lambda_j, \lambda_k, I)} [P_j \circ P_k]. \end{aligned} \tag{44}$$

In the above process, the following notation for a function $f = \tilde{f}(x, y, z)$ is used:

$$f|_{(a,b,c)} = \tilde{f}(a, b, c)$$

with any given numbers (a, b, c).

Next, let $X = P_{ij} = n_i \vee n_j$ and (ijk) again a permutation of (123). The eigenvalues of the tensor $\bar{V} = V + \alpha P_{ij}$ are given by

$$\begin{cases} \bar{\lambda}_i = \frac{1}{2}(\lambda_i + \lambda_j + p), \\ \bar{\lambda}_j = \frac{1}{2}(\lambda_i + \lambda_j - p), \\ \bar{\lambda}_k = \lambda_k, \quad p = \sqrt{(\lambda_i - \lambda_j)^2 + 4\alpha^2} \end{cases} \tag{45}$$

and the corresponding eigenprojections by

$$\begin{cases} \bar{P}_i = p^{-1}\big(\alpha P_{ij} + (\lambda_i - \bar{\lambda}_j)\, P_i + (\lambda_j - \bar{\lambda}_j)\, P_j\big), \\ \bar{P}_j = -p^{-1}\big(\alpha P_{ij} + (\lambda_i - \bar{\lambda}_i)\, P_i + (\lambda_j - \bar{\lambda}_i)\, P_j\big), \\ \bar{P}_k = P_k\,. \end{cases} \tag{46}$$

Hence, by using the above facts and the symmetry condition (42) we get

$$\Phi(V + \alpha P_{ij}) = 2\tilde{g}(\bar{\lambda}_i, \lambda_k, I)[\bar{P}_i \circ P_k] + 2\tilde{g}(\bar{\lambda}_j, \lambda_k, I)[\bar{P}_j \circ P_k] + 2\tilde{g}(\bar{\lambda}_i, \bar{\lambda}_j, I)\,[\bar{P}_i \circ \bar{P}_j],$$

where

$$[\bar{P}_i \circ P_k] = \frac{\alpha}{p}[P_k \circ P_{ij}] + \frac{\lambda_i - \bar{\lambda}_j}{p}[P_i \circ P_k] + \frac{\lambda_j - \bar{\lambda}_j}{p}[P_j \circ P_k],$$

$$[\bar{P}_j \circ P_k] = -\frac{\alpha}{p}[P_k \circ P_{ij}] - \frac{\lambda_i - \bar{\lambda}_i}{p}[P_i \circ P_k] - \frac{\lambda_j - \bar{\lambda}_i}{p}[P_j \circ P_k],$$

$$\begin{aligned} [\bar{P}_i \circ \bar{P}_j] =& \frac{\alpha^2}{p^2}([P_i \circ P_i] + [P_j \circ P_j] - [P_{ij} \circ P_{ij}]) \\ &- \frac{\alpha(\lambda_i - \lambda_j)}{p^2}([P_i \circ P_{ij}] - [P_j \circ P_{ij}]) \\ &+ \frac{(\lambda_i - \lambda_j)^2 + 2\alpha^2}{p^2}[P_i \circ P_j]. \end{aligned}$$

Utilizing the above results, we derive

$$\begin{aligned} \frac{1}{2}\frac{\partial \Gamma}{\partial V} : P_{ij} &= \frac{1}{2}\lim_{\alpha \to 0} \frac{\Phi(V + \alpha P_{ij}) - \Phi(V)}{\alpha} \\ &= \beta_{ijk}[P_k \circ P_{ij}] + \beta_{ik}[P_i \circ P_k] + \beta_{jk}[P_j \circ P_k] + \beta_{ij}[P_i \circ P_j] \\ &\quad + \gamma_{ij}([P_i \circ P_i] + [P_j \circ P_j] - [P_{ij} \circ P_{ij}]) + \eta_{ij}([P_i \circ P_{ij}] - [P_j \circ P_{ij}]), \end{aligned}$$

where

$$\beta_{ijk} = \lim_{\alpha \to 0} \frac{\tilde{g}(\bar{\lambda}_i, \lambda_k, I) - \tilde{g}(\bar{\lambda}_j, \lambda_k, I)}{p} = \frac{\tilde{g}(\lambda_i, \lambda_k, I) - \tilde{g}(\lambda_j, \lambda_k, I)}{\lambda_i - \lambda_j},$$

$$\beta_{ik} = \lim_{\alpha \to 0} \frac{1}{\alpha}\left(\frac{\lambda_i - \bar{\lambda}_j}{p}\, \tilde{g}(\bar{\lambda}_i, \lambda_k, I) - \frac{\lambda_i - \bar{\lambda}_i}{p}\, \tilde{g}(\bar{\lambda}_j, \lambda_k, I) - \tilde{g}(\lambda_i, \lambda_k, I)\right) = 0,$$

$$\beta_{jk} = \lim_{\alpha \to 0} \frac{1}{\alpha}\left(\frac{\lambda_j - \bar{\lambda}_j}{p}\, \tilde{g}(\bar{\lambda}_j, \lambda_k, I) - \frac{\lambda_j - \bar{\lambda}_i}{p}\, \tilde{g}(\bar{\lambda}_j, \lambda_k, I) - \tilde{g}(\lambda_j, \lambda_k, I)\right) = 0,$$

$$\gamma_{ij} = \lim_{\alpha \to 0} \frac{\alpha}{p^2}\tilde{g}(\bar{\lambda}_i, \bar{\lambda}_j, I) = 0,$$

$$\eta_{ij} = -\lim_{\alpha \to 0} \frac{\lambda_i - \lambda_j}{p^2}\, \tilde{g}(\bar{\lambda}_i, \bar{\lambda}_j, I) = -\frac{\tilde{g}(\lambda_i, \lambda_j, I)}{\lambda_i - \lambda_j}.$$

In evaluating some of the above limits, the following asymptotic formula is useful:

$$p = |\lambda_i - \lambda_j| + o(\alpha),$$

where

$$\lim_{\alpha \to 0} \frac{o(\alpha)}{\alpha} = 0.$$

Thus, from the above results, we arrive at

$$\frac{1}{2}\frac{\partial \tilde{\Gamma}}{\partial V} : \boldsymbol{P}_{ij} = \frac{\tilde{g}(\lambda_i, \lambda_k, I) - \tilde{g}(\lambda_j, \lambda_k, I)}{\lambda_i - \lambda_j}[\boldsymbol{P}_k \circ \boldsymbol{P}_{ij}] - \frac{\tilde{g}(\lambda_i, \lambda_j, I)}{\lambda_i - \lambda_j}([\boldsymbol{P}_i \circ \boldsymbol{P}_{ij}] - [\boldsymbol{P}_j \circ \boldsymbol{P}_{ij}]). \tag{47}$$

4 The main result and some remarks

Theorem A. Let $\boldsymbol{\Omega}^*$ be a spin tensor defined by the spin function $h(x, y, z)$ through (17) and $\overset{\circ}{\boldsymbol{\tau}}{}^*$ the objective corotational stress rate defined by this spin tensor (see (16) with $\boldsymbol{S} = \boldsymbol{\tau}$). Then the hypoelastic equation of grade zero

$$\overset{\circ}{\boldsymbol{\tau}}{}^* = \lambda(\mathrm{tr}\boldsymbol{D})\,\boldsymbol{I} + 2\mu\boldsymbol{D} = \boldsymbol{H}_0 : \boldsymbol{D} \tag{48}$$

is exactly integrable to define an elastic relation if and only if the spin $\boldsymbol{\Omega}^*$ is the logarithmic spin $\boldsymbol{\Omega}^{\log}$ defined by the spin function $h^{\log}(x/y)$ given by (23), i.e. if and only if the corotational stress rate $\overset{\circ}{\boldsymbol{\tau}}{}^*$ is the logarithmic stress rate $\overset{\circ}{\boldsymbol{\tau}}{}^{\log}$ (see (24) with $\boldsymbol{S} = \boldsymbol{\tau}$). When the stress rate $\overset{\circ}{\boldsymbol{\tau}}{}^*$ is the logarithmic stress rate and $\boldsymbol{\tau}$ is the Kirchhoff stress, the integrable-exactly hypoelastic equation (48) defines the isotropic hyperelastic relation

$$\boldsymbol{\tau} = \lambda\big(\ln(\det \boldsymbol{V})\big)\,\boldsymbol{I} + 2\mu \ln \boldsymbol{V}, \tag{49}$$

where det $\boldsymbol{V}$ is the determinant, i.e., the third principal invariant, of the left stretch tensor $\boldsymbol{V}$.

Proof. The sufficiency is obviously true, since it may easily be shown that the isotropic linear elasticity tensor $\boldsymbol{H}_0$ (see (2)) fulfils the integrability condition (3.8) and (3.19) presented in [51]. In what follows, we prove that the necessity is true.

Let the hypoelastic equation (48) be exactly integrable to deliver an isotropic elastic relation

$$\boldsymbol{\tau} = \phi(\boldsymbol{h}), \qquad \boldsymbol{h} = \ln \boldsymbol{V}. \tag{50}$$

In the above relation $\boldsymbol{h}$ may be replaced by any other strain measure $e = f(\boldsymbol{V})$ (see Eq. (12)), since there is a one-to-one isotropic relation between the logarithmic measure $\boldsymbol{h}$ and any other strain measure e. Here the use of the logarithmic strain measure $\boldsymbol{h}$ (see (13)) will simplify the subsequent account. Then by applying the chain rule (37) for corotational rates we infer

$$\overset{\circ}{\boldsymbol{\tau}}{}^* = \frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{h}} : \overset{\circ}{\boldsymbol{h}}{}^*, \tag{51}$$

where $\overset{\circ}{\boldsymbol{\tau}}{}^*$ and $\overset{\circ}{\boldsymbol{h}}{}^*$ are the objective corotational rates of the Eulerian stress $\boldsymbol{\tau}$ and the logarithmic strain $\boldsymbol{h}$ defined by a spin tensor $\boldsymbol{\Omega}^*$ of the form (17) through (16). Substituting (22) into (51), we obtain

$$\overset{\circ}{\boldsymbol{\tau}}{}^* = \left(\frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{h}} : \boldsymbol{\Gamma}\right) : \boldsymbol{D}. \tag{52}$$

Hence (48.2) and (52) yield:

$$\left(\frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{h}} : \boldsymbol{\Gamma}\right) : \boldsymbol{D} = \boldsymbol{H}_0 : \boldsymbol{D}$$

for each stretching $\boldsymbol{D} \in Sym$. The latter is equivalent to

$$\frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{h}} : \boldsymbol{\Gamma} = \boldsymbol{H}_0. \tag{53}$$

Since $\boldsymbol{H}_0$ is invertable, $\boldsymbol{\Gamma}$ must also be invertable, i.e., Eq. (33) must be fulfilled. The latter implies that the spin tensor $\boldsymbol{\Omega}^*$ can not be the twirl tensor of the principal frame $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)$ of the left stretch tensor $\boldsymbol{V}$.

Under the condition (33) we may recast (53) in the form

$$\frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{h}} = \boldsymbol{H}_0 : \boldsymbol{\Gamma}^{-1},$$

where the inverse $\boldsymbol{\Gamma}^{-1}$ is given by (31). By using (2) and (34) we further convert the latter to the form

$$\frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{h}} = \boldsymbol{H}_0 + 2\mu\tilde{\boldsymbol{\Gamma}} \quad (\equiv \hat{\boldsymbol{\Gamma}}), \tag{54}$$

where the fourth-order tensor $\tilde{\boldsymbol{\Gamma}}$ is given by

$$\tilde{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma}^{-1} - \boldsymbol{\Upsilon} = \sum_{\sigma \neq \tau}^{m} \tilde{g}(\lambda_\sigma, \lambda_\tau, I)[\boldsymbol{P}_\sigma \circ \boldsymbol{P}_\tau] \tag{55}$$

and

$$g = \tilde{g}(x, y, z) = (\ln x - \ln y)^{-1}\left(\frac{x^2 + y^2}{x^2 - y^2} + h(x, y, z)\right)^{-1} - 1 \tag{56}$$

with the spin function $h(x, y, z)$ defining the spin tensor $\boldsymbol{\Omega}^*$. In deriving (54), the equality

$$\boldsymbol{I} : \boldsymbol{\Gamma}^{-1} = \boldsymbol{I}$$

is used.

Now we restrict ourselves to the case when $\boldsymbol{V}$ has three distinct eigenvalues, i.e., $m = 3$. Let λ_1, λ_2, λ_3 be the three distinct eigenvalues of $\boldsymbol{V}$ and $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ three corresponding orthonormal eigenvectors. Then we identify (55)–(56) with (40)–(41) and (56). According to the application of Vainberg's general theorem for potential operators in elasticity (see, e.g., Marsden and Hughes [28], and Simo and Pister [41]), Eq. (54) holds if and only if the gradient of the fourth-order tensor, $\hat{\boldsymbol{\Gamma}}$, given by the right-hand side of Eq. (54) with respect to $\boldsymbol{h}$, keeps unchanged when its last two pairs of indices are exchanged, i.e.,

$$\left(\frac{\partial \hat{\boldsymbol{\Gamma}}}{\partial \boldsymbol{h}}\right)_{ijklrs} = \left(\frac{\partial \hat{\boldsymbol{\Gamma}}}{\partial \boldsymbol{h}}\right)_{ijrskl}, \tag{57}$$

i.e.,

$$\left(\frac{\partial \tilde{\boldsymbol{\Gamma}}}{\partial \boldsymbol{h}}\right)_{ijklrs} = \left(\frac{\partial \tilde{\boldsymbol{\Gamma}}}{\partial \boldsymbol{h}}\right)_{ijrskl}. \tag{58}$$

To apply the results derived in Sect. 3, we need to establish the relationship between the two gradients $\partial\tilde{\boldsymbol{\Gamma}}/\partial\boldsymbol{h}$ and $\partial\tilde{\boldsymbol{\Gamma}}/\partial\boldsymbol{V}$. The $\tilde{\boldsymbol{\Gamma}}$ in the gradient $\partial\tilde{\boldsymbol{\Gamma}}/\partial\boldsymbol{V}$ is the fourth-order tensor-valued function of the left stretch $\boldsymbol{V}$ as given by (55)–(56), i.e., $\tilde{\boldsymbol{\Gamma}} = \check{\boldsymbol{\Gamma}}(\boldsymbol{V})$, while the $\tilde{\boldsymbol{\Gamma}}$ in the gradient

$\partial\tilde{\Gamma}/\partial \boldsymbol{h}$ is the fourth-order tensor-valued function of the logarithmic strain $\boldsymbol{h}$ through the composition

$$\tilde{\Gamma} = \breve{\Gamma}(\boldsymbol{V}), \qquad \boldsymbol{V} = e^{\boldsymbol{h}}.$$

With this fact in mind and by using the chain rule and applying (32) and (40) and (27)–(28), we infer that the foregoing relationship is given by

$$\begin{aligned}\frac{\partial\tilde{\Gamma}}{\partial \boldsymbol{h}} &= \frac{\partial\tilde{\Gamma}}{\partial \boldsymbol{V}} : \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{h}} \\ &= \sum_{i,j=1}^{3} \frac{\lambda_i - \lambda_j}{\ln \lambda_i - \ln \lambda_j} \frac{\partial\tilde{\Gamma}}{\partial \boldsymbol{V}} : [\boldsymbol{P}_i \circ \boldsymbol{P}_j] \\ &= \sum_{i=1}^{3} \lambda_i \left(\frac{\partial\tilde{\Gamma}}{\partial \boldsymbol{V}} : \boldsymbol{P}_i\right) \otimes \boldsymbol{P}_i + \frac{1}{4}\sum_{i \neq j}^{3} \frac{\lambda_i - \lambda_j}{\ln \lambda_i - \ln \lambda_j} \left(\frac{\partial\tilde{\Gamma}}{\partial \boldsymbol{V}} : \boldsymbol{P}_{ij}\right) \otimes \boldsymbol{P}_{ij}.\end{aligned} \tag{59}$$

In deriving the last equality above (note the factor 1/4), the notations

$$\boldsymbol{P}_i = \boldsymbol{n}_i \otimes \boldsymbol{n}_i, \qquad \boldsymbol{P}_{ij} = \boldsymbol{n}_i \vee \boldsymbol{n}_j$$

and the identity (28) with $(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4) = (\boldsymbol{n}_i, \boldsymbol{n}_i, \boldsymbol{n}_j, \boldsymbol{n}_j)$ are used.

Finally, choosing the Eulerian triad $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)$ as a Cartesian basis and applying the formulae (44) and (47), from (58)–(59) we derive

$$\left(\frac{\partial\tilde{\Gamma}}{\partial \boldsymbol{h}}\right)_{121211} = \left(\frac{\partial\tilde{\Gamma}}{\partial \boldsymbol{h}}\right)_{121112} \Longrightarrow x\left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\right) = -\frac{g}{2(\ln x - \ln y)} \tag{60}$$

when $(x, y, z) = (\lambda_1, \lambda_2, I)$; and

$$\left(\frac{\partial\tilde{\Gamma}}{\partial \boldsymbol{h}}\right)_{121233} = \left(\frac{\partial\tilde{\Gamma}}{\partial \boldsymbol{h}}\right)_{123312} \Longrightarrow \frac{\partial g}{\partial z} = 0 \tag{61}$$

when $(x, y, z) = (\lambda_1, \lambda_2, I)$; and

$$\begin{cases} \left(\dfrac{\partial\tilde{\Gamma}}{\partial \boldsymbol{h}}\right)_{132312} = \left(\dfrac{\partial\tilde{\Gamma}}{\partial \boldsymbol{h}}\right)_{131223} \Longrightarrow \\ \dfrac{\tilde{g}(\lambda_1, \lambda_3, I) - \tilde{g}(\lambda_2, \lambda_3, I)}{\ln \lambda_1 - \ln \lambda_2} = \dfrac{\tilde{g}(\lambda_1, \lambda_2, I) - \tilde{g}(\lambda_1, \lambda_3, I)}{\ln \lambda_2 - \ln \lambda_3}, \end{cases} \tag{62}$$

as well as other similar results. The above conditions must hold for any $\lambda_1, \lambda_2, \lambda_3 > 0$. Hence, we have

$$\begin{cases} x\left(\dfrac{\partial g}{\partial x} + \dfrac{\partial g}{\partial z}\right) = -\dfrac{1}{2}\dfrac{g}{\ln x - \ln y}, \\ \dfrac{\partial g}{\partial z} = 0, \end{cases} \tag{63}$$

for any $x, y, z > 0$. The second equation above implies that the function $g = \tilde{g}(x, y, z)$ is independent of z, i.e.,

$$g = \hat{g}(x, y). \tag{64}$$

Assume that

$$g \neq 0. \tag{65}$$

Then, from (63.1) and (64), i.e.,

$$\frac{1}{g}\frac{\partial g}{\partial x} = -\frac{1}{2(\ln x - \ln y)}\frac{\partial(\ln x - \ln y)}{\partial x}, \tag{66}$$

we obtain

$$\ln|g| = -\frac{1}{2}\ln|\ln x - \ln y| + \psi(y). \tag{67}$$

Since $g = \hat{g}(x, y)$ is a symmetric function, i.e.,

$$\hat{g}(x,y) = \hat{g}(y,x),$$

from (67) we infer

$$\psi(y) = c = \text{const}.$$

Hence, we arrive at

$$g = \frac{\delta e^c}{\sqrt{|\ln x - \ln y|}}, \qquad \delta \in \{1, -1\}. \tag{68}$$

Then, using (64) and reformulating (62,2) as

$$\hat{g}(\lambda_1,\lambda_2)(\ln\lambda_1 - \ln\lambda_2) + \hat{g}(\lambda_2,\lambda_3)(\ln\lambda_2 - \ln\lambda_3) = \hat{g}(\lambda_1,\lambda_3)(\ln\lambda_1 - \ln\lambda_3),$$

and then substituting (68) into the latter and using $\delta e^c \neq 0$, we deduce

$$H(\lambda_1 - \lambda_2)\sqrt{\left|\ln\frac{\lambda_1}{\lambda_2}\right|} + H(\lambda_2 - \lambda_3)\sqrt{\left|\ln\frac{\lambda_2}{\lambda_3}\right|} = H(\lambda_1 - \lambda_3)\sqrt{\left|\ln\frac{\lambda_1}{\lambda_3}\right|}. \tag{69}$$

Here $H(x)$ is used to represent the function

$$H(x) = \begin{cases} +1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

Without loss of generality, we set

$$\lambda_1 > \lambda_2 > \lambda_3 (> 0).$$

Then (69) becomes

$$\sqrt{\ln\frac{\lambda_1}{\lambda_2}} + \sqrt{\ln\frac{\lambda_2}{\lambda_3}} = \sqrt{\ln\frac{\lambda_1}{\lambda_3}}. \tag{70}$$

Let

$$p = \ln\frac{\lambda_1}{\lambda_2}, \qquad q = \ln\frac{\lambda_2}{\lambda_3}, \qquad r = \ln\frac{\lambda_1}{\lambda_3}.$$

Then

$$\frac{p}{r} + \frac{q}{r} = 1, \qquad p > 0, \quad q > 0, \quad r > 0,$$

and hence the following inequality holds (note that $0 < w < 1$):

$$\begin{cases} \left(\frac{p}{r}\right)^\alpha + \left(\frac{q}{r}\right)^\alpha = w^\alpha + (1-w)^\alpha > 1, & -\infty < \alpha < 1, \quad \text{esp. } \alpha = \frac{1}{2}, \\ \left(\frac{p}{r}\right)^\alpha + \left(\frac{q}{r}\right)^\alpha = w^\alpha + (1-w)^\alpha < 1, & 1 < \alpha < +\infty, \end{cases} \tag{71}$$

which contradicts the equality (70).

Thus, the assumption (65) is not true and hence we conclude that $g = 0$. This and (56) produce (23), i.e., $\mathbf{\Gamma}^{-1} = \mathbf{\Upsilon}$, i.e., $\mathbf{\Gamma} = \mathbf{\Upsilon}$. The latter and (22) results in the defining equation (25) for the logarithmic spin $\mathbf{\Omega}^{\log}$.

This completes the proof for Theorem A. □

Remarks. In addition to the objective corotational rates, the other commonly-known objective rates include Oldroyd rate [34], Cotter-Rivlin rate [4] and Truesdell rate [44]–[45]. Simo and Pister [41] showed that the hypoelastic equation (48) with Truesdell stress rate is integrable only when

$$\lambda + \mu = 0.$$

Moreover, according to Sansour and Bednarczyk [40], the hypoelastic equation (48) with Oldroyd stress rate is integrable only when

$$\lambda = 0.$$

It seems that neither of the above two conditions with the Lamé elastic constants $\lambda > 0$ and $\mu > 0$ is reasonable. Thus, neither of the two hypoelastic equations (48) with Truesdell stress rate and Oldroyd stress rate is exactly integrable, if the material constants λ and μ therein are chosen as the Lamé elastic constants, as has been done earlier.

Finally, the hypoelastic equation (48) with Cotter-Rivlin stress rate is of the form

$$\overset{\circ}{\boldsymbol{\tau}}{}^{*} = (\dot{\boldsymbol{\tau}} + \boldsymbol{\tau}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{\tau}) + \boldsymbol{\tau}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{\tau} = \lambda(\mathrm{tr}\boldsymbol{D})\boldsymbol{I} + 2\mu\boldsymbol{D}. \tag{72}$$

The latter can be regarded as a particular form of the equation (48) in Sansour and Bednarczyk [40] with

$$\alpha_6 = -2, \qquad \alpha_0 = \lambda, \qquad \alpha_4 = 2\mu, \qquad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_5 = 0,$$

i.e.,

$$L_v\boldsymbol{\tau} = \lambda(\mathrm{tr}\boldsymbol{D})\boldsymbol{I} + 2\mu\boldsymbol{D} - 2(\boldsymbol{\tau}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{\tau}), \tag{73}$$

where the Lie derivative (see, e.g., Marsden and Hughes [28] for a detailed account of this aspect)

$$L_v\boldsymbol{\tau} = (\dot{\boldsymbol{\tau}} + \boldsymbol{\tau}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{\tau}) - \boldsymbol{\tau}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{\tau}$$

is just the Oldroyd stress rate. From one of the integrability conditions derived by Sansour and Bednarczyk [40], i.e., Eq. (56) therein, we know that the rate equation (73), i.e., (72), is not integrable.

On the other hand, in finite inelastic deformations, other kinds of objective corotational rates can be defined by means of *plastic spins*. This is done by replacing the spin tensor $\mathbf{\Omega}^*$ in (16) with $\boldsymbol{W} - \boldsymbol{W}^p$, where $\boldsymbol{W}^p$ is a plastic spin (see, e.g. Dafalias [6]–[7], Loret [27] and Zbib and Aifantis [56]–[57]). It is evident that, for any process of elastic deformation, each such objective corotational rate is reduced to the Zaremba-Jaumann-Noll rate, for which the corresponding rate equation (48) for elastic response has been shown to be non-integrable.

Combining the above facts and Theorem A, we may further conclude that, to achieve an integrable-exactly hypoelastic equation of grade zero, the logarithmic stress rate is the only choice among all possible objective corotational stress rates and other well-known objective stress rates. Thus, the Eulerian rate type finite deformation elastoplasticity models based upon the logarithmic stress rates, suggested by these authors in a succeeding article [3], are not only the first, but also unique, self-consistent ones of their kinds, in the sense of compatibility of the applied Eulerian rate type formulation of elastic behaviour with the notion of elasticity.

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Added in proof: As has been indicated, the unique integrable-exactly zeroth-grade hypoelastic equation, i.e., Eq. (48) with the logarithmic stress rate $\overset{\circ}{\boldsymbol{\tau}}{}^{*} \equiv \overset{\circ}{\boldsymbol{\tau}}{}^{\log}$, yields the isotropic hyperelastic relation (49), which is a natural generalization of small deformation Hooke's law with the replacement of the infinitesimal strain by the Hencky's logarithmic strain $\ln \boldsymbol{V}$. In a convincing study by L. Anand [58], [59], it is shown that the hyperelastic relation (49) is in good agreement with experiments for a wide class of materials for moderately large deformations.

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