Math. Z. 192, 283-295 (1986)

Mathematische Zeitschrift

Existence of Holomorphic Functions on Almost Complex Manifolds

O. Muškarov

Institute of Mathematics, Bulgarian Academy of Sciences, 1090 Sofia, Bulgaria

Introduction

Given an almost complex manifold (M, J) (J being the almost complex structure) one can define the operator $\bar{\partial}$ which maps the space of (p, q)-forms on (M, J) into the space of $(p, q+1)$-forms [3]. It is well-known that the almost complex structure J is integrable if and only if $\bar{\partial}^2=0$. So in the general case $\bar{\partial}^2 \neq 0$ and this is one of the main reasons which make difficult even the local study of the $\bar{\partial}$-equation on a non-integrable almost complex manifold. The first step in this direction is to consider the problem of local existence of functions satisfying the equation $\bar{\partial} f=0$. We call these functions *holomorphic* on (M, J) [8, 15]. Not much is known on the $\bar{\partial}$-equation even in this simplest case. The celebrated Newlander-Nirenberg theorem [14] states that each point of M has a neighbourhood in which there exist $1/2 \dim M$ (functionally) independent holomorphic functions if and only if the Nijenhuis tensor of J vanishes. On the other hand Hermann [8] and Calabi [2] have constructed examples of almost complex manifolds which even locally have no holomorphic functions except constants. Recently, Dimiev [4] has given a sufficient condition for the local existence of independent holomorphic functions on a real-analytic (M, J).

The main point of the present paper is to give a general criterion for the local existence of k $(0 \leqq k \leqq 1/2 \dim M)$ independent holomorphic functions on an arbitrary (smooth) almost complex manifold, which will englobe all these results. More precisely we show in Sect. 2 that the local existence of independent holomorphic functions on (M, J) is equivalent to the local existence of a subbundle of the tangent bundle of M, having some special properties (Theorem 2.1). The key result for a crucial step in our proof is a theorem of Treves [17], which in turn depends on the Newlander-Nirenberg theorem.

Given an almost complex manifold (M, J) and a point x in M, let $m(x)$ be the maximal number of holomorphic functions which are independent at x. In general $m(x)$ is not constant, hence we introduce the following:

Definition. The almost complex manifold (M, J) is said to be of (*constant*) *type* m if $m(x)=m$ for all $x \in M$.

This is a more restrictive condition than Spencer's in [15] who defines the type of (M, J) to be the minimum value of $m(x)$ when x ranges over M. In our terminology (M, J) is an integrable almost complex manifold if and only if it is of type $1/2 \dim M$. A main purpose of this paper is to investigate various classes of almost complex manifolds which have been previously studied from a differential-geometric point of view, and show they are of constant type.

Accordingly, in Sect. 3 we give a simple criterion for an almost complex manifold to be of type 0 (Theorem 3.1). Furthermore, we show how the results of Hermann and Calabi mentioned above can be obtained by applying this criterion (Examples 3.2 and 3.3).

Section four is devoted to the most interesting class of almost Hermitian manifolds, namely that of nearly Kähler manifolds [6]. In this case the Nijenhuis tensor satisfies some extra conditions (Lemma 4.1) which allows us to obtain an explicit formula for the type of a nearly Kähler manifold (Theorem 4.2).

In Sect. 5 we reduce the question of determining the type of a reductive homogeneous almost complex space to a purely algebraic problem (Theorem 5.1). As an application we discuss Thurston's example [16] of a compact 4-dimensional almost Kähler manifold which does not admit a Kähler structure and show it is of type 1 (Example 5.3).

Finally in Sect. 6 we examine the almost complex manifolds $(T(M), J)$, where $T(M)$ is the tangent bundle of a smooth manifold M and J is the almost complex structure on $T(M)$ defined by a flat connection D of M [5]. We determine the type of $(T(M), J)$ by means of the torsion tensor of D (Theorem 6.1).

1. Preliminaries

In this paper, manifolds and tensor fields are assumed to be of class $\mathscr{C}^\infty$ unless otherwise specified.

1.1. Let M be an almost complex manifold with almost complex structure J, i.e. J is an automorphism of the tangent bundle $T(M)$ of M, such that $J^2 = -I$, where I is the identity automorphism of $T(M)$. Denote by $\mathbb{C}T(M)$ (resp. $\mathbb{C}T^*(M)$) the complexification of the tangent (resp. cotangent) bundle of M. Extending J by complex-linearity to an automorphism of $\mathbb{C}T(M)$ we can write

$$\mathbb{C}T(M) = T(M)^{1,0} \oplus T(M)^{0,1},$$

where $T(M)^{1,0}$ and $T(M)^{0,1}$ are the complex subbundles of $\mathbb{C}T(M)$ associated respectively with the $(+i)$ and $(-i)$-eigenspaces of J $(i^2 = -1)$. Also

$$\mathbb{C}T^*(M) = T^*(M)^{1,0} \oplus T^*(M)^{0,1},$$

where $T^*(M)^{1,0}$ and $T^*(M)^{0,1}$ are the dual bundles of $T(M)^{1,0}$ and $T(M)^{0,1}$ respectively. For any complex vector field Z on M we denote by $P_{1,0}(Z)$ (resp. $P_{0,1}(Z)$) the $(1,0)$ (resp. $(0,1)$)-component of Z, so that

$$P_{1,0}(Z) = 1/2(Z - iJZ), \qquad P_{0,1}(Z) = 1/2(Z + iJZ). \tag{1.1}$$

Note that Z is a $(1,0)$ (resp. $(0,1)$)-vector field if and only if $Z=X-iJX$ (resp. $Z=X+iJX$), where X is a real vector field on M (cf. [11]).

1.2. Let f be a complex-valued $\mathscr{C}^\infty$ function on M. We denote by $\bar{\partial} f$ the $(0,1)$-component of the 1-form df, where d is the exterior derivative. Recall that f is said to be a holomorphic function on (M,J) provided $\bar{\partial} f=0$, or equivalently if df is a $(1,0)$-form on (M,J) (cf. [8, 15]). Let $f_1, f_2, \ldots, f_k$ be holomorphic functions defined on a neighbourhood of a point $x\in M$. Then these functions are said to be (functionally) independent at x if the $(1,0)$-forms $df_1, df_2, \ldots, df_k$ are $\mathbb{C}$-linearly independent at x [15]. It is well-known (cf. loc. cit.) that the almost complex structure J is integrable if and only if at any point of M there exist $1/2 \dim M$ independent holomorphic functions.

1.3. Denote by $\mathfrak{X}(M)$ the Lie algebra of all real vector fields on M. The Nijenhuis tensor of J is the tensor field N defined by

$$N(X,Y)=[X,Y]+J[JX,Y]+J[X,JY]-[JX,JY],$$

where $X, Y\in\mathfrak{X}(M)$ and $[.,.]$ is the Lie bracket. It is easily verified that N satisfies the following identities:

$$N(X,Y)=-N(Y,X), \qquad N(X,JY)=-JN(X,Y) \tag{1.2}$$

for all $X, Y\in\mathfrak{X}(M)$. The Newlander-Nirenberg theorem [14] states that the almost complex structure J is integrable if and only if $N=0$.

1.4. Given a complex subbundle $\mathscr{V}$ of $\mathbb{C}T(M)$ we denote by $\Gamma(\mathscr{V})$ the space of all smooth sections of $\mathscr{V}$ and by $\mathscr{V}(x)$ the fibre of $\mathscr{V}$ over $x\in M$. According to classical terminology, $\mathscr{V}$ is called *involutive* if $[\Gamma(\mathscr{V}),\Gamma(\mathscr{V})]\subset\Gamma(\mathscr{V})$, i.e. given any two sections X, Y of $\mathscr{V}$ the Lie bracket $[X,Y]$ is also a section of $\mathscr{V}$. Now let $\mathscr{W}$ be a complex subbundle of $\mathbb{C}T^*(M)$. According to Treves [17] $\mathscr{W}$ is said to be closed if for any local chart $(U,\varphi_1,\ldots,\varphi_k)$ of $\mathscr{W}$ and for any smooth section ψ of $\mathscr{W}$ over U, there exist k smooth 1-forms in U, $\psi_1,\ldots,\psi_k$, such that $d\psi=\varphi_1\wedge\psi_1+\ldots+\varphi_k\wedge\psi_k$. Let $\mathscr{W}^\perp$ be the orthogonal vector bundle of $\mathscr{W}$ with respect to the duality between tangent and cotangent vectors. Then $\mathscr{W}^\perp$ is the complex subbundle of $\mathbb{C}T(M)$ with fibres:

$$\mathscr{W}^\perp(x)=\{v\in\mathbb{C}T_x(M);\ \psi(v)=0 \text{ for all } \psi\in\mathscr{W}(x)\}.$$

It is known (cf. [17, Chap. I, Proposition 1.1]) that $\mathscr{W}$ is closed if and only if its orthogonal bundle $\mathscr{W}^\perp$ is involutive.

2. Existence of Holomorphic Functions and IJ-Bundles

Let (M,J) be an almost complex manifold with Nijenhuis tensor N. For any point $x\in M$ we define

$$\mathscr{L}\mathscr{N}(x)=\text{Span}\{N_x(a,b);\ a,b\in T_x(M)\}$$

and set $\operatorname{rank}_x N = \dim_{\mathbb{R}} \mathscr{L}\mathscr{N}(x)$. Notice that $\operatorname{rank}_x N$ is an even integer since $\mathscr{L}\mathscr{N}(x)$ is a J-invariant subspace of $T_x(M)$ (see (1.2)). If $\operatorname{rank} N$: $x \mapsto \operatorname{rank}_x N$ is a constant function we denote by $\mathscr{L}\mathscr{N}$ the subbundle of $T(M)$ with fibres $\mathscr{L}\mathscr{N}(x)$, and call it the *Nijenhuis bundle* of (M, J).

Definition. Let $\mathscr{V}$ be a subbundle of $T(M)$. We say that $\mathscr{V}$ is an IJ-bundle if the following properties are satisfied:

(i) $\mathscr{V}$ is J-invariant, i.e. each fibre $\mathscr{V}(x)$ of $\mathscr{V}$ is a J-invariant subspace of $T_x(M)$.

(ii) $\mathscr{L}\mathscr{N}(x) \subset \mathscr{V}(x)$ for all $x \in M$.

(iii) $\mathscr{V}$ is involutive.

(iv) $[X, Y] + J[X, JY] \in \Gamma(\mathscr{V})$ for all $X \in \Gamma(\mathscr{V})$ and all $Y \in \mathfrak{X}(M)$.

We now state our main result.

Theorem 2.1. *Let (M, J) be an almost complex manifold and let $x \in M$ be an arbitrary point. Then in a neighbourhood of x there exist k independent holomorphic functions if and only if on some neighbourhood of x there exists an IJ-bundle of fibre dimension* $\dim M - 2k$.

We devide the proof of the Theorem into two lemmas.

Lemma 2.2. *The local existence of k independent holomorphic functions on (M, J) is equivalent to the local existence of a closed subbundle of $T^*(M)^{1,0}$ with complex fibre dimension k.*

Proof of Lemma 2.2. Let $f_1, f_2, \ldots, f_k$ be independent holomorphic functions on a neighbourhood U of x. Then it is easy to verify that the subbundle of $T^*(U)^{1,0}$ generated by $df_1, df_2, \ldots, df_k$ is closed.

Conversely, let $\mathscr{V}$ be a closed subbundle of $T^*(U)^{1,0}$ with complex fibre dimension k. Then the complex conjugate bundle $\bar{\mathscr{V}}$ of $\mathscr{V}$ is a subbundle of $T^*(U)^{0,1}$. In particular $\mathscr{V} \cap \bar{\mathscr{V}} = 0$ and by a result of Treves ([17, Chap. I, Theorem 1.1]) it follows (contracting U if necessary) that $\mathscr{V}$ is generated by 1-forms $df_1, df_2, \ldots, df_k$, where $f_1, f_2, \ldots, f_k$ are smooth functions on U. Since $df_1, df_2, \ldots, df_k$ are $(1,0)$-forms it follows that $f_1, f_2, \ldots, f_k$ are independent holomorphic functions on U.

The lemma above reduces the question of local existence of independent holomorphic functions on (M, J) to the description of the closed subbundles of $T^*(M)^{1,0}$. To do this we need a further piece of notation. Let $\mathscr{V}$ be a complex subbundle of $T^*(M)^{1,0}$. Since the $(1,0)$-forms are annihilated by the $(0,1)$-vectors we can write

$$\mathscr{V}^{\perp} = \mathscr{V}^{1,0} \oplus T(M)^{0,1},$$

where $\mathscr{V}^{1,0}$ is a complex subbundle of $T(M)^{1,0}$. Denote by $\mathscr{V}_{\mathscr{R}}$ the subbundle of $T(M)$ with fibres:

$$\mathscr{V}_{\mathscr{R}}(x) = \{v \in T_x(M);\ v - iJv \in \mathscr{V}^{1,0}(x)\}.$$

Note that $\mathscr{V}_{\mathscr{R}}$ is a J-invariant subbundle of $T(M)$ since $\mathscr{V}^{1,0}$ is a complex subbundle of $T(M)^{1,0}$.

Lemma 2.3. *The correspondence* $\mathscr{V} \mapsto \mathscr{V}_{\mathscr{R}}$ *is a bijection between the set of all closed subbundles of* $T^*(M)^{1,0}$ *with complex fibre dimension k and the set of all* IJ*-bundles on* (M, J) *with real fibre dimension* $\dim M - 2k$.

Proof of Lemma 2.3. Let $\mathscr{V}$ be a closed subbundle of $T^*(M)^{1,0}$. Then the orthogonal bundle $\mathscr{V}^{\perp}$ of $\mathscr{V}$ is involutive and therefore we have the following inclusion relations:

$$\begin{aligned} P_{1,0}[\Gamma(\mathscr{V}^{1,0}), \Gamma(\mathscr{V}^{1,0})] &\subset \Gamma(\mathscr{V}^{1,0}) \\ P_{1,0}[\Gamma(\mathscr{V}^{1,0}), \Gamma(T(M)^{0,1})] &\subset \Gamma(\mathscr{V}^{1,0}) \\ P_{1,0}[\Gamma(T(M)^{0,1}), \Gamma(T(M)^{0,1})] &\subset \Gamma(\mathscr{V}^{1,0}). \end{aligned} \tag{2.1}$$

Any section Z of $T(M)^{0,1}$ can be written as $Z = X + iJX$, where $X \in \mathfrak{X}(M)$ (cf. Sect. 1). Similarly, any section Z of $\mathscr{V}^{1,0}$ can be written in the form $Z = X - iJX$, where $X \in \Gamma(\mathscr{V}_{\mathscr{R}})$. Then by (1.1) we have

$$\begin{aligned} 2P_{1,0}[X+iJX, Y+iJY] &= [X+iJX, Y+iJY] - iJ[X+iJX, Y+iJY] \\ &= N(X, Y) - iJN(X, Y), \end{aligned}$$

where N is the Nijenhuis tensor of J. Hence the last inclusion of (2.1) is satisfied if and only if $N(X, Y) \in \Gamma(\mathscr{V}_{\mathscr{R}})$ for all $X, Y \in \mathfrak{X}(M)$. This is equivalent to the condition $\mathscr{L}\mathscr{N}(x) \subset \mathscr{V}_{\mathscr{R}}(x)$ for all $x \in M$. Using the same arguments for the remaining inclusions of (2.1) we see that $\mathscr{V}^{\perp}$ is involutive if and only if the bundle $\mathscr{V}_{\mathscr{R}}$ has the following properties:

(a) $\mathscr{V}_{\mathscr{R}}$ is J-invariant,

(b) $\mathscr{L}\mathscr{N}(x) \subset \mathscr{V}_{\mathscr{R}}(x)$ for all $x \in M$,

(c) $[X, Y] - [JX, JY] \in \Gamma(\mathscr{V}_{\mathscr{R}})$ for all $X, Y \in \Gamma(\mathscr{V}_{\mathscr{R}})$,

(d) $[X, Y] + J[X, JY] \in \Gamma(\mathscr{V}_{\mathscr{R}})$ for all $X \in \Gamma(\mathscr{V}_{\mathscr{R}})$ and all $Y \in \mathfrak{X}(M)$.

To prove that $\mathscr{V}_{\mathscr{R}}$ is an IJ-bundle it remains to show that it is involutive. Let $X, Y \in \Gamma(\mathscr{V}_{\mathscr{R}})$. Then by (d) we have $[JX, Y] + J[JX, JY] \in \Gamma(\mathscr{V}_{\mathscr{R}})$ and $[Y, JX] - J[Y, X] \in \Gamma(\mathscr{V}_{\mathscr{R}})$ which imply $[X, Y] + [JX, JY] \in \Gamma(\mathscr{V}_{\mathscr{R}})$ since $\mathscr{V}_{\mathscr{R}}$ is J-invariant. Hence by (c) we get $[X, Y] \in \Gamma(\mathscr{V}_{\mathscr{R}})$ as required.

Conversely, if we are given an IJ-bundle $\mathscr{W}$ (with real fibre dimension $\dim M - 2k$), we can define an unique complex subbundle $\mathscr{V}$ of $T^*(M)^{1,0}$ (with complex fibre dimension k), such that $\mathscr{V}_{\mathscr{R}} = \mathscr{W}$. Since $\mathscr{W}$ is an IJ-bundle it follows that $\mathscr{V}_{\mathscr{R}}$ has the properties (a), (b), (c), (d). Hence $\mathscr{V}^{\perp}$ is involutive and therefore $\mathscr{V}$ is closed.

Theorem 2.1 follows by combining Lemma 2.2 and Lemma 2.3.

As a consequence of Theorem 2.1 we obtain the following useful criterion for an almost complex manifold to be of constant type, in the sence of the Introduction.

Theorem 2.4. *Let* (M, J) *be an almost complex manifold with Nijenhuis tensor* N *and let* $\operatorname{rank} N = \dim M - 2m$ *be a constant function. Then* (M, J) *is of type* m *if and only if the Nijenhuis bundle* $\mathscr{L}\mathscr{N}$ *has the following properties:*

(i) $\mathscr{L}\mathscr{N}$ *is involutive,*

(ii) $[X, Y] + J[X, JY] \in \Gamma(\mathscr{L}\mathscr{N})$ *for all* $X \in \Gamma(\mathscr{L}\mathscr{N})$ *and all* $Y \in \mathfrak{X}(M)$.

Proof. By Theorem 2.1 (M, J) is of type m if and only if $\mathscr{L}\mathscr{N}$ is an IJ-bundle. This is equivalent to conditions (i) and (ii) since $\mathscr{L}\mathscr{N}$ is J-invariant (see (1.2)).

3. Strict Almost Complex Manifolds

The almost complex manifolds which are the furtherst away from the complex manifolds are those on which every locally defined holomorphic function is constant. In our terminology these are the almost complex manifolds of type 0. In this section we propose a simple criterion for an almost complex manifold to be of type 0.

Definition. An almost complex manifold (M, J) is called *strict* provided $\{x \in M;\ \operatorname{rank}_x N = \dim M\}$ is a dense subset of M.

Let (M, J) be a strict almost complex manifold. Then the only (local) IJ-bundle on (M, J) is the tangent bundle $T(M)$. Hence by Theorem 2.1 we get the following

Theorem 3.1. *Any strict almost complex manifold is of type* 0.

In the next example we generalize slightly a result of Calabi stated without proof in [2, p. 429].

Example 3.2. Let M be an orientable hypersurface of $\mathbb{R}^7$ equipped with the Calabi almost complex structure J [2] and let N be the Nijenhuis tensor of J. For any point $x \in M$ denote by A_x the symmetric transformation of $T_x(M)$ determined by the second fundamental form of M and an unit normal vector field to M [11]. Then $K_x = -A_x + J_x A_x J_x$ is a symmetric Hermitian transformation of $T_x(M)$ with respect to the restriction of the usual metric of $\mathbb{R}^7$ and the complex structure of $T_x(M)$ determined by J_x. Hence K_x has three real eigenvalues, say $k_1(x)$, $k_2(x)$ and $k_3(x)$. Denote by k the following function on M:

$$k(x) = (k_1(x) + k_2(x))(k_2(x) + k_3(x))(k_3(x) + k_1(x)).$$

In [12] we have shown that $\{x \in M;\ \operatorname{rank}_x N = \dim M\} = \{x \in M;\ k(x) \neq 0\}$. Suppose further that M is connected, real-analytic and that the function k is not identically zero. Since

$$24\,k(x) = (\operatorname{Trace} K_x)^3 - 4 \operatorname{Trace} K_x^3$$

it follows that k is a real-analytic function on M and the principle of analytic continuation implies that $\{x \in M;\ k(x) \neq 0\}$ is a dense subset of M. Hence (M, J) is a strict almost complex manifold and therefore of type 0. We should note that if M is compact then the function k is not identically zero. This follows by the fact that in this case there is a point x in M, such that the transformation A_x of $T_x(M)$ is definite (positive or negative, depending on the orientation) [11]. Hence any compact real-analytic hypersurface of $\mathbb{R}^7$ is an almost complex manifold of type 0.

Example 3.3. Let $M = G/K$ be a homogeneous space equipped with a G-invariant almost complex structure J. Then the Nijenhuis tensor N of J is a G-

invariant tensor field on M and therefore $\operatorname{rank} N$ is a constant function. We now assume that M is isotropy irreducible. Since $\mathscr{L}\mathscr{N}(o)$ $(o=\{K\}\in G/K)$ is an invariant subspace of the linear isotropy representation of K at o it follows that either $\operatorname{rank} N=0$ or $\operatorname{rank} N=\dim M$. Thus by Theorem 3.1 we obtain the following result of Hermann ([8, Theorem 3.1]): *Every isotropy irreducible homogeneous almost complex space is either integrable or of type* 0.

A typical example of an isotropy irreducible homogeneous almost complex space of type 0 is the sphere $S^6=G_2/SU(3)$. Other examples can be found in Wolf [18].

4. Nearly Kähler Manifolds

Let M be an almost Hermitian manifold with metric g, Riemannian connection D and almost complex structure J. Then we have $g(JX,JY)=g(X,Y)$ for all $X,Y\in\mathfrak{X}(M)$. Recall [6] that M is said to be a nearly Kähler manifold if $D_X(J)(Y)+D_Y(J)(X)=0$ for all $X,Y\in\mathfrak{X}(M)$. In this section we determine the type of a nearly Kähler manifold by means of the function $\operatorname{rank} N$. We first prove a technical lemma.

Lemma 4.1. *Let M be a connected nearly Kähler manifold with Nijenhuis tensor N. Then*

(i) $\operatorname{rank} N$ *is a constant function.*

(ii) *For the Nijenhuis bundle $\mathscr{L}\mathscr{N}$ and all $X\in\mathfrak{X}(M)$, $Y\in\Gamma(\mathscr{L}\mathscr{N})$ one has $D_X Y\in\Gamma(\mathscr{L}\mathscr{N})$.*

Proof. Statement (i) is known [13], but we include a proof for the sake of compliteness. For any almost Hermitian manifold one has the following well-known identities (cf. [19]):

$$N(X,Y)=-D_X(J)(JY)+D_{JY}(J)(X)-D_{JX}(J)(Y)+D_Y(J)(JX)$$
$$D_X(J)(JY)=-JD_X(J)(Y)$$

for all $X,Y\in\mathfrak{X}(M)$. In particular, if M is a nearly Kähler manifold, we get easily

$$N(X,Y)=4JD_X(J)(Y) \tag{4.1}$$

$$g(N(X,Y),Z)=g(N(Y,Z),X) \tag{4.2}$$

for all $X,Y,Z\in\mathfrak{X}(M)$. The Kähler form of M is the 2-form F given by $F(X,Y)=g(JX,Y)$. Using (4.1) it is not hard to check that

$$dF(X,Y,Z)=3/4\,g(N(X,Y),JZ). \tag{4.3}$$

Denote by $\operatorname{rank}_x dF$ the rank of the 3-form dF at a point $x\in M$. Then by a result of Kirichenko [10] $\operatorname{rank}_x dF$ is constant. On the other hand (4.3) implies $\operatorname{rank}_x N=\operatorname{rank}_x dF$ and (i) follows.

To prove (ii) we shall use an identity for the curvature operator R_{XY} $(X,Y\in\mathfrak{X}(M))$ of a nearly Kähler manifold. In [6] Gray has shown that

$$g(R_{XY}Z, T) - g(R_{XY}JZ, JT) = g(D_X(J)(Y), D_Z(J)(T))$$

for all $X, Y, Z, T \in \mathfrak{X}(M)$. Then from (4.1) and (4.2) it follows that

$$R_{XY}Z + JR_{XY}JZ = -D_Z(J)(D_X(J)(Y)) \in \Gamma(\mathscr{L}\mathscr{N}).$$

By the very definition of R_{XY} we have

$$R_{XY}Z + JR_{XY}JZ = -D_X(J)(JD_YZ) + D_Y(J)(JD_XZ) + D_{[X,Y]}(J)(JZ) + JD_XD_Y(J)(Z) - JD_YD_X(J)(Z).$$

The two identities above, together with (4.1), imply that

$$D_XD_Y(J)(Z) - D_YD_X(J)(Z) \in \Gamma(\mathscr{L}\mathscr{N})$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Taking symmetric sum in X, Y, Z we obtain, after some simplifications, that $D_XD_Y(J)(Z) \in \Gamma(\mathscr{L}\mathscr{N})$. Then (ii) is a consequence of the fact that the Nijenhuis bundle $\mathscr{L}\mathscr{N}$ is locally generated by vector fields of the form $D_Y(J)(Z)$ (see (4.1)).

Now we are in a position to prove the main result of this section.

Theorem 4.2. *Let M be a connected nearly Kähler manifold with Nijenhuis tensor N. Then* $\operatorname{rank} N$ *is a constant function and M is an almost complex manifold of type* $1/2(\dim M - \operatorname{rank} N)$.

Proof. We have shown in Lemma 4.1 that $\operatorname{rank} N$ is a constant function, so it suffices to prove that the Nijenhuis bundle $\mathscr{L}\mathscr{N}$ satisfies the conditions of Theorem 2.4. From Lemma 4.1, (ii) we have $D_XY \in \Gamma(\mathscr{L}\mathscr{N})$ and $D_YX \in \Gamma(\mathscr{L}\mathscr{N})$ for all $X, Y \in \Gamma(\mathscr{L}\mathscr{N})$. Hence $[X, Y] = D_XY - D_YX \in \Gamma(\mathscr{L}\mathscr{N})$ and therefore $\mathscr{L}\mathscr{N}$ is involutive. Now let $X \in \Gamma(\mathscr{L}\mathscr{N})$ and $Y \in \mathfrak{X}(M)$. Then from Lemma 4.1, (ii) and (4.1) it follows that

$$[X, Y] + J[X, JY] = -D_X(J)(JY) - D_YX - JD_{JY}X \in \Gamma(\mathscr{L}\mathscr{N}).$$

The theorem follows.

The next corollary is an immediate consequence of Theorem 4.2, formula (4.1) and ([7, Theorems 5.2 and 5.3])

Corollary 4.3. *Let M be a connected non-Kähler nearly Kähler manifold. Then*

(i) *If* $\dim M = 6$ *then M is of type* 0.

(ii) *If* $\dim M = 8$ *then M is of type* 1.

5. Homogeneous Almost Complex Spaces

Let $M = G/K$ be a homogeneous space on which the connected Lie group G acts effectively. We fix a direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K respectively and $\mathfrak{m}$ is a vector subspace of $\mathfrak{g}$ which may be identified with the rangent space to M at $o = \{K\} \in G/K$. For the sake of simplicity, we shall always assume that M is reductive in the sence that $\mathfrak{m}$ is an $ad(K)$-invariant subspace of $\mathfrak{g}$. We also identify the G-invariant almost complex structures on M with $ad(K)$-invariant endomorphisms J of $\mathfrak{m}$, such that $J^2 X = -X$ for all $X \in \mathfrak{m}$ [11].

Let $(M = G/K, J)$ be a homogeneous almost complex space with Nijenhuis tensor N. Denote the subspace $\mathscr{L}\mathscr{N}(o)$ of $\mathfrak{m}$ by $\mathfrak{l}\mathfrak{n}$. Then $\operatorname{rank} N \equiv \dim_{\mathbb{R}} \mathfrak{l}\mathfrak{n}$ since N is a G-invariant tensor field on M.

Definition. A vector subspace $\mathfrak{h}$ of $\mathfrak{m}$ is called IJ-subspace if:

(i) $\mathfrak{h}$ is both $ad(K)$-invariant and J-invariant,

(ii) $\mathfrak{l}\mathfrak{n} \subset \mathfrak{h}$,

(iii) $[\mathfrak{h}, \mathfrak{h}]_{\mathfrak{m}} \subset \mathfrak{h}$,

(iv) $[X, Y]_{\mathfrak{m}} + J[X, JY]_{\mathfrak{m}} \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, $Y \in \mathfrak{m}$.

(The subscript denotes the component in $\mathfrak{m}$.)

Denote by $d(M, J)$ the real dimension of the intersection of all IJ-subspaces of $\mathfrak{m}$. From (i) it follows that $d(M, J)$ is an even integer. Note that the almost complex structure J is integrable if and only if $d(M, J) = 0$ [11]. Moreover the following holds.

Theorem 5.1. *Let (M, J) be a reductive homogeneous almost complex space. Then (M, J) is of type* $1/2(\dim M - d(M, J))$.

Proof. Let $m = m(o)$ be the maximal number of holomorphic functions which are independent at o. Since J is a G-invariant almost complex structure and G acts transitively on M it follows easily that (M, J) is an almost complex manifold of type m. An examination of the proof of Theorem 2.1 shows that the local existence of k independent holomorphic functions on (M, J) is equivalent to the existence of a G-invariant IJ-bundle on (M, J) with fibre dimension $\dim M - 2k$. On the other hand it is not hard to verify that the correspondence $\mathscr{V} \mapsto \mathscr{V}(o)$ is a bijection between the set of all G-invariant IJ-bundles on (M, J) and the set of all IJ-subspaces of $\mathfrak{m}$. These two remarks imply that $\dim M - 2m = d(M, J)$ and the result follows.

To illustrate the Theorem we consider an example.

Example 5.2. Let $U(4)$ be the unitary group of order 4. Consider the complex flag manifold $M = G/K$, where $G = U(4)$ and $K = U(1) \times U(1) \times U(1) \times U(1)$. Let $\mathfrak{g} = \mathfrak{u}(4)$ be the Lie algebra of $U(4)$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where

$$\mathfrak{k} = \left\{ \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix}; \quad a_{ii} \in \mathbb{C},\ 1 \leqq i \leqq 4 \right\}$$

is the Lie algebra of K and

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -\bar{a}_{12} & 0 & a_{23} & a_{24} \\ -\bar{a}_{13} & -\bar{a}_{23} & 0 & a_{34} \\ -\bar{a}_{14} & -\bar{a}_{24} & -\bar{a}_{34} & 0 \end{pmatrix}; \quad a_{ij} \in \mathbb{C},\ 1 \leqq i < j \leqq 4 \right\}$$

is the tangent space to M at o ($\bar{a}_{ij}$ is the complex conjugate of a_{ij}). Define $\mathfrak{m}_{12}$ by

$$\mathfrak{m}_{12} = \left\{ \begin{pmatrix} 0 & a_{12} & 0 & 0 \\ -\bar{a}_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad a_{12} \in \mathbb{C} \right\};$$

$\mathfrak{m}_{13}, \mathfrak{m}_{14}, \mathfrak{m}_{23}, \mathfrak{m}_{24}, \mathfrak{m}_{34}$ are defined in an analogous way. Thus we have $\mathfrak{m} = \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \ldots \oplus \mathfrak{m}_{34}$. Let $\varepsilon = (\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{14}, \varepsilon_{23}, \varepsilon_{24}, \varepsilon_{34})$, where $\varepsilon_{ij} = \pm 1$. An $U(4)$-invariant almost complex structure $J(\varepsilon)$ on M is given by

$$J(\varepsilon) = \varepsilon_{12} J_{12} \oplus \varepsilon_{13} J_{13} \oplus \ldots \oplus \varepsilon_{34} J_{34},$$

where $J_{pq}\colon \mathfrak{m}_{pq} \to \mathfrak{m}_{pq}$ are the linear maps $J_{pq}(a_{pq}) = (i\,a_{pq})$, $i^2 = -1$. For any triple (p, q, r) $(1 \leqq p < q < r \leqq 4)$ we denote

$$\varepsilon_{pqr} = \varepsilon_{pq} - \varepsilon_{pr} + \varepsilon_{qr} - \varepsilon_{pq} \cdot \varepsilon_{pr} \cdot \varepsilon_{qr}.$$

A messy but straightforward calculation shows that one has the following three cases:

(a) $\varepsilon_{pqr} = 0$ for all triples (p, q, r). Then $J(\varepsilon)$ is an integrable almost complex structure.

(b) There is exactly one triple, say (p, q, r), such that $\varepsilon_{pqr} \neq 0$. Then $\mathfrak{n} = \mathfrak{m}_{pq} \oplus \mathfrak{m}_{pr} \oplus \mathfrak{m}_{qr}$ and $\mathfrak{n}$ is an $IJ(\varepsilon)$-subspace of $\mathfrak{m}$. Hence $d(M, J(\varepsilon)) = 6$ and by Theorem 5.1 it follows that $(M, J(\varepsilon))$ is an almost complex manifold of type 3.

(c) In all other cases $d(M, J(\varepsilon)) = 12$ and hence $(M, J(\varepsilon))$ is an almost complex manifold of type 0.

As a special case of Theorem 5.2 we obtain the following

Corollary 5.3. *Let (M, J) be a nonintegrable homogeneous almost complex space with* $\dim M = 4$. *Then (M, J) is an almost complex manifold of type* 1 *if and only if:*

(i) $[\mathfrak{n}, \mathfrak{n}]_{\mathfrak{m}} \subset \mathfrak{n}$,

(ii) $[X, Y]_{\mathfrak{m}} + J[X, JY]_{\mathfrak{m}} \subset \mathfrak{n}$ for all $X \in \mathfrak{n}$ and all $Y \in \mathfrak{m}$.

Proof. From (1.2) it follows that $\dim \mathfrak{n} = 2$. Hence by Theorem 5.1 (M, J) is of type 1 if and only if $\mathfrak{n}$ is an IJ-subspace of $\mathfrak{m}$.

Example 5.4. In 1976 Thurston [16] constructed a compact 4-dimensional almost Kähler manifold W which does not carry any Kähler structure. We briefly recall the definition of W following Abbena [1]. Let G be the closed subgroup of $GL(4, \mathbb{C})$ defined by

$$G = \left\{ \begin{pmatrix} 1 & x & y & 0 \\ 0 & 1 & z & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \exp(2\pi i t) \end{pmatrix}; \quad x, y, z, t \in \mathbb{R} \right\}$$

and let K be the discrete subgroup of G consisting of all matrices of G with integer elements. The vector fields $e_1=\frac{\partial}{\partial x}$, $e_2=\frac{\partial}{\partial y}+x\frac{\partial}{\partial z}$, $e_3=\frac{\partial}{\partial z}$, $e_4=\frac{\partial}{\partial t}$ are left-invariant and form a basis of the Lie algebra $\mathfrak{g}$ of G. The Lie multiplication table of $\mathfrak{g}$ is given by

$$\begin{aligned} &[e_1, e_2]=e_3 \quad && [e_2, e_3]=0 \\ &[e_1, e_3]=0 && [e_2, e_4]=0 \\ &[e_1, e_4]=0 && [e_3, e_4]=0. \end{aligned} \tag{5.1}$$

Let J be the left-invariant almost complex structure on G defined by $Je_1=-e_4$, $Je_2=e_3$, $Je_3=-e_2$, $Je_4=e_1$. In particular J is invariant under K and therefore J determines an invariant almost complex structure $\tilde{J}$ on $W=G/K$. To determine the type of $(W, \tilde{J})$ we use Corollary 5.3. Since K is a discrete group the tangent space to W at a point may be identified with $\mathfrak{g}$. Then an easy computation, involving (5.1) shows that In is the vector subspace of $\mathfrak{g}$ generated by e_2 and e_3, and that In satisfies the conditions of Corollary 5.3. Hence $(W, \tilde{J})$ is an almost complex manifold of type 1.

6. The Tangent Bundle of a Manifold with Flat Connection

Let M be a smooth manifold endowed with a linear connection D. Dombrowski [5] has shown that D determines an almost complex structure on the tangent bundle $T(M)$ of M, which we denote by J. In this section we examine the almost complex manifolds obtained in this manner when D is a flat connection (i.e. the curvature tensor of D vanishes).

We briefly recall the definition of J following Dombrowski [5]. For any $X\in\mathfrak{X}(M)$ let $X^h, X^v\in\mathfrak{X}(T(M))$ be the horizontal and vertical liftings of X respectively. Then J is defined by $JX^h=X^v$, $JX^v=-X^h$ for all $X\in\mathfrak{X}(M)$. Denote by T and R the torsion and curvature tensors of D respectively. For any point x in M we define

$$\mathscr{L}\mathscr{T}(x)=\mathrm{Span}\{T_x(a,b);\ a,b\in T_x(M)\}$$

and set $\operatorname{rank}_x T=\dim\mathscr{L}\mathscr{T}(x)$. If $\operatorname{rank} T$: $x\mapsto\operatorname{rank}_x T$ is a constant function we denote by $\mathscr{L}\mathscr{T}$ the subbundle of $T(M)$ with fibres $\mathscr{L}\mathscr{T}(x)$. It is well-known [5] that J is an integrable almost complex structure if and only if $T=0$ and $R=0$. As a partial generalization we have the following

Theorem 6.1. *Let M be a smooth manifold with flat connection D and let* $\operatorname{rank} T=\dim M-t$ *be a constant function. Then the almost complex manifold $(T(M), J)$ is of type t if and only if $D_X Y\in\Gamma(\mathscr{L}\mathscr{T})$ for all $X\in\mathfrak{X}(M)$, $Y\in\Gamma(\mathscr{L}\mathscr{T})$.*

Proof. Since $R=0$ we have the following formulas (cf. [5]):

$$[X^h, Y^h]=[X, Y]^h \tag{6.1}$$

$$[X^h, Y^v]=(D_X Y)^v \tag{6.2}$$

$$N(X^h, Y^h)=-(T(X, Y))^h \tag{6.3}$$

$$N(X^h, Y^v)=(T(X, Y))^v \tag{6.4}$$

for all $X, Y \in \mathfrak{X}(M)$, where N is the Nijenhuis tensor of J. From (6.3) and (6.4) it follows that $\operatorname{rank} N = 2 \operatorname{rank} T = 2(\dim M - t)$ is a constant function. Furthermore

$$\mathscr{L}\mathscr{N} = \mathscr{L}\mathscr{T}^h \oplus \mathscr{L}\mathscr{T}^v,$$

where $\mathscr{L}\mathscr{T}^h$ and $\mathscr{L}\mathscr{T}^v$ are the horizontal and vertical liftings of the bundle $\mathscr{L}\mathscr{T}$, respectively. Now from (6.1) and (6.2) it is not hard to verify that the Nijenhuis bundle $\mathscr{L}\mathscr{N}$ of $(T(M), J)$ satisfies the conditions of Theorem 2.4 if and only if $D_X T(Y, Z) \in \Gamma(\mathscr{L}\mathscr{T})$ for all $X, Y, Z \in \mathfrak{X}(M)$. The theorem then follows by the fact that the bundle $\mathscr{L}\mathscr{T}$ is locally generated by vector fields of the form $T(Y, Z)$, where $Y, Z \in \mathfrak{X}(M)$.

If M is connected and the torsion tensor T has vanishing covariant derivative (then T is said to be parallel) it is obvious that the conditions of Theorem 6.1 are fulfilled. Thus we obtain

Corollary 6.2. *Let M be a connected manifold and let D be a flat connection with parallel torsion T. Then* $\operatorname{rank} T$ *is a constant function and the almost complex manifold $(T(M), J)$ is of type* $\dim M - \operatorname{rank} T$.

A natural candidate for a manifold satisfying the conditions of Corollary 6.2 is any connected Lie group with $(-)$-connection of E. Cartan [11]. Moreover, by a result of Kamber-Tondeur [9] every connected manifold which admits a complete flat connection with parallel torsion is a coset space $M = G/\pi$, where G is a connected and simply connected Lie group and π is a subgroup of the affine group of G acting properly discontinuously and without fixed points on G. Note that in this case the type of the almost complex manifold $(T(M), J)$ can be determined by means of the Lie algebra $\mathfrak{g}$ of G. In fact, by the arguments of [9] it follows that $\operatorname{rank} T = \dim[\mathfrak{g}, \mathfrak{g}]$ and therefore $(T(M), J)$ is of type $\dim \mathfrak{g} - \dim[\mathfrak{g}, \mathfrak{g}]$.

References

1. Abbena, E.: An example of an almost Kähler manifold which is not Kählerian. Boll. Unione Mat. Ital. (6) **3-A**, 383–392 (1984)
2. Calabi, E.: Constructions and properties of some 6-dimensional almost complex manifolds. Trans. Am. Math. Soc. **87**, 407–438 (1958)
3. Chern, S.S.: Complex manifolds without potential theory. New York: Van Nostrand Reinhold Company 1967
4. Dimiev, S.: Propriétés locales des fonctions presque-holomorphes. In: Analytic functions (Blazejewko 1982), pp. 102–117. Lecture Notes in Math., vol. **1039**. Berlin-Heidelberg-New York: Springer 1983
5. Dombrowski, P.: On the geometry of the tangent bundle. J. Reine Angew. Math. **210**, 73–88 (1962)
6. Gray, A.: Nearly Kähler manifolds. J. Differ. Geom. **4**, 283–309 (1970)
7. Gray, A.: The structure of nearly Kähler manifolds. Math. Ann. **223**, 233–248 (1976)
8. Hermann, R.: Compact homogeneous almost complex spaces of positive characteristic. Trans. Am. Math. Soc. **83**, 471–481 (1956)
9. Kamber, F.W., Tondeur, Ph.: Flat manifolds with parallel torsion. J. Differ. Geom. **2**, 385–389 (1968)
10. Kirichenko, V.F.: K-spaces of maximal rank. Mat. Zametki **22**, 465–476 (1977) [Russian]

11. Kobayashi, S., Nomizu, K.: Foundations of differential geometry, vol. II. New York: John Wiley & Sons, Inc. (Interscience Division) 1969
12. Muškarov, O.: Almost complex manifolds without almost holomorphic functions. C.R. Acad. Bulg. Sci. **9**, 1225–1228 (1981)
13. Muškarov, O.: Complex submanifolds in almost complex manifolds. (to appear in Simon Stevin)
14. Newlander, A., Nirenberg, L.: Complex analytic coordinates in almost complex manifolds. Ann. Math. **65**, 391–404 (1954)
15. Spencer, D.C.: Potential theory and almost complex manifolds. In: Lectures on functions of a complex variable, pp. 15–43. University of Michigan Press 1955
16. Thurston, W.P.: Some examples of symplectic manifolds. Proc. Am. Math. Soc. **55**, 467–468 (1976)
17. Treves, F.: Approximation and representation of functions and distributions annihilated by a system of complex vector fields. École Polytechniques, Centre de Mathematiques, Palaiseau, 1981
18. Wolf, J.: The geometry and structure of isotropy irreducible homogeneous spaces. Acta Math. **120**, 59–148 (1968)
19. Yano, K.: Differential geometry on complex and almost complex spaces. New York: Pergamon Press 1965

Received April 1, 1985