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Modeling axisymmetric flow through a converging channel with an arbitrary yield condition

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Summary. A system of ordinary differential equations describing the velocity, strain and stress fields of an isotropic rigid perfectly plastic material flowing through an axisymmetrical converging channel was derived. The plastic flow behavior was assumed to be described by an arbitrary yield condition and its associated flow rule. The solution of this problem was applied to the case of a yield function suitable for isotropic FCC polycrystals such as aluminum alloys. The singularity of the strain rate field at the channel wall where maximum friction forces occur was discussed and the influence of the yield surface shape on the velocity, strain and stress fields was investigated.

1 Introduction

Flow through a channel is one of the classical problems in solid and fluid mechanics. Applied to the case of plastic material, this problem is the simpliest approximation of extrusion processes. There are only a few exact analytical solutions for a rigid, perfectly plastic material under axially symmetric conditions. These solutions can be used to calculate the stress and plastic deformation which occur during the flow of the material through an infinite converging channel (Sokolovski [1]; Shield [2]; Brovman [3]), the flow of a plastic material along a rigid fiber (Spencer [4]) and the steady penetration of a rigid cone into an infinite plastic space (Durban and Fleck [5]). The Von Mises and Tresca yield conditions were adopted in all of these solutions, and a constant shearing stress was assumed on the friction surfaces. These solutions showed that if the shearing stress is equal to the shear yield stress, the shear strain rate as well as the effective strain rate tend to infinity on the friction surfaces. It was shown that this is a general feature near the surfaces with maximum friction and a velocity jump for a rigid perfectly plastic material. The asymptotic behavior of the velocity field was found under plane strain conditions by Sokolovsky [6], under axially symmetric conditions by Alexandrov and Druyanov [7] and Druyanov and Alexandrov [8] for a Von Mises material, and Alexandrov and Richmond [9] for a Tresca material. The asymptotic behavior in the general case of an arbitrary three-dimensional flow was derived by Alexandrov [10], [11] with the Von Mises yield condition. Because the effective strain rate influences many material properties such as hardening, softening and fracture and physical fields such as temperature, it is important to know whether the singular behavior near the friction surface with maximum friction and a velocity jump is still valid for a material following a yield condition other than Tresca or Von Mises.

In classical time-independent flow theory of plasticity, the yield condition serves as a potential for the strain rates, i.e., the strain rate is normal to the yield surface. Hecker [12] reviewed many biaxial and triaxial experiments on metals and concluded that, though the normality rule could not be proven, it was never violated. For polycrystalline materials, Bishop and Hill [13] demonstrated that if dislocation glide obeys the Schmid rule, i.e., slip occurs in crystallographic planes and directions for which the resolved shear stress reaches a critical value, the yield condition was also a potential. Therefore, for metals and alloys, the classical flow theory seems to be a good approximation of the plastic behavior. It has been used with yield conditions different from Tresca and Von Mises for isotropic and anisotropic materials (for instance Hill [14]; Hershey [15]; Hosford [16]; Bassani [17]; Hill [18]; Logan and Hosford [19]; Budianski [20]; Hill [21]; Barlat, Lege and Brem [22]; Karafillis and Boyce [23]).

The material behavior has a major impact on the prediction of properties such as plastic flow localization, fracture and, more generally, on stability or bifurcation phenomena. For instance, in a thin sheet, the prediction of plastic flow localization assuming either an imperfection in the material (MK theory, Marciniak and Kuczynski [24]) or a singularity (developing vertex) on the yield surface (Stören and Rice [25]) strongly depends on the yield surface shape. Barlat [26] computed the forming limits in a thin sheet for Von Mises and Tresca materials using the MK theory. In the biaxial stretching range where the sheet is stretched by the same amount in any direction, the Von Mises limit strains were four to five times larger than the Tresca limit strains. Bate [27] and Barlat [26] computed the yield surface of FCC metals with the Taylor [28]/Bishop and Hill [13] polycrystal model (TBH) and used it to predict the forming limit of FCC sheets. The limit strains were in between the forming limits computed with either the Tresca or the Von Mises potentials and in better agreement with the experimental trends.

In this paper, the plastic flow in a converging channel is studied assuming that the material behavior obeys the classical flow rule of plasticity with an arbitrary isotropic convex yield function independent of the hydrostatic stress. It is also assumed that this function is differentiable with respect to its arguments. In the first Sections of this paper, the general governing equations are derived. Because the yield condition has such a tremendous impact on material properties, it is important to select and use yield conditions that best describe the material behavior. For FCC polycrystals such as aluminum alloys, Hershey [15] and Hosford [16] proposed an isotropic yield condition which can approach the theoretical polycrystal yield condition very well. Therefore, for a numerical example, this yield condition is combined with the general governing equations of the converging channel problem. Results pertaining to Von Mises, Tresca-like and FCC (such as aluminum alloys) materials are discussed.

2 Statement of the problem

It is assumed that the material flows through a conical circular channel of sufficient length so that the steady flow in the channel is not affected by its extremities. The angle of the conical channel is denoted by 2α and the material is assumed to behave like a rigid perfectly plastic body. A friction force between the channel wall and the material counteracts the plastic flow. The corresponding friction stress is assumed to be $k\tau$, where τ is the flow stress in shear and k is a constant coefficient ($0 \leq k \leq 1$). A spherical coordinate system r, θ and φ (Fig. 1) is used to describe the problem. The material is assumed to be isotropic and, as a result, the velocity, strain and stress fields exhibit axial symmetry. Therefore, as for the plane strain channel problem (Hill [29]), one angle denoted by ψ is sufficient to characterize the rotation between the principal reference frame and the spherical coordinate system as illustrated in Fig. 1.

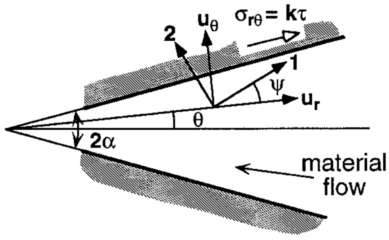

Fig. 1. Sketch of the converging channel with 1 and 2 being the principal stress axes

3 Relationships between stress components

Due to the symmetry of the converging channel problem, the two components $t_{\theta\varphi}$ and $t_{\varphi r}$ of the stress (or rate of deformation) tensor **t** are equal to zero. The relationship between the principal components and the components of **t** expressed in the spherical coordinate system are given by the usual tensor transformation formulae,

$$t_1 = \frac{1}{2}(t_r + t_\theta) + \frac{1}{2}(t_r - t_\theta)\cos 2\psi + t_{r\theta}\sin 2\psi, \tag{1}$$

$$t_2 = \frac{1}{2}(t_r + t_\theta) - \frac{1}{2}(t_r - t_\theta)\cos 2\psi - t_{r\theta}\sin 2\psi, \tag{2}$$

$$t_3 = t_\varphi, \tag{3}$$

$$0 = \frac{1}{2}(t_\theta - t_r)\sin 2\psi + t_{r\theta}\cos 2\psi. \tag{4}$$

These equations can be applied to the case of the stress tensor. σ_r is larger than σ_θ because a material elongates along the channel. Moreover, the friction of the die on the material is opposed to the flow motion. As a result, the angle ψ can be defined without ambiguity as a function of the stress components expressed in the spherical coordinate system (Eq. (4)),

$$\operatorname{tg} 2\psi = \frac{2\sigma_{r\theta}}{\sigma_r - \sigma_\theta} \geq 0 \qquad \text{or} \qquad 0 \leq \psi \leq \pi/4, \tag{5}$$

and, using trigonometric relations,

$$\cos 2\psi = \frac{\sigma_r - \sigma_\theta}{\sqrt{(\sigma_r - \sigma_\theta)^2 + 4\sigma_{r\theta}^2}} \qquad \text{and} \qquad \sin 2\psi = \frac{2\sigma_{r\theta}}{\sqrt{(\sigma_r - \sigma_\theta)^2 + 4\sigma_{r\theta}^2}}. \tag{6}$$

It is possible to eliminate ψ from the principal stress expressions (Eqs. (1) and (2)),

$$\sigma_1 = \frac{1}{2}(\sigma_r + \sigma_\theta) + \frac{1}{2}\sqrt{(\sigma_r - \sigma_\theta)^2 + 4\sigma_{r\theta}^2}, \tag{7}$$

$$\sigma_2 = \frac{1}{2}(\sigma_r + \sigma_\theta) - \frac{1}{2}\sqrt{(\sigma_r - \sigma_\theta)^2 + 4\sigma_{r\theta}^2}. \tag{8}$$

In the following, $\sigma = (\sigma_r + \sigma_\theta + \sigma_\varphi)/3$ will denote the mean stress and P_k the difference between two of the principal stresses (or twice the maximum shear stresses in the principal axes frame). P_k can be expressed with the deviatoric stress components from Eqs. (3), (7) and (8),

$$P_1 = \sigma_2 - \sigma_3 = \frac{1}{2}(s_r + s_\theta - 2s_\varphi) - \frac{1}{2}\sqrt{(s_r - s_\theta)^2 + 4s_{r\theta}^2}, \tag{9}$$

$$P_2 = \sigma_3 - \sigma_1 = \frac{1}{2}(2s_\varphi - s_r - s_\theta) - \frac{1}{2}\sqrt{(s_r - s_\theta)^2 + 4s_{r\theta}^2}, \tag{10}$$

$$P_3 = \sigma_1 - \sigma_2 = \sqrt{(s_r - s_\theta)^2 + 4s_{r\theta}^2}. \tag{11}$$

For convenience in this problem, $P = s_r + s_\theta = -s_\varphi$ will be used as an independent stress variable. The quantities P_1 and P_2 can be expressed as a function of P and P_3 only,

$$P_1 = \frac{3P - P_3}{2}, \qquad P_2 = \frac{-3P - P_3}{2}. \tag{12}$$

The set of variables σ, P, P_3 and $s_{r\theta}$ will be used throughout this paper to characterize the stress state in the channel.

4 Relationship between velocity and strain components

The plastic flow is assumed to be radial with the velocity $v_r = v$. The components of the rate of deformation tensor are

$$\varepsilon_r = \frac{\partial v}{\partial r}, \qquad \varepsilon_\theta = \varepsilon_\varphi = \frac{v}{r} \qquad \text{and} \qquad \varepsilon_{r\theta} = \frac{1}{2r}\frac{\partial v}{\partial \theta}, \tag{13}$$

and because of plastic incompressibility, v obeys the following differential equation:

$$\frac{\partial v}{\partial r} + \frac{2v}{r} = 0. \tag{14}$$

Integration of this equation leads to

$$v = \frac{v_0(\theta)}{r^2}, \tag{15}$$

and the components of the rate of deformation can be rewritten as

$$\varepsilon_r = -\frac{2v_0(\theta)}{r^3}, \qquad \varepsilon_\theta = \varepsilon_\varphi = -\frac{\varepsilon_r}{2} = \frac{v_0(\theta)}{r^3} \qquad \text{and} \qquad \varepsilon_{r\theta} = \frac{1}{2r^3}\frac{dv_0(\theta)}{d\theta}. \tag{16}$$

Because the material is isotropic, the tensor transformation equations (Eqs. (1) to (4)) are also valid with the same angle ψ for the strain rate components. Equation (4) applied to the rate of deformation results in

$$\operatorname{tg} 2\psi = \frac{2\varepsilon_{r\theta}}{\varepsilon_r - \varepsilon_\theta} = \frac{4\varepsilon_{r\theta}}{3\varepsilon_r} \geq 0 \tag{17}$$

and, using trigonometric relations,

$$\cos 2\psi = \frac{3\varepsilon_r}{\sqrt{9{\varepsilon_r}^2 + 16\varepsilon_{r\theta}^2}} \qquad \text{and} \qquad \sin 2\psi = \frac{4\varepsilon_{r\theta}}{\sqrt{9{\varepsilon_r}^2 + 16\varepsilon_{r\theta}^2}}. \tag{18}$$

5 Material behavior

The material is assumed to be rigid perfectly plastic and to obey the classical flow theory of plasticity with an isotropic yield condition $\phi = \phi_0$ and the associated flow rule. The relationship between principal stresses and strain rates can be expressed with the variable P_k,

$$\varepsilon_i = \lambda\frac{\partial \phi}{\partial \sigma_i} = \lambda\frac{\partial \phi}{\partial P_k}\frac{\partial P_k}{\partial \sigma_i} = \lambda\phi_k\frac{\partial P_k}{\partial \sigma_i} \qquad \text{with} \qquad \phi_k = \frac{\partial \phi}{\partial P_k} \tag{19}$$

or, taking Eqs. (9) to (11) into account,

$$\varepsilon_1 = \lambda(\phi_3 - \phi_2), \qquad \varepsilon_2 = \lambda(\phi_1 - \phi_3), \qquad \varepsilon_3 = \lambda(\phi_2 - \phi_1). \tag{20}$$

It is possible to introduce the function $F = (\varepsilon_1 + \varepsilon_2)/(\varepsilon_1 - \varepsilon_2)$ which may be expressed as a function of ϕ_i or ψ by means of Eqs. (1), (2), (13) and (20) in the following form:

$$F = \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1 - \varepsilon_2} = \frac{\phi_1 - \phi_2}{2\phi_3 - \phi_1 - \phi_2} = \frac{\varepsilon_r}{\sqrt{9\varepsilon_r{}^2 + 16\varepsilon_{r\theta}^2}} = \frac{1}{3}\cos 2\psi. \tag{21}$$

Using the relations between velocity and strain rate, Eq. (16), Eq. (21) results in

$$\frac{1}{v_0(\theta)}\frac{dv_0(\theta)}{d\theta} = \pm\sqrt{\left(\frac{2\phi_3 - \phi_1 - \phi_2}{\phi_1 - \phi_2}\right)^2 - 9} = \pm 3\,\mathrm{tg}\,2\psi. \tag{22}$$

A consequence of Eqs. (21) and (22) is that both ϕ_k and P_k are functions of only one variable, ψ or θ and, by implication from Eqs. (9) to (12), s_r, s_θ, s_φ and P can also be assumed to be functions of ψ or θ only. The two variables P and P_3 that are used to describe the stress state in the channel can be expressed as a function of ψ using the yield condition and Eq. (21) which, for the present calculations, are more conveniently used in differential forms,

$$d\phi = \phi_1\,dP_1 + \phi_2\,dP_2 + \phi_3\,dP_3 = d\phi_0 = 0, \tag{23}$$

$$dF = F_1\,dP_1 + F_2\,dP_2 + F_3\,dP_3 = -\frac{2}{3}\sin 2\psi\,d\psi \qquad \text{with} \qquad F_j = \frac{\partial F}{\partial P_j}. \tag{24}$$

Differentiating P_k as a function of P and P_3 only (Eq. (12)) and substituting in Eq. (23) leads to a system of two differential equations that can be used to calculate P and P_3 as a function of ψ,

$$\begin{aligned} \frac{dP}{d\psi} &= -\frac{4}{3}\sin 2\psi \frac{1}{3(F_1 - F_2) - (2F_3 - F_1 - F_2)\cos 2\psi} \\ \frac{dP_3}{d\psi} &= -\cos 2\psi \frac{dP}{d\psi}. \end{aligned} \tag{25}$$

6 Equilibrium equations

Due to the axial symmetry of the channel, the stress components $\sigma_{\varphi\theta}, \sigma_{\theta\varphi}$ equal zero, and the equilibrium equations in spherical coordinates reduce to

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{1}{r}\{2\sigma_r - \sigma_\theta - \sigma_\varphi + \sigma_{r\theta}\cot\theta\} = 0, \tag{26}$$

$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{1}{r}\{(\sigma_\theta - \sigma_\varphi)\cot\theta + 3\sigma_{r\theta}\} = 0. \tag{27}$$

Using the decomposition of the stresses into the deviatoric components and the mean pressure, Eq. (26) can be written as

$$\frac{\partial\sigma}{\partial r} + \frac{c(\theta)}{r} = 0, \qquad \text{where} \qquad c(\theta) = \frac{\partial s_{r\theta}}{\partial\theta} + 3s_r + s_{r\theta}\cot\theta \tag{28}$$

and, by integration,

$$\sigma = -c(\theta)\ln r + A(\theta)\,. \tag{29}$$

To satisfy Eq. (27), it is necessary to impose $c(\theta) = -C$, where C is an arbitrary constant. Then, the two equilibrium conditions (Eqs. (26) and (27)) become

$$C + \frac{\partial s_{r\theta}}{\partial \theta} + 3s_r + s_{r\theta}\cot\theta = 0\,, \tag{30}$$

$$\frac{\partial s_\theta}{\partial r} + \frac{dA}{d\theta} + (s_\theta - s_\varphi)\,\cot\theta + 3s_{r\theta} = 0\,. \tag{31}$$

Using the variables $P = -s_\varphi$, P_3 defined by Eq. (11), $\cos 2\psi$ and $\sin 2\psi$ defined by Eq. (6), it is possible to express $s_{r\theta}$ and s_r,

$$s_{r\theta} = \frac{P_3 \sin 2\psi}{2} \qquad \text{and} \qquad s_r = \frac{P + P_3\cos 2\psi}{2}\,. \tag{32}$$

Substituting in the equilibrium equation (Eq. (30)), and using the differential system (Eqs. (25)) to express the stresses P and P_3 as a function of ψ only, it is possible to express θ as a function of ψ in a differential form,

$$\frac{d\theta}{d\psi} = -\frac{2P_3\cos 2\psi\,\mathrm{tg}\,\theta + \dfrac{4}{3}\dfrac{\cos 2\psi \sin^2 2\psi\,\mathrm{tg}\,\theta}{[3(F_1 - F_2) - (2F_3 - F_1 - F_2)\,\cos 2\psi]}}{(2C + 3P + 3P_3\cos 2\psi)\,\mathrm{tg}\,\theta + P_3 \sin 2\psi}\,. \tag{33}$$

7 Analysis of equations

Equations (25) and Eq. (33) form a differential system of three equations with three functions, θ, P and P_3, to determine as a function of ψ. This system is valid for any smooth isotropic yield condition which appears in the system through the function F_j. In order to solve this system, three boundary conditions are required and one condition is needed to determine the value of C. Because of the axial symmetry of the channel, $s_{r\theta} = 0$ at $\theta = 0$ which gives $\psi = 0$ from Eq. (32). In addition, because of this symmetry, $s_\theta = s_\varphi$ at $\theta = \psi = 0$. The stress state on the channel axis is equivalent to uniaxial tension superimposed with a hydrostatic pressure and, therefore,

$$\frac{s_r}{\sigma_y} = \frac{s_\theta}{\sigma_y} = -\frac{s_\varphi}{\sigma_y} = -\frac{1}{3}\,, \qquad \frac{s_{r\theta}}{\sigma_y} = 0 \tag{34}$$

where σ_y is the uniaxial flow stress. Therefore, the boundary condition for Eq. (25) is

$$\frac{P_3}{\sigma_y} = \frac{3P}{\sigma_y} = 1\,. \tag{35}$$

However, for this case, the point $\theta = 0, \psi = 0$ in Eq. (33) is critical and a direct numerical procedure is not possible at this point. To overcome this problem, a linear approximation of this equation is considered, assuming that $3(F_1 - F_2) - 2(F_3 - F_1 - F_2) \neq 0$ at this critical point. This inequality may be checked for the boundary condition Eq. (25) and a specific yield condition. Small quantities ψ^* and θ^* can be introduced instead of ψ and θ near the critical point. Then,

$$\sin 2\psi^* \approx 2\psi^*\,, \qquad \mathrm{tg}\,\theta^* \approx \theta^* \qquad \text{and} \qquad \frac{d\psi}{d\theta} \approx \frac{\psi^*}{\theta^*}\,, \tag{36}$$

and Eq. (33) leads to

$$\frac{d\theta^*}{d\psi^*} = -\frac{\theta^*}{c_1\theta^* + \psi^*} \qquad \text{where} \qquad c_1 = (2C + 3P + 3P_3)/2P_3. \tag{37}$$

The values of P and P_3 are taken at a critical point from the boundary conditions, Eq. (35). Equation (37) has the general solution

$$\psi^* = \frac{\mathcal{D}}{\theta^*} - \frac{c_1\theta^*}{2} \tag{38}$$

where $\mathcal{D}$ is an arbitrary constant which must be taken $\mathcal{D} = 0$ to satisfy the boundary condition $\psi^* = 0$ at $\theta^* = 0$. Then, Eq. (38) gives the solution near the critical point,

$$\psi = -\frac{\theta}{2}\left(C_1 + \frac{3P(0)}{2P_3(0)} + \frac{3}{2}\right) \qquad \text{with} \qquad C_1 = \frac{C}{P_3(0)}. \tag{39}$$

Taking into account the boundary condition for P and P_3, Eq. (39) leads to

$$\psi = -\frac{\theta}{2}(C_1 + 2) \tag{40}$$

where C_1 is defined by the friction factor on the wall.

The only quantity that has to be known to solve this problem is the final value of ψ. This can be obtained from the boundary condition on the wall of the channel where the shear stress is equal to $k\tau$. Combining with Eqs. (6) and (11), the final value ψ_f is given by

$$\psi_f = \frac{1}{2}\arcsin\left[\frac{2k\tau}{P_3(\psi_f)}\right]. \tag{41}$$

In this equation, P_3 (defined by the differential system of Eqs. (25)) and τ (the simple shear yield stress) can be calculated for a given yield function. The value of ψ_f can, therefore, be obtained for a given value of k describing the boundary condition on the channel wall.

To summarize, the differential system describing the stress distribution in the channel is defined by Eqs. (25) and (33). Equation (25) is solved first to find the stresses as functions of ψ and the interval of integration $[0, \psi_f]$ for the variable ψ from Eq. (41). Equation (33) is solved to obtain the relationship between θ and ψ. This final result is obtained in an iterative manner, the constant C being adjusted to satisfy the boundary condition $\theta = \alpha$ at ψ_f. Then, the deviatoric stresses can be obtained from Eq. (32) with $s_\varphi = -P$ and $s_\theta = -(s_r + s_\varphi)$. Because plastic flow does not depend on the hydrostatic pressure and there is no boundary condition for the normal stresses, $A(\theta)$ cannot be defined uniquely. The velocity can be obtained with a prescribed mass flux or value of the velocity V_0 at any point, for example at $\theta = 0$, by solving Eq. (22) which can be rewritten as

$$\begin{aligned}\frac{dv_0}{d\psi}(\psi) &= \pm 3v_0(\psi)\,\mathrm{tg}\,2\psi\frac{d\theta}{d\psi}\\ &= \mp 3v_0(\psi)\sin 2\psi\frac{2P_3\,\mathrm{tg}\,\theta + \dfrac{4}{3}\dfrac{\sin^2 2\psi\,\mathrm{tg}\,\theta}{[3(F_1 - F_2) - (2F_3 - F_1 - F_2)\cos 2\psi]}}{(2C + 3P + 3P_3\cos 2\psi)\,\mathrm{tg}\,\theta + P_3\sin 2\psi}.\end{aligned} \tag{42}$$

This differential equation can be solved simultaneously with the main system of equations (Eqs. (25) and (33)). In Eq. (42), $v_0(\psi)$ is negative (see Fig. 1) and, because of friction on the channel wall, its absolute value decreases when ψ increases. Since $d\theta/d\psi$ is positive, the correct signs after the first and second equalities of Eq. (42) are minus and plus, respectively.

As an example, these equations are solved in the next Section for a given yield function and for the friction shear stress on the channel wall corresponding to the value of $k = 1$ (maximum friction shear stress). In this case, it follows from Eq. (41) that $\psi_f = \pi/4$.

8 Numerical example

As mentioned earlier, the Hershey [15]–Hosford [16] isotropic yield function is used as an application of the general solution for the converging channel,

$$\phi = \Phi_m = \mid \sigma_1 - \sigma_2 \mid^m + \mid \sigma_2 - \sigma_3 \mid^m + \mid \sigma_3 - \sigma_1 \mid^m = \mid P_1 \mid^m + \mid P_2 \mid^m + \mid P_3 \mid^m = 2{\sigma_y}^m. \tag{43}$$

This yield function reduces to Tresca yield condition when the parameter m is equal to 1 or to ∞, and to the Von Mises yield function when m is equal to 2 and 4. Moreover, this function leads to a very good approximation of the yield surface calculated with the Taylor [28] polycrystal model for BCC and FCC materials when the exponent is equal to 6 and 8, respectively. In the Taylor model, all the grains are assumed to be subject to the same macroscopic strain imposed on the polycrystal. In the case of FCC metals, plastic deformation is assumed to occur by dislocation glide on $\{1\,1\,1\}$ crystallographic planes and in $\langle 1\,\bar{1}\,0 \rangle$ directions. In order to accommodate any imposed deformation, Bishop and Hill [13] showed that the stress state in each grain corresponds to the stress on one of the 56 vertices of the single crystal yield surface. The active vertex is defined by the maximum plastic work principle. For an isotropic polycrystal, the grain orientation distribution is uniform in orientation space and the overall stress acting on the polycrystal results from averaging the stress over all the grain orientations. The resulting polycrystal yield surface is represented in the π-plane in Fig. 2. This figure also shows the yield surface obtained with Eq. (43) for $m = 20$ (near Tresca), $m = 8$ and $m = 2$ (Von Mises). For $m = 8$, the corresponding yield surface is in excellent agreement with the polycrystal yield surface computed for isotropic FCC materials.

The numerical solution of the equations derived in the previous Sections using the yield condition (Eq. (43)) with $m = 2, 8$ and 20 is discussed in the next Section.

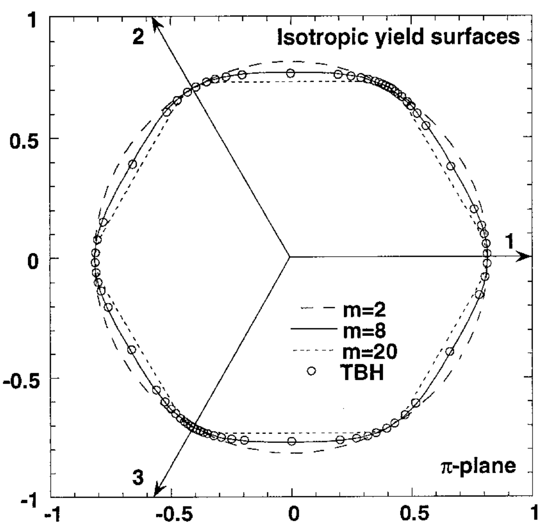

Fig. 2. Isotropic yield surfaces in the π-plane predicted with the yield function exponent $m = 2, 8$ and 20 and with the Taylor/Bishop and Hill (TBH) polycrystal model. Stresses are normalized by the uniaxial flow stress σ_y.

9 Results and discussion

The deviatoric stresses normalized by the uniaxial flow stress σ_y are represented as functions of θ in Figs. 3a, 3b and 3c for m, the yield function exponent, equal to 2 (Von Mises), 8 (FCC material) and 20 (near Tresca), respectively. The stresses, particularly the component s_φ, vary significantly for the different values of m used in this numerical application. For illustration,

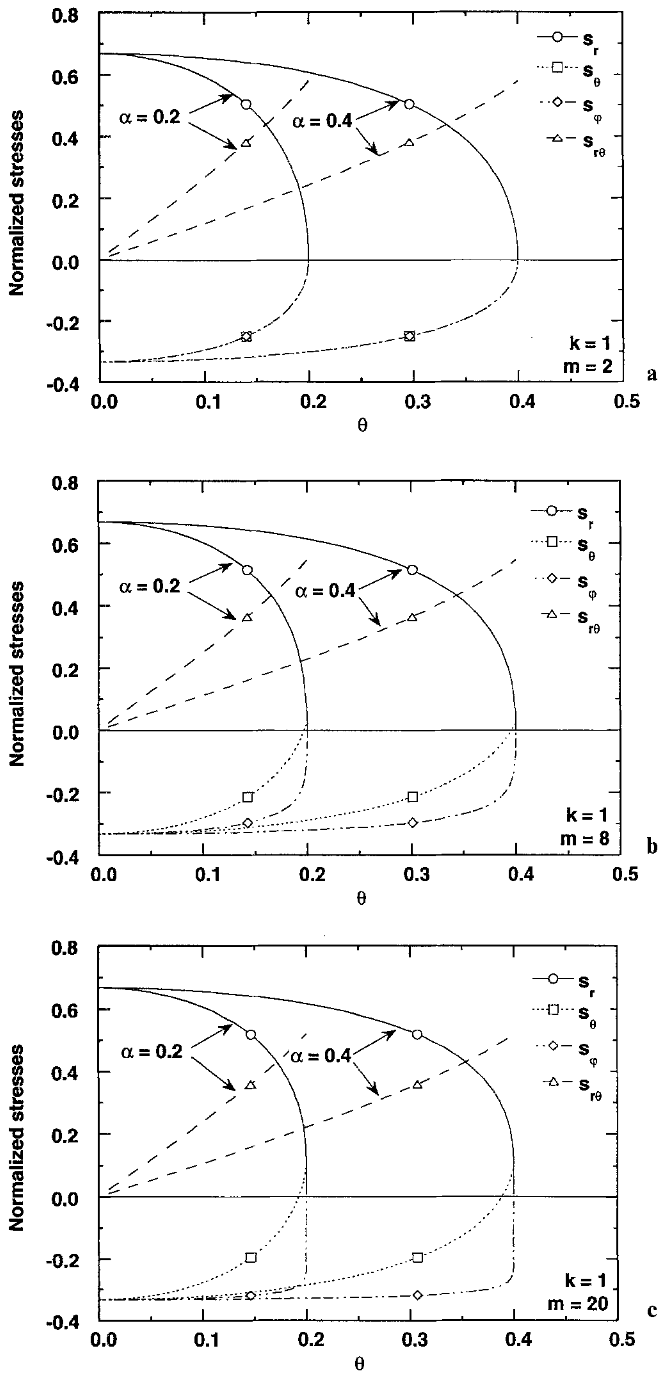

Fig. 3. Deviatoric stresses normalized by the uniaxial flow stress σ_y as a function of θ for two values of the channel semi-angle α (0.2 and 0.4). **a** $m = 2$; **b** $m = 8$; **c** $m = 20$

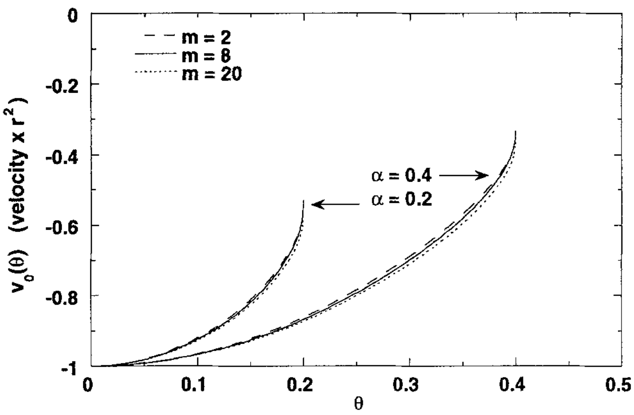

Fig. 4. Velocity as a function of the angle θ for two values of the channel semi-angle α (0.2 and 0.4) and for 3 values of the yield function exponent m (2, 8 and 20)

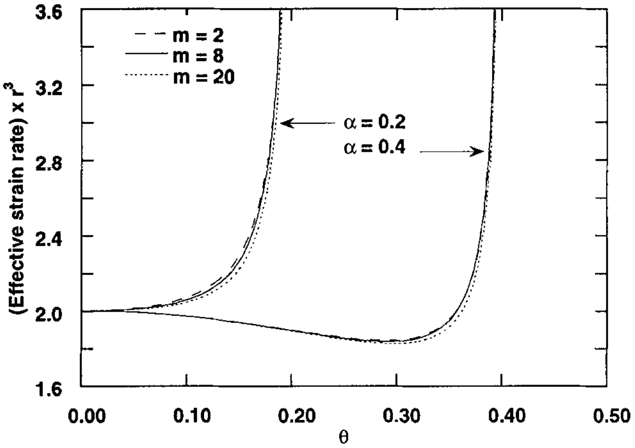

Fig. 5. Strain rate as a function of the angle θ for two values of the channel semi-angle α (0.2 and 0.4) and for 3 values of the yield function exponent m (2, 8 and 20)

two values of α, 0.2 and 0.4, were used to compute the stresses in these figures. In order to find the hydrostatic component of the stress, Eq. (29) can be used. However, the value of the function $A(\theta)$ cannot be determined in the case of an infinite channel. Nevertheless, Eq. (29) is useful for an approximate analysis of extrusion and wire-drawing processes.

Finally, the radius-independent part of the velocity profile v_0, defined by the differential equation in Eq. (42), was computed as a function of α and m, assuming that the absolute value of v_0 on the channel axis ($\theta = 0$) was 1. Figure 4 shows that v_0 depends on the channel semi-angle α but does not vary very much with m. Moreover the shear strain rate $\varepsilon_{r\theta}$ as well as the effective strain rate $\bar{\varepsilon}$ approach infinity for $\psi = \pi/4$, i.e., when the friction stress on the channel wall is maximum, equal to the shear yield stress. The behavior of these quantities is independent of the prescribed value v_0 on the channel axis. The effective strain rate is plotted as a function of θ in Fig. 5. It can be shown that the velocity component tangent to the friction surface follows an inverse square rule near this surface.

10 Conclusions

The solution of the problem of flow through a converging axisymmetric channel for an isotropic, incompressible, rigid-perfectly plastic material obeying the classical flow theory of plasticity with an arbitrary isotropic yield condition was found. It is shown that, in the case of a rough channel wall where maximum friction forces occur (material in contact with the wall yields with the shear yield stress), the effective strain rate approaches infinity. This result has two important practical consequences. First, because of the asymptotic behavior of the velocity near the surface with maximum friction forces, special boundary elements near the surface are required when numerical codes such as those based on finite element methods are developed. Second, because the effective strain rate influences many material properties and physical fields, the material in the surface layers may be subject to structural transformations, which may require material models and constitutive equations different from the bulk properties.

When the exponent m of the Hershey-Hosford yield criterion increases from 2 to 8 and 20, leading to criteria evolving from Von Mises to FCC-like and Tresca-like criteria, respectively, the deviatoric stresses near the channel wall change very quickly. Moreover, the deviatoric component s_θ changes its sign which may lead to some effects in the surface layer such as, depending on the hydrostatic pressure, loss of contact between material and channel wall that would not occur if the Von Mises yield condition was adopted.

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