Acta Mechanica 150, 23–37 (2001)

ACTA MECHANICA

Fully developed flow of a modified second grade fluid with temperature dependent viscosity

M. Massoudi and **T. X. Phuoc**, Pittsburgh, Pennsylvania

(Received May 11, 2000)

Summary. In this paper we will study the fully developed flow of a modified (and sometimes referred to as the generalized) second grade fluid down an inclined plane. The reasons for using such a model for the flow of non-Newtonian fluids are (i) the capability of predicting the normal stress differences and (ii) allowing for the possibility of shear dependent viscosity. The boundary value problem is solved numerically, and the special case of constant viscosity amends itself an exact solution (as previously reported in the literature) which serves as a test case to check the accuracy of our numerical scheme. The velocity and temperature profiles are obtained for various dimensionless numbers, for the case where the viscosity is also a function of temperature.

1 Introduction

Two distinct features of many non-Newtonian fluids are either the fact that many of them exhibit normal stress differences or the fact that their viscosity depends on the shear rate. Of course, there are other features such as yield stress, time-dependency, history effects and other nonlinear issues; in these cases more complex constitutive relations should be used. Perhaps the simplest model which can predict the normal stress differences is the second grade fluid, or the Rivlin-Ericksen fluid of grade two (cf. Rivlin and Ericksen [22], or Truesdell and Noll [32]). This model has been used and studied extensively (cf. Dunn and Fosdick [6]), and is a special case of fluids of differential type (cf. Dunn and Rajagopal [7]). Though this model is relatively simple (even though higher order terms are introduced) its viscosity is assumed to be constant, and thus it cannot be used for fluids where experimental data indicate shear dependent viscosity. At the same time, one of the most widely used non-Newtonian models in the field of engineering is the so-called "Power-law" model (cf. Bird et al. [5] or Slattery [28]), which allows for the viscosity to depend on the velocity gradient. This model has been extensively used in coal-water slurries (cf. Shook and Roco [27]). However, this model cannot predict the normal stress effects which could lead to phenomena like "die-swell" and "rod-climbing", (cf. Schowalter [25]) which are manifestations of the stresses that develop orthogonal to planes of shear.

One of the main areas of interest in energy related processes, such as power plants, atomization, alternative fuels, etc., is the use of slurries, specifically coal-water or coal-oil slurries, as the primary fuel. Some studies indicate that the viscosity of coal-water mixtures depends not only on the volume fraction of solids and the mean size and the size distribution of the coal, but also on the shear rate, since the slurry behaves as a shear-thinning fluid (cf. Roh et al.

[23], Papachristodoulou and Trass [18], Tsai and Knell [34]). At the same time, there are studies which indicate that preheating the fuel results in better performance (cf. Tsai et al. [33], Saeki and Usui [24]), and as a result of such heating, the viscosity changes. A similar situation, i.e. efficient heating or cooling of a liquid occurs in the flow of a thin film along a solid surface which is kept at a constant temperature. Astarita et al. [4] studied the fully developed flow of a non-Newtonian film along a plane surface. Gupta [11] studied the flow of a second grade fluid down an inclined plane. He also performed a linearized stability analysis. Stability analysis of non-Newtonian fluids, especially second grade fluids has received much attention (cf. Dunn and Fosdick [6], Straughan [29], [30]). In recent years, heat transfer to a falling fluid film has been a subject of extensive research (cf. Rao [20], Andersson and Shang [3], Shang and Andersson [26]). In most of these studies, the non-Newtonian fluid is represented using the power-law model. There are very few studies where the effects of viscous dissipation are included, even though this has been shown to be very important in many cases such as polymer processing (cf. Winter [35]).

Man et al. [15] and Man [16] proposed models where the shear viscosity (and also the normal stress coefficients) of a second grade fluid would also depend on the shear rate. Therefore this new "modified" or "generalized" second grade fluid is not only capable of predicting normal stress differences, but it can be used for shear-thinning and shear-thickening fluids also. Gupta and Massoudi [12] proposed a model based on Man's work [16] where in addition to the above effects they suggested that the shear viscosity can also depend on temperature. Therefore, their model becomes very useful for cases when the effect of viscous dissipation and the effect of temperature on viscosity cannot be ignored. Franchi and Straughan [10] have done a stability analysis for a modified second grade fluid where the viscosity is assumed to be a linear function of temperature.

In this paper we will study the fully developed flow of a non-Newtonian fluid down a heated inclined plane. The constitutive relation is that of the modified second grade fluid where the viscosity is assumed to depend on temperature (cf. Gupta and Massoudi [12]).

2 Constitutive relation

Fluids of differential type form an important class of non-Newtonian fluids. The thermodynamics and stability of fluids of second and third grade have been studied extensively by Dunn and Fosdick [6] and Fosdick and Rajagopal [8], and a recent review article by Dunn and Rajagopal [7] sheds light on many of the interesting and challenging issues in the fluids of differential type. The generalized (or modified) form of the second grade fluid is where the shear viscosity (or all the rheological properties, i.e., the normal stress coefficient, as well) depend on the shear rate.

The constitutive relation for the modified second grade fluid is given by (cf. Man [16])

$$\boldsymbol{T} = -p\mathbf{1} + \mu\pi^{m/2}\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1{}^2 , \tag{1}$$

where $\boldsymbol{T}$ is the Cauchy stress tensor; p is the indeterminate part of the stress due to the constraint of incompressibility, μ is the coefficient of viscosity, and α_1 and α_2 are material moduli usually referred to as the normal stress coefficients. The kinematical tensors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are the first and the second Rivlin-Ericksen tensors (cf. Rivlin and Ericksen [22]), respectively, and are given by

$$A_1 = L + L^T, \tag{2}$$

$$A_1 = \frac{dA_1}{dt} + A_1 L + L^T A_1, \tag{3}$$

$$L = \nabla v, \tag{4}$$

where v denotes the velocity field, ∇ is the gradient operator, and d/dt is the material time derivative which is defined as

$$\frac{d(.)}{dt} = \frac{\partial(.)}{\partial t} + [\nabla(.)]v, \tag{5}$$

where $\partial/\partial t$ is the partial time derivative. In Eq. (1), μ, m, α_1 and α_2 are material functions which only depend on temperature; when $m < 0$, the fluid is "shear-thinning", while if $m > 0$, the fluid is "shear-thickening". When $m = 0$, Eq. (1) reduces to the standard second grade fluid of Rivlin-Ericksen. At the same time, if $\alpha_1 = \alpha_2 = 0$, Eq. (1), reduces to the generalized power-law model. In Eq. (1), $\pi \equiv (1/2)\,\mathrm{tr}\,(A_1{}^2)$. The constitutive equation (1), proposed by Man et al. [14] has been used by Man [16] to study the non-steady channel flow of ice and by Gupta and Massoudi [12] to study the flow between two heated plates.

3 Governing equations

One of the most widely studied problems, due to its significant industrial applications and ease of computation, is the flow down an inclined plane. When the fluid is assumed to be represented by a power-law model, with a constant viscosity, and where fully developed conditions are assumed, it is possible to obtain an exact solution for the velocity field (cf. Bird et al. [5, p. 217]). For the general two-dimensional cases, including heat transfer effects, Andersson and co-workers have studied this problem extensively. The stability of a liquid film, where the constitutive equation is that of a second grade fluid, was studied by Gupta [11], where again for a fully developed condition an exact solution was obtained. Recently Akyildiz [1] has studied the flow of a third grade fluid flowing down a vertical wall which is oscillating longitudinally. In most of the studies, the effects of temperature dependent viscosity and viscous dissipation are ignored. In the present study, we will try to look into these issues, assuming that the constitutive relation for the stress is that of a modified second grade fluid, with a temperature dependent viscosity.

For a fully developed flow, we seek velocity and temperature fields of the form

$$\begin{aligned} v &= u(y)\, i, \\ \theta &= \theta(y), \end{aligned} \tag{6}$$

where i is the unit vector in the x-direction (the direction of the flow), while y is normal to the inclined plane, with α being the angle of inclination. Substituting Eqs. (1) and (6) into the balance of linear momentum,

$$\varrho \frac{dv}{dt} = \mathrm{div}\, T + \varrho b, \tag{7}$$

and using the fact that the fluid can undergo only isochoric motions (incompressibility constraint), i.e. div $\boldsymbol{v} = 0$, we obtain

$$\frac{\partial}{\partial x}\left[-p+\alpha_2\left(\frac{du}{dy}\right)^2\right]+\frac{\partial}{\partial y}\left(\mu\left[\left|\frac{du}{dy}\right|^2\right]^{m/2}\frac{du}{dy}\right)+\varrho g\sin\alpha=0\,, \tag{8}$$

$$\frac{\partial}{\partial y}\left[-p+(2\alpha_1+\alpha_2)\left(\frac{du}{dy}\right)^2\right]-\varrho g\cos\alpha=0\,, \tag{9}$$

$$\frac{\partial p}{\partial z}=0\,. \tag{10}$$

If we define a modified pressure p^* through (cf. Rajagopal and Gupta [19])

$$p^*=p-(2\alpha_1+\alpha_2)\left(\frac{du}{dy}\right)^2, \tag{11}$$

Eqs. (8)–(10) become

$$\frac{\partial p^*}{\partial x}=\frac{\partial}{\partial y}\left[\mu\left(\left|\frac{du}{dy}\right|^2\right)^{m/2}\frac{du}{dy}\right]+\varrho g\sin\alpha\,, \tag{12}$$

$$\frac{\partial p^*}{\partial y}=-\varrho g\cos\alpha\,, \tag{13}$$

$$\frac{\partial p^*}{\partial z}=0\,. \tag{14}$$

If we set $m=0$ in Eqs. (12)–(13), we obtain the equations of motion for a second grade fluid down an inclined plane given by Gupta [11]. For a pressure-driven flow, using Eq. (11) would indicate that $\partial p^*/\partial y=\partial p^*/\partial z=0$, and thus Eq. (12) can be integrated directly. In the present problem, however, the pressure gradient in the normal direction is not equal to zero. Let us look at the boundary conditions for this problem. At the surface of the plane, i.e. at $y=0$, we impose the no-slip condition. That is,

$$\text{at}\quad y=0:\quad u=0\,. \tag{15}$$

At the free surface, we impose the traction-free condition. That is,

$$\text{at}\quad y=h:\quad t_x=t_y=0\,. \tag{16}$$

Now,

$$\boldsymbol{t}=\boldsymbol{T}^T\boldsymbol{n}\,, \tag{17}$$

which implies

$$(t_x)_{y=h}=(T_{xy}n_y)_{y=h}=\mu\left[\left|\frac{du}{dy}\right|^2\right]^{m/2}\frac{du}{dy}=0 \tag{18}$$

and

$$(t_y)_{y=h}=(T_{yy}n_y)_{y=h}=-p+(2\alpha_1+\alpha_2)\left(\frac{du}{dy}\right)^2=0\,. \tag{19}$$

Comparing (19) and (11) indicates that

$$p^* = 0 \qquad \text{at the free surface}\,. \tag{20}$$

Now, Eq. (13) implies that for a given angle of inclination, α, we have

$$\frac{\partial p^*}{\partial y} = K_1\,, \tag{21}$$

where K_1 is a constant. Integrating Eq. (21), we have

$$p^* = K_1 y + C_1(x)\,. \tag{22}$$

Applying Eq. (20) to Eq. (22), we have

$$(p^*)_{y=h} = 0 = K_1 h + C_1(x) \to C_1(x) = -K_1 h = \text{const}\,. \tag{23}$$

Now, differentiating Eq. (22) with respect to x we have

$$\frac{\partial p^*}{\partial x} = 0\,. \tag{24}$$

Therefore, Eq. (12) becomes

$$\frac{\partial}{\partial y}\left(\mu\left[\left|\frac{du}{dy}\right|^2\right]^{m/2}\frac{du}{dy}\right) = -\varrho g \sin\,\alpha\,. \tag{25}$$

Integrating with respect to y, we have

$$\mu\left(\left|\frac{du}{dy}\right|^2\right)^{m/2}\frac{du}{dy} = -y\varrho g \sin\,\alpha + C_2\,. \tag{26}$$

Now, applying Eq. (18), we have

$$(t_x)_{y=h} = 0 \to -\varrho g h \sin\,\alpha + C_2 = 0 \to C_2 = +\varrho g h \sin\,\alpha\,, \tag{27}$$

and therefore Eq. (26) becomes:

$$\mu\left|\frac{du}{dy}\right|^m \frac{du}{dy} = -\varrho g(y-h)\sin\,\alpha\,. \tag{28}$$

In order to solve this equation, we need to know how the viscosity changes as a function of temperature, and therefore we also have to consider the balance of energy which is

$$\varrho\frac{d\varepsilon}{dt} = \boldsymbol{T}.\boldsymbol{L} - \operatorname{div}\boldsymbol{q} + \varrho r\,, \tag{29}$$

where ε is the specific internal energy, $\boldsymbol{q}$ is the heat flux vector, and r is the radiant heating. We assume that $\boldsymbol{q}$ is given by Fourier's conduction law,

$$\boldsymbol{q} = -k\nabla\theta\,, \tag{30}$$

where θ is the temperature and k is thermal conductivity, which is assumed to be constant. Now the specific internal energy ε is related to the specific Helmholtz free energy through (cf. Dunn and Fosdick [6]):

$$\varepsilon = \psi + \theta\eta = \varepsilon(\theta, \boldsymbol{A}_1, \boldsymbol{A}_2) = \hat{\varepsilon}(y)\,, \tag{31}$$

where η is the specific entropy. It follows from Eq. (6) that

$$\frac{d\varepsilon}{dt} = 0\,, \tag{32}$$

and the energy equation reduces to (cf. Gupta and Massoudi [12])

$$\mu(\theta)\left|\frac{du}{dy}\right|^{m+2} + k\,\frac{d^2\theta}{dy^2} = 0\,. \tag{33}$$

The boundary conditions for this equation are:

$$\theta = \theta_1 \quad \text{at} \quad y = 0\,, \tag{34}$$

$$\theta = \theta_2 \quad \text{at} \quad y = h\,, \tag{35}$$

where $\theta_2 > \theta_1$. The first term in Eq. (33) represents viscous dissipation which could be very important for the flow of many polymers and viscous oils. The basic equations for the present problem are the momentum equation, given by Eq. (28) subject to the boundary condition (15), and the energy equation, given by Eq. (33) subject to the boundary conditions (34) and (35). These equations are coupled, and the form of the momentum equation depends on the form of the viscosity function. We will discuss these and the dimensionless numbers pertinent to this problem in the next section.

4 Nondimensionalization

Let us introduce the following nondimensional parameters:

$$\begin{aligned} &\theta^* = \frac{\theta - \theta_1}{\theta_2 - \theta_1}\,, \qquad \mu^* = \frac{\mu}{\mu_o}\,, \\ &\xi = \frac{y}{h}\,, \qquad u^* = \frac{u}{V}\,, \end{aligned} \tag{36}$$

where μ_o and V are some reference viscosity and velocity, respectively. With these, the dimensionless form of the energy equation becomes

$$\frac{d^2\theta^*}{d\xi^2} + \mu^*\Gamma\left|\frac{du^*}{d\xi}\right|^{m+2} = 0\,, \tag{37}$$

where

$$\Gamma = \mu_o\left|\frac{V}{h}\right|^{m+2}\frac{h^2}{k(\theta_2 - \theta_1)}\,, \tag{38}$$

and the boundary conditions become

$$\theta(0) = 0\,, \qquad \theta(1) = 1\,. \tag{39}$$

Now, Γ can be re-written as

$$\Gamma = \frac{V^2}{c_p(\theta_2 - \theta_1)}\,\frac{c_p\mu}{k}\left(\frac{V}{h}\right)^m\,, \tag{40}$$

where we can define Eckert and Prandtl numbers (cf. Etemad and Mujumdar [9]):

$$\mathrm{Ec} = \frac{V^2}{c_p(\theta_2 - \theta_1)}\,, \tag{41}$$

$$\mathrm{Pr} = \frac{c_p \mu}{k}\left(\frac{V}{h}\right)^m, \tag{42}$$

and $\Gamma = \mathrm{Ec}\,\mathrm{Pr}$ is also known as the Brinkman number.

For the viscosity, we assume that the fluid obeys the Reynolds viscosity law (used in many lubrication problems (cf. Reynolds [21], Szeri and Rajagopal [31]), where

$$\mu^* = e^{-M\theta^*}\,; \qquad M = n(\theta_2 - \theta_1)\,; \tag{43}$$

n is a constant. With this, the dimensionless form of the momentum equation (Eq. (28)) becomes

$$\frac{du^*}{d\xi} = \beta^{\frac{1}{1+m}}\left[(1-\xi)\,e^{M\theta^*}\right]^{\frac{1}{1+m}}(\sin\,\alpha)^{\frac{1}{1+m}}\,, \tag{44}$$

and Eq. (37) now becomes

$$\frac{d^2\theta^*}{d\xi^2} + \Gamma\beta^{\frac{m+2}{m+1}}\,e^{\frac{M\theta^*}{1+m}}\,(1-\xi)^{\frac{m+2}{m+1}} = 0\,, \tag{45}$$

where

$$\beta = \frac{\varrho g h^{m+2}}{\mu_o V^{m+1}}\,, \tag{46}$$

$$\beta' = \frac{\varrho g h^{m+2}}{\mu_o V^{m+1}}\sin\,\alpha\,. \tag{47}$$

Now, the boundary condition for the velocity in Eq. (44) becomes

$$u^*(0) = 0\,, \tag{48}$$

recalling that the condition at the free surface has already been used. Now β can be re-written as

$$\beta = \left(\frac{gh}{V^2}\right)\frac{\varrho h^{m+1}V^{1-m}}{\mu_o}\,, \tag{49}$$

where we can define

$$Fr = \frac{V^2}{gh}\,, \tag{50}$$

$$\mathrm{Re} = \frac{\varrho h^{m+1}V^{1-m}}{\mu_o}\,. \tag{51}$$

And therefore, β is the ratio of the viscous effects to the gravitational effects. For power-law fluids, the Reynolds number is typically given as

$$\mathrm{Re} = \frac{\varrho L^n V^{2-n}}{K}\,, \tag{52}$$

where n is the power-law index, and K is the coefficient of consistency (cf. Andersson and Shang [3]). Comparing Eq. (52) with Eq. (51), it is clear that the parameter m in this model is related to the power-law index n:

$$m = n - 1\,. \tag{53}$$

In the next section, we will present the solution to the problem for two different cases; the first case is when the viscosity is not a function of temperature, and the second case is when the viscosity-temperature relationship is given by the Reynolds model.

5 Numerical solutions

5.1 Constant viscosity $(M = 0) : \mu(\theta) = \mu_o$

In the case of constant viscosity Eq. (44) becomes independent of the temperature, and it can be re-written as

$$\frac{du^*}{d\xi} = \beta^{\frac{1}{1+m}} (1 - \xi)^{\frac{1}{1+m}} (\sin\,\alpha)^{\frac{1}{1+m}}\,. \tag{54}$$

Equation (54) subject to condition (48) describes the velocity distribution in the ξ-direction under the effects of β, and power m. The solution is possible only when $m \neq -1$ and $m \neq -2$. Thus, using the boundary condition (48), Eq. (54) is integrated, and we obtain the following solution for the dimensionless velocity:

$$u^* = \beta^{\frac{1}{m+1}} \left(\frac{m+1}{m+2}\right) \left[1 - (1 - \xi)^{\frac{m+2}{m+1}}\right] (\sin\,\alpha)^{\frac{1}{1+m}}\,; \quad m > -1\,. \tag{55}$$

Now, when $m = 0$ (i.e., when we have a second-grade fluid with constant shear viscosity), Eq. (55) becomes:

$$u^* = \frac{\beta}{2}\,(2\xi - \xi^2)\,\sin\,\alpha\,, \tag{56}$$

$$u = \frac{\varrho g h^2 \sin\alpha}{2\mu_o}\,(2\xi - \xi^2)\,. \tag{57}$$

This expression was given by Gupta [11]. Figure 1 shows the dependence of the dimensionless velocity profiles using power m as a parameter and keeping β' constant. The results are given for $m = -0.5, 0$, and 0.5 when $\beta = 4.0$ and $\alpha = 90°$. It is clear that as m increases the curves shift from right to left. That is, for the case of shear thinning, where m is negative, the velocity looks similar to that of plug flow. For the case of shear thickening, where m is positive, however, the flow seems to be more retarded near the wall. We can also see that the maximum velocity, which is achieved at the free surface, has a higher value for $m = -0.5$ (shear-thinning fluid) than for $m = 0.5$ (shear-thickening fluid).

The effects of the parameter β and the inclination angle α are shown in Figs. 2 and 3, respectively, for two values of the exponent m. As Fig. 2a, b shows, for a given α, as β increases, the velocity profiles change from an almost uniform (flat) profile to a fully-developed-type profile. For smaller β's, the profiles are more flat. It is interesting to note that for the same β (for example $\beta = 4.0$) and the same α, the maximum velocity for the case $m = -0.25$ (Fig. 2a) is greater than the maximum velocity for the case $m = 0.25$ (Fig. 2b).

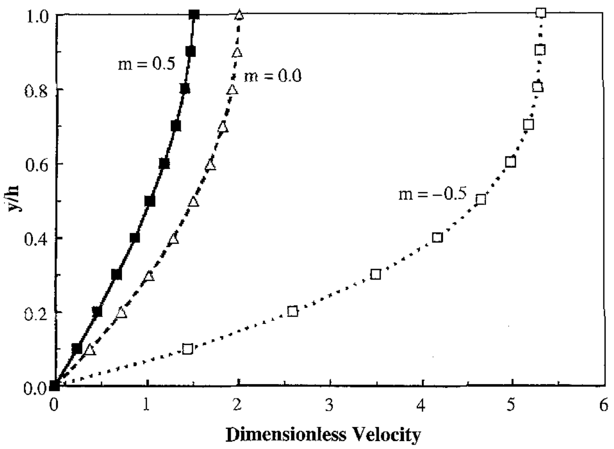

Fig. 1. Typical dimensionless velocity profiles calculated for $\beta = 4.0$ and $\alpha = 90°$ with three values of m, (–0.5, 0, and 0.5). The dimensionless velocity u^* was defined by Eq. (36) (constant viscosity, $M = 0.0$)

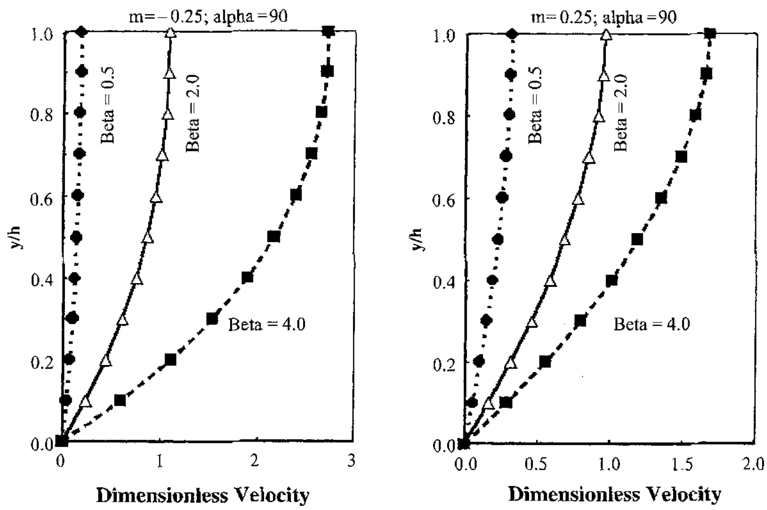

Fig. 2. Effect of parameter β on the dimensionless velocity profiles

That is, for a shear-thinning fluid, the flow achieves a higher velocity at the free surface. For a given β, similar effects are observed when α increases. Again, for the case of a shear-thinning fluid, $m = -0.25$ and $\alpha = 90°$ (for example), the maximum velocity is almost 1.7 times the maximum velocity for a shear-thickening fluid with $m = 0.25$.

Figures 4 and 5 show the results for the gradient of the dimensionless velocity at the wall, $(du^*/d\xi)_{\xi=0}$, as a function of β and α, respectively, for $m = -0.25, 0$, and 0.5. For a given m, $(du^/d\xi)_{\xi=0}$ increases as β or α increases. This is more drastic when β is less than 1. For a given β, the effects of the exponent m on $(du^*/d\xi)_{\xi=0}$ are shown in Fig. 6. When β is less than 1, $(du^*/d\xi)_{\xi=0}$ initially increases as m increases, and then it flattens out. When β is greater than one, $(du^*/d\xi)_{\xi=0}$ initially decreases as m increases, and gradually it becomes a constant. The dimensionless temperature profiles for the constant viscosity case are also given in Fig. 7. As we can see from Fig. 7a, for a shear-thinning fluid, $m = -0.25$ and a given β, the temperature does not vary much as we change Γ. That is, if we consider Γ as a measure of viscous dissipation, we can see that its effect is more observable for a shear-thickening fluid (see Fig. 7b) than for a shear-thinning fluid. In either case, the temperature profile is almost linear.

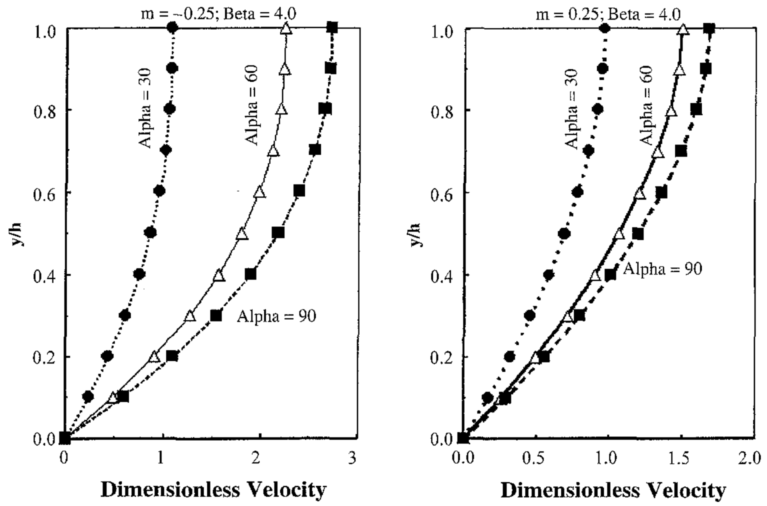

Fig. 3. Effect of parameter inclination angle α on the dimensionless velocity profiles

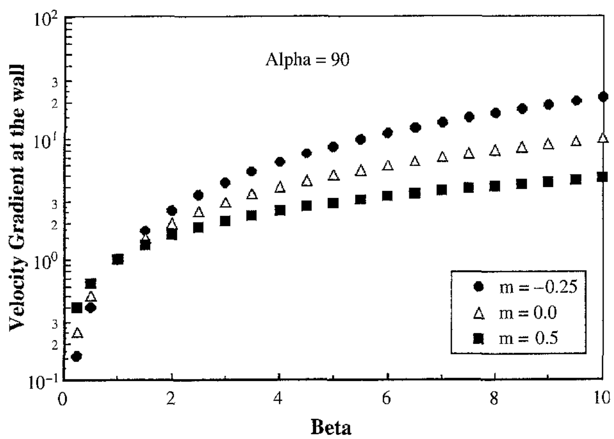

Fig. 4. Dimensionless velocity gradient at the surface as a function of parameter β: Effect of the power m (constant viscosity case, $M = 0$)

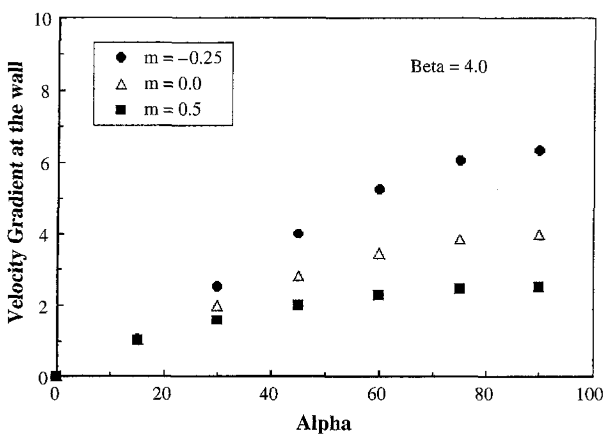

Fig. 5. Dimensionless velocity gradient at the surface as a function of inclination α: Effect of the power m (constant viscosity case, $M = 0$)

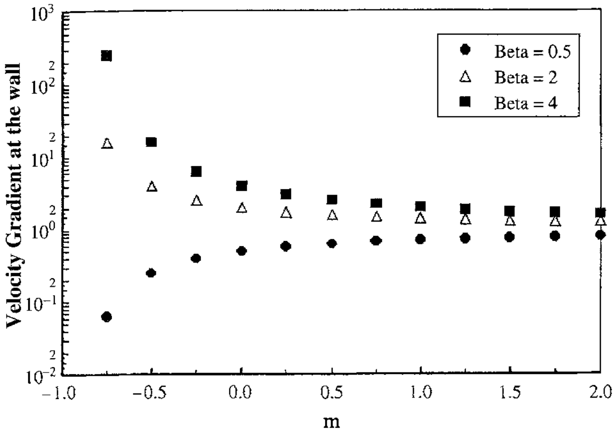

Fig. 6. Dimensionless velocity gradient at the surface of the plate as a function of power m: effect of β (constant viscosity case, $M = 0.0$; $\alpha = 90°$)

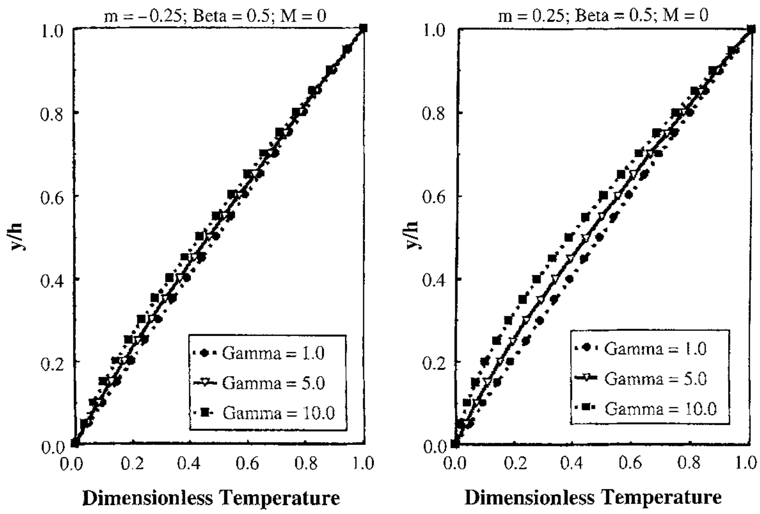

Fig. 7. Dimensionless temperature profiles (constant viscosity; $\alpha = 90°$)

5.2 Variable viscosity

When the viscosity depends on the temperature, Eqs. (44) and (45) must be solved simultaneously. Typical results for the velocity profiles are shown in Figs. 8 to 10 for both shear-thinning and shear-thickening fluids with three values of M. As shown in Fig. 8, we can see that as M increases, the velocity seems to remain the same near the wall and then, at a distance of approximately $\xi = 0.2$ for $m = -0.25$ and $\xi = 0.25$ for $m = 0.25$, the velocity profiles begin to change rapidly, reaching their maximum values at the free stream. We can see that at the same distance from the wall (for example $\xi = 0.8$), and for the same values of $\beta = 1$ and $\Gamma = 1$ the velocity is higher for larger values of M. This is true whether the fluid is shear-thinning or shear-thickening. When β and Γ are both equal to 1, the magnitude of the velocity near the free stream is seen to be almost the same for both shear-thinning and shear-thickening fluids as we can see from Fig. 8. However, this is not the case when either β or Γ is not equal to one. For example, Fig. 9 reveals that with $\beta = 0.5$, for shear-thickening fluids, the dimensionless velocity u^* at the free surface is found to be 0.32, 0.44, and 0.52

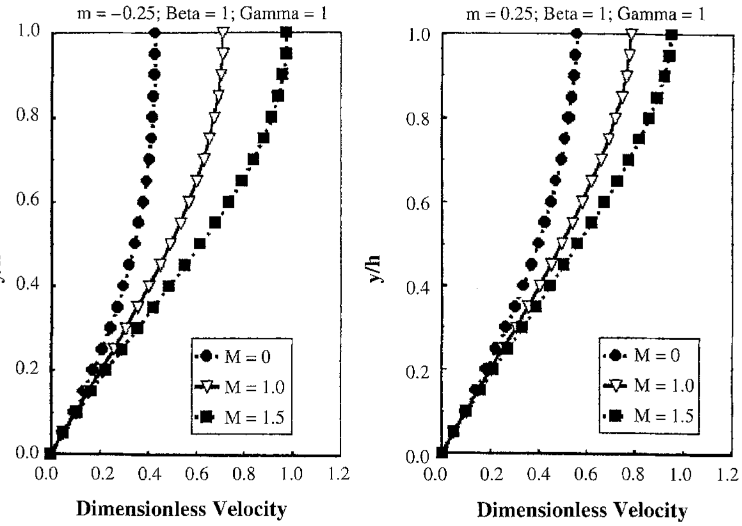

Fig. 8. Effects of temperature-dependent viscosity on the dimensionless velocity profile ($\alpha = 90°$)

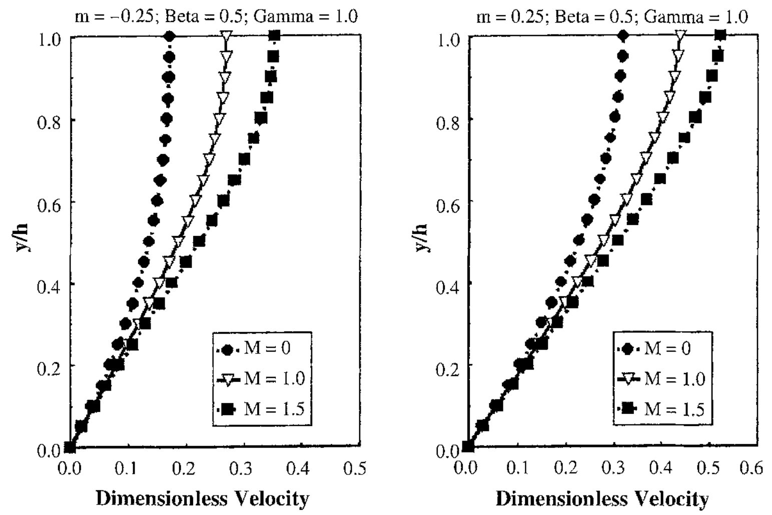

Fig. 9. Effects of temperature-dependent viscosity on the dimensionless velocity profile ($\alpha = 90°$)

for M being equal to 0, 1.0, and 1.5, respectively, while for shear-thinning fluids the value of the velocity at the free surface is 0.17, 0.26, and 0.35 for the same values of M. Figure 10 shows a similar trend when $\Gamma = 2$. Comparing Figs. 8 and 10 we can see that as Γ changes from 1 to 2, the free surface velocity for a shear-thinning fluid ($m = -0.25$) and for a given $M = 1.5$ is much higher for the larger value of Γ, whereas for a shear-thickening fluid ($m = 0.25$) and for the same $M = 1.5$ the free stream velocity almost remains the same regardless of the value of Γ.

From the energy equation (Eq. (45)), it is clear that when either β or Γ is equal to zero the temperature variation is linear throughout the flow domain. However, when either β or Γ is not equal to zero or when the viscosity is temperature dependent, the temperature distribution becomes highly nonlinear as shown in Fig. 11. In this case the calculations reveal the development of a thermal boundary layer near the wall. For $\Gamma = 1.0$, the temperature

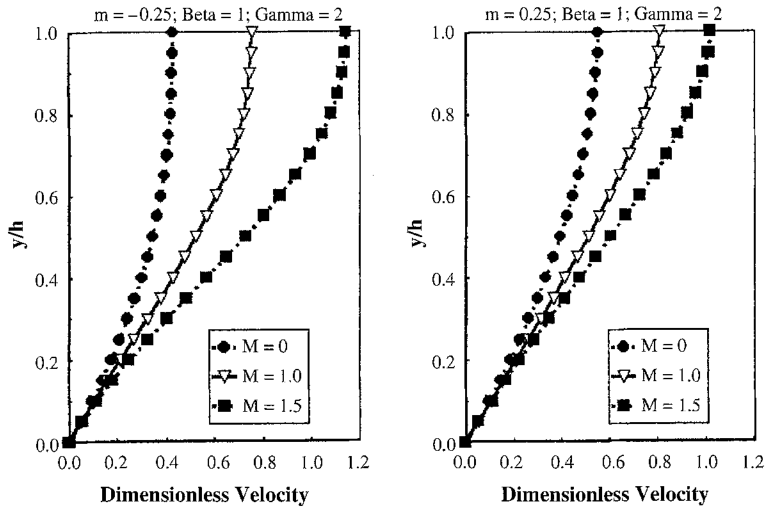

Fig. 10. Effects of temperature-dependent viscosity on the dimensionless velocity profile ($\alpha = 90°$)

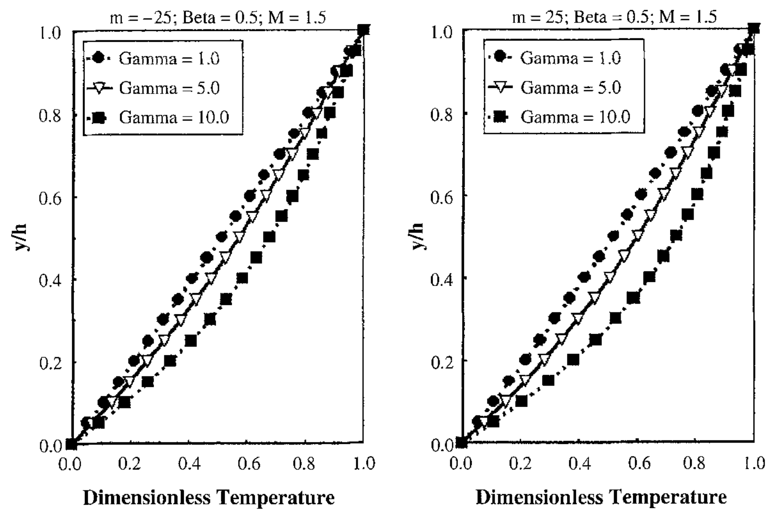

Fig. 11. Dimensionless temperature profiles ($\alpha = 90°$)

profiles seem to be almost linear for $m < 0$ or $m > 0$, for a given β and M. This effect is also seen in the case of constant viscosity. We can see that as Γ increases, i.e. as the effect of viscous dissipation increases, the temperature rises within the fluid. That is, for a given height, for example $\xi = 0.4$, when $\Gamma = 1.0, \theta^* \approx 0.4$, while when $\Gamma = 10.0, \theta^* \approx 0.6$ for a shear-thickening fluid ($m = 0.25$), Fig. 11b. A similar trend is also observed for the shear-thinning case (Fig. 11a).

6 Conclusions

The fully developed film flow of a non-Newtonian fluid, modeled as a modified second grade fluid with a temperature dependent viscosity, down an inclined plane is studied. The results in terms of dimensionless velocity and temperature profiles are presented for various dimensionless numbers. The significant parameters in this problem are the angle of inclination α, a dimensionless number β which is related to the Reynolds and Froude numbers, a dimensionless number Γ, a measure of viscous dissipation, which is related to the Eckert and Prandtl numbers, an exponent m which is related to the power-law index n (for a power-law fluid model), and an exponent M which represents the variation of viscosity with temperature. Due to the kinematics of the flow, the normal stress coefficients, α_1 and α_2, are absorbed in the pressure term, and for this particular problem they do not affect the velocity and temperature fields directly. The case where viscosity is not a function of temperature, i.e., $M = 0$, and where the viscosity is also not shear dependent, i.e., $m = 0$, provides an exact solution for the velocity profile given by Gupta [11] who also considered the stability of such a flow. This exact solution provides a test case for our numerical scheme. We feel that the model proposed here is a general and an important model for non-Newtonian fluids, since it includes the effects of normal stress differences, shear dependent viscosity, and temperature dependent viscosity.

Acknowledgement

We are grateful to Dr. B. Straughan for his helpful and encouraging comments.

References

[1] Akyildiz, F. T.: A note on the flow of a non-Newtonian fluid film. Int. J. Non-Linear Mech. **33,** 1061–1067 (1998).
[2] Al-Besharah, J. M., Akashah, S. A., Mumford, C. J.: The effect of temperature and pressure on the viscosities of crude oils and their mixtures. Ind. Eng. Chem. Res. **28,** 213–221 (1989).
[3] Andersson, H. I., Shang, D. Y.: An extended study of the hydrodynamics of gravity-driven film flow of power-law fluids. Fluid Dyn. Res. **22,** 345–357 (1998).
[4] Astarita, G., Marrucci, G., Palumbo, G.: Non-Newtonian gravity flow along inclined plane surfaces. Ind. Engng Chem. Fund. **34,** 333–339 (1964).
[5] Bird, R. B., Armstrong, R. C., Hassager, J.: Dynamics of polymeric liquids, vol. 1. New York: Wiley 1977.
[6] Dunn, J. E., Fosdick, R. L.: Thermodynamics stability, and boundedness of fluids of complexity 2 and fluids of second grade. Arch. Rat. Mech. Anal. **56,** 191–252 (1974).
[7] Dunn, J. E., Rajagopal, K. R.: Fluids of differential type: critical review and thermodynamic analysis. Int. J. Engng Sci. **33,** 689–729 (1995).
[8] Fosdick, R. L., Rajagopal, K. R.: Thermodynamics and stability of fluids of third grade. Proc. R. Soc. London A **339,** 351–377 (1980).
[9] Etemad, S., GH., Mujumdar, A. S.: Effect of variable viscosity and viscous dissipation on laminar heat transfer of a power-law fluid in the entrance region of a semi-circular duct. Int. J. Heat. Mass Transfer **38,** 2225–2238 (1995).
[10] Franchi, H., Straughan, B.: Stability and nonexistence results in the generalized theory of a fluid of second grade. J. Math. Anal. Appl. **180,** 122–137 (1993).
[11] Gupta, A. S.: Stability of a visco-elastic liquid film flowing down an inclined plane. J. Fluid Mech. **28,** 17–28 (1967).
[12] Gupta, G., Massoudi, M.: Flow of a generalized second grade fluid between heated plates. Acta Mech. **99,** 21–33 (1993).
[13] Kumari, M., Pop, I., Takhar, H. S.: Free-convection boundary-layer flow of a non-Newtonian fluid along a vertical wavy surface. Int. J. Heat Fluid Flow **18,** 625–631 (1997).
[14] Man, C. S., Shields, D. H., Kjartanson, B., Sun, Q. K.: Creep of ice as a fluid of complexity 2: the pressure meter problem. Proc. 10th CANCAM, 1, pp. 347–348 (1985).

[15] Man, C. S., Sun, Q. K.: On the significance of normal stress effects in the flow of glaciers. J. Glaciology **33,** 268–273 (1987).
[16] Man, C. S.: Nonsteady channel flow of ice as a modified second-order fluid with power-law viscosity. Arch. Rat. Mech. Anal. **119,** 35–57 (1992).
[17] Massoudi, M., Christie, I.: Effects of variable viscosity and viscous dissipation on the flow of a third grade fluid. Int. J. Non-Linear Mech. **30,** 687–699 (1995).
[18] Papachristodoulou, G., Trass, O.: Coal slurry fuel technology. Canadian J. Chem. Engng **65,** 177–201 (1987).
[19] Rajagopal, K. R., Gupta, A. S.: An exact solution for the flow of a non-Newtonian fluid past an infinite porous plate. Meccanica **19,** 158–160 (1984).
[20] Rao, B. K.: Heat transfer to a falling power-law fluid film. Int. J. Heat Fluid Flow **20,** 429–436 (1999).
[21] Reynolds, O.: On the theory of lubrication and its application to Mr. Beuchamp Tower's experiment, including an experimental determination of the viscosity of olive oil. Phil. Trans. R. Soc **177,** 157–234 (1886).
[22] Rivlin, R. S., Ericksen, J. L.: Stress deformation relations for isotropic materials. J. Rat. Mech. Anal. **4,** 323–425 (1955).
[23] Roh, N. S., Shin, D. Y., Kim, D. C., Kim, J. D.: Rheological behavior of coal-water mixtures: 1. Effects of coal type, loading and particle size. Fuel **74,** 1220–1225 (1995).
[24] Saeki, T., Usui, H.: Heat transfer characteristics of coal-water mixtures. Canadian J. Chem. Engng **73,** 400–404 (1995).
[25] Schowalter, W. R.: Mechanics of non-Newtonian fluids. New York: Academic Press 1978.
[26] Shang, D. Y., Andersson, H. I.: Heat transfer in gravity-driven film flow of power-law fluids. Int. J. Heat Mass Transfer **42,** 2085–2099 (1999).
[27] Shook, C. A., Roco, M. C.: Slurry flow: principles and practices. Boston: Butterworth-Heinemann 1991.
[28] Slattery, J. C.: Advanced transport phenomena. New York: Cambridge University Press 1999.
[29] Straughan, B.: Energy stability in the Bénard problem for a fluid of second grade. J. Appl. Math. Phys. (ZAMP) **34,** 502–508 (1983).
[30] Straughan, B.: The energy method, stability, and nonlinear convection. New York: Springer 1992.
[31] Szeri, A. Z., Rajagopal, K. R.: Flow of a non-Newtonian fluid between heated parallel plates. Int. J. Non-Linear Mech. **20,** 91–101 (1985).
[32] Truesdell, C., Noll, W.: The non-linear field theories of mechanics, 2nd ed. New York: Springer 1992.
[33] Tsai, C. Y., Novack, M., Roffe, G.: Rheological and heat transfer characteristics of flowing coal-water mixtures. DOE Report, DOE/MC/23255-2763, December 1988.
[34] Tsai, S. C., Knell, E. W.: Viscosity and rheology of coal water slurry. Fuel **65,** 566–571 (1986).
[35] Winter, H. H.: Viscous dissipation in shear flows of molten polymers. Adv. Heat Transfer **13,** 205–267 (1977).

Authors' address: M. Massoudi and T. X. Phuoc, U.S. Department of Energy, National Energy Technology Laboratory, P.O. Box 10940, Pittsburgh PA, 15236, U.S.A. (E-mail: Massoudi@netl.doe.gov)