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Note

Slip flow past an approximate spheroid

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Summary. Creeping axisymmetric slip flow past a spheroid whose shape deviates slightly from that of a sphere is investigated. An exact solution is obtained to the first order in the small parameter characterizing the deformation. As an application, the case of flow past an oblate spheroid is considered and the drag experienced by it is evaluated. Special well-known cases are deduced and some observations made.

Notation

$A_n, B_n, C_n, D_n, E_n, F_n, b_2, d_2,$	– Constants
$a, b,$	– radii of spheres
β	– coefficient of sliding fraction
D	– drag
$\in, \alpha_m$	– parameters characterizing the deformation of the sphere
c	– $a(1 + \in)$
μ	– viscosity coefficient
θ	– $\frac{\mu\beta}{a}$
σ	– dimensionless coordinate $\frac{r}{a}$
I_n	– Gegenbauer function
P_n	– Legendre function
Ψ	– Stream function
U	– stream velocity at infinity

1 Introduction

The problem of the Stokes symmetrical flow due to the translation of an approximate spherical solid particle in an unbounded fluid medium was first investigated by Sampson [1]. The case of asymmetric flows has been examined independently by Brenner [2] and Acrivos and Taylor [3]. In all cases the authors assumed the no-slip condition. It is of some interest to examine the possibility that the fluid may slip at the surface. In the case of the perfect sphere this problem was solved by Basset [4].

In this brief Note the problem of symmetric flow past a spheroid whose shape deviates slightly from that of a sphere is examined under the assumption of slip at the surface. An explicit expression is obtained for the stream function associated with the flow field to the first order in the small parameter characterizing the deformation. As an application, we consider the flow past an oblate spheroidal particle and determine the drag experienced by it. Special known cases are then deduced.

2 Statement and solution of the problem

We consider the problem of slow, steady axisymmetrical flow of an incompressible fluid past a spheroid whose shape varies slightly from that of a sphere and which is assumed to be macroscopically at rest in an otherwise uniform stream of speed U in the direction of the negative z-axis in the absence of body forces. We refer the motion to a spherical co-ordinate system (r, θ, φ). The stream function characterizing this type of flow is given by [5]

$$\psi(r, \theta) = \sum_{n=2}^{\infty} (A_n r^n + B_n r^{-n+1} + C_n r^{n+2} + D_n r^{-n+3})\, I_n(\zeta) \tag{2.1}$$

where $\zeta = \cos\theta$ and $I_n(\zeta)$ is the Gegenbauer function related to the Legendre function $P_n(\zeta)$ by the relation

$$I_n(\zeta) = \frac{P_{n-2}(\zeta) - P_n(\zeta)}{2n-1}, \quad n \geqq 2.$$

In particular

$$I_2(\zeta) = \frac{1}{2}(1-\zeta^2),\ I_3(\zeta) = \frac{1}{2}\zeta(1-\zeta^2).$$

These functions have the following special property [5]:

$$I_m I_2 = -\frac{(m-2)(m-3)}{2(2m-1)(2m-3)} I_{m-2} + \frac{m(m-1)}{(2m+1)(2m-3)} I_m - \frac{(m+1)(m+2)}{2(2m-1)(2m+1)} I_{m+2},$$
$$m \geqq 2. \tag{2.2}$$

The stream function ψ is related to the velocity field $(u_r, u_\theta, 0)$ by the usual relationship

$$u_r = -\frac{1}{r^2 \sin\theta} \frac{\partial\psi}{\partial\theta}, \quad u_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{2.3}$$

We take the surface S of a spheroid to be of the form $r = a[1 + f(\theta)]$. The orthogonality of the Gegenbauer functions permit us, under general circumstances, to assume the expansion $f(\theta) = \sum_{k=1}^{\infty} \alpha_k I_k(\zeta)$. Hence, we can take S to be

$$r = a[1 + \alpha_m I_m(\zeta)] \tag{2.4}$$

and neglect terms of $0(\alpha_m^2)$. Our main problem is to determine the flow field.

Using the condition

$$\psi \to \frac{1}{2} U r^2 \sin^2\theta \quad \text{as} \quad r \to \infty, \tag{2.5}$$

we see that we can write (2.1) in the form

$$\frac{\Psi}{Ua^2} = \left[\sigma^2 + \frac{b_2}{\sigma} + d_2\sigma\right] I_2(\zeta) + \sum_{n=3}^{\infty} [E_n \sigma^{-n+1} + F_n \sigma^{-n+3}]\, I_n(\zeta) \tag{2.6}$$

where $\sigma = \frac{r}{a}$. The only coefficients which contribute to flow past a sphere are b_2 and d_2. Consequently all other coefficients in (2.6) are of $0(\alpha_m)$. Hence, except where b_2 and d_2 are encountered, we may take the surface to be $\sigma = 1$.

The unknown coefficients appearing in (2.6) must be determined from the boundary conditions. The kinematic condition of impenetrability at the surface demands that we take

$$\Psi = 0 \quad \text{on } S. \tag{2.7}$$

As regards the slip condition we use the most plausible hypothesis [4] that the tangential velocity of fluid relative to the solid at a point on its surface is proportional to the tangential stress $t_{r\theta}$ prevailing at that point. In our case this hypothesis takes the form

$$t_{r\theta} = \beta u_\theta \quad \text{on } S \tag{2.8}$$

where the constant β is the coefficient of sliding friction. Utilizing the impenetrability condition and the fact that

$$t_{r\theta} = \mu\left[\frac{1}{r}\frac{\partial u_r}{\partial\theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right],$$

one can rewrite (2.8) in the form

$$\mu r\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\psi}{\partial r}\right) = \beta\frac{1}{r}\frac{\partial\psi}{\partial r} \quad \text{on } S. \tag{2.9}$$

With the aid of (2.4), the boundary conditions (2.7) and (2.9) lead respectively to the following:

$$0 = (1 + b_2 + d_2)\, I_2(\zeta) + (2 - b_2 + d_2)\, \alpha_m I_m(\zeta)\, I_2(\zeta) + \sum_{n=3}^{\infty} (E_n + F_n)\, I_n(\zeta), \tag{2.10}$$

$$0 = [(2 - b_2 + d_2) + \theta(2 - 4b_2 + 2d_2)]\, I_2(\zeta) + [(2 + 2b_2) + \theta(12b_2 - 2d_2)\, \alpha_m I_m(\zeta)\, I_2(\zeta) + \sum_{n=3}^{\infty} E_n(1 - n)\, \{1 + \theta\,(n + 2)\} + F_n(3 - n)\, \{1 + \theta n\}]\, I_n(\zeta), \tag{2.11}$$

where $\theta = \frac{\mu}{\beta a}$.

The leading terms in the above system must vanish. Hence,

$$1 + b_2 + d_2 = 0, \quad (2 - b_2 + d_2) + \theta(2 - 4b_2 + d_2) = 0. \tag{2.12}$$

From (2.12) we see that

$$b_2 = \frac{1}{6\theta + 2}, \quad d_2 = -\frac{6\theta + 3}{6\theta + 2}. \tag{2.13}$$

Substituting these into (2.10) and (2.11) gives respectively

$$\frac{6\theta}{6\theta + 2}\alpha_m I_m I_2 + \sum_{n=3}^{\infty} (E_n + F_n)\, I_n = 0 \tag{2.14}$$

and

$$\frac{6(2\theta^2+5\theta+1)}{6\theta+2}\alpha_m I_m I_2 + \sum_{n=3}^{\infty} \left[E_n(1-n)\left\{1+\theta(n+2)\right\} + F_n(3-n)(1+\theta n)\right] I_n = 0. \tag{2.15}$$

In solving (2.14) and (2.15) with the aid of (2.2) we see that E_n, F_n vanish for all n except when n has the values $m-2$, m and $m+2$. The surviving coefficients are

$$E_{m-2} = \frac{\alpha_m(m-2)(m-3)}{2(2m-1)(2m-3)} \frac{[\gamma - \alpha(5-m)(1+\theta(m-2)]}{(10\theta - 2 - 4\theta m)},$$

$$F_{m-2} = \frac{\alpha_m(m-2)(m-3)}{2(2m-1)(2m-3)} \frac{[\alpha(3-m)(1+\theta m) - \gamma]}{(10\theta - 2 - 4\theta m)},$$

$$E_m = \frac{\alpha_m m(m-1)}{(2m+1)(2m-3)} \frac{[\alpha(3-m)(1+\theta m) - \gamma]}{(2\theta - 2 - 4\theta m)},$$

$$F_m = \frac{\alpha_m m(m-1)}{(2m+1)(2m-3)} \frac{[\gamma - \alpha(1-m)(1+m\theta+2\theta)]}{(2\theta - 2 - 4\theta m)},$$

$$E_{m+2} = \frac{\alpha_m(m+1)(m+2)}{2(2m-1)(2m+1)} \frac{[\gamma - \alpha(1-m)(1+m\theta+2\theta)]}{(-6\theta - 2 - 4\theta m)},$$

$$F_{m+2} = \frac{\alpha_m(m+1)(m+2)}{2(2m-1)(2m+1)} \frac{[-\gamma - \alpha(1+m)(1+m\theta+4\theta)]}{(-6\theta - 2 - 4\theta m)},$$

$$\text{where } \alpha = \frac{6\theta}{6\theta+2} \text{ and } \gamma = \frac{6(2\theta^2+5\theta+1)}{6\theta+2}. \tag{2.16}$$

We have thus determined the field for the flow past an approximate sphere when there is slip on the surface. It is given by

$$\frac{\Psi}{Ua^2} = \left[\sigma^2 + \frac{b_2}{\sigma} + d_2\sigma\right] I_2(\zeta) + [E_{m-2}\sigma^{-m+3} + F_{m-2}\sigma^{-m+5}] I_{m-2}(\zeta)$$
$$+ [E_m\sigma^{-m+1} + F_m\sigma^{-m+3}] I_m(\zeta) + [E_{m+2}\sigma^{-m-1} + F_{m+2}\sigma^{-m+1}] I_{m+2}(\zeta) \tag{2.17}$$

where the coefficients are given by (2.13) and (2.16).

3 Application to a spheroid

As an application of the foregoing analysis we now consider the particular case of slip flow past an oblate spheroid whose equation we take as

$$\frac{x^2+y^2}{c^2} + \frac{z^2}{c^2(1-\epsilon)^2} = 1. \tag{3.1}$$

As before we neglect terms of $0(\epsilon^2)$ in which case (3.1) becomes in polar form

$$r = a[1 + 2\epsilon I_2(\zeta)] \quad \text{or} \quad \sigma = 1 + 2\epsilon I_2(\zeta), \tag{3.2}$$

where $a = c(1 - \epsilon)$. In order to apply the results of the previous Section we must take $m = 2$ and $\alpha_m = 2\epsilon$. Substituting into (2.17) and simplifying gives the stream function as

$$\Psi = Uc^2\left[\left\{\left(\frac{r}{c}\right)^2 + A\frac{r}{c} + B\frac{c}{r}\right\}\right] I_2(\zeta) + \left\{E_4\left(\frac{c}{r}\right)^3 + F_4\frac{c}{r}\right\} I_4(\zeta), \tag{3.3}$$

where from (2.13) and (2.16)

$$\begin{aligned} A &= d_2(1-\epsilon) + F_2 = -\frac{3}{2}\frac{2\mu+\beta c}{3\mu+\beta c} + \frac{3\epsilon}{10(3\mu+\beta c)^2}\{6\mu^2 + 6\mu\beta c + \beta^2c^2\},\\ B &= b_2(1-3\epsilon) + E_2 = \frac{\beta c}{2(3\mu+\beta c)} - \frac{3\epsilon}{10(3\mu+\beta c)^2}\{24\mu^2 + 44\mu\beta c + 9\beta^2c^2\},\\ E_4 &= -\frac{6\epsilon}{5}\frac{6\mu^2 + 6\mu\beta c + \beta^2c^2}{(7\mu+\beta c)(3\mu+\beta c)},\\ F_4 &= -\frac{6\epsilon}{5}\frac{(-16\mu^2 + 2\mu\beta c + \beta^2c^2)}{(3\mu+\beta c)(7\mu+\beta c)}. \end{aligned} \tag{3.4}$$

We now focus on an important physical feature of the flow – the force experienced by the spheroid.

The evaluation of this drag is most readily done by the application of the elegant formula derived by Payne and Pell [6]. In our case of slow, steady axisymmetric flow past the oblate spheroid, the formula gives the drag D experienced as

$$D = 8\pi\mu \lim_{r\to\infty} \frac{\Psi - \Psi_\infty}{r\sin^2\theta}, \tag{3.5}$$

where Ψ_∞ is the stream function corresponding to the fluid motion at infinity. Here

$$\Psi_\infty = \frac{1}{2}Ur^2\sin^2\theta = Ur^2 I_2(\zeta). \tag{3.6}$$

Substitution of (3.3) and (3.6) into (3.5) gives

$$D = 4\pi\mu c U A.$$

Utilizing (2.13), (3.4) and simplifying, we obtain the drag,

$$D = -6\pi\mu c U\left[\frac{2\mu+\beta c}{3\mu+\beta c} - \frac{\epsilon}{5(3\mu+\beta c)^2}\{6\mu^2 + 6\mu\beta c + \beta^2c^2\}\right]. \tag{3.7}$$

The following special cases can be deduced immediately:

(a) *No-slip flow past an oblate spheroid*
Here $\beta \to \infty$ and so

$$D = -6\pi\mu c U\left(1 - \frac{1}{5}\epsilon\right), \tag{3.8}$$

a result obtained by Happel and Brenner [5].

(b) *Perfect slip flow past an oblate spheroid*
Here $\beta = 0$ and so

$$D = -4\pi\mu c U\left(1 - \frac{1}{5}\epsilon\right) \tag{3.9}$$

This is a new result.

(c) *Slip flow past a sphere*
Here $\varepsilon = 0$ and so

$$D = -6\pi\mu c U\left(\frac{\beta c + 2\mu}{\beta c + 3\mu}\right), \tag{3.10}$$

a result previously obtained [5].

(d) *No-slip flow past a sphere*
Here $\epsilon = 0$, $\beta \to \infty$ giving

$$D = -6\pi\mu c U = -C_W \frac{\rho}{2} U^2 \pi c^2$$

where $C_W = \frac{24}{\mathrm{Re}}$, $\mathrm{Re} = \frac{U2c}{\nu}$.
This is the well-known Stokes Formula.

We now make the following observations:

(i) The force, in the general case, exerted on the oblate spheroid given by (3.7) is smaller than that experienced by a sphere of radius equal to the equatorial radius of the spheroid.
(ii) For both no-slip and perfect slip flow past an oblate spheroid, the reduction factor of this force is the same. It is $1/5\ \epsilon$.
(iii) A sphere of radius

$$b = \left(1 - \frac{1}{3}\epsilon\right)c \tag{3.11}$$

would have the same volume as our sphere (3.1) and the resistance of this sphere from (3.10) is

$$D = 6\pi\mu b U\left(\frac{\beta b + 2\mu}{\beta b + 3\mu}\right). \tag{3.12}$$

Substituting (3.11) into (3.7) gives after some simplification

$$D = -6\pi\mu b U\left[\left(1 + \frac{2\varepsilon}{15}\right)\frac{\beta b + 2\mu}{\beta b + 3\mu} + \frac{2\varepsilon}{15}\frac{\mu\beta b}{(3\mu + \beta b)^2}\right]. \tag{3.13}$$

Comparison of (3.12) with (3.13) shows that a sphere of equal volume experiences a smaller resistance than the oblate spheroid.

(iv) It was seen [5] that the drag on a gaseous spherical bubble rising slowly through a liquid is identical to that for flow past a solid sphere at whose surface perfect slip occurs. It is therefore not surprising to see that (3.9) gives the drag on a gaseous oblate spheroidal bubble [7].

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