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Nonlinear Streaming due to the Oscillatory Stretching of a Sheet in a Viscous Fluid

By

C. Y. Wang, East Lansing, Michigan

With 5 Figures

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Summary

An elastic sheet is stretched back and forth in a viscous fluid. The problem is governed by a nondimensional parameter S which represents the relative magnitude of frequency to stretching rate. The Navier-Stokes equations are solved by matched asymptotic expansions for large S. Due to nonlinearity there exists boundary layers of $O(S^{-1/2})$. The unsteady oscillatory flow contains both basic and higher harmonic oscillations. The induced steady streamlines show a saddle like flow which is different from that of acoustic streaming.

Introduction

The flow due to the two dimensional, steady stretching of a flat sheet in a viscous fluid was first studied by Crane [1]. The axisymmetric and three dimensional stretching was considered by Wang [2]. In general, the tangential velocity decays exponentially in a boundary layer of constant thickness. If the velocities are reversed (shrinking) the boundary layer is destroyed and no similarity solution exists. Physically vorticity is being transported inwards towards the origin and is thus highly sensitive to the edge conditions at infinity. The solution, if any, will be non unique.

What would be the flow if the sheet is periodically stretched back and forth? If the frequency is low the flow is quasi-steady. As noted before, the shrinking solution does not exist. The situation is different for high frequency oscillatory stretching studied in this paper. The results may be applied to the flow caused by the tensional vibrations of a stretched elastic sheet.

Formulation

The unsteady Navier-Stokes equations are

$$u_t + uu_x + vu_y = -\frac{1}{\varrho} p_x + \nu \nabla^2 u \tag{1}$$

$$v_t + uv_x + vv_y = -\frac{1}{\varrho} p_y + \nu \nabla^2 v \tag{2}$$

$$u_x + v_y = 0. \tag{3}$$

Here (u, v) are velocity components in (x, y) directions respectively, ϱ is the density, ν is the kinematic viscosity and p is the pressure. Fig. 1 shows the sheet at $y = 0$ stretched with a velocity

$$u = bx \cos \omega t, \qquad v = 0. \tag{4}$$

Here b is the maximum rate of stretch with dimensions $(\text{time})^{-1}$ and ω is the frequency. We shall assume

$$S \equiv \frac{\omega}{b} \equiv \frac{1}{\varepsilon} \gg 1 \tag{5}$$

where ε is small. Any particle path on the sheet is

$$\begin{aligned} x &= x_0 \exp\left(\frac{1}{S} \sin \omega t\right) \\ &= x_0[1 + \varepsilon \sin \omega t + O(\varepsilon^2)]. \end{aligned} \tag{6}$$

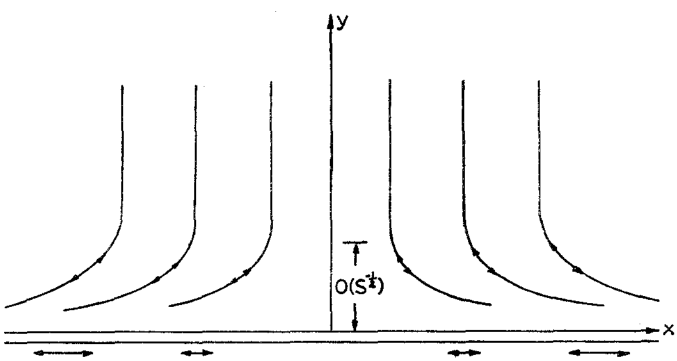

Fig. 1. Oscillatory stretching of a elastic sheet

Thus large S implies small amplitude oscillations relative to x_0. The boundary conditions suggest the similarity transform

$$u = bxf_\eta(\eta, \tau), \qquad v = -\sqrt{b\nu}\, f(\eta, \tau), \qquad p = p(\eta, \tau) \tag{7}$$

$$\eta \equiv y \sqrt{\frac{b}{\nu}}, \qquad \tau \equiv t\omega. \tag{8}$$

The governing equations become

$$S f_{\eta\tau} + f_\eta^2 - f f_{\eta\eta} = f_{\eta\eta\eta} \tag{9}$$

$$\frac{p}{\varrho} = \nu\omega \int f_\tau \, d\eta - \frac{v^2}{2} + \nu v_y + \text{const.} \tag{10}$$

The boundary conditions are

$$f_\eta(0, \tau) = \cos\tau, \qquad f(0, \tau) = 0, \qquad f_\eta(\infty, \tau) = 0. \tag{11}$$

Since an exact solution does not exist we seek a perturbation solution of Eqs. (9), (11) for large S. The pressure variation can be obtained after the velocity field is determined.

High Frequency Decomposition in the Boundary Layer

For high frequency oscillatory viscous flows, the method of Wang [3] is useful. Let an overbar and a tilde denote the steady and periodic parts of a function. Let

$$f = \bar{f}(\eta) + \tilde{f}(\eta, \tau). \tag{12}$$

Eq. (9) separates into

$$S\tilde{f}_{\eta\tau} + 2\bar{f}_\eta \tilde{f}_\eta + \widetilde{(\tilde{f}_\eta^2)} - \bar{f}\tilde{f}_{\eta\eta} - \tilde{f}\bar{f}_{\eta\eta} - \widetilde{\tilde{f}\tilde{f}_{\eta\eta}} = \tilde{f}_{\eta\eta\eta} \tag{13}$$

$$(\bar{f}_\eta)^2 - \overline{\bar{f}\bar{f}_{\eta\eta}} + \overline{(\tilde{f}_\eta)^2} - \overline{\tilde{f}\tilde{f}_{\eta\eta}} = \bar{f}_{\eta\eta\eta}. \tag{14}$$

The steady part of the unsteady products in Eq. (14) is equivalent to the "Reynolds stress" in the study of turbulent mean flow. Now for large S, Eq. (13) shows the existence of an unsteady boundary layer of order $S^{-1/2}$. Since from Eq. (11) $\tilde{f}_\eta$ is of order one, $\tilde{f}$ must be of order $S^{-1/2}$ in the boundary layer. In order to balance the Reynolds stress in the steady boundary layer, $\bar{f}$ must be of order $S^{-3/2}$ in Eq. (14). Thus for the boundary layer let

$$\tilde{f} = \sqrt{\varepsilon}\,[f_0(\zeta, \tau) + \varepsilon \tilde{f}_1(\zeta, \tau) + \cdots] \tag{15}$$

$$\bar{f} = \varepsilon \sqrt{\varepsilon}\,[\bar{f}_1(\zeta) + \cdots] \tag{16}$$

$$\zeta \equiv \eta/\sqrt{\varepsilon}. \tag{17}$$

The leading orders of Eqs. (13), (14) become

$$\tilde{f}_{0\zeta\tau} = \tilde{f}_{0\zeta\zeta\zeta} \tag{18}$$

$$\tilde{f}_{1\zeta\tau} + (\widetilde{\tilde{f}_{0\zeta}^2}) - (\widetilde{\tilde{f}_0 \tilde{f}_{0\zeta\zeta}}) = \tilde{f}_{1\zeta\zeta\zeta} \tag{19}$$

$$(\overline{\tilde{f}_{0\zeta}^2}) - (\overline{\tilde{f}_0 \tilde{f}_{0\zeta\zeta}}) = \bar{f}_{1\zeta\zeta\zeta} \tag{20}$$

with the boundary conditions

$$\tilde{f}_{0\zeta}(0, \tau) = \cos\tau, \qquad \tilde{f}_0(0, \tau) = 0 \tag{21}$$

$$\tilde{f}_{1\zeta}(0, \tau) = 0, \qquad \tilde{f}_1(0, \tau) = 0 \tag{22}$$

$$\bar{f}_{1\zeta}(0) = 0, \qquad \bar{f}_1(0) = 0. \tag{23}$$

The third boundary condition is obtained from matching with the outer flow. The solution to Eqs. (18), (21) is

$$\tilde{f}_0 = \frac{(1-i)}{\sqrt{2}} \, 1 - e^{\left[\frac{-(1+i)}{\sqrt{2}}\zeta\right]} e^{i\tau} \tag{24}$$

where the real part of the product is implied. Eq. (19) gives

$$\tilde{f}_{1\zeta\zeta\zeta} - \tilde{f}_{1\zeta\tau} = \frac{1}{2}(\tilde{f}_{0\zeta}^2 - \tilde{f}_0 \tilde{f}_{0\zeta\zeta\zeta}) = \frac{1}{2} e^{\frac{-(1+i)}{\sqrt{2}}\zeta} e^{2i\tau}. \tag{25}$$

The solution is

$$\tilde{f}_1 = \left[\frac{(1+i)}{-2\sqrt{2}} e^{\frac{-(1+i)}{\sqrt{2}}\zeta} + \frac{(1+i)}{4} e^{-(1+i)\zeta} + \frac{(1+i)}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)\right] e^{2i\tau}, \tag{26}$$

Eq. (20) can be written as

$$\bar{f}_{1\zeta\zeta\zeta} = \frac{1}{2}(\tilde{f}_{0\zeta}\tilde{f}_{0\zeta}^* - \tilde{f}_0{}^* \tilde{f}_{0\zeta\zeta}) = \frac{1}{2} e^{-\sqrt{2}\zeta} + \frac{i}{2} e^{\frac{-(1+i)}{\sqrt{2}}\zeta}, \tag{27}$$

where the asterisk denotes the complex conjugate. The solution is

$$\bar{f}_1 = \frac{1}{-4\sqrt{2}} e^{-\sqrt{2}\zeta} - \frac{(1-i)}{2\sqrt{2}} e^{\frac{-(1+i)}{\sqrt{2}}\zeta} + \frac{3}{4\sqrt{2}} - \frac{3}{4}\zeta. \tag{28}$$

The Outer Flow

The boundary layer induces motion to the interior fluid. Eqs. (24), (26) show as $\zeta \to \infty$

$$\tilde{f} \to \sqrt{\varepsilon}\left[\frac{(1-i)}{\sqrt{2}} e^{i\tau} + \varepsilon \frac{(1+i)}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) e^{2i\tau} + \cdots\right]. \tag{29}$$

This is an oscillatory vertical flow due to displacement thickness. We set, for the outer flow

$$\tilde{f} = \sqrt{\varepsilon}\left(\tilde{F}_0(\eta, \tau) + \varepsilon\tilde{F}_1(\eta, \tau) + \cdots\right] \tag{30}$$

into Eq. (13). The leading orders are

$$\tilde{F}_{0\eta\tau} = 0 \tag{31}$$

$$\tilde{F}_{1\eta\tau} = \tilde{F}_{0\eta\eta\eta}. \tag{32}$$

The solution after matching is the vertical flow

$$\tilde{F}_0 = \frac{(1-i)}{\sqrt{2}}\, e^{i\tau}, \qquad \tilde{F}_1 = \frac{(1+i)}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) e^{2i\tau}. \tag{33}$$

Eq. (28) shows asymptotically for large ζ

$$\bar{f} \to \varepsilon\sqrt{\varepsilon}\left(\frac{3}{4\sqrt{2}} - \frac{3}{4}\,\zeta\right) + \cdots \sim \varepsilon\sqrt{\varepsilon}\,\frac{3}{4\sqrt{2}} - \varepsilon\,\frac{3}{4}\,\eta + \cdots. \tag{34}$$

Thus we set for the outer steady flow

$$\bar{f} = \varepsilon\bar{F}_0(\eta) + \varepsilon\sqrt{\varepsilon}\,\bar{F}_0(\eta) + \cdots. \tag{35}$$

Eq. (14) becomes

$$\bar{F}_{0\eta\eta\eta} = 0, \qquad \bar{F}_{1\eta\eta\eta} = 0. \tag{36}$$

The least singular solution is saddle like flow $\bar{F}_0$ and the vertical flow $\bar{F}_1$

$$\bar{F}_0 = -\frac{3}{4}\,\eta, \qquad \bar{F}_1 = \frac{3}{4\sqrt{2}}. \tag{37}$$

The composite uniformly valid solution is thus

$$\begin{aligned}\tilde{f} &= \frac{1}{\sqrt{S}}\,\frac{(1-i)}{\sqrt{2}}\,(1-E)\,e^{i\tau} \\ &\quad + \frac{1}{S\sqrt{S}}\,\frac{(1+i)}{2\sqrt{2}}\left(\frac{1}{\sqrt{2}}\,E^{\sqrt{2}} - E + 1 - \frac{1}{\sqrt{2}}\right) e^{2i\tau} + O(S^{-5/2})\end{aligned} \tag{38}$$

$$\bar{f} = \frac{-3}{4S}\,\eta + \frac{1}{S\sqrt{S}\,4\sqrt{2}}\,[3 - e^{-\sqrt{2S}\eta} - 2(1-i)\,E] + O(S^{-5/3}) \tag{39}$$

$$E \equiv e^{\frac{-(1+i)}{\sqrt{2}}\sqrt{S}\eta} \tag{40}$$

Results and Discussion

Fig. 2 shows the highest order unsteady flow $\sqrt{\varepsilon}\tilde{f}_0$ for $S = 10$. Locally it represents an oscillatory Stokes layer of thickness $(S^{-1/2})$. The velocities are in phase with the stretching motion. Fig. 3 shows the unsteady flow induced by the nonlinear interaction. The frequency is doubled while the amplitude is much smaller. The oscillatory horizontal velocity is confined to the boundary layer.

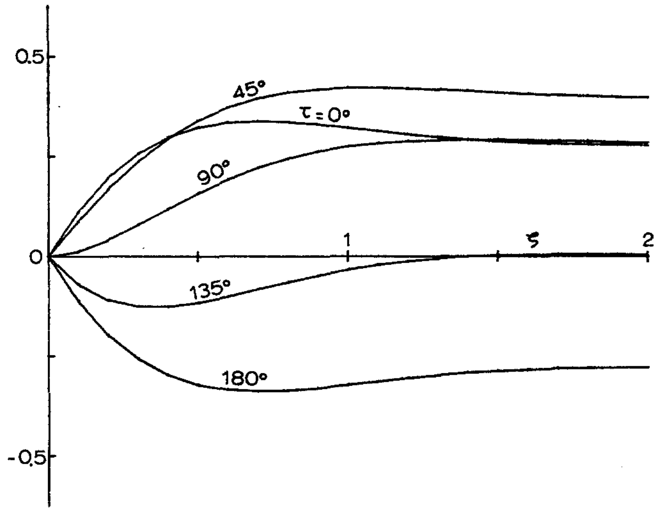

Fig. 2. The primary unsteady flow $S^{-1/2}\tilde{f}_0$ has same frequency as the oscillatory stretching ($S = 10$)

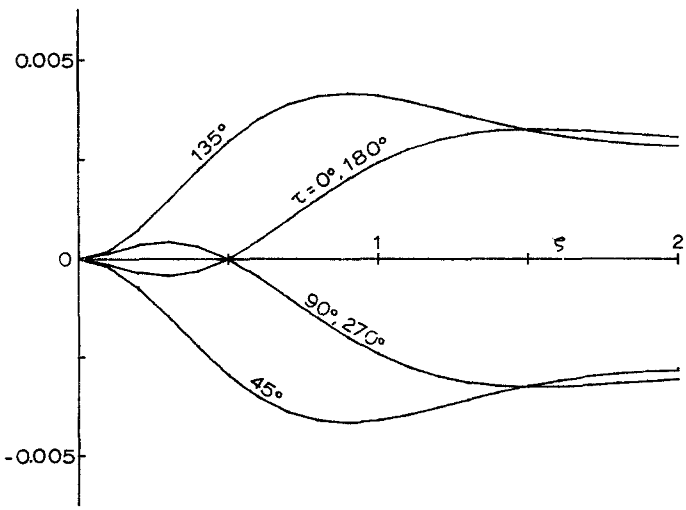

Fig. 3. The secondary unsteady flow $S^{-3/2}\tilde{f}_1$ has twice the frequency of oscillatory stretching ($S = 10$)

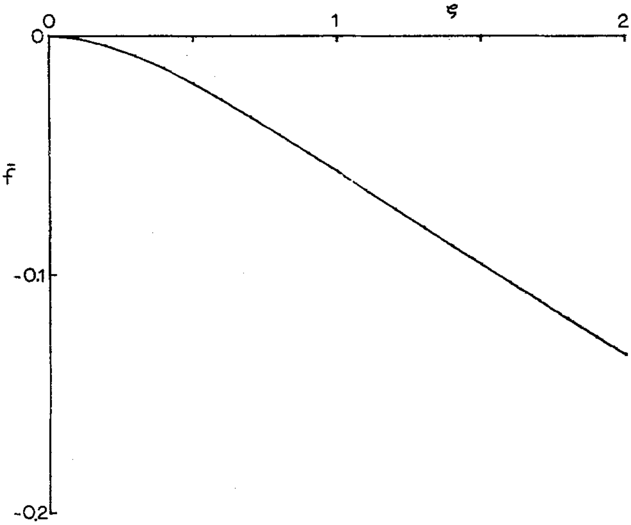

Fig. 4. The secondary steady flow $\bar{f}$ for $S = 10$

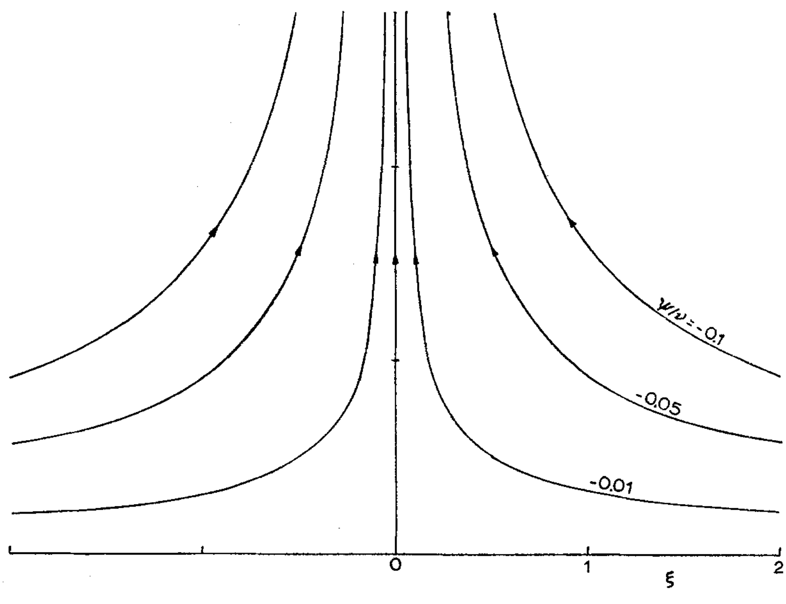

Fig. 5. The induced steady streamlines

Fig. 4 shows the steady flow $\bar{f}$ decreases monotonically with η reflecting the leading term in Eq. (39). The magnitude of $\bar{f}$ decreases with increased S. The steady stream function can be defined as

$$\psi = \sqrt{b\nu}\, x \bar{f}(\zeta) \tag{41}$$

or

$$\frac{\psi}{\nu} = \xi \bar{f}(\eta), \quad \xi \equiv x \sqrt{\frac{b}{\nu}}. \tag{42}$$

Fig. 5 shows the steady streamlines for $S = 10$. The wider spacing near $\eta = 0$ reflects the velocity is being brought to zero on the surface. The inward flow is caused by the steady part of the Reynolds stress which exists inside a boundary layer of order $(S^{-1/2})$. Outside the boundary layer is a saddle like inviscid flow. The induced steady flow is extremely important in heat and mass transfer from the surface. It provides a unidirectional convection which greatly enhances the transfer process.

The generation of a steady flow from purely oscillatory boundary conditions has been noted before. In all previous cases studied the boundary is a non moving solid surface while the fluid at infinity oscillates due to an unsteady pressure gradient. The phenomenon is called "acoustic streaming" since it was first observed near small objects in an acoustical field. In the present paper the oscillatory flow is not caused by an extremal pressure gradient but through the stretching of the boundary. Although the basic mechanism for nonlinear streaming is still due to the Reynolds stress in the Stokes layer, the secondary streaming motion is different. For example the steady streamlines near a two dimensional stagnation point in acoustic streaming, [3], show a pair of recirculation cells while those due to oscillatory stretching studied in this paper do not exhibit cellular recirculation, (Fig. 5).

Lastly we mention that the phenomenon studied in this paper would occur in other oscillatory stretching problems, such as axisymmetric oscillatory stretching of a surface or the longitudinal oscillatory stretching of a cylinder.

References

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C. Y. Wang
Department of Mathematics
Michigan State University
East Lansing, MI 48824
U.S.A.