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Large Axisymmetric Deformation of a Non-Linear Viscoelastic Circular Membrane

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With 8 Figures

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Summary — Zusammenfassung

Large Axisymmetric Deformation of a Non-Linear Viscoelastic Circular Membrane. The problem of a viscoelastic circular membrane deforming under its own weight is considered. The membrane, clamped in a horizontal plane at its boundary, is formed from a non-linear viscoelastic material whose behaviour is modelled by a non-linear single integral constitutive equation. The sag of the membrane is considered quasi-static and provision is made in the model for making the relaxation time stretch-dependent. The analysis leads to a system of three equations, two of which are first-order non-linear partial-differential integral equations while the third is an algebraic relation. Time descretization enables the differential equations to be solved as ordinary differential equations which may be dealt with by a familiar step-wise integration process.

Numerical solutions are obtained for membranes with both stretch-dependent and stretch-independent relaxation times.

Große, axialsymmetrische Durchbiegung einer nichtlinearen, viskoelastischen Kreismembran. Das Deformationsproblem einer viscoelastischen Kreismembran unter Eigengewichtsbelastung wird untersucht. Die in ihrer horizontalen Mittelfläche am Rand eingespannte Membran ist aus nichtlinearem, viskoelastischem Material, dessen Verhalten durch nichtlineare, konstitutive Einfachintegrale beschrieben wird. Die Absenkung der Membran wird quasistatisch angeommen, die Relaxationszeit wurde in das Modell dehnungsabhängig eingeführt. Die mathematische Beschreibung führt auf ein System von drei Gleichungen, von denen zwei nichtlineare, partielle Differentialgleichungen erster Ordnung sind, während die dritte eine algebraische Beziehung darstellt. Diskretisierung in der Zeit ermöglicht eine Lösung der Differentialgleichungen, die mit einem bekannten, schrittweisen Integrationsverfahren gelöst werden können.

Numerische Ergebnisse werden sowohl für dehnungsabhängige als auch für dehnungsunabhängige Relaxationszeiten erhalten.

1. Introduction

The stretching of viscoelastic membranes may lead to considerable distortion from their original configuration. It is the intention of this paper to study the deformation of a non-linearly viscoelastic circular membrane which sags from supports at its boundary under its own weight.

Elastic membranes have been treated extensively in the literature. Green and Adkins [1], for example cover the theory of non-linearly elastic membranes com-

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prehensively, including axiallysymmetric membranes as a particular case of the more general theory of thin elastic membranes. The treatment of viscoelastic membrane problems is comparatively rare. A number of non-linear viscoelastic membrane problems have been solved by Wineman [2], [3], [4] and Feng [5] has adopted a similar approach to Wineman's in his consideration of a bonded viscoelastic toroidal membrane. Finally, there is a recent paper by Buckley and Green [6] which studies small deformations of a non-linear viscoelastic tube due to combinations of axial force, axial couple and excess pressure.

The papers of Wineman have particular relevance to the present work. The constitutive equations for the membrane material in [2] and [3], as here, are based on the single-integral representation developed by Pipkin and Rogers [7], and the actual material used here is that chosen by Wineman in the same two papers. Furthermore, Wineman's formulation of the membrane problem in [4] in terms of radial and circumferential stretch ratios as dependent variables has been used in large measure in the following analysis.

2. Formulation of the Theory

A cylindrical polar co-ordinate system $\varrho = \varrho(r, t)$, θ, $z = z(r, t)$ is chosen for convenience, where $(r, \theta, 0)$ are co-ordinates in the membrane mid-surface for $t \leqq 0$. By the symmetry of the deformation the particle at the central point for $t \leqq 0$ remains at the deformed central point for all time. The distance of the point $P(\varrho, \theta, z)$ from the deformed central point measured along the membrane mid-surface in the meridional direction is s; for $t \leqq 0$, $s = r$. At P the tangent in the meridional direction makes the angle ψ with the horizontal plane and the extension ratios in the meridional, circumferential and normal directions are, respectively,

$$\lambda_1 = \frac{\partial s}{\partial r}(r, t), \tag{1}$$

$$\lambda_2 = \frac{\varrho}{r}(r, t), \tag{2}$$

and

$$\lambda_3 = (\lambda_1 \lambda_2)^{-1}. \tag{3}$$

Eq. (3) expresses the assumption that the membrane material is incompressible.

Let w_0 be the weight per unit area of unstretched membrane and $T = T(r, t)$ and $N = N(r, t)$ be the meridional and hoop forces per unit length of membrane mid-surface respectively at time $t > 0$.

Two independent force balance equations may be derived by considering the equilibrium of an element in the deformed surface.

The equation for vertical force components is

$$\frac{\partial}{\partial r}(T\varrho \sin \psi) = w_0 r,$$

which, when integrated, gives

$$T\varrho \sin \psi = \frac{1}{2} w_0 r^2. \tag{4}$$

Similarly, force equilibrium in the meridional direction at P gives

$$\frac{\partial}{\partial r}(T\varrho) = rw_0 \sin\psi + N \frac{\partial s}{\partial r}\cos\psi. \tag{5}$$

The elimination of ψ is achieved by using the familiar relation

$$\cos\psi = \partial\varrho/\partial s = \eta/\lambda_1, \tag{6}$$

where

$$\eta = \frac{\partial\varrho}{\partial r}, \tag{7}$$

together with

$$\sin\psi = \frac{(\lambda_1^2 - \eta^2)^{1/2}}{\lambda_1}. \tag{8}$$

Eqs. (7) and (8), with Eq. (2), transform Eqs. (4) and (5) to

$$\lambda_2 T = \frac{1}{2} w_0 r \lambda_1 (\lambda_1^2 - \eta^2)^{-1/2} \tag{9}$$

and

$$\lambda_1 \frac{\partial}{\partial r}(r\lambda_2 T) = rw_0(\lambda_1^2 - \eta^2)^{1/2} + N\eta\lambda_1 \tag{10}$$

respectively.

As has already been noted by Wineman [4] the most natural dependent variables in axisymmetric membrane problems are λ_1, λ_2 and η. The third equation to accompany Eqs. (9) and (10) is found through a compatibility condition formed by eliminating ϱ between λ_2 and η. From Eq. (2)

$$\frac{\partial\lambda_2}{\partial r} = \frac{\eta - \lambda_2}{r}. \tag{11}$$

The non-zero components of stress per unit deformed area are in the meridional and circumferential directions, being denoted by σ_1 and σ_2 respectively. The meridional and hoop forces may be obtained in terms of the current membrane thickness $h = h_0\lambda_3$, where h_0 is the thickness of the unstretched membrane, and the stress components through

$$T = h_0\lambda_3\sigma_1 \quad \text{and} \quad N = h_0\lambda_3\sigma_2. \tag{12}$$

Substituting Eqs. (12) in Eqs. (9) and (10) leads to

$$\lambda_3\sigma_1 = \frac{w_0 r \lambda_1}{2h_0\lambda_2(\lambda_1^2 - \eta^2)^{1/2}}, \tag{13}$$

and

$$\frac{\partial}{\partial r}(\lambda_3\sigma_1) = \frac{w_0\lambda_3}{h_0}(\lambda_1^2 - \eta^2)^{1/2} + \frac{\lambda_1\lambda_3^2}{r}\eta(\sigma_2 - \sigma_1) \tag{14}$$

where λ_3 may be expressed in terms of λ_1 and λ_2 by Eq. (3).

Now σ_1 and σ_2 may be written as functions of λ_1 and λ_2 through a constitutive relation, and when this is done Eqs. (11), (13) and (14) provide a system of partial differential equations for the dependent variables $\lambda_1(r, t)$, $\lambda_2(r, t)$ and $\eta(r, t)$. The boundary conditions for the system are obtained by considering the stretch ratios at the centre of the membrane and the fixed hoop stretch ratio of the membrane boundary. For particles at $P(\varrho, \theta, z)$ near the centre of the membrane,

$$\lim_{r\to 0} \frac{\varrho}{r} = \frac{\partial \varrho}{\partial r} \quad \text{and} \quad \lim_{r\to 0} \frac{\varrho}{s} = 1 .$$

Thus, by Eqs. (1), (2) and (6) respectively,

$$\lambda_1(0, t) = \lambda_2(0, t) = \eta(0, t) . \tag{15}$$

Also, since particles at $P(R, \theta, 0)$ are clamped in their original horizontal position the other boundary condition is

$$\lambda_2(R, t) = 1 . \tag{16}$$

The instantaneous position of a particle at $P(\varrho, \theta, z)$ on the deformed membrane mid-surface can be found once λ_1, λ_2 and η are known across the membrane.

By Eq. (2),

$$\varrho = \lambda_2 r . \tag{17}$$

Also, since $\partial z/\partial r = (\partial z/\partial s), (\partial s/\partial r) = \sin \psi \cdot \lambda_1$,

$$z = \int_0^r (\lambda_1^2 - \eta^2)^{1/2} \, dr \tag{18}$$

by Eq. (8).

3. Constitutive Equation and Specific Material

The single integral constitutive equation developed by Pipkin and Rogers [7] will be used to model the behaviour of an isotropic non-linearly viscoelastic solid. If X_A and $x_i(X_A, t)$ are the coordinates of a particle in its initial or reference state and at time t respectively, the three-dimensional relation for an incompressible material takes the form

$$\Pi_{AB}(t) = \int_{-\infty}^{t} d_G R_{AB}[\boldsymbol{G}(\tau), t - \tau] - p X_{A,k} X_{B,k} . \tag{19}$$

Here, Π_{AB} is given in terms of the Cauchy stress components σ_{ij} by

$$\Pi_{AB} = X_{A,i} X_{B,j} \sigma_{ij} ,$$

which may be inverted to give $\sigma_{ij} = x_{i,A} x_{j,B} \Pi_{AB}$.

$\boldsymbol{R}$ is the one-step relaxation function, and

$$\boldsymbol{G} = \boldsymbol{F}^T \boldsymbol{F} ,$$

the right Cauchy-Green strain measure, where $\boldsymbol{F}$ is the deformation gradient tensor defined by

$$F_{iA} = \frac{\partial x_i}{\partial X_A} \equiv x_{i,A} .$$

The quantity p is an arbitrary scalar arising as a reaction to material incompressibility.

The operator d_G is defined by

$$d_G f_{AB}(G) = \frac{\partial f_{AB}}{\partial G_{PQ}} G'_{PQ}(\tau)\, d\tau$$

so after inversion and an integration by parts Eq. (19) becomes

$$\sigma(t) = -p\boldsymbol{I} + \boldsymbol{F}(t) \left\{ \boldsymbol{R}[\boldsymbol{G}(t), 0^+] + \int_{0^+}^{t} \partial_t \boldsymbol{R}[\boldsymbol{G}(\tau), t - \tau]\, d\tau \right\} \boldsymbol{F}^T(t) \tag{20}$$

where $\partial_t \equiv \partial/\partial(t - \tau)$.

Rivlin [8] has shown that for isotropic materials $\boldsymbol{R}$ may be expressed in the form

$$\boldsymbol{R}[\boldsymbol{G}, t] = \phi_0 \boldsymbol{I} + \phi_1 \boldsymbol{G} + \phi_2 \boldsymbol{G}^2 \tag{21}$$

where ϕ_0, ϕ_1 and ϕ_2 are scalar functions of t and the invariants of $\boldsymbol{G}$ defined by

$$\begin{aligned} I_1 &= \operatorname{tr} \boldsymbol{G}, \\ I_2 &= \frac{1}{2} [(\operatorname{tr} \boldsymbol{G})^2 - \operatorname{tr} \boldsymbol{G}^2], \\ I_3 &= \det \boldsymbol{G}, \end{aligned} \tag{22}$$

and det $\boldsymbol{G}$ denotes the determinant. The specific non-linear viscoelastic material which will be used to illustrate the numerical method of solution of Eqs. (11), (13) and (14) was introduced by Wineman [3] and subsequently by Feng [5]. The forms chosen for ϕ_0, ϕ_1 and ϕ_2 lead to analytical brevity whilst indicating some of the problems arising in investigations of this complexity.

The Mooney [9] material of elasticity theory is combined with the standard linear solid of viscoelasticity theory to give

and
$$\begin{aligned} \phi_0 &= (1 + \alpha I_1)\, \hat{R}[I_1, t], \\ \phi_1 &= -\alpha \hat{R}[I_1, t], \\ \phi_2 &= 0, \end{aligned} \tag{23}$$

where α is a dimensionless constant and

$$\hat{R}[I_1, t] = C_0[(1 - \gamma)\, e^{-\hat{p}[I_1]t} + \gamma] = C_0 C[I_1, t] \tag{24}$$

with $\gamma = C_\infty/C_0$, the ratio of the long-term elastic modulus to the instantaneous modulus at $t = 0^+$. Here, $\{\hat{p}[I_1]\}^{-1}$ represents a strain-dependent relaxation time and

$$\hat{p}[I_1] = \frac{1}{\tau_R} [1 + \beta(I_1 - 3)] \tag{25}$$

where β is a positive dimensionless constant and τ_R is the relaxation time in the linear case.

Since the membrane deformation is axisymmetric for all $t \geqq 0^+$ the principal directions of the stresses and stretch ratios at each point on the membrane mid-surface are known to be in the meridional, circumferential and normal directions at each time t. The deformation gradient is, therefore,

$$\boldsymbol{F} = \delta_{ij} \lambda_i, \qquad i, j = 1, 2, 3 \quad \text{(no summation over } i\text{)}.$$

The condition σ_{33} $(= \sigma_3) = 0$ enables the pressure p to be determined from Eqs. (20), (21), (25) and (26). The non-zero stresses σ_{11} $(\equiv \sigma_1)$ and σ_{22} $(\equiv \sigma_2)$ are then given by

$$\frac{\sigma_1(t)}{C_0} = F_1[\lambda(t)] + \int_{0^+}^{t} F_2[\lambda(t), \lambda(\tau)]\, \partial_t C[I_1(\tau), t - \tau]\, d\tau, \tag{26}$$

and

$$\frac{\sigma_2(t)}{C_0} = F_3[\lambda(t)] + \int_{0^+}^{t} F_4[\lambda(t), \lambda(\tau)]\, \partial_t C[I_1(\tau), t - \tau]\, d\tau. \tag{27}$$

Here, $\lambda(t)$ denotes the triplet $\big(\lambda_1(t), \lambda_2(t), \lambda_3(t)\big)$ and

$$\begin{aligned} F_1[\lambda(t)] &= \big(\lambda_1^2(t) - \lambda_3^2(t)\big)\big(1 + \alpha\lambda_2^2(t)\big), \\ F_2[\lambda(t), \lambda(\tau)] &= \big(\lambda_1^2(t) - \lambda_3^2(t)\big)\big(1 + \alpha\lambda_2^2(\tau)\big) + \alpha\big(\lambda_1^2(t)\,\lambda_3^2(\tau) - \lambda_3^2(t)\,\lambda_1^2(\tau)\big). \end{aligned} \tag{28}$$

The functions F_3 and F_4 are obtained from Eqs. (28) by

$$\begin{aligned} F_3[\lambda_1(t), \lambda_2(t), \lambda_3(t)] &= F_1[\lambda_2(t), \lambda_1(t), \lambda_3(t)], \\ F_4[\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_1(\tau), \lambda_2(\tau), \lambda_3(\tau)] &= F_2[\lambda_2(t), \lambda_1(t), \lambda_3(t), \lambda_2(\tau), \lambda_1(\tau), \lambda_3(\tau)]. \end{aligned} \tag{29}$$

The dependence of all stresses and stretch ratios on position has been omitted for the sake of clarity in presentation.

The constitutive Eqs. (26) and (27) are now substituted into the membrane equilibrium Eqs. (13) and (14) which, together with the compatibility condition (11), form a system of non-linear partial differential equations for $\lambda_1(r, t)$, $\lambda_2(r, t)$ and $\eta(r, t)$. Some simplification of these equations is achieved by defining a new quantity

$$\mu \equiv \lambda_3\sigma_1, \tag{30}$$

and using a rearrangement of Eq. (13) for η in terms of λ_1, λ_2 and μ to rewrite Eqs. (11) and (14) as

$$\frac{\partial\lambda_2}{\partial r} = \frac{1}{r}\left[\lambda_1\left\{1 - \left(\frac{w_0 r}{2h_0\lambda_2\mu}\right)^2\right\}^{1/2} - \lambda_2\right], \tag{31}$$

and

$$\frac{\partial\mu}{\partial r} = \frac{w_0^2 r\lambda_1\lambda_3}{2h_0^2\lambda_2\mu} + \frac{\lambda_1^2\lambda_3^2}{r}\left\{1 - \left(\frac{w_0 r}{2h_0\lambda_2\mu}\right)^2\right\}^{1/2}(\sigma_2 - \sigma_1) \tag{32}$$

respectively.

Dimensionless variables are defined as follows:

$$\begin{gathered} r^* = r/R, \qquad h_0^* = h_0/R, \qquad \varrho^* = \varrho/R, \qquad z^* = z/R, \\ t^* = t/\tau_R, \qquad \tau^* = \tau/\tau_R, \qquad \mu^* = \mu/w_0, \qquad \sigma_i^* = \sigma_i/w_0, \qquad i = 1, 2, 3. \end{gathered}$$

Eq. (30) remains unchanged when it is transformed to dimensionless form and the asterisks are omitted, as they are in the following equations. Eqs. (31) and (32) take the nondimensional forms

$$\frac{\partial\lambda_2}{\partial r} = \frac{\lambda_1}{r}\left\{1 - \left(\frac{r}{2h_0\lambda_2\mu}\right)^2\right\}^{1/2} - \frac{\lambda_2}{r}, \tag{33}$$

and

$$\frac{\partial \mu}{\partial r} = \frac{r}{2h_0^2\lambda_2^2\mu} + \frac{1}{r\lambda_2^2}\left\{1 - \left(\frac{r}{2h_0\lambda_2\mu}\right)^2\right\}^{1/2} (\sigma_2 - \sigma_1), \tag{34}$$

where λ_3 has been eliminated using Eq. (3). Similarly, the stress Eqs. (26) and (27) become

$$\frac{\sigma_1}{C_1} = F_1[\lambda(t)] + \int_{0^+}^{t} F_2[\lambda(t), \lambda(\tau)]\, \partial_t C[I_1(\tau), t - \tau]\, d\tau, \tag{35}$$

and

$$\frac{\sigma_2}{C_1} = F_3[\lambda(t)] + \int_{0^+}^{t} F_4[\lambda(t), \lambda(\tau)]\, \partial_t C[I_1(\tau), t - \tau]\, d\tau \tag{36}$$

where $C_1 = C_0/w_0$, the functions F_1, F_2, F_3 and F_4 are given by Eqs. (28) and (29) and

$$C[I_1(\tau), t - \tau] = (1 - \gamma) \exp\left[-\left\{1 + \beta\left(I_1(\tau) - 3\right)\right\} \cdot (t - \tau)\right] + \gamma. \tag{37}$$

For completeness at this stage the first-boundary condition, Eqs. (15), although unchanged in form, is

$$\lambda_1(0, t) = \lambda_2(0, t) = \eta(0, t) \tag{38}$$

and the second, Eq. (16), becomes

$$\lambda_2(1, t) = 1. \tag{39}$$

4. The Development of the Solution

The numerical method used for solving the system of Eqs. (30), (33) and (34) subject to the boundary conditions (38) and (39) is similar to the method used in papers to which reference has already been made [2], [4], [5]. The integrals in the expressions (35) and (36) for the stresses at time t are approximated by applying the trapezoidal rule with $(n - 1)$ unequal intervals, (t_k, t_{k+1}), $k = 1, 2, \ldots, n - 1$ where $t_1 = 0$ and $t_n = t$. If the solution is known up to and including time t_{n-1} Eqs. (30), (33) and (34) then constitute a system of one algebraic equation and two ordinary differential equations for λ_1, λ_2 and μ as functions of r at time $t_n = t$.

The differential equations are solved for λ_2 and μ using a fourth order Runge-Kutta routine whilst the algebraic equation is solved for λ_1 at each step by an iterative method. Since the problem involves boundary values at the two end points it is necessary to employ a "shooting" method. Thus, an estimate is made of the values of λ_2 $(= \lambda_1)$ at $r = 0$ and this allows an estimated value of μ to be obtained from Eq. (30). The integration then proceeds up to the end point $r = 1$ and if the boundary condition (39) is not satisfied it is necessary to repeat the process with new estimates at $r = 0$.

Full details of the numerical procedures are given in Roberts [10].

5. Results and Discussion

The Mooney model parameter α is chosen as 0.1 and the viscoelastic properties of the membrane material are determined by the values of γ and β. Attention here is concentrated on materials which become softer as time progresses, so that

$\gamma < 1$. The effects of a strain-dependent relaxation time as defined by Eq. (25) are assessed by varying β from zero.

The results for the sag of a circular membrane under its own weight are presented in Figs. 1—8. The membrane profile history shown in Fig. 1 is for the material $C_1 = 288$, $\gamma = 1/3$, $\beta = 0$. This shows an initial sag at the centre of 30% of the membrane radius, increasing to almost 50% as $t \to \infty$. The stretch ratios and stresses as they vary with respect to r are shown in Figs. 2 and 3 respectively for the same membrane. Fig. 4 demonstrates the "overshoot" phenomenon found by both Wineman [3] and Feng [5] with similar materials. Here the history of z

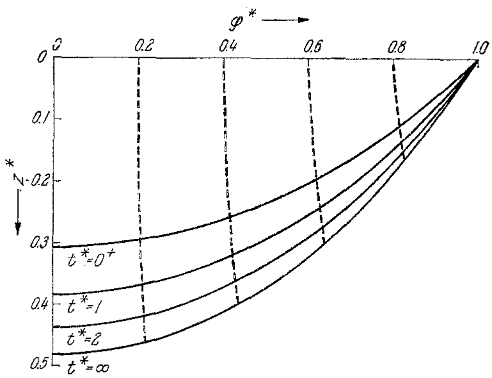

Fig. 1. Membrane profile history. $C_1 = 288$, $\gamma = 1/3$, $\beta = 0$. Dashed lines denote particle paths

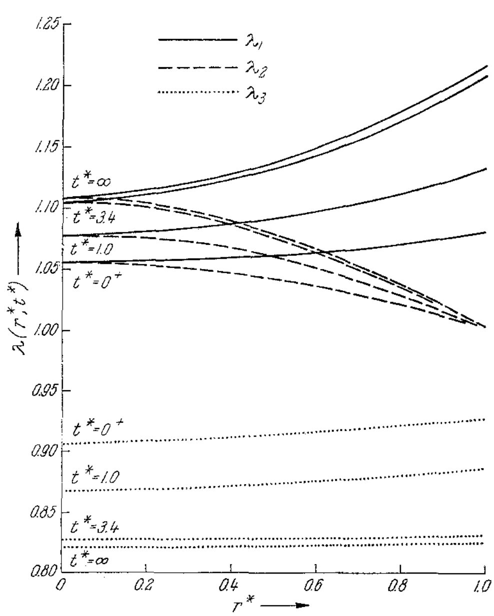

Fig. 2. Histories of stretch ratios. $C_1 = 288$, $\gamma = 1/3$, $\beta = 0$

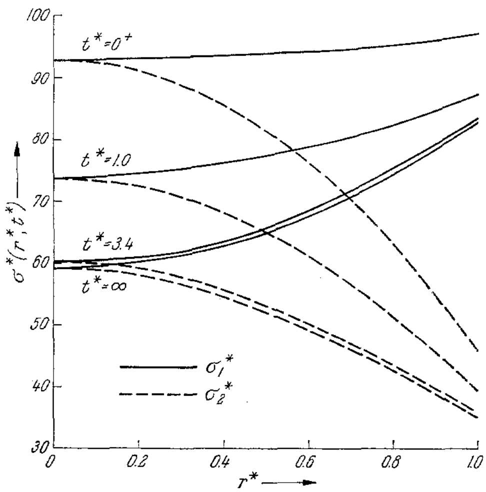

Fig. 3. Stress histories. $C_1 = 288$, $\gamma = 1/3$, $\beta = 0$

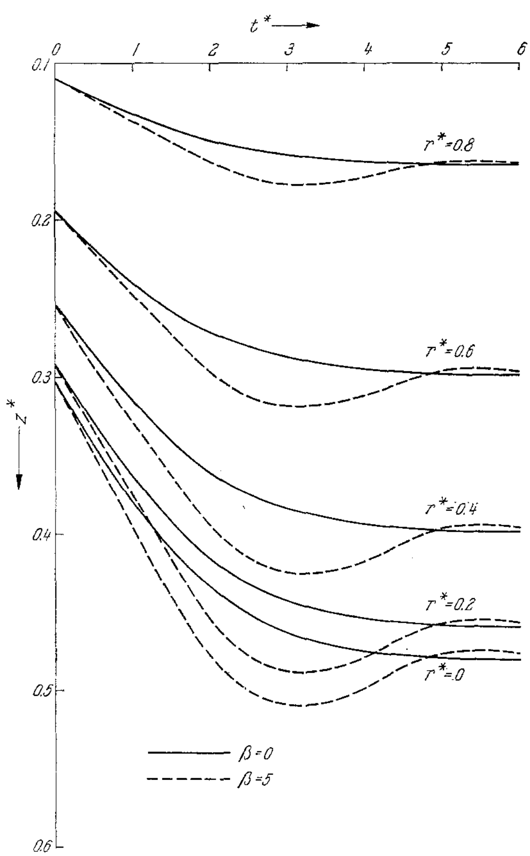

Fig. 4. Deformation history. $C_1 = 288$, $\gamma = 1/3$, $\beta = 0.5$

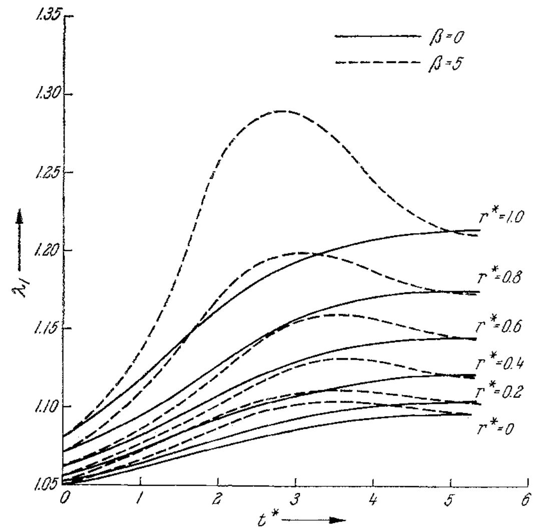

Fig. 5. History of λ_1. $C_1 = 288$, $\gamma = 1/3$, $\beta = 0.5$

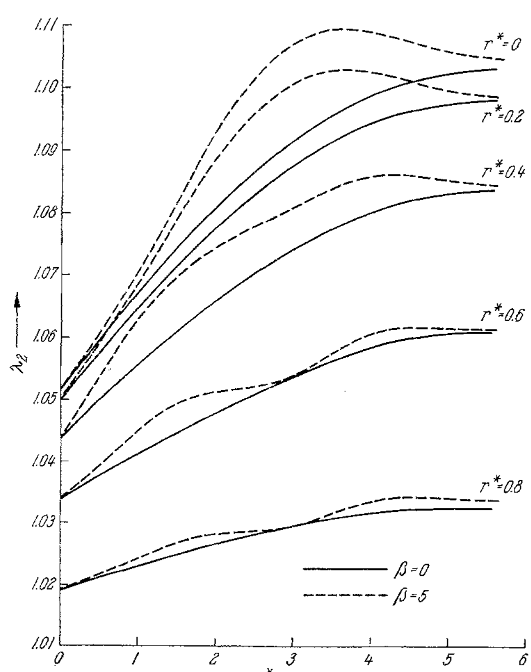

Fig. 6. History of λ_2. $C_1 = 288$, $\gamma = 1/3$, $\beta = 0.5$

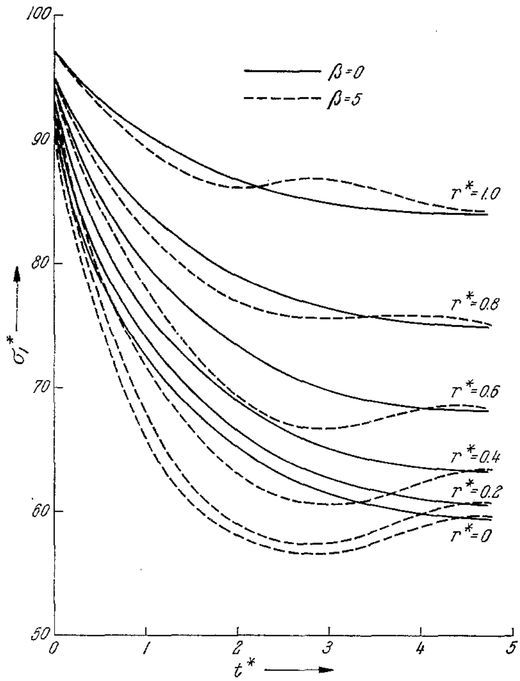

Fig. 7. History of σ_1*. $C_1 = 288$, $\gamma = 1/3$, $\beta = 0.5$

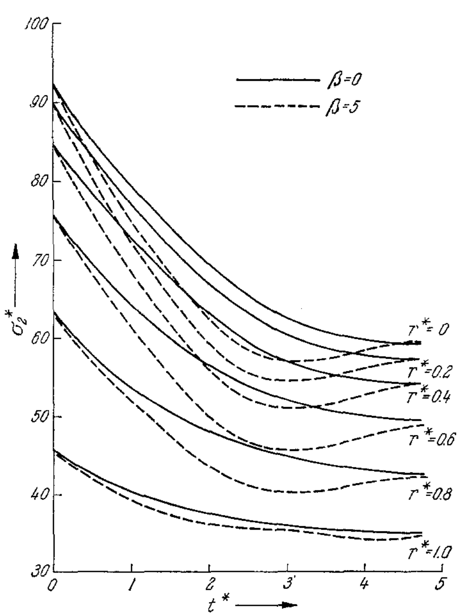

Fig. 8. History of σ_2*. $C_1 = 288$, $\gamma = 1/3$, $\beta = 0.5$

for particles at $r = 0\ (0.2)\ 1.0$ for $\beta = 0$ is compared with the highly non-linear viscoelastic case when $\beta = 5$. The value of β does not affect initial or final values of the variables (see also Figs. 5—8) but takes them towards their residual values at different rates depending on the size of β. For relatively large β the variables overshoot their residual values, a phenomenon which may be explained in the case of the sagging membrane by considering the re-distribution of weight as the membrane stretches. Any initial stretch at the centre puts more weight on the particles nearer the boundary. The amount of creep which follows is amplified by β through the form of $\hat{p}[I_1]$, (Eq. (25)), and the viscoelastic effects at the outer particles of the membrane work themselves out rapidly, leaving only an elastic component to affect the final deformation which, for $\beta = 5$, is a contraction across the entire membrane.

For the membrane problem considered here the overshoot phenomenon presents no instability problems since all stresses are decreasing for most of the time. In Wineman's spinning membrane problem [3] and Feng's toroidal membrane problem [5], however, the principal stress resultants increase and therefore could give rise to instability.

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